Asymptotic behaviour of a cylindrical elastic structure periodically reinforced along identical fibers

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Abstract

We describe the asymptotic behaviour of a cylindrical elastic body, reinforced along identical ε-periodically distributed fibers of size r_ε, with $0 < r_\varepsilon < \varepsilon$, filled in with some different elastic material, when this small parameter ε goes to 0. The case of small deformations and small strains is considered. We exhibit a critical size of the fibers and a critical link between the radius of the fibers and the size of the Lamé coefficients of the reinforcing elastic material. Epi-convergence arguments are used in order to prove this asymptotic behaviour. The proof is essentially based on the construction of appropriate test-functions.

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1 Introduction

The purpose of this work is to determine the asymptotic behaviour of an elastic material periodically reinforced by means of identical fibers filled in with some isotropic and homogeneous elastic material. In the first part, the fibers are longitudinally distributed inside the elastic material. The limit law is derived, studying the convergence of the elastic energy, and we exhibit a critical size of the fibers and a critical size of the Lamé coefficients

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of the reinforcing fibers. In the last part of this work, we suppose that the fibers are transversally distributed and we exhibit the limit law, which still involves a critical size and a critical size of the Lamé coefficients of the fibers, but working in a different limit functional space. These configurations intend to modelize, for example, the behaviour of a strap reinforced by means of identical fibers which are longitudinally or transversally disposed inside the strap.

Let ω be a bounded, smooth and open subset of $\mathbf{R}^2$ and $\Omega = \omega \times]0, L[\subset \mathbf{R}^3$, where L is positive. Γ_1 denotes the lower basis of Ω : $\Gamma_1 = \omega \times \{0\}$, Γ_2 its upper basis : $\Gamma_2 = \omega \times \{L\}$ and Σ its lateral surface : $\Sigma = \partial\omega \times]0, L[$.

Let ε be some positive real. In the first part of this work, we dispose inside Ω longitudinal fibers. More precisely, for every $k = (k_1, k_2)$ in $\mathbf{Z}^2$, we define the square : $Y_\varepsilon^k = (\varepsilon k_1, \varepsilon k_2) +]-\varepsilon/2, \varepsilon/2[^2$. Then we denote by Y_ε the union of all the ε-cells Y_ε^k included in ω : $Y_\varepsilon = \cup_{k \in K(\varepsilon)} Y_\varepsilon^k$. Choosing a parameter r_ε smaller than ε, we consider the disk D_ε^k of radius r_ε contained in Y_ε^k and the cylinder $T_\varepsilon^k = D_\varepsilon^k \times]0, L[$. T_ε denotes the union $\cup_k T_\varepsilon^k$ of the cylinders T_ε^k contained in Ω. Thus $\overline{T_\varepsilon} \cap \Sigma$ is empty. The total number of such cylinders contained in Ω (that is the cardinal of $K(\varepsilon)$) is equivalent to $|\omega| / \varepsilon^2$, with $|\omega| = area\,(\omega)$. The domain $\Omega_\varepsilon = \Omega \backslash \overline{T_\varepsilon}$ is supposed to be the reference configuration of some linear elastic, homogeneous and isotropic material, thus satisfying the following Hooke's law

$$\sigma_{ij}(u) = \lambda e_{mm}(u)\,\delta_{ij} + 2\mu e_{ij}(u), \qquad i, j, m = 1, 2, 3, \tag{1}$$

where the summation convention has been used with respect to repeated indices, λ and μ are the Lamé coefficients of the material, satisfying : $\mu > 0$ and $\lambda \geq 0$, δ_{ij} is Kronecker's symbol and $e(u)$ is the linearized deformation tensor, the components of which are given by : $e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$.

Figure 1: The domain Ω and the cylinders T_ε^k.

We suppose that T_ε is the reference configuration of some linear elastic, homogeneous and isotropic material satisfying Hooke's law

$$\sigma_{ij}^\varepsilon(u) = \lambda^\varepsilon e_{mm}(u)\,\delta_{ij} + 2\mu^\varepsilon e_{ij}(u), \qquad i, j, m = 1, 2, 3, \tag{2}$$

where the Lamé coefficients $\lambda^{\varepsilon} \geq 0$ and $\mu^{\varepsilon} > 0$ depend on ε and satisfy

$$\exists c > 0,\ \forall \varepsilon > 0 : \mu^{\varepsilon} \geq c. \tag{3}$$

The structure Ω built with these two elastic materials is submitted to some volumic forces the density of which $f = (f_1, f_2, f_3)$ belongs to $L^2(\Omega, \mathbf{R}^3)$. We suppose that the structure is held fixed along Γ_1 and that the tractions are equal to 0 on the rest of the boundary : $\sigma_{ij}(u^{\varepsilon}) n_j = 0$, $i, j = 1, 2, 3$, where n is the unit outer normal to the boundary. Let us introduce the functional F^{ε} defined on $H^1(\Omega, \mathbf{R}^3)$ by:

$$F^{\varepsilon}(u) = \begin{cases} \displaystyle\int_{\Omega_{\varepsilon}} \sigma_{ij}(u) e_{ij}(u)\, dx + \int_{T_{\varepsilon}} \sigma_{ij}^{\varepsilon}(u) e_{ij}(u)\, dx & \text{if } u \in H^1_{\Gamma_1}(\Omega, \mathbf{R}^3) \\ +\infty & \text{otherwise,} \end{cases} \tag{4}$$

with : $H^1_{\Gamma_1}(\Omega, \mathbf{R}^3) = \{u \in H^1(\Omega, \mathbf{R}^3) \mid u = 0 \text{ on } \Gamma_1\}$. The problem under consideration can be associated to the minimization problem involving the functional F^{ε}, as indicated in the following

Lemma 1 *1. The minimization problem:*

$$\min_{u \in H^1(\Omega, \mathbf{R}^3)} \left\{ F^{\varepsilon}(u) - 2\int_{\Omega} f.u dx \right\}, \tag{5}$$

admits a unique solution u^{ε} belonging to $H^1_{\Gamma_1}(\Omega, \mathbf{R}^3)$ and which satisfies the variational formulation:

$$\int_{\Omega_{\varepsilon}} \sigma_{ij}(u^{\varepsilon}) e_{ij}(u)\, dx + \int_{T_{\varepsilon}} \sigma_{ij}^{\varepsilon}(u^{\varepsilon}) e_{ij}(u)\, dx = \int_{\Omega} f.u dx, \quad \forall u \in H^1_{\Gamma_1}\left(\Omega, \mathbf{R}^3\right) \tag{6}$$

and is a weak solution of the problem:

$$\begin{cases} -\sigma_{ij,j}(u^{\varepsilon}) &= f_i & \text{in } \Omega_{\varepsilon} \\ -\sigma_{ij,j}^{\varepsilon}(u^{\varepsilon}) &= f_i & \text{in } T_{\varepsilon} \\ u^{\varepsilon} &= 0 & \text{on } \Gamma_1 \\ \sigma_{ij}(u^{\varepsilon}) n_j &= 0 & \text{on } \partial\Omega \setminus \Gamma_1. \end{cases} \tag{7}$$

2. *The sequence $(u^{\varepsilon})_{\varepsilon}$ is bounded in $H^1(\Omega, \mathbf{R}^3)$.*

3. *Assume that : $\sup_{\varepsilon}(-\varepsilon^2 \ln(r_{\varepsilon})) < +\infty$. Then, $\sup_{\varepsilon}\left(\left(\int_{T_{\varepsilon}} |u^{\varepsilon}|^2 dx\right) / |T_{\varepsilon}|\right)$ is finite and if $R^{\varepsilon}(u^{\varepsilon})$ is the rescaled restriction of u^{ε} to the fibers defined by:*

$$R^{\varepsilon}(u^{\varepsilon}) = \frac{|\Omega|}{|T_{\varepsilon}|} u^{\varepsilon} \mathbf{1}_{T_{\varepsilon}}, \tag{8}$$

where $|\Omega|$ means the volume of Ω and $\mathbf{1}_{T_{\varepsilon}}$ denotes the characteristic function of T_{ε}, the sequence $(R^{\varepsilon}(u^{\varepsilon}))_{\varepsilon}$ is bounded in $L^1(\mathbf{R}^3, \mathbf{R}^3)$.

Proof. *1.* Because λ^{ε} is nonnegative, we write for every u in $H^1_{\Gamma_1}(\Omega, \mathbf{R}^3)$

$$F^{\varepsilon}(u) \geq \inf(2\mu, 2\mu^{\varepsilon}) \int_{\Omega} e_{ij}(u)\, e_{ij}(u)\, dx \geq C \inf(2\mu, 2\mu^{\varepsilon}) \int_{\Omega} |\nabla u|^2 \, dx,$$

using the classical Korn's inequality, because u vanishes on Γ_1. The hypothesis (3) and this inequality imply that F^{ε} is coercive on $H^1(\Omega, \mathbf{R}^3)$. Moreover, F^{ε} is lower semi-continuous for the weak topology of $H^1_{\Gamma_1}(\Omega, \mathbf{R}^3)$ and is not identically equal to $+\infty$. Thus, classical convex analysis results imply the existence and the uniqueness of a minimizer u^{ε} of F^{ε} on $H^1_{\Gamma_1}(\Omega, \mathbf{R}^3)$, which satisfies the variational formulation (6) and, thus, is a weak solution of (7).

2. We observe that : $F^{\varepsilon}(u^{\varepsilon}) - 2\int_{\Omega} f.u^{\varepsilon} dx \leq F^{\varepsilon}(0) = 0$, which implies, using the preceding inequality, that ■

$$C \inf(2\mu, 2\mu^{\varepsilon}) \int_{\Omega} |\nabla u^{\varepsilon}|^2 \, dx \leq 2 \|f\|_{L^2(\Omega)} \|u^{\varepsilon}\|_{L^2(\Omega)}.$$

Using Poincaré's inequality, we thus deduce that $(u^{\varepsilon})_{\varepsilon}$ is bounded in $H^1_{\Gamma_1}(\Omega, \mathbf{R}^3)$.

3. Before proving this assertion, let us first recall the following estimate, which has been proved in [6]

Lemma 2 *There exists some positive constant C such that, for every u in $H^1(\Omega, \mathbf{R}^3)$, one has :*

$$\frac{1}{|T_{\varepsilon}|} \int_{T_{\varepsilon}} u^2 dx \leq C \left(\int_{\Omega} |\nabla u|^2 \, dx - \varepsilon^2 \ln(r_{\varepsilon}) + \varepsilon^2 \right). \tag{9}$$

Proof. We first define : $u'(r, \theta, z) := u(\varepsilon k_1 + r\cos(\theta), \varepsilon k_2 + r\cos(\theta), z)$, in the fiber centred at $(\varepsilon k_1, \varepsilon k_2)$. Then, we observe that, for every $r_1 \leq r_2 < \varepsilon/2$

$$u'(r_2, \theta, z) - u'(r_1, \theta, z) = (r_2 - r_1) \int_0^1 \frac{\partial u'}{\partial r}((1-t) r_1 + t r_2) \frac{\sqrt{(1-t) r_1 + t r_2}}{\sqrt{(1-t) r_1 + t r_2}} dt$$

$$\Rightarrow (u'(r_2, \theta, z) - u'(r_1, \theta, z))^2 \leq (\ln(r_2) - \ln(r_1)) \int_{r_1}^{r_2} \left(\frac{\partial u'}{\partial r} \right)^2 r dr.$$

Defining : $f(r) = \sum_{k \in K(\varepsilon)} \int_0^L \int_0^{2\pi} (u')^2 (r, \theta, z)\, d\theta dz$, the previous inequality implies : $f(r_1) \leq 2 f(r_2) + 2 \|\nabla u\|^2_{L^2(\Omega, \mathbf{R}^3)} \ln(r_2/r_1)$, which implies, for every r_2 in $[\varepsilon/4, \varepsilon/2]$

$$\begin{aligned}
\frac{1}{|T_{\varepsilon}|} \int_{T_{\varepsilon}} u^2 dx &= \frac{1}{|T_{\varepsilon}|} \int_0^{r_{\varepsilon}} f(r)\, r dr \\
&\leq \frac{2}{|T_{\varepsilon}|} \int_0^{r_{\varepsilon}} \left(f(r_2) + \|\nabla u\|^2_{L^2(\Omega, \mathbf{R}^3)} (\ln(r_2) - \ln(r)) \right) r dr \\
&\leq \frac{C\varepsilon^2}{(r_{\varepsilon})^2} \left(f(r_2)(r_{\varepsilon})^2 + \|\nabla u\|^2_{L^2(\Omega, \mathbf{R}^3)} \left((r_{\varepsilon})^2 - \frac{(r_{\varepsilon})^2}{2} \ln(r_{\varepsilon}) + \frac{(r_{\varepsilon})^2}{4} \right) \right) \\
&\leq C \left(f(r_2)\varepsilon^2 + \|\nabla u\|^2_{L^2(\Omega, \mathbf{R}^3)} \varepsilon^2 - \frac{\varepsilon^2}{2} \ln(r_{\varepsilon}) + \frac{\varepsilon^2}{4} \right) \\
&\leq C \left(4 f(r_2)\varepsilon r_2 + \|\nabla u\|^2_{L^2(\Omega, \mathbf{R}^3)} \varepsilon^2 - \frac{\varepsilon^2}{2} \ln(r_{\varepsilon}) + \frac{\varepsilon^2}{4} \right)
\end{aligned}$$

and then, taking the mean value of this inequality with respect to r_2 in $[\varepsilon/4, \varepsilon/2]$ ■

$$\begin{aligned}\frac{1}{|T_\varepsilon|}\int_{T_\varepsilon} u^2 dx &\leq C\left(16\int_{\varepsilon/4}^{\varepsilon/2} f(r)\, r dr + \|\nabla u\|^2_{L^2(\Omega,\mathbf{R}^3)}\varepsilon^2 - \frac{\varepsilon^2}{2}\ln(r_\varepsilon) + \frac{\varepsilon^2}{4}\right)\\ &\leq C\left((16+\varepsilon^2)\|\nabla u\|^2_{L^2(\Omega,\mathbf{R}^3)} - \frac{\varepsilon^2}{2}\ln(r_\varepsilon) + \frac{\varepsilon^2}{4}\right). \qquad \square\end{aligned}$$

Coming back to the proof of Lemma 1, we observe that Lemma 2 implies that $\sup_\varepsilon\left(\left(\int_{T_\varepsilon}|u^\varepsilon|^2 dx\right)/|T_\varepsilon|\right)$ is finite, as soon as $\sup_\varepsilon(-\varepsilon^2\ln(r_\varepsilon)) < +\infty$. Then, using Cauchy-Schwarz inequality, we finally prove that the quantity $\left(\int_{\mathbf{R}^3}|R^\varepsilon(u^\varepsilon)|\, dx\right)_\varepsilon$ is bounded, which ends the proof of Lemma 1. □

In the sequel, we will assume that the hypothesis $\sup_\varepsilon(-\varepsilon^2\ln(r_\varepsilon)) < +\infty$ is always satisfied.

Our purpose is to describe the asymptotic behaviour of $(u^\varepsilon)_\varepsilon$ and that of $(R^\varepsilon(u^\varepsilon))_\varepsilon$, when ε goes to 0. This will be obtained using epi-convergence arguments, that is studying the asymptotic behaviour of the sequence $(F^\varepsilon)_\varepsilon$, when ε goes to 0. We will first suppose that the coefficients λ_o and μ_o, defined by

$$\lambda_o = \lim_{\varepsilon\to 0}\frac{\lambda^\varepsilon(r_\varepsilon)^2}{\varepsilon^2},\ \mu_o = \lim_{\varepsilon\to 0}\frac{\mu^\varepsilon(r_\varepsilon)^2}{\varepsilon^2}. \tag{10}$$

are finite and μ_o is positive. Thanks to the properties of the epi-convergence, we then derive the asymptotic behaviour of the solution in many other cases.

This kind of reinforcement problems follows earlier works like [2], [3], [6], for example. However, the works [2] and [3] were dealing with scalar problems (also involving the p-laplacian operator). The work [6] is dealing with linear elasticity problems but assuming another scaling of the coefficients, which will be described later on in the present work. The work [4] deals with the homogenization of composite media evoking the vectorial case. See also [5] for similar phenomena in a quite general situation.

2 Construction and study of the test-functions

We define

$$\begin{aligned} D &= \left\{(y_1,y_2)\in\mathbf{R}^2 \mid (y_1)^2+(y_2)^2<1\right\}\\ D(r,r') &= \left\{(y_1,y_2)\in\mathbf{R}^2 \mid r^2<(y_1)^2+(y_2)^2<r'^2\right\}\\ S_r &= \left\{(y_1,y_2)\in\mathbf{R}^2 \mid (y_1)^2+(y_2)^2=r^2\right\}\end{aligned}$$

for $0<r<r'$, and for every $k=(k_1,k_2)$ in $\mathbf{Z}^2$

$$\begin{aligned} B^k_\varepsilon &= \left\{(x_1,x_2,x_3) \mid (x_1-k_1\varepsilon)^2+(x_2-k_2\varepsilon)^2<(s_\varepsilon)^2,\ x_3\in\,]0,L[\right\}\\ C^k_\varepsilon &= \left\{(x_1,x_2,x_3) \mid (r_\varepsilon)^2<(x_1-k_1\varepsilon)^2+(x_2-k_2\varepsilon)^2<(s_\varepsilon)^2,\ x_3\in\,]0,L[\right\},\end{aligned}$$

choosing s_ε such that

$$\lim_{\varepsilon\to0}\frac{s_\varepsilon}{\varepsilon}=0=\lim_{\varepsilon\to0}\frac{r_\varepsilon}{s_\varepsilon}=0=\lim_{\varepsilon\to0}\varepsilon^2\ln^2 s_\varepsilon.$$

Finally, we denote: $B_\varepsilon=\cup_k B_\varepsilon^k$, $C_\varepsilon=\cup_k C_\varepsilon^k$.

We introduce the solution $w^m=(w_1^m,w_2^m)$, $m=1,2$, of the linear plane elasticity problems

$$\left\{\begin{array}{rcll}\sigma_{ij,j}\left(w^m\right)(y) & = & 0 & \forall y\in\mathbf{R}^2\backslash\overline{D},\ i,\ j=1,2\\ w^m(y) & = & 0 & \text{on } S_1\\ w_m^m(y) & \simeq & -\ln|y|+Cte & \text{when } |y|\to\infty\\ \left|w_p^m\right|(y) & \leq & Cte & \text{when } \left\{\begin{array}{l}p=2 \text{ if } m=1\\ p=1 \text{ if } m=2,\end{array}\right.\end{array}\right. \tag{11}$$

where: $\sigma_{ij}\left(w^m\right)=\lambda e_{ij}\left(w^m\right)+2\mu e_{ij}\left(w^m\right)$. Thanks to the potential theory methods, described for example in [7], the solution w^m of (11) can be computed as

$$\left\{\begin{array}{rcl}w_1^1(y_1,y_2) & = & -\ln|y|+\dfrac{(y_2)^2-(y_1)^2}{2\kappa|y|^2}-\dfrac{(y_2)^2-(y_1)^2}{2\kappa|y|^4}\\ w_2^1(y_1,y_2) & = & \dfrac{y_2y_1}{\kappa|y|^2}-\dfrac{y_2y_1}{\kappa|y|^4}\\ w_1^2(y_1,y_2) & = & \dfrac{y_2y_1}{\kappa|y|^2}-\dfrac{y_2y_1}{|y|^4}\\ w_2^2(y_1,y_2) & = & -\ln|y|-\dfrac{(y_2)^2-(y_1)^2}{2\kappa|y|^2}+\dfrac{(y_2)^2-(y_1)^2}{2\kappa|y|^4},\end{array}\right.$$

with : $\kappa=(\lambda+3\mu)/(\lambda+\mu)$. We also introduce the function $w(y_1,y_2)=-\ln|y|$, which is harmonic in $\mathbf{R}^2\setminus\{0\}$ and verifies the following properties

$$w_{|S_1}=0,\ \lim_{|y|\to\infty}\frac{w(y_1,y_2)}{\ln|y|}=-1,\ \int_{S_1}\frac{\partial w}{\partial n}d\sigma=2\pi.$$

Let us observe that

Lemma 3 *One has the following convergences:*

1. $\displaystyle\lim_{R\to+\infty}\frac{1}{\ln R}\int_{D(1,R)}\sigma_{ij}\left(w^m\right)e_{ij}\left(w^l\right)dy=\frac{2\pi\mu(1+\kappa)}{\kappa}\delta_{lm}.$

2. $\displaystyle\lim_{R\to+\infty}\frac{1}{\ln R}\int_{D(1,R)}|\nabla w|^2dy=2\pi,$

Proof. The proof is trivial. □ ■

Using the solutions of these plane problems, we now build the functions w_ε^{mk}, for every $k=(k_1,k_2)$ as

$$
w_\varepsilon^{\alpha k}(x_1,x_2) = \frac{-1}{\ln r_\varepsilon}\begin{pmatrix} w_1^\alpha\left(\dfrac{x_1-k_1\varepsilon}{r_\varepsilon},\dfrac{x_2-k_2\varepsilon}{r_\varepsilon}\right) \\ w_2^\alpha\left(\dfrac{x_1-k_1\varepsilon}{r_\varepsilon},\dfrac{x_2-k_2\varepsilon}{r_\varepsilon}\right) \\ 0 \end{pmatrix}
$$

$$
w_\varepsilon^{3k}(x_1,x_2) = \frac{-1}{\ln r_\varepsilon}\begin{pmatrix} 0 \\ 0 \\ w\left(\dfrac{x_1-k_1\varepsilon}{r_\varepsilon},\dfrac{x_2-k_2\varepsilon}{r_\varepsilon}\right) \end{pmatrix},
$$

$\alpha = 1$, 2. These functions w_ε^{mk} satisfy the following properties.

Lemma 4 *There exist two positive constants C_0 and C_1, independant of ε, such that:*

1. $\left|e_m - w_\varepsilon^{mk}\right|^2 \le C_0 \dfrac{\ln^2\left(R_\varepsilon^k\right)+1}{\ln^2(r_\varepsilon)}$, *in* B_ε^k,

2. $\left|\dfrac{\partial w_\varepsilon^{mk}}{\partial x_i}\right|^2 \le \dfrac{C_1}{(R_\varepsilon^k)^2\ln^2(r_\varepsilon)}$, *in* B_ε^k, $i = 1,2,3$,

where e_m is the m-th vector of the canonical basis of $\mathbf{R}^3$ and

$$
\left(R_\varepsilon^k\right)^2 = (x_1-k_1\varepsilon)^2 + (x_2-k_2\varepsilon)^2 .
$$

Proof. Immediate, thanks to the expression of w_ε^{mk}. □ ■

Lemma 5 *If $\gamma := \lim_{\varepsilon\to 0}\left(-1/\left(\varepsilon^2 \ln r_\varepsilon\right)\right)$ is finite, then:*

1. *For every m and l, one has*

$$
\lim_{\varepsilon\to 0}\int_{B_\varepsilon}\sigma_{ij}\left(w_\varepsilon^{mk}\right)e_{ij}\left(w_\varepsilon^{lk}\right)dx = \begin{cases} \dfrac{2\pi\gamma\mu(1+\kappa)}{\kappa}\left|\Omega\right|\delta_{lm} & m,l=1,2 \\ 0 & l=3,\ m=1,2 \\ 2\pi\gamma\mu\left|\Omega\right| & m,l=3. \end{cases}
$$

2. *Let φ be any element of $C^1\left(\overline{\Omega}\right)$. Then*

$$
\lim_{\varepsilon\to 0}\int_{B_\varepsilon}\sigma_{ij}\left(w_\varepsilon^{mk}\right)e_{ij}\left(w_\varepsilon^{lk}\right)\varphi dx = \begin{cases} \dfrac{2\pi\gamma\mu(1+\kappa)}{\kappa}\delta_{lm}\displaystyle\int_\Omega \varphi dx & m,l=1,2 \\ 0 & l=3,\ m=1,2 \\ 2\pi\gamma\mu\displaystyle\int_\Omega \varphi dx & m,l=3. \end{cases}
$$

3. Let φ_ε^k be the truncation function defined by

$$\varphi_\varepsilon^k(x) = \varphi_\varepsilon^k(x_1, x_2) = \begin{cases} \dfrac{-4}{3(s_\varepsilon)^2}\left(\left(R_\varepsilon^k\right)^2 - (s_\varepsilon)^2\right) & \text{if } \dfrac{s_\varepsilon}{2} \le R_\varepsilon^k \le s_\varepsilon \\ 1 & \text{if } R_\varepsilon^k \le \dfrac{s_\varepsilon}{2} \\ 0 & \text{if } R_\varepsilon^k \ge s_\varepsilon \end{cases} \tag{12}$$

and z_ε^m the function defined by

$$z_\varepsilon^m(x) = \begin{cases} \varphi_\varepsilon^k(x)\left(e_m - w_\varepsilon^{mk}\right)(x) & \forall x \in B_\varepsilon^k, \forall k \\ 0 & \forall x \in \Omega \backslash \overline{B_\varepsilon}. \end{cases} \tag{13}$$

Then $(z_\varepsilon^m)_{|T_\varepsilon} = e_m$, $(z_\varepsilon^m)_\varepsilon$ converges to 0 in the weak topology of $H^1(\Omega, \mathbf{R}^3)$ and

$$\lim_{\varepsilon\to 0}\int_\Omega \sigma_{ij}(z_\varepsilon^m)\, e_{ij}\left(z_\varepsilon^l\right) dx = \begin{cases} \dfrac{2\pi\gamma\mu(1+\kappa)}{\kappa}|\Omega|\,\delta_{lm} & \text{if } m,l = 1,2 \\ 0 & \text{if } l = 3,\ m = 1,2 \\ 2\pi\gamma\mu|\Omega| & \text{if } m,l = 3. \end{cases}$$

Proof. *1.* Using Hooke's law, the above expression of w_ε^{mk} and the estimates given in Lemma 4, one has, for $m, l = 1, 3$

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} \sigma_{ij}\left(w_\varepsilon^{mk}\right) e_{ij}\left(w_\varepsilon^{lk}\right) dx = \frac{|\Omega|}{\varepsilon^2 \ln^2 r_\varepsilon}\int_{D(1, s_\varepsilon/r_\varepsilon)} \sigma_{ij}(w^m)\, e_{ij}\left(w^l\right) dy_1 dy_2 + o_\varepsilon,$$

where: $y_1 = (x_1 - k_1\varepsilon)/r_\varepsilon$, $y_2 = (x_2 - k\varepsilon)/r_\varepsilon$, σ_{ij} and e_{ij} respectively denote the stress and the deformation tensors in the plane, with the Lamé coefficients λ and μ and $\lim_{\varepsilon\to 0} o_\varepsilon = 0$. One deduces from Lemma 3, through the definition of s_ε that

$$\lim_{\varepsilon\to 0}\frac{-1}{\ln r_\varepsilon}\int_{D(1, s_\varepsilon/r_\varepsilon)} \sigma_{ij}(w^m)\, e_{ij}\left(w^l\right) dy_1 dy_2 = \frac{2\pi\mu(1+\kappa)}{\kappa}\delta_{ml},$$

the other cases being treated in a similar way. We conclude, using the definition of γ.
2. The smoothness of φ implies that for every (x_1, x_2, x_3) in C_ε^k we have : $\varphi(x_1, x_2, x_3) = \varphi(k_1\varepsilon, k_2\varepsilon, x_3) + O\left(R_\varepsilon^k\right)$, which implies

$$\begin{aligned} &\int_{C_\varepsilon} \sigma_{ij}\left(w_\varepsilon^{mk}\right) e_{ij}\left(w_\varepsilon^{lk}\right) \varphi dx \\ &= \frac{1}{\varepsilon^2 \ln^2 r_\varepsilon}\left(\int_{D(1, s_\varepsilon/r_\varepsilon)} \sigma_{ij}(w^m)\, e_{ij}\left(w^l\right) dy_1 dy_2 \left(\sum_k \varepsilon^2 \int_0^L \varphi(k_1\varepsilon, k_2\varepsilon, x_3)\, dx_3\right)\right) + o_\varepsilon. \end{aligned}$$

But the smoothness of φ also implies

$$\lim_{\varepsilon\to 0}\sum_k \varepsilon^2 \int_0^L \varphi(k_1\varepsilon, k_2\varepsilon, x_3)\, dx_3 = \int_\Omega \varphi dx,$$

from which we conclude, using the first assertion.
3. We observe that $\varphi_\varepsilon^k \equiv 0$ in $\Omega \setminus \overline{B_\varepsilon}$ and $w_\varepsilon^{mk} \equiv 0$ in T_ε. Then we compute

$$\begin{aligned}\int_\Omega \sigma_{ij}\left(z_\varepsilon^m\right) e_{ij}\left(z_\varepsilon^l\right) dx \; = \; & \sum_k \int_{C_\varepsilon^k} \sigma_{ij}\left(w_\varepsilon^{mk}\right) e_{ij}\left(w_\varepsilon^{lk}\right)\left(\varphi_\varepsilon^k\right)^2 dx \\ & -2\sum_k \int_{C_\varepsilon^k \cap \left\{s_\varepsilon/2<R_\varepsilon^k<s_\varepsilon\right\}} \sigma_{ij}\left(w_\varepsilon^{mk}\right) \frac{\partial \varphi_\varepsilon^k}{\partial x_i}\left(e_l - w_\varepsilon^{lk}\right)_j dx \\ & +\sum_k \int_{C_\varepsilon^k \cap \left\{s_\varepsilon/2<R_\varepsilon^k<s_\varepsilon\right\}} \left(e_m - w_\varepsilon^{mk}\right)_i \frac{\partial \varphi_\varepsilon^k}{\partial x_i}\left(e_l - w_\varepsilon^{lk}\right)_j \frac{\partial \varphi_\varepsilon^k}{\partial x_j} dx.\end{aligned}$$

Thanks to Lemma 4 and to the definition of φ_ε^k, one can prove that the two last sums are respectively bounded by : $C\left|\ln s_\varepsilon\right| / \left(\varepsilon^2 \ln^2 r_\varepsilon\right)$ and $C \ln^2 s_\varepsilon / \left(\varepsilon^2 \ln^2 r_\varepsilon\right)$. These two upper bounds converge to 0, because γ is finite and thanks to the choice of s_ε. Moreover, the first term of the preceding equality can be computed as

$$\begin{aligned}\int_{C_\varepsilon^k} \sigma_{ij}\left(w_\varepsilon^{mk}\right) e_{ij}\left(w_\varepsilon^{lk}\right)\left(\varphi_\varepsilon^k\right)^2 dx = & \int_{C_\varepsilon^k} \sigma_{ij}\left(w_\varepsilon^{mk}\right) e_{ij}\left(w_\varepsilon^{lk}\right) dx \\ & + \int_{C_\varepsilon^k \cap \left\{s_\varepsilon/2<R_\varepsilon^k<s_\varepsilon\right\}} \sigma_{ij}\left(w_\varepsilon^{mk}\right) e_{ij}\left(w_\varepsilon^{lk}\right)\left(\left(\varphi_\varepsilon^k\right)^2 - 1\right) dx\end{aligned}$$

and using the definition (12) of φ_ε^k we get

$$\begin{aligned}& \left| \int_{C_\varepsilon^k \cap \left\{s_\varepsilon/2<R_\varepsilon^k<s_\varepsilon\right\}} \sigma_{ij}\left(w_\varepsilon^{mk}\right) e_{ij}\left(w_\varepsilon^{lk}\right)\left(\left(\varphi_\varepsilon^k\right)^2 - 1\right) dx \right| \\ & \qquad \leq \int_{C_\varepsilon^k \cap \left\{s_\varepsilon/2<R_\varepsilon^k<s_\varepsilon\right\}} \sigma_{ij}\left(w_\varepsilon^{mk}\right) e_{ij}\left(w_\varepsilon^{lk}\right) dx.\end{aligned}$$

Thanks to the estimates of Lemma 4, we deduce

$$\lim_{\varepsilon \to 0} \sum_k \int_{C_\varepsilon^k \cap \left\{s_\varepsilon/2<R_\varepsilon^k<s_\varepsilon\right\}} \sigma_{ij}\left(w_\varepsilon^{mk}\right) e_{ij}\left(w_\varepsilon^{lk}\right) dx = 0,$$

which implies

$$\lim_{\varepsilon \to 0} \int_\Omega \sigma_{ij}\left(z_\varepsilon^m\right) e_{ij}\left(z_\varepsilon^l\right) dx = \lim_{\varepsilon \to 0} \int_{C_\varepsilon} \sigma_{ij}\left(w_\varepsilon^{mk}\right) e_{ij}\left(w_\varepsilon^{lk}\right) dx.$$

One concludes using the first assertion. Because $\left(z_\varepsilon^m\right)_{|\Gamma_1} = 0$, there exists some positive constant C such that

$$\int_\Omega \left|\nabla z_\varepsilon^m\right|^2 dx \leq C \int_\Omega \sigma_{ij}\left(z_\varepsilon^m\right) e_{ij}\left(z_\varepsilon^m\right) dx.$$

Hence $\left(z_\varepsilon^m\right)_\varepsilon$ is bounded in $H^1\left(\Omega, \mathbf{R}^3\right)$, which implies that a subsequence still denoted $\left(z_\varepsilon^m\right)_\varepsilon$ converges to some z^* in the weak topology of $H^1\left(\Omega, \mathbf{R}^3\right)$ and in the strong topology of $L^2\left(\Omega, \mathbf{R}^3\right)$. We observe that $z_\varepsilon^m = 0$ in $\Omega \backslash \overline{B_\varepsilon}$ and because the sequence of characteristic functions of $\Omega \backslash \overline{B_\varepsilon}$ converges to 1 in the strong topology of $L^2(\Omega)$, we infer that $z^* = 0$. Hence $\left(z_\varepsilon^m\right)_\varepsilon$ converges to 0 in the weak topology of $H^1\left(\Omega, \mathbf{R}^3\right)$. □ ■

3 Convergence

We define the topology τ which will be used throughout this paragraph as

$$u_\varepsilon \underset{\varepsilon\to 0}{\overset{\tau}{\rightharpoonup}} (u,v) \Leftrightarrow \left\{ \begin{array}{l} u_\varepsilon \underset{\varepsilon\to 0}{\overset{w\text{-}H^1(\Omega,\mathbf{R}^3)}{\rightharpoonup}} u \\ \text{and} : \forall\varphi\in C_0^0(\mathbf{R}^3) : \displaystyle\int_\Omega R^\varepsilon(u_\varepsilon)\varphi dx \underset{\varepsilon\to 0}{\to} \int_\Omega v\varphi dx, \end{array}\right.$$

where $w\text{-}H^1(\Omega,\mathbf{R}^3)$ stands for the weak topology of $H^1(\Omega,\mathbf{R}^3)$ and R^ε is the rescaled restriction operator defined in (8).

Our main result reads as follows

Theorem 6 *Suppose that $\gamma = \lim_{\varepsilon\to 0}(-1/(\varepsilon^2 \ln r_\varepsilon))$ is finite, λ_o and μ_o are finite and μ_o is positive. Then, the sequence $(F^\varepsilon)_\varepsilon$ epi-converges in the topology τ to the functional F^o defined on $H^1(\Omega,\mathbf{R}^3)\times L^1(\Omega,\mathbf{R}^3)$ by:*

$$F^o(u,v) = \left\{ \begin{array}{ll} \displaystyle\int_\Omega \sigma_{ij}(u)e_{ij}(u)dx + 2\pi\gamma\int_\Omega (v-u)^t A(v-u)dx + \pi E_o\int_\Omega (e_{33}(v))^2 dx, & \\ \qquad\qquad\qquad\qquad\qquad \text{if } (u,v)\in H^1_{\Gamma_1}(\Omega,\mathbf{R}^3)\times V & \\ +\infty \qquad\qquad\qquad\qquad\qquad \text{otherwise,} & \end{array}\right. \tag{14}$$

using the summation convention with respect to repeated indices and where A is the diagonal matrix with : $A_{11} = \mu(1+\kappa)/\kappa = A_{22}$ and $A_{33} = \mu$, where $\kappa = (\lambda+3\mu)/(\lambda+\mu)$, $E_o = \mu_o(3\lambda_o + 2\mu_o)/(\lambda_o+\mu_o)$ and V denotes the subspace

$$V = \left\{ v\in L^2\left(\Omega,\mathbf{R}^3\right) \mid v_{3|\Gamma_1} = 0,\ e_{33}(v)\in L^2(\Omega)\right\}.$$

As a consequence of this theorem and of the properties of the epi-convergence (see [1] for a definition and the main properties of this notion of convergence well-fitted to the description of the asymptotic behaviour of the solution of minimization problems), one gets the following asymptotic behaviour, when ε goes to 0, of the solution u^ε of (5)

Corollary 7 *Under the hypotheses of Theorem 6, the solution u^ε of (5) converges, in the topology τ, to the solution (u^o,v^o) in the space $H^1_{\Gamma_1}(\Omega,\mathbf{R}^3)\times V$ of the following problem*

$$\left\{ \begin{array}{rcll} -\sigma_{ij,j}(u^o) - 2\gamma\pi A_{ij}(v^o-u^o)_j & = & f_i & \text{in } \Omega,\ i=1,2,3 \\ u^o & = & 0 & \text{on } \Gamma_1 \\ \sigma_{ij}(u^o)n_j & = & 0 & \text{on } \partial\omega\times]0,L[\cup\Gamma_2 \\ & & & \quad i,j=1,2,3 \\ E_o\dfrac{\partial}{\partial x_3}(e_{33}(v^o)) & = & 2\gamma\mu(v^o-u^o)_3 & \text{in } \Omega \\ v^o & = & 0 & \text{on } \Gamma_1 \\ (u^o)_\alpha & = & (v^o)_\alpha & \text{in } \Omega,\ \alpha=1,2 \\ e_{33}(v^o) & = & 0 & \text{on } \Gamma_2. \end{array}\right. \tag{15}$$

(u^o, v^o) *is the unique solution of the minimization problem*

$$\min\left\{F^o(u,v) - 2\int_\Omega f.udx \mid u \in H^1_{\Gamma_1}\left(\Omega, \mathbf{R}^3\right),\ v \in V\right\}.$$

Moreover, the convergence of the linked energies : $\lim_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) = F^o(u^o, v^o)$ *holds true.*

Remark 8 *In the expression of the limit functional F^o, the term $\pi E_o \int_\Omega (e_{33}(v))^2 dx$ can be interpreted as the "pure influence" of the fibers, due to their longitudinal repartition, on the asymptotic behaviour. The term $2\pi\gamma \int_\Omega (v-u)^t A(v-u)\,dx$ can be interpreted as the mixed influence of the fibers and of the elastic material (for example, shearing effect of the fibers on the material, for the term $2\pi\gamma\mu \int_\Omega (v_3 - u_3)^2 dx$).*

of Theorem 6. This proof will be decomposed in two main parts, corresponding to the verification of the two assertions of the epi-convergence. As a first step, let us verify : *For every u in $H^1_{\Gamma_1}(\Omega, \mathbf{R}^3)$ and for every v in V, there exists a sequence $(u^o_\varepsilon)_\varepsilon$ of elements of $H^1_{\Gamma_1}(\Omega, \mathbf{R}^3)$ converging to (u,v) in the topology τ and such that :* $\limsup_{\varepsilon\to 0} F^\varepsilon(u^o_\varepsilon) \leq F^o(u,v)$.

Let us first choose any element u of $C^1\left(\overline{\Omega}, \mathbf{R}^3\right) \cap H^1_{\Gamma_1}(\Omega, \mathbf{R}^3)$ and any element v of $C^2\left(\overline{\Omega}, \mathbf{R}^3\right) \cap V$. For every $k = (k_1, k_2)$, we define the function $\mathcal{R}_\varepsilon(v)$ in B^k_ε by its three components as follows:

$$\begin{cases} (\mathcal{R}_\varepsilon(v))_\alpha(x_1,x_2,x_3) &= v_\alpha(k_1\varepsilon, k_2\varepsilon, x_3) \\ & \quad -\dfrac{\lambda^\varepsilon}{2(\mu^\varepsilon+\lambda^\varepsilon)}(x_\alpha - k_\alpha\varepsilon)\dfrac{\partial v_3}{\partial x_3}(k_1\varepsilon, k_2\varepsilon, x_3) \\ (\mathcal{R}_\varepsilon(v))_3(x_1,x_2,x_3) &= v_3(k_1\varepsilon, k_2\varepsilon, x_3) - (x_1 - k_1\varepsilon)\dfrac{\partial v_1}{\partial x_3}(k_1\varepsilon, k_2\varepsilon, x_3) \\ & \quad -(x_2 - k_2\varepsilon)\dfrac{\partial v_2}{\partial x_3}(k_1\varepsilon, k_2\varepsilon, x_3). \end{cases}$$

■

Let us choose some smooth function ψ_ε identically equal to 1 (resp. to 0) in $\Omega \setminus \overline{\Sigma_{2\varepsilon}}$ (resp. in Σ_ε), with : $\Sigma_\varepsilon = \{x \in \Omega \mid d(x, \Gamma_1) < \varepsilon\}$. We define:

$$\begin{aligned} u^o_\varepsilon &= (1-\psi_\varepsilon)u + \psi_\varepsilon\left((e_m - z^m_\varepsilon)u_m + z^m_\varepsilon(\mathcal{R}_\varepsilon(v))_m\right) \\ &= u - \psi_\varepsilon z^m_\varepsilon\left(u_m - (\mathcal{R}_\varepsilon(v))_m\right), \end{aligned} \tag{16}$$

where u_m and $(\mathcal{R}_\varepsilon(v))_m$ are the m-th components of u and $\mathcal{R}_\varepsilon(v)$ in the canonical basis $(e_m)_{m=1,2,3}$ of $\mathbf{R}^3$ and z^m_ε is defined in (13). One has the following estimates.

Lemma 9 *1. There exists some positive constant C independant of ε such that*

$$\begin{aligned} |u^o_\varepsilon|(x) &\leq C & \forall x \in \Omega \\ |\nabla\mathcal{R}_\varepsilon(v)|(x) &\leq C & \forall x \in B_\varepsilon \\ |\mathcal{R}_\varepsilon(v) - v|(x) &\leq Cr_\varepsilon & \forall x \in T_\varepsilon \\ |\mathcal{R}_\varepsilon(v) - v|(x) &\leq Cs_\varepsilon & \forall x \in B_\varepsilon. \end{aligned}$$

2. u_ε^o *belongs to* $H^1_{\Gamma_1}(\Omega, \mathbf{R}^3)$, $(u_\varepsilon^o)_\varepsilon$ *converges to* (u,v) *in the above defined topology* τ.

Proof. *1.* Because v belongs to $L^\infty(\Omega, \mathbf{R}^3)$, together with its first order derivatives, we get, in every B_ε^k : $|\mathcal{R}_\varepsilon(v)| \le C$ and $|\nabla \mathcal{R}_\varepsilon(v)| \le C'$, where C and C' are positive constants. Using Lemma 4, we get : $|u_\varepsilon^o| \le C$, in Ω. One has, for every $k = (k_1, k_2)$

$$\begin{aligned} \left|(\mathcal{R}_\varepsilon(v) - v)_\alpha\right|_{|T_\varepsilon^k} &\le |v_\alpha(k_1\varepsilon, k_2\varepsilon, x_3) - v_\alpha(x_1,x_2,x_3)| \\ &\quad + \frac{\lambda^\varepsilon}{2(\mu^\varepsilon + \lambda^\varepsilon)} \left|(x_\alpha - k_\alpha\varepsilon)\frac{\partial v_3}{\partial x_3}(k_1\varepsilon, k_2\varepsilon, x_3)\right| \\ &\le Cr_\varepsilon, \end{aligned}$$

because v belongs to $C^1\left(\overline{\Omega}, \mathbf{R}^3\right)$ and using the hypotheses on λ^ε and μ^ε. Similarly, we have : $\left|(\mathcal{R}_\varepsilon(v) - v)_3\right|_{|T_\varepsilon^k} \le Cr_\varepsilon$, and : $\left|\mathcal{R}_\varepsilon(v) - v\right|_{|B_\varepsilon^k} \le Cs_\varepsilon$, for every k.

2. Observe that u_ε^o belongs to $H^1_{\Gamma_1}(\Omega, \mathbf{R}^3)$ because u vanishes on Γ_1 and ψ_ε also vanishes on Γ_1. Furthermore, there exists some constant C_m such that one has in B_ε

$$\begin{aligned} |\nabla u_\varepsilon^o| &\le \left|\nabla u_m(e_m - z_\varepsilon^m) + z_\varepsilon^m \nabla\left(\mathcal{R}_\varepsilon(v)\right)_m + \left((\mathcal{R}_\varepsilon(v))_m - u_m\right)\nabla z_\varepsilon^m\right| \\ &\le C_m\left(|\nabla u_m| + \varepsilon\,|\nabla z_\varepsilon^m| + |\nabla z_\varepsilon^m|\,|v_m - u_m|\right), \end{aligned} \tag{17}$$

for some constant C_m, thanks to the preceding estimates. We then compute

$$\int_\Omega |\nabla u_\varepsilon^o|^2\,dx = \int_{\Omega\setminus\overline{B_\varepsilon}} |\nabla u_\varepsilon^o|^2\,dx + \int_{B_\varepsilon} |\nabla u_\varepsilon^o|^2\,dx \tag{18}$$

Thanks to (17) and to Lemma 5 one has

$$\begin{aligned} \int_{B_\varepsilon} |\nabla u_\varepsilon^o|^2\,dx &\le C'_m\left(\int_{B_\varepsilon} |\nabla u_m|^2\,dx + \varepsilon\int_{B_\varepsilon} |\nabla z_\varepsilon^m|^2\,dx + \int_{B_\varepsilon} |v_m - u_m|^2\,|\nabla z_\varepsilon^m|^2\,dx\right) \\ &\le C, \end{aligned}$$

where C is some positive constant independant of ε. Furthermore, because z_ε^m outside B_ε

$$\int_{\Omega\setminus\overline{B_\varepsilon}} |\nabla u_\varepsilon^o|^2\,dx \underset{\varepsilon\to 0}{\to} \int_\Omega |\nabla u|^2\,dx.$$

This proves that $(u_\varepsilon^o)_\varepsilon$ converges to u in the weak topology of $H^1(\Omega, \mathbf{R}^3)$. Let φ be any element of $C_0^1(\mathbf{R}^3, \mathbf{R}^3)$. We have, because : $(z_\varepsilon^m)_{|T_\varepsilon} = e_m$

$$\begin{aligned} \int_\Omega \varphi R^\varepsilon(u_\varepsilon^o)\,dx &= \frac{|\Omega|}{|T_\varepsilon|}\int_{T_\varepsilon} \varphi u_\varepsilon^o dx \\ &= \frac{|\Omega|}{|T_\varepsilon|}\int_{T_\varepsilon} \varphi \mathcal{R}_\varepsilon(v)\,dx \\ &= \frac{|\Omega|\left|T_\varepsilon^k \cap \omega\right|}{|T_\varepsilon|\,\varepsilon^2} \sum_k \varepsilon^2 \int_0^L \varphi(k_1\varepsilon, k_2\varepsilon, x_3)\, v(k_1\varepsilon, k_2\varepsilon, x_3)\,dx_3 + o_\varepsilon, \end{aligned}$$

φ and v being continuously differentiable and $\left|T_\varepsilon^k \cap \omega\right|$ being independant of k. We have, thanks to the smoothness of φ and v

$$\lim_{\varepsilon\to 0}\sum_k \varepsilon^2 \int_0^L \varphi\left(k_1\varepsilon, k_2\varepsilon, x_3\right) v\left(k_1\varepsilon, k_2\varepsilon, x_3\right) dx_3 = \int_\Omega \varphi v dx$$

and we observe that : $\lim_{\varepsilon\to 0}\left(|\Omega|\left|T_\varepsilon^k \cap \omega\right|\right) / \left(|T_\varepsilon| \varepsilon^2\right) = 1$. This proves that the sequence $\left(u_\varepsilon^o\right)_\varepsilon$ converges to (u, v) in the above defined topology τ. □ ■

For every u in $C^1\left(\overline{\Omega}, \mathbf{R}^3\right) \cap H^1_{\Gamma_1}\left(\Omega, \mathbf{R}^3\right)$ and every v in $C^1\left(\overline{\Omega}, \mathbf{R}^3\right)$, we compute

$$F^\varepsilon\left(u_\varepsilon^o\right) = \int_{\Omega\setminus\overline{C_\varepsilon\cup T_\varepsilon}} \sigma_{ij}(u)\, e_{ij}(u)\, dx + \int_{C_\varepsilon} \sigma_{ij}\left(u_\varepsilon^o\right) e_{ij}\left(u_\varepsilon^o\right) dx + \int_{T_\varepsilon} \sigma_{ij}^\varepsilon\left(\mathcal{R}_\varepsilon(v)\right) e_{ij}\left(\mathcal{R}_\varepsilon(v)\right) dx. \tag{19}$$

Because the characteristic function of $\Omega\setminus\overline{C_\varepsilon \cup T_\varepsilon}$ converges to 1 in the strong topology of $L^2(\Omega)$, the first integral of (19) immediately leads to

$$\lim_{\varepsilon\to 0}\int_{\Omega\setminus\overline{C_\varepsilon\cup T_\varepsilon}} \sigma_{ij}(u)\, e_{ij}(u)\, dx = \int_\Omega \sigma_{ij}(u)\, e_{ij}(u)\, dx. \tag{20}$$

Let us study the second integral of (19). One has, using the definition (16) of the test-function u_ε^o

$$\begin{aligned}\int_{C_\varepsilon} &\sigma_{ij}\left(u_\varepsilon^o\right) e_{ij}\left(u_\varepsilon^o\right) dx \\ &= \int_{C_\varepsilon} \sigma_{ij}(u)\, e_{ij}(u)\, dx + 2\int_{C_\varepsilon} \sigma_{ij}(u)\, e_{ij}\left(z_\varepsilon^m\left(\left(\mathcal{R}_\varepsilon(v)\right)_m - u_m\right)\right) dx \\ &\quad + \int_{C_\varepsilon} \sigma_{ij}\left(z_\varepsilon^m\left(\left(\mathcal{R}_\varepsilon(v)\right)_m - u_m\right)\right) e_{ij}\left(z_\varepsilon^l\left(\left(\mathcal{R}_\varepsilon(v)\right)_l - u_l\right)\right) dx.\end{aligned} \tag{21}$$

The second integral of the right hand side of (21) converges to 0, because $\left(z_\varepsilon^m\right)_\varepsilon$ converges to 0 in the weak topology of $H^1_{\Gamma_1}\left(\Omega, \mathbf{R}^3\right)$ and thanks to the estimates of Lemma 8. The third integral of this right hand side of (21) can be computed as

$$\begin{aligned}&\int_{C_\varepsilon} \sigma_{ij}\left(z_\varepsilon^m\left(v_m - u_m\right)\right) e_{ij}\left(z_\varepsilon^l\left(v_l - u_l\right)\right) dx \\ &\quad +2\int_{C_\varepsilon} \sigma_{ij}\left(z_\varepsilon^m\left(\left(\mathcal{R}_\varepsilon(v)\right)_m - u_m\right)\right) e_{ij}\left(z_\varepsilon^l\left(v_l - u_l\right)\right) dx \\ &\quad +\int_{C_\varepsilon} \sigma_{ij}\left(z_\varepsilon^m\left(\left(\mathcal{R}_\varepsilon(v)\right)_m - v_m\right)\right) e_{ij}\left(z_\varepsilon^l\left(\left(\mathcal{R}_\varepsilon(v)\right)_l - v_l\right)\right) dx.\end{aligned} \tag{22}$$

Thanks to Lemmas 5 and 9, the two last integrals of (22) converge to 0 and the first integral of (22) is equal to

$$\int_\Omega \sigma_{ij}\left(z_\varepsilon^m\right) e_{ij}\left(z_\varepsilon^l\right)\left(v_m - u_m\right)\left(v_l - u_l\right) dx + o_\varepsilon,$$

with $\lim_{\varepsilon\to 0} o_\varepsilon = 0$, because $(z_\varepsilon^m)_\varepsilon$ converges to 0 in the weak topology of $H^1_{\Gamma_1}(\Omega, \mathbf{R}^3)$. One deduces from Lemma 5 and the smoothness of u and v that

$$\lim_{\varepsilon\to 0}\int_\Omega \sigma_{ij}(z_\varepsilon^m)\, e_{ij}\left(z_\varepsilon^l\right)(v_m - u_m)(v_l - u_l)\, dx = 2\pi\gamma\int_\Omega (v-u)^t A\,(v-u)\, dx. \tag{23}$$

In order to study the third integral of (19), one observes that the above expression of $\mathcal{R}_\varepsilon(v)$ implies

$$\begin{aligned}
Tr\left(e\left(\mathcal{R}_\varepsilon(v)\right)\right) &= \frac{\mu^\varepsilon}{\mu^\varepsilon+\lambda^\varepsilon}\frac{\partial v_3}{\partial x_3}(k_1\varepsilon, k_2\varepsilon, x_3) - (x_\alpha - k_\alpha\varepsilon)\frac{\partial^2 v_\alpha}{\partial x_3^2}(k_1\varepsilon, k_2\varepsilon, x_3)\\
\sigma^\varepsilon_{11}\left(\mathcal{R}_\varepsilon(v)\right) &= -\lambda^\varepsilon(x_\alpha - k_\alpha\varepsilon)\frac{\partial^2 v_\alpha}{\partial x_3^2}(k_1\varepsilon, k_2\varepsilon, x_3)\\
\sigma^\varepsilon_{22}\left(\mathcal{R}_\varepsilon(v)\right) &= -\lambda^\varepsilon(x_\alpha - k_\alpha\varepsilon)\frac{\partial^2 v_\alpha}{\partial x_3^2}(k_1\varepsilon, k_2\varepsilon, x_3)\\
\sigma^\varepsilon_{12}\left(\mathcal{R}_\varepsilon(v)\right) &= 0\\
\sigma^\varepsilon_{33}\left(\mathcal{R}_\varepsilon(v)\right) &= \mu^\varepsilon\frac{2\mu^\varepsilon+3\lambda^\varepsilon}{\mu^\varepsilon+\lambda^\varepsilon}\frac{\partial v_3}{\partial x_3}(k_1\varepsilon, k_2\varepsilon, x_3)\\
&\qquad -(2\mu^\varepsilon+\lambda^\varepsilon)(x_\alpha - k_\alpha\varepsilon)\frac{\partial^2 v_\alpha}{\partial x_3^2}(k_1\varepsilon, k_2\varepsilon, x_3)\\
\sigma^\varepsilon_{\alpha 3}\left(\mathcal{R}_\varepsilon(v)\right) &= -\mu^\varepsilon(x_\alpha - k_\alpha\varepsilon)\frac{\partial^2 v_\alpha}{\partial x_3^2}(k_1\varepsilon, k_2\varepsilon, x_3).
\end{aligned}$$

One easily proves that all the terms of the third integral of (19) converge to 0 except the following one

$$\begin{aligned}
&\int_{T_\varepsilon}\sigma^\varepsilon_{33}\left(\mathcal{R}_\varepsilon(v)\right) e_{33}\left(\mathcal{R}_\varepsilon(v)\right) dx\\
&\qquad = \frac{\pi\mu^\varepsilon (r_\varepsilon)^2}{\varepsilon^2}\frac{2\mu^\varepsilon+3\lambda^\varepsilon}{\mu^\varepsilon+\lambda^\varepsilon}\sum_k \varepsilon^2\int_0^L\left(\frac{\partial v_3}{\partial x_3}\right)^2(k_1\varepsilon, k_2\varepsilon, x_3)\, dx_3 + o_\varepsilon\\
&\qquad \underset{\varepsilon\to 0}{\to} \pi E_o\int_\Omega\left(e_{33}(v)\right)^2 dx,
\end{aligned}$$

with the above definition of E_o. Thus, we get, for this third integral of (19)

$$\lim_{\varepsilon\to 0}\int_{T_\varepsilon}\sigma^\varepsilon_{ij}\left(\mathcal{R}_\varepsilon(v)\right) e_{ij}\left(\mathcal{R}_\varepsilon(v)\right) dx = \pi E_o\int_\Omega\left(e_{33}(v)\right)^2 dx. \tag{24}$$

From (20), (23) and (24), we thus derive : $\lim_{\varepsilon\to 0} F^\varepsilon(u^o_\varepsilon) = F^o(u,v)$.

We conclude the verification of this first assertion, using a density argument and the diagonalization argument contained in [1, Corollary 1.18]. Indeed, for every u in $H^1_{\Gamma_1}(\Omega, \mathbf{R}^3)$, there exists a sequence $(u^n, v^n)_n$ in $\left(C^1\left(\overline{\Omega}, \mathbf{R}^3\right)\cap H^1_{\Gamma_1}(\Omega, \mathbf{R}^3)\right)\times\left(C^2\left(\overline{\Omega}, \mathbf{R}^3\right)\cap V\right)$ converging to (u,v) in the strong topology of the space $H^1(\Omega, \mathbf{R}^3)\times V$. Thanks to Lemma 9, $((u^n)^o_\varepsilon)_\varepsilon$ converges to (u^n, v^n) in the topology τ and

$$\lim_{n\to+\infty}\lim_{\varepsilon\to 0} F^\varepsilon\left((u^n)^o_\varepsilon\right) = \lim_{n\to+\infty} F^o(u^n, v^n) = F^o(u,v).$$

The space $H^1(\Omega,\mathbf{R}^3)\times L^1(\Omega,\mathbf{R}^3)$ is metrizable for the topology τ. One deduces from [1, Corollary 1.18], the existence of a subsequence $\left(\left(u^{n(\varepsilon)}\right)^o_\varepsilon\right)_\varepsilon$ converging to u in the weak topology of $H^1_{\Gamma_1}(\Omega,\mathbf{R}^3)$, such that $\left(R^\varepsilon\left(v^{n(\varepsilon)}\right)\right)_\varepsilon$ converges to v in the weak* topology of $L^1(\Omega,\mathbf{R}^3)$ and : $\limsup_{\varepsilon\to 0} F^\varepsilon\left(\left(u^{n(\varepsilon)}\right)^o_\varepsilon\right)\le F^o(u,v)$. This ends the verification of the first assertion.

Let us now prove the second assertion of the epi-convergence, that is : *For every sequence $(u_\varepsilon)_\varepsilon$ of elements of $H^1_{\Gamma_1}(\Omega,\mathbf{R}^3)$, converging to (u,v) in the topology τ, then v belongs to V, satisfies : $v=0$, on Γ_1, and : $\liminf_{\varepsilon\to 0} F^\varepsilon(u_\varepsilon)\ge F^o(u,v)$.*

Let $(u^n)_n$ be any sequence of smooth functions in $C^1\left(\overline{\Omega},\mathbf{R}^3\right)\cap H^1_{\Gamma_1}(\Omega,\mathbf{R}^3)$ converging to u in the strong topology of $H^1(\Omega,\mathbf{R}^3)$ and $(v^n)_n$ be any sequence of smooth functions in $C^2\left(\overline{\Omega},\mathbf{R}^3\right)\cap V$ converging to v in the strong topology of V. Let us suppose that $\sup_\varepsilon F^\varepsilon(u_\varepsilon)<+\infty$, otherwise the assertion is trivially satisfied. Under these hypotheses, one proves

Lemma 10 *$(u_\varepsilon)_\varepsilon$ is bounded in $H^1_{\Gamma_1}(\Omega,\mathbf{R}^3)$ and the sequence $(R^\varepsilon(u_\varepsilon))_\varepsilon$ converges in the weak* topology of $L^1(\Omega,\mathbf{R}^3)$ to some v belonging to V.*

Proof. We use some argument similar to [2, Lemme A1], defining:

$$\Phi_\varepsilon=e_{33}(u_\varepsilon),\ \delta_\varepsilon=\frac{|\Omega|}{|T_\varepsilon|}\mathbf{1}_{T_\varepsilon}dx,\ \delta=\mathbf{1}_\Omega dx.$$

δ_ε and δ are two bounded Radon measures such that $(\delta_\varepsilon)_\varepsilon$ converges weakly to δ in the sense of measures. We then compute

$$\begin{aligned}\int_{\mathbf{R}^3}|\Phi_\varepsilon|\,\delta_\varepsilon &\le \left(\int_{\mathbf{R}^3}|\Phi_\varepsilon|^2\,\delta_\varepsilon\right)^{1/2}\sqrt{|T_\varepsilon|}\\ &\le \frac{C}{\sqrt{|T_\varepsilon|}}\left(\int_{T_\varepsilon}|\Phi_\varepsilon|^2\,dx\right)^{1/2}\le C\left(\sup_\varepsilon F^\varepsilon(u_\varepsilon)\right)^{1/2}<+\infty,\end{aligned}$$

because $(\lambda^\varepsilon|T_\varepsilon|)_\varepsilon$ and $(\mu^\varepsilon|T_\varepsilon|)_\varepsilon$ have finite limits. Hence, the sequence $(\Phi_\varepsilon\delta_\varepsilon)_\varepsilon$ of measures has uniformly bounded variations. One can extract some subsequence, still denoted by $(\Phi_\varepsilon\delta_\varepsilon)_\varepsilon$, which converges to some measure Φ. For every φ in $C^o_c(\mathbf{R}^3)$, we write Fenchel's inequality

$$\int_{\mathbf{R}^3}|\Phi_\varepsilon|^2\,\delta_\varepsilon\ge 2\int_{\mathbf{R}^3}\Phi_\varepsilon\varphi\delta_\varepsilon-\int_{\mathbf{R}^3}\varphi^2\delta_\varepsilon,$$

which implies

$$\liminf_{\varepsilon\to 0}\int_{\mathbf{R}^3}|\Phi_\varepsilon|^2\,\delta_\varepsilon\ge 2\langle\Phi,\varphi\rangle-\int_{\mathbf{R}^3}\varphi^2\delta,$$

where $\langle .,.\rangle$ means the duality product between measures and functions, from which we deduce that : $\sup\left\{\langle\Phi,\varphi\rangle\mid\varphi\in C^o_c(\mathbf{R}^3),\ \|\varphi\|_{L^2(\Omega)}\le 1\right\}<+\infty$. Riesz's representation theorem implies the existence of some χ in $L^2_\delta(\Omega)$ such that for every φ in $C^o_c(\mathbf{R}^3)$: $\langle\Phi,\varphi\rangle = \int_{\mathbf{R}^3}\chi\varphi\delta = \int_\Omega \chi\varphi dx$. For every φ in $C^1_0(\Omega)$, one has

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{|\Omega|}{|T_\varepsilon|}\int_{T_\varepsilon} e_{33}(u_\varepsilon)\varphi dx &= \int_\Omega \chi\varphi dx \\ &= -\lim_{\varepsilon\to 0}\frac{|\Omega|}{|T_\varepsilon|}\int_{T_\varepsilon}\frac{\partial\varphi}{\partial x_3}(u_\varepsilon)_3\, dx \\ &\underset{\varepsilon\to 0}{\to} -\int_\Omega \frac{\partial\varphi}{\partial x_3}v_3 dx = \int_\Omega \varphi e_{33}(v)\, dx.\end{aligned}$$

We thus get : $\int_\Omega (\chi\varphi - \varphi e_{33}(v))\, dx = 0$, which implies that $e_{33}(v)$ $(=\chi)$ belongs to $L^2(\Omega)$.

In order to prove that v_i belongs to $L^2(\Omega)$, for $i=1,2,3$, we repeat the above argument with $\Phi_{\varepsilon,i} = (u_\varepsilon)_i$ instead of $\Phi_\varepsilon = e_{33}(u_\varepsilon)$ and we use the estimates of Lemma 1 3.

In order to prove that v_3 is equal to 0 on Γ_1, let us take any function φ in $C^1(\overline{\Omega})$ taking the form: $\varphi(x) = \theta(x_1,x_2)\psi(x_3)$, with $\psi(0)=1$, $\psi(L)=0$, θ in $C^\infty(\omega)$. We first compute

$$\begin{aligned}&\int_\Omega \frac{\partial v_3}{\partial x_3}\varphi dx \\ &\quad= -\int_\Omega \frac{\partial\varphi}{\partial x_3}v_3 dx + \lim_{\varepsilon\to 0}\frac{|\Omega|}{|T_\varepsilon|}\int_{T_\varepsilon}\begin{pmatrix} (\varphi(u_\varepsilon)_3)(x_1,x_2,L) \\ -(\varphi(u_\varepsilon)_3)(x_1,x_2,0)\end{pmatrix} dx_1dx_2 \\ &\quad= -\int_\Omega \frac{\partial\varphi}{\partial x_3}v_3 dx,\end{aligned}$$

thanks to the boundary conditions verified by φ and u_ε. Moreover, using Green's formula, we get

$$\int_\Omega \frac{\partial v_3}{\partial x_3}\varphi dx = -\int_\Omega \frac{\partial\varphi}{\partial x_3}v_3 dx + \int_\omega \theta(x_1,x_2)\, v_3(x_1,x_2,0)\, dx_1dx_2,$$

which implies

$$\int_\omega \theta(x_1,x_2)\, v_3(x_1,x_2,0)\, dx_1dx_2 = 0 \Rightarrow v_3(x_1,x_2,0) = 0.$$

Thus v belongs to V. □ ■

In order to prove this second assertion, we write the subdifferential inequality for the first term of $F^\varepsilon(u_\varepsilon)$

$$\begin{aligned}\int_{\Omega\backslash\overline{C_\varepsilon\cup T_\varepsilon}}\sigma_{ij}(u_\varepsilon)\, e_{ij}(u_\varepsilon)\, dx \geq &\int_{\Omega\backslash\overline{C_\varepsilon\cup T_\varepsilon}}\sigma_{ij}\left((u^n)^o_\varepsilon\right) e_{ij}\left((u^n)^o_\varepsilon\right) dx \\ &+2\int_{\Omega\backslash\overline{C_\varepsilon\cup T_\varepsilon}}\sigma_{ij}(u^n)\, e_{ij}\left(u_\varepsilon - (u^n)^o_\varepsilon\right) dx,\end{aligned}$$

where $(u^n)^o_\varepsilon$ is associated to u^n through (16). The sequence $\left((u^n)^o_\varepsilon\right)_\varepsilon$ converges to u^n in the weak topology of $H^1(\Omega,\mathbf{R}^3)$, thanks to Lemma 9, and coincides with u_n in $\Omega\backslash\overline{C_\varepsilon\cup T_\varepsilon}$.

Thus, $\left(e_{ij}\left(u_\varepsilon-\left(u^n\right)_\varepsilon^o\right)\right)_\varepsilon$ converges to $e_{ij}\left(u-u^n\right)$ in the weak topology of $L^2\left(\Omega\right)$, for $i,j=1,2,3$. The sequence of characteristic functions of $\Omega\backslash\overline{C_\varepsilon\cup T_\varepsilon}$ converges to 1 in the strong topology of $L^2\left(\Omega\right)$. This implies the following convergence

$$\liminf_{\varepsilon\to 0}\int_{\Omega\backslash\overline{C_\varepsilon\cup T_\varepsilon}}\sigma_{ij}\left(u_\varepsilon\right)e_{ij}\left(u_\varepsilon\right)dx\geq\int_\Omega\sigma_{ij}\left(u^n\right)e_{ij}\left(u^n\right)dx +2\int_\Omega\sigma_{ij}\left(u^n\right)e_{ij}\left(u-u^n\right)dx.$$

Letting n increase to $+\infty$ we get, using the convergence of $\left(u^n\right)_n$ to u in the strong topology of $H^1\left(\Omega,\mathbf{R}^3\right)$

$$\liminf_{\varepsilon\to 0}\int_{\Omega\backslash\overline{C_\varepsilon\cup T_\varepsilon}}\sigma_{ij}\left(u_\varepsilon\right)e_{ij}\left(u_\varepsilon\right)dx\geq\int_\Omega\sigma_{ij}\left(u\right)e_{ij}\left(u\right)dx. \tag{25}$$

We then write the subdifferential inequality for the second term of $F^\varepsilon\left(u_\varepsilon\right)$

$$\int_{C_\varepsilon}\sigma_{ij}\left(u_\varepsilon\right)e_{ij}\left(u_\varepsilon\right)dx\geq\int_{C_\varepsilon}\sigma_{ij}\left(\left(u^n\right)_\varepsilon^o\right)e_{ij}\left(\left(u^n\right)_\varepsilon^o\right)dx +2\int_{C_\varepsilon}\sigma_{ij}\left(\left(u^n\right)_\varepsilon^o\right)e_{ij}\left(u_\varepsilon-\left(u_\varepsilon^n\right)_\varepsilon^o\right)dx,$$

with

$$2\int_{C_\varepsilon}\sigma_{ij}\left(\left(u^n\right)_\varepsilon^o\right)e_{ij}\left(u_\varepsilon-\left(u^n\right)_\varepsilon^o\right)dx=2\int_{C_\varepsilon}\sigma_{ij}\left(u^n\right)e_{ij}\left(u_\varepsilon-\left(u^n\right)_\varepsilon^o\right)dx +2\int_{C_\varepsilon}\sigma_{ij}\left(z_\varepsilon^m\left(\left(\mathcal{R}_\varepsilon\left(v^n\right)\right)_m-\left(u^n\right)_m\right)\right)e_{ij}\left(u_\varepsilon-\left(u^n\right)_\varepsilon^o\right)dx.$$

We immediately get : $\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\sigma_{ij}\left(u^n\right)e_{ij}\left(u_\varepsilon-\left(u^n\right)_\varepsilon^o\right)dx=0$, because the sequence $\left(e_{ij}\left(u_\varepsilon-\left(u^n\right)_\varepsilon^o\right)\right)_\varepsilon$ converges to $e_{ij}\left(u-u^n\right)$ in the weak topology of $L^2\left(\Omega\right)$, for $i,j=1,2,3$ and the sequence of characteristic functions of C_ε converges to 0 in the strong topology of $L^2\left(\Omega\right)$. The second term of the last equality can be computed as

$$\begin{aligned}&\int_{C_\varepsilon}\sigma_{ij}\left(z_\varepsilon^m\left(\left(\mathcal{R}_\varepsilon\left(v^n\right)\right)_m-\left(u^n\right)_m\right)\right)e_{ij}\left(u_\varepsilon-\left(u^n\right)_\varepsilon^o\right)dx\\ &\quad=\int_{C_\varepsilon}\sigma_{ij}\left(z_\varepsilon^m\right)\left(\left(\mathcal{R}_\varepsilon\left(v^n\right)\right)_m-\left(u^n\right)_m\right)e_{ij}\left(u_\varepsilon-\left(u^n\right)_\varepsilon^o\right)dx\\ &\qquad+\int_{C_\varepsilon}a_{ijst}\left(z_\varepsilon^m\right)_s\frac{\partial\left(\left(\mathcal{R}_\varepsilon\left(v^n\right)\right)_m-\left(u^n\right)_m\right)}{\partial x_t}e_{ij}\left(u_\varepsilon-\left(u^n\right)_\varepsilon^o\right)dx,\end{aligned}$$

writing : $\sigma_{ij}=a_{ijst}e_{st}$. We observe that

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon}a_{ijst}\left(z_\varepsilon^m\right)_s\frac{\partial\left(\left(\mathcal{R}_\varepsilon\left(v^n\right)\right)_m-\left(u^n\right)_m\right)}{\partial x_t}e_{ij}\left(u_\varepsilon-\left(u^n\right)_\varepsilon^o\right)dx=0,$$

because $(z_\varepsilon^m)_\varepsilon$ converges to 0 in the strong topology of $L^2(\Omega, \mathbf{R}^3)$, $|\nabla(\mathcal{R}_\varepsilon(v^n) - u^n)| \le C_n$, in C_ε, and $(e_{ij}(u_\varepsilon - (u^n)_\varepsilon^o))_\varepsilon$ converges to $e_{ij}(u - u^n)$ in the weak topology of $L^2(\Omega)$, for $i, j = 1, 2, 3$. Then, we compute, using the definition of z_ε^m

$$\begin{aligned}
&\int_{C_\varepsilon} \sigma_{ij}(z_\varepsilon^m)\left((\mathcal{R}_\varepsilon(v^n))_m - (u^n)_m\right) e_{ij}\left(u_\varepsilon - (u^n)_\varepsilon^o\right) dx \\
&\quad = -\sum_k \int_{C_\varepsilon^k} \sigma_{ij}\left(w_\varepsilon^{mk}\right) e_{ij}\left(u_\varepsilon - (u^n)_\varepsilon^o\right)\left((\mathcal{R}_\varepsilon(v^n))_m - (u^n)_m\right)\varphi_\varepsilon^k dx \\
&\qquad + \sum_k \int_{C_\varepsilon^k} a_{ijst}\left((\mathcal{R}_\varepsilon(v^n))_m - (u^n)_m\right)\left(e_m - w_\varepsilon^{mk}\right)_s \frac{\partial \varphi_\varepsilon^k}{\partial x_t} e_{ij}\left(u_\varepsilon - (u^n)_\varepsilon^o\right) dx.
\end{aligned}$$

But, for every k, one has, thanks to the definition (12) of φ_ε^k and using Lemmas 4 and 9 assertion *1.*

$$\begin{aligned}
&\left|\int_{C_\varepsilon^k} \left((\mathcal{R}_\varepsilon(v^n))_m - (u^n)_m\right)\left(e_m - w_\varepsilon^{mk}\right)_s \frac{\partial \varphi_\varepsilon^k}{\partial x_t} e_{ij}\left(u_\varepsilon - (u^n)_\varepsilon^o\right) dx\right| \\
&\qquad \le \frac{C_n |\ln s_\varepsilon|}{\varepsilon^2 |\ln r_\varepsilon|} \int_{C_\varepsilon^k \cap \left\{s_\varepsilon/2 < R_\varepsilon^k < s_\varepsilon\right\}} R_\varepsilon^k \left|\nabla\left(u_\varepsilon - (u^n)_\varepsilon^o\right)\right| dx.
\end{aligned}$$

This implies, because $(u_\varepsilon)_\varepsilon$ and $((u^n)_\varepsilon^o)_\varepsilon$ are bounded in $H^1(\Omega, \mathbf{R}^3)$

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \left|\sum_k \int_{C_\varepsilon^k} a_{ijst}\left((\mathcal{R}_\varepsilon(v^n))_m - (u^n)_m\right)\left(e_m - w_\varepsilon^{mk}\right)_s \frac{\partial \varphi_\varepsilon^k}{\partial x_t} e_{ij}\left(u_\varepsilon - (u^n)_\varepsilon^o\right) dx\right| \\
&\qquad \le \limsup_{\varepsilon \to 0} \frac{C_n |\ln s_\varepsilon|}{\varepsilon |\ln r_\varepsilon|} \left(\int_\Omega \left|\nabla\left(u_\varepsilon - (u^n)_\varepsilon^o\right)\right|^2 dx\right)^{1/2} = 0,
\end{aligned}$$

because γ is finite and using the properties of s_ε. Similarly, we estimate, using Lemma 4

$$\begin{aligned}
&\left|\sum_k \int_{C_\varepsilon^k} \sigma_{ij}\left(w_\varepsilon^{mk}\right) e_{ij}\left(u_\varepsilon - (u^n)_\varepsilon^o\right)\left((\mathcal{R}_\varepsilon(v^n))_m - (v^n)_m\right)\varphi_\varepsilon^k dx\right| \\
&\qquad \le \frac{C_n \sqrt{s_\varepsilon}}{|\ln(r_\varepsilon)|} \left(\int_\Omega \left|\nabla\left(u_\varepsilon - (u^n)_\varepsilon^o\right)\right|^2 dx\right)^{1/2} \underset{\varepsilon \to 0}{\to} 0,
\end{aligned}$$

because γ is finite. We then have to compute the limit of the remaining term

$$\begin{aligned}
&\sum_k \int_{C_\varepsilon^k} \sigma_{ij}\left(w_\varepsilon^{mk}\right) e_{ij}\left(u_\varepsilon - (u^n)_\varepsilon^o\right)\left((v^n)_m - (u^n)_m\right)\varphi_\varepsilon^k dx \\
&\quad = -\sum_k \int_{C_\varepsilon^k} \sigma_{ij,j}\left(w_\varepsilon^{mk}\right)\left(u_\varepsilon - (u^n)_\varepsilon^o\right)_i \left((v^n)_m - (u^n)_m\right)\varphi_\varepsilon^k dx \\
&\qquad -\sum_k \int_{C_\varepsilon^k} \sigma_{ij}\left(w_\varepsilon^{mk}\right)\left(u_\varepsilon - (u^n)_\varepsilon^o\right)_i \frac{\partial\left(\left((v^n)_m - (u^n)_m\right)\varphi_\varepsilon^k\right)}{\partial x_j} dx \\
&\qquad + \sum_k \int_{\partial T_\varepsilon^k} \sigma_{ij}\left(w_\varepsilon^{mk}\right) n_j \left(u_\varepsilon - (u^n)_\varepsilon^o\right)_i \left((v^n)_m - (u^n)_m\right) dx.
\end{aligned}$$

Using the estimates of Lemma 4, we prove that the second term above converges to 0. Using the properties of w_ε^{mk}, the first term above is equal to 0. Then, the properties of w_ε^{mk} and the convergence of $\left(u_\varepsilon - (u^n)_\varepsilon^o\right)_\varepsilon$ to $u - u^n$ in the weak topology of $H^1(\Omega, \mathbf{R}^3)$ imply

$$\lim_{\varepsilon\to 0}\sum_k \int_{C_\varepsilon^k} \sigma_{ij}\left(w_\varepsilon^{mk}\right) e_{ij}\left(u_\varepsilon - (u^n)_\varepsilon^o\right)\left((v^n)_m - (u^n)_m\right)\varphi_\varepsilon^k dx = 2\pi\gamma \int_\Omega (v^n - u^n)^t A\,(u - u^n)\,dx.$$

We let n increase to $+\infty$ and get

$$\liminf_{\varepsilon\to 0}\sum_k \int_{C_\varepsilon^k} \sigma_{ij}\left(w_\varepsilon^{mk}\right) e_{ij}\left(u_\varepsilon - (u^n)_\varepsilon^o\right)\left((v^n)_m - (u^n)_m\right)\varphi_\varepsilon^k dx \geq 0,$$

which implies, using the computations of the first assertion

$$\liminf_{\varepsilon\to 0}\int_{C_\varepsilon} \sigma_{ij}(u_\varepsilon)\, e_{ij}(u_\varepsilon)\, dx \geq 2\pi\gamma \int_\Omega (v - u)^t A\,(v - u)\,dx. \tag{26}$$

We finally observe that for the third term of $F^\varepsilon(u_\varepsilon)$, one has

$$\int_{T_\varepsilon} \sigma_{ij}^\varepsilon(u_\varepsilon)\, e_{ij}(u_\varepsilon)\, dx \geq \mu^\varepsilon \frac{2\mu^\varepsilon + 3\lambda^\varepsilon}{\mu^\varepsilon + \lambda^\varepsilon} \int_{T_\varepsilon} \left(e_{33}(u_\varepsilon)\right)^2 dx.$$

Indeed, one can easily verify that for every x, y, z in $\mathbf{R}$, one has

$$\lambda^\varepsilon (x + y + z)^2 + 2\mu^\varepsilon \left(x^2 + y^2 + z^2\right) \geq \mu^\varepsilon \frac{2\mu^\varepsilon + 3\lambda^\varepsilon}{\mu^\varepsilon + \lambda^\varepsilon} z^2.$$

We then use the computations given in Lemma 10, which imply, because μ_o and λ_o are finite

$$\liminf_{\varepsilon\to 0}\int_{T_\varepsilon} \sigma_{ij}^\varepsilon(u_\varepsilon)\, e_{ij}(u_\varepsilon)\, dx \geq \pi E_o \int_\Omega \left(e_{33}(v)\right)^2 dx. \tag{27}$$

One deduces from (25)-(27)

$$\liminf_{\varepsilon\to 0} F^\varepsilon(u_\varepsilon) \geq \int_\Omega \sigma_{ij}(u)\, e_{ij}(u)\, dx + 2\pi\gamma \int_\Omega (v - u)^t A\,(v - u)\,dx + \pi E_o \int_\Omega \left(e_{33}(v)\right)^2 dx,$$

which concludes the proof. □

3.1 Other situations

The other situations given by different values of the parameters γ or λ_o or μ_o are summarized in the

Proposition 11 *1. If* λ_o *and* μ_o *are equal to* 0*, then* $(u^\varepsilon)_\varepsilon$ *converges in the topology* τ *to the solution* (u_o^o, v_o^o) *of the minimization problem associated to the functional* F_o^o *defined in a similar way than (14), but with* $\lambda_o = \mu_o = 0$*.*

2. If γ *is equal to* $+\infty$*, one obtains* $u^{o\infty} = v^{o\infty}$ *in* Ω *and* $F^{o\infty}$ *only depends on* u

$$F^{o\infty}(u) = \int_\Omega \sigma_{ij}(u)\, e_{ij}(u)\, dx + \pi E_o \int_\Omega (e_{33}(u))^2\, dx.$$

Proof. *1.* This case corresponds to a situation where the Lamé coefficients λ^ε and μ^ε of the reinforcing material are smaller than the critical ones given in (10), that is given by

$$\lambda_c^\varepsilon = \frac{c\varepsilon^2}{(r_\varepsilon)^2},\ \mu_c^\varepsilon = \frac{c\varepsilon^2}{(r_\varepsilon)^2},$$

for every positive and small c, but preserving the critical radius r_ε of the fibers given through γ. Let F_c^ε be the functional defined in (4) but with these critical Lamé coefficients. Thanks to the property of the epi-convergence, we get, for every (u, v) in $H^1_{\Gamma_1}(\Omega, \mathbf{R}^3) \times V$

$$\begin{aligned} F_o^o(u,v) \leq F_c^o(u,v) = \int_\Omega \sigma_{ij}(u)\, e_{ij}(u)\, dx + \pi c E_o \int_\Omega (e_{33}(v))^2\, dx \\ +2\pi\gamma \int_\Omega (v-u)^t A (v-u)\, dx. \end{aligned}$$

This inequality being true for every positive c, we get, letting c go to 0

$$F_o^o(u,v) \leq \int_\Omega \sigma_{ij}(u)\, e_{ij}(u)\, dx + 2\pi\gamma \int_\Omega (v-u)^t A (v-u)\, dx.$$

In order to establish the reverse inequality, we observe that, for every sequence $(u_\varepsilon)_\varepsilon$ converging to (u, v) in the above-defined topology τ, one has

$$F^\varepsilon(u_\varepsilon) \geq \int_{\Omega\setminus\overline{B_\varepsilon}} \sigma_{ij}(u_\varepsilon)\, e_{ij}(u_\varepsilon)\, dx + \int_{C_\varepsilon} \sigma_{ij}(u_\varepsilon)\, e_{ij}(u_\varepsilon)\, dx,$$

thus omitting the integral involving the fibers T_ε. We then adapt the proof of the second assertion in the Theorem 6 in order to conclude
2. We again observe that this situation corresponds to a case where the Lamé coefficients of the reinforcing material are still given by (10) but where the radius of the fibers is larger than the critical one, that is $r_\varepsilon \geq \exp(-1/C\varepsilon^2)$, for every positive C. The functional F^ε is thus larger than the functional $F^{\varepsilon C}$ given by (4), but with the radius $\exp(-1/C\varepsilon^2)$. The comparison principle implies that for every (u, v) in $H^1_{\Gamma_1}(\Omega, \mathbf{R}^3) \times V$

$$\begin{aligned} F^{o\infty}(u,v) \geq F^{oC}(u,v) = \int_\Omega \sigma_{ij}(u)\, e_{ij}(u)\, dx + \pi E_o \int_\Omega (e_{33}(v))^2\, dx \\ +2\pi\gamma C \int_\Omega (v-u)^t A (v-u)\, dx. \end{aligned}$$

Letting C increase to $+\infty$, we observe that $F^{o\infty}(u,v)$ is finite if and only if the integral $\int_\Omega (v-u)^t A(v-u)\,dx = 0$, which implies : $u=v$, in Ω. The reverse inequality is still obtained adapting the proof of Theorem 6 (first part) but with $v=u$. □ ■

Let us now examine the special case when $\lambda_o = \mu_o = +\infty$. As a special subcase, [6] have considered the case when $\gamma = +\infty$ and

$$\frac{\lambda^\varepsilon (r_\varepsilon)^4}{\varepsilon^2} \underset{\varepsilon\to 0}{\to} \lambda_1,\ \frac{\mu^\varepsilon (r_\varepsilon)^4}{\varepsilon^2} \underset{\varepsilon\to 0}{\to} \mu_1, \tag{28}$$

with positive and finite λ_1 and μ_1. We now adapt their result considering

Proposition 12 *Suppose that the above hypothesis (28) holds true and γ belongs to $]0,+\infty]$. Then, the sequence $(u^\varepsilon)_\varepsilon$ converges in the topology τ, to the solution (u^1,v^1) of*

$$\min_{H^1_{\Gamma_1}(\Omega,\mathbf{R}^3)\times V'} \left(\begin{array}{r} \displaystyle\int_\Omega \sigma_{ij}(u)\,e_{ij}(u)\,dx + 2\pi\gamma\int_\Omega (v-u)^t A(v-u)\,dx \\ \displaystyle +\frac{\pi E_1}{4}\int_\Omega \left(\left(\frac{\partial^2 v_1}{\partial x_3^2}\right)^2 + \left(\frac{\partial^2 v_2}{\partial x_3^2}\right)^2\right) dx \end{array} \right),$$

with $E_1 = \mu_1(3\lambda_1+2\mu_1)/(\lambda_1+\mu_1)$ and

$$V' = \left\{ v_\alpha \in L^2\left(\omega, H^2(0,L)\right) \mid v_{|\Gamma_1} = 0,\ v_3 = 0 \right\}.$$

Proof. We proceed in a similar way to [6]. Indeed, we first follow their method in order to prove the following estimates

$$\frac{1}{|T_\varepsilon|}\int_{T_\varepsilon} |u^\varepsilon|\,dx < C,\ \frac{1}{|T_\varepsilon|}\int_{T_\varepsilon} |u^\varepsilon|^2\,dx < C,\ \frac{1}{(r_\varepsilon)^2|T_\varepsilon|}\int_{T_\varepsilon} |e_{ij}(u^\varepsilon)|^2\,dx < C,$$

where C is independant of ε. For every smooth v in $C^2\left(\overline{\Omega},\mathbf{R}^3\right)\cap V'$, we set

$$\left\{ \begin{array}{l} (\mathcal{R}_{\varepsilon 1}(v))_1(x_1,x_2,x_3) = v_1(k_1\varepsilon,k_2\varepsilon,x_3) \\ \qquad -\dfrac{\lambda^\varepsilon}{2(\mu^\varepsilon+\lambda^\varepsilon)}\dfrac{(x_1-k_1\varepsilon)^2-(x_2-k_2\varepsilon)^2}{2}\dfrac{\partial^2 v_1}{\partial x_3^2}(k_1\varepsilon,k_2\varepsilon,x_3) \\ \qquad -\dfrac{\lambda^\varepsilon}{2(\mu^\varepsilon+\lambda^\varepsilon)}\dfrac{(x_1-k_1\varepsilon)(x_2-k_2\varepsilon)}{2}\dfrac{\partial^2 v_2}{\partial x_3^2}(k_1\varepsilon,k_2\varepsilon,x_3) \\ (\mathcal{R}_{\varepsilon 1}(v))_2(x_1,x_2,x_3) = v_2(k_1\varepsilon,k_2\varepsilon,x_3) \\ \qquad -\dfrac{\lambda^\varepsilon}{2(\mu^\varepsilon+\lambda^\varepsilon)}\dfrac{(x_1-k_1\varepsilon)^2-(x_2-k_2\varepsilon)^2}{2}\dfrac{\partial^2 v_2}{\partial x_3^2}(k_1\varepsilon,k_2\varepsilon,x_3) \\ \qquad -\dfrac{\lambda^\varepsilon}{2(\mu^\varepsilon+\lambda^\varepsilon)}\dfrac{(x_1-k_1\varepsilon)(x_2-k_2\varepsilon)}{2}\dfrac{\partial^2 v_1}{\partial x_3^2}(k_1\varepsilon,k_2\varepsilon,x_3) \\ (\mathcal{R}_{\varepsilon 1}(v))_3(x_1,x_2,x_3) = -(x_1-k_1\varepsilon)\dfrac{\partial v_1}{\partial x_3}(k_1\varepsilon,k_2\varepsilon,x_3) \\ \qquad -(x_2-k_2\varepsilon)\dfrac{\partial v_2}{\partial x_3}(k_1\varepsilon,k_2\varepsilon,x_3). \end{array} \right.$$

The verification of the first assertion of the epi-convergence is obtained computing the energy of the test-function associated to this $\mathcal{R}_{\varepsilon 1}(v)$. The verification of the second assertion follows the same lines as in Theorem 6. $\square$ $\blacksquare$

Remark 13 *The extra term occuring in the energy functional described in Proposition 12 corresponds to the flexion of the fibers.*

Remark 14 *In the case $\gamma = 0$, one can still prove that $\left(\int_{T_\varepsilon} |(u_\varepsilon)_3| \, dx / |T_\varepsilon|\right)_\varepsilon$ is bounded, writing : $u_\varepsilon(s) = \int_0^s \partial (u_\varepsilon)_3 / \partial x_3 dt$ and using some trivial arguments. Thus Lemma 10 still implies the existence of $e_{33}(v)$ in $L^2(\Omega)$, with $v_3 = 0$ on Γ_1. We conjecture that the limit functional is*

$$F^{oo}(u,v) = \int_\Omega \sigma_{ij}(u) e_{ij}(u) \, dx + \pi E_o \int_\Omega (e_{33}(v))^2 \, dx.$$

4 Further extensions

4.1 The case of a almost non-periodic distribution of fibers

Let $\widetilde{\omega}$ be some open subset of $\mathbf{R}^2$ and θ be a C^1-diffeomorphism from $\widetilde{\omega}$ to ω. We define the following almost non-periodic distribution of non-homogeneous fibers as follows. The fibers are defined as

$$T_\varepsilon^k = \left\{(x_1,x_2,x_3) \mid (x_1 - \theta_1(k_1\varepsilon, k_2\varepsilon))^2 + (x_2 - \theta_2(k_1\varepsilon, k_2\varepsilon))^2 < (r_\varepsilon)^2, \, x_3 \in]0, L[\right\}.$$

Replacing $(k_1\varepsilon, k_2\varepsilon)$ by $\theta(k_1\varepsilon, k_2\varepsilon)$ in the local test-functions and adapting the proof of Theorem 6, one can prove

Theorem 15 *Suppose that γ is positive and finite and the nonhomogeneous material filling in the fibers satisfies the usual conditions of symmetry, uniform ellipticity and continuity and*

$$\sup_{x_3 \in [0,L], \varepsilon > 0} \left| \frac{(r_\varepsilon)^2}{\varepsilon^2} a_{ijkl}^\varepsilon(x_3) \right| < +\infty, \; \frac{(r_\varepsilon)^2}{\varepsilon^2} a_{ijkl}^\varepsilon(x_3) \underset{\varepsilon \to 0}{\to} a_{ijkl}^o(x_3), \; a.e. \text{ in } \Omega.$$

Then, the sequence $(F^\varepsilon)_\varepsilon$ epi-converges in the topology τ to the functional F^o defined on $H^1(\Omega, \mathbf{R}^3) \times L^1(\Omega, \mathbf{R}^3)$ by:

$$F^o(u,v) = \begin{cases} \int_\Omega \sigma_{ij}(u) e_{ij}(u) \, dx + 2\pi\gamma \int_\Omega (v-u)^t A (v-u) \left|\nabla \theta^{-1}\right| (x_1,x_2) \, dx \\ \qquad + \pi \int_\Omega E^o(x_3) e_{33}(v) e_{33}(v) \left|\nabla \theta^{-1}\right| (x_1,x_2) \, dx, \\ \qquad\qquad\qquad\qquad \text{if } (u,v) \in H_{\Gamma_1}^1(\Omega, \mathbf{R}^3) \times V \\ +\infty \qquad\qquad\qquad\qquad \text{otherwise,} \end{cases}$$

where $E^o(x_3)$ is Young's modulus associated to $a_{ijkl}^o(x_3)$.

4.2 The case of tranverse fibers

Let us assume in this paragraph that ω is the disk centred at the origin and of radius $R > 0$ of $\mathbf{R}^2$. Choose any R^* in $]0, R]$ and positive ε and r_ε such that : $0 < 2r_\varepsilon < \varepsilon < 1$. For every k in $\mathbf{Z}$, we introduce the torus T_ε^k defined as

$$T_\varepsilon^k = \left\{ (x_1, x_2, x_3) \in \mathbf{R}^3 \mid \left(R^* - \sqrt{(x_1)^2 + (x_2)^2} \right)^2 + (x_3 - k\varepsilon)^2 < (r_\varepsilon)^2 \right\}.$$

T_ε denotes the union $\bigcup_{k=-n(\varepsilon)}^{k=n(\varepsilon)} T_\varepsilon^k$ of the tori T_ε^k ε-periodically distributed along the surface : $\Sigma_{R^*} = \left\{ (x_1)^2 + (x_2)^2 = (R^*)^2 ,\ x_3 \in]0, L[\right\}$ and contained in $\Omega = \omega \times]0, L[$. We suppose that $\overline{T_\varepsilon} \cap \Gamma_1$ and $\overline{T_\varepsilon} \cap \Gamma_2$ are empty. The number $n(\varepsilon)$ of such tori contained in Ω is equivalent to L/ε.

We define the topology τ^* as

$$u_\varepsilon \underset{\varepsilon \to 0}{\overset{\tau^*}{\rightharpoonup}} (u, v) \Leftrightarrow u_\varepsilon \underset{\varepsilon \to 0}{\overset{w\text{-}H^1(\Omega, \mathbf{R}^3)}{\rightharpoonup}} u$$
$$\text{and : } \forall \varphi \in C_c^o(\mathbf{R}^3) : \int_{\Sigma_{R^*}} \left(R^{\varepsilon^*}(u_\varepsilon)\, \varphi \right)_{|\Sigma_{R^*}} d\sigma \underset{\varepsilon \to 0}{\rightarrow} \int_{\Sigma_{R^*}} (v\varphi)_{|\Sigma_{R^*}} d\sigma,$$

where R^{ε^*} is defined by : $R^{\varepsilon^*}(u) = |\Sigma_{R^*}|\, u \mathbf{1}_{T_\varepsilon} / |T_\varepsilon|$. We introduce the space

$$V^* = \left\{ \begin{array}{c} v = (v_r, v_\theta, v_{x_3}) : [0, 2\pi] \times]0, L[\to \mathbf{R}^3 \mid v_\alpha \in L^2(]0, 2\pi[\times]0, L[), \\ v_\alpha(0, .) = v_\alpha(2\pi, .),\ \alpha = r, \theta, x_3,\ \dfrac{\partial v_\theta}{\partial \theta} + v_r \in L^2(]0, 2\pi[\times]0, L[). \end{array} \right\}$$

Figure 2: The cylinder Ω and the tori T_ε^k.

Following similar arguments to the ones presented in the previous parts, we prove

Theorem 16 *Suppose that* $\gamma^* = \lim_{\varepsilon\to 0}(-1/(\varepsilon \ln r_\varepsilon))$ *is finite,* λ_o^* *and* μ_o^* *are finite and* μ_o^* *is positive, with:*

$$\lambda_o^* = \lim_{\varepsilon\to 0}\frac{\lambda^\varepsilon (r_\varepsilon)^2}{\varepsilon},\ \mu_o^* = \lim_{\varepsilon\to 0}\frac{\mu^\varepsilon (r_\varepsilon)^2}{\varepsilon}.$$

Then, the sequence $(F^\varepsilon)_\varepsilon$ *epi-converges in the topology* τ^* *to the functional* F^{o^*} *defined on* $H^1(\Omega, \mathbf{R}^3) \times L^1(\Omega, \mathbf{R}^3)$ *by:*

$$F^{o^*}(u,v) = \begin{cases} \displaystyle\int_\Omega \sigma_{ij}(u)\, e_{ij}(u)\, dx + \pi E_o^* \int_0^{2\pi}\int_0^L \left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)^2 (R^*, \theta, x_3)\, d\theta dx_3 \\ \displaystyle\quad +2\pi\gamma^* R^* \int_0^{2\pi}\int_0^L \left(v - u_{|\Sigma_{R^*}}\right)^t A \left(v - u_{|\Sigma_{R^*}}\right)(R^*, \theta, x_3)\, d\theta dx_3, \\ \hfill \text{if } (u,v) \in H^1_{\Gamma_1}(\Omega, \mathbf{R}^3) \times V^* \\ +\infty \hfill \text{otherwise,} \end{cases}$$

with A *as in Theorem 6 and* $E_o^* = \mu_o^*(3\lambda_o^* + 2\mu_o^*)/(\lambda_o^* + \mu_o^*)$.

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