Homogenization of the Vlasov equation and of the Vlasov - Poisson system with a strong external magnetic field

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Abstract: Motivated by the difficulty arising in the numerical simulation of the movement of charged particles in presence of a large external magnetic field, which adds an additional time scale and thus imposes to use a much smaller time step, we perform in this paper a homogenization of the Vlasov equation and the Vlasov-Poisson system which yield approximate equations describing the mean behavior of the particles. The convergence proof is based on the two scale convergence tools introduced by N'Guetseng and Allaire. We also consider the case where, in addition to the magnetic field, a large external electric field orthogonal to the magnetic field and of the same magnitude is applied.

Keywords: Vlasov-Poisson equations, kinetic equations, homogenization, gyrokinetic approximation, multiple time scales, two scale convergence.

Abbreviated title: Homogenization of the Vlasov equation

1 Introduction

In many kind of devices involving charged particles, like electron guns, diodes or tokamaks, a large external magnetic field needs to be applied in order to keep the particles on the desired tracks. In Particle-In-Cell (PIC) simulations of such devices, this large external magnetic field obviously needs to be taken into account when pushing the particles. However, due to the magnitude of the concerned field this often adds a new time scale to the simulation and thus a stringent restriction on the time step. In order to get rid of this additional time scale, we would like to find approximate equations, where only the gross behavior implied by the external field would be retained and which could be used in a numerical simulation.

The trajectory of a particle in a constant magnetic field $\mathbf{B}$ is a helicoid along the magnetic field lines with a radius proportional to the inverse of the magnitude of $\mathbf{B}$. Hence, when this field becomes very large the particle gets trapped along the magnetic field lines. However due to the fast oscillations around the apparent trajectory, its apparent velocity is smaller than the actual one. This result has been known for some time as the "guiding center" approximation, and the link between the real and the apparent velocity is well known in terms of $\mathbf{B}$. We refer to Lee [13] and Dubin et *al* [5] for a complete physical viewpoint review on this subject. In the case of a cloud of particles, the movement of which is described by the Vlasov-Poisson equations, the situation is less clear as in the one particle case because of the non linearity of the problem. In particular, the question of knowing if the mutual influence of the particles can be expressed in terms of their apparent motion or if the oscillation generates additional effects is important.

In this paper, we deduce the "guiding center" approximation in the framework of partial differential equations via an homogenization process on the linear Vlasov equation. Then, we show that in a cloud of particles the mutual influence of the particles can be expressed in term of their apparent motion without any additionnal terms. This is provided applying an homogenization process on the Vlasov-Poisson system, similar to the one used for the linear Vlasov equation and using the regularity of the charge density.

Hence, we apply first a homogenization process to the linear Vlasov equation with a strong and constant external magnetic field. In other words for a constant vector $\mathcal{M} \in \mathbb{S}^2$, we consider the following equation:

$$\begin{cases} \dfrac{\partial f^\varepsilon}{\partial t} + \mathbf{v} \cdot \nabla_x f^\varepsilon + (\mathbf{E}^\varepsilon + \mathbf{v} \times (\mathbf{B}^\varepsilon + \dfrac{\mathcal{M}}{\varepsilon})) \cdot \nabla_v f^\varepsilon = 0, \\ f^\varepsilon_{|t=0} = f_0. \end{cases} \tag{1.1}$$

In this equation $f^\varepsilon \equiv f^\varepsilon(t, \mathbf{x}, \mathbf{v})$ with $t \in [0, T)$, for any $T \in \mathbb{R}^+$, $\mathbf{x} \in \mathbb{R}^3_x$ and $\mathbf{v} \in \mathbb{R}^3_v$. For convenience, we introduce the notations $\Omega = \mathbb{R}^3_x \times \mathbb{R}^3_v$, $\mathcal{O} = [0, T) \times \mathbb{R}^3_x$ and $\mathcal{Q} = [0, T) \times \Omega$. The initial data satisfies

$$f_0 \geq 0, \quad 0 < \int_\Omega f_0^2 \, d\mathbf{x} \, d\mathbf{v} < \infty. \tag{1.2}$$

The electric field $\mathbf{E}^\varepsilon \equiv \mathbf{E}^\varepsilon(t, \mathbf{x})$ and the magnetic field $\mathbf{B}^\varepsilon \equiv \mathbf{B}^\varepsilon(t, \mathbf{x})$ are defined on $\mathcal{O}$, both bounded in $L^\infty(0, T; L^2(\mathbb{R}^3_x))$ and satisfy

$$\mathbf{E}^\varepsilon \to \mathbf{E} \text{ in } L^\infty(0, T; L^2_{loc}(\mathbb{R}^3_x)) \text{ strong}, \tag{1.3}$$

and

$$\mathbf{B}^\varepsilon \to \mathbf{B} \text{ in } L^\infty(0,T; L^2_{loc}(\mathbb{R}^3_x)) \text{ strong,} \tag{1.4}$$

for any $T \in \mathbb{R}^+$.
With those conditions, for every ε there exists a unique solution of the equation (1.1) $f^\varepsilon \in L^\infty(0,T;\, L^2(\Omega))$. We characterize here the equation satisfied by the limit f (in some weak topologies) of $(f^\varepsilon)_\varepsilon$.
The first main result of the paper is the following:

THEOREM 1.1 *Under assumptions (1.2)-(1.4), the sequence $(f^\varepsilon)_\varepsilon$ of solutions of the Vlasov equation (1.1) satisfies, for any $T \in \mathbb{R}^+$,*

$$f^\varepsilon \rightharpoonup f \text{ in } L^\infty(0,T; L^2(\Omega)) \text{ weak} - *.$$

Moreover, denoting for any vector $\mathbf{v}$, $\mathbf{v}_\parallel = (\mathbf{v}\cdot\mathcal{M})\mathcal{M}$, f is the unique solution of:

$$\begin{cases} \dfrac{\partial f}{\partial t} + \mathbf{v}_\parallel \cdot \nabla_x f + (\mathbf{E}_\parallel + \mathbf{v}\times\mathbf{B}_\parallel)\cdot\nabla_v f = 0, \\ f_{|t=0} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f_0(\mathbf{x}, \mathbf{u}(\mathbf{v},\tau))\,d\tau, \end{cases} \tag{1.5}$$

where $\mathbf{u}(\mathbf{v},\tau)$ is the rotation of angle τ around $\mathcal{M}$ applied to $\mathbf{v}$ (see (2.7) for more details).

The deduction of this Theorem uses the notion of two scale convergence introduced by N'Guetseng [15] and Allaire [2]. This convergence result is the following:

THEOREM 1.2 *(N'Guetseng [15] and Allaire [2]) If a sequence f^ε is bounded in $L^\infty(0,T;L^2(\Omega))$, then for every period θ, there exists a θ-periodic profile $F_\theta(t,\tau,\mathbf{x},\mathbf{v}) \in L^\infty(0,T;L^\infty_\theta(\mathbb{R}_\tau;L^2(\Omega)))$ such that for all $\psi_\theta(t,\tau,\mathbf{x},\mathbf{v})$ regular, with compact support with respect to $(t,\mathbf{x},\mathbf{v})$ and θ-periodic with respect to τ we have, up to a subsequence,*

$$\int_{\mathcal{Q}} f^\varepsilon \psi^\varepsilon_\theta \,dt\,d\mathbf{x}\,d\mathbf{v} \to \int_{\mathcal{Q}}\int_0^\theta F_\theta \psi_\theta\,d\tau\,dt\,d\mathbf{x}\,d\mathbf{v}. \tag{1.6}$$

Above, $L^\infty_\theta(\mathbb{R}_\tau)$ stands for the space of functions being $L^\infty(\mathbb{R})$ and being θ-periodic and $\psi^\varepsilon_\theta \equiv \psi_\theta(t,\frac{t}{\varepsilon},\mathbf{x},\mathbf{v})$.

The profile F_θ is called the θ-periodic two scale limit of f^ε and the link between F_θ and the weak$-*$ limit f is given by

$$\int_0^\theta F_\theta(t,\tau,\mathbf{x},\mathbf{v})\,d\tau = f(t,\mathbf{x},\mathbf{v}). \tag{1.7}$$

This result has been used with success in the context of homogenization of transport equation with periodically oscillating coefficients by E [6] and Alexandre and Hamdache [1] and in the context of kinetic equations with strong and periodically oscillating coefficients in Frénod [7] and Frénod and Hamdache [8].
Here, since the strong magnetic field induces periodic oscillations in f^ε, the two scale limit describes well its asymptotic behavior. Therefore, it is the right tool to tackle homogenization of equation (1.1), and Theorem 1.1 is a consequence of the following result concerning the two scale limit of f^ε.

Theorem 1.3 *Under assumptions (1.2)-(1.4), the* 2π*-periodic two scale limit* $F \in L^\infty(0,T;$ $L^\infty_{2\pi}(I\!R_\tau; L^2(\Omega)))$ *of* f^ε *is the unique solution of*

$$\begin{cases} \dfrac{\partial F}{\partial \tau} + (\mathbf{v} \times \mathcal{M}) \cdot \nabla_v F = 0, \\ \dfrac{\partial F}{\partial t} + \mathbf{v}_\| \cdot \nabla_x F + (\mathbf{E}_\| + \mathbf{v} \times \mathbf{B}_\|) \cdot \nabla_v F = 0, \\ F_{|t=0} = \dfrac{1}{2\pi} f_0(\mathbf{x}, \mathbf{u}(\mathbf{v},\tau)), \end{cases} \tag{1.8}$$

where $\mathbf{u}(\mathbf{v},\tau)$ *is the rotation of angle* τ *around* $\mathcal{M}$ *applied to* $\mathbf{v}$ *(see (2.7) for more details).*

Remark 1.1 This last Theorem can be generalised to the case of a non uniform strong magnetic field, see Remark 2.2.

Remark 1.2 The method we develop enables also to deduce the homogenized equations when a strong external electric field $\frac{\mathcal{N}}{\varepsilon}$ is added in equation (1.1). In this case we get, in addition a $\mathcal{N} \times \mathcal{M}$ drift.This drift shares a lot with the one used in the "guiding center" approximation. See Theorems 3.1 and 3.2 for more details.

Secondly, as a relatively direct application of the former results, we may characterize the asymptotic behavior of a sequence $(f^\varepsilon, \mathbf{E}^\varepsilon)$ of solutions of the following Vlasov-Poisson system

$$\begin{cases} \dfrac{\partial f^\varepsilon}{\partial t} + \mathbf{v} \cdot \nabla_x f^\varepsilon + (\mathbf{E}^\varepsilon + \mathbf{v} \times \dfrac{\mathcal{M}}{\varepsilon}) \cdot \nabla_v f^\varepsilon = 0, \\ f^\varepsilon_{|t=0} = f_0, \\ \mathbf{E}^\varepsilon = -\nabla u^\varepsilon, \;\; -\Delta u^\varepsilon = \rho^\varepsilon, \end{cases} \tag{1.9}$$

with

$$\rho^\varepsilon = \int_{I\!R^3_v} f^\varepsilon \, d\mathbf{v},$$

where f_0 is assumed to satisfy

$$f_0 \geq 0, \quad f_0 \in L^1 \cap L^2(\Omega), \text{ and } 0 < \int_\Omega f_0(1 + |v|^2)\, d\mathbf{x}\, d\mathbf{v} < \infty. \tag{1.10}$$

The second main result of the paper is the following:

Theorem 1.4 *Under assumption (1.10), for each* ε*, there exists a solution* $(f^\varepsilon, \mathbf{E}^\varepsilon)$ *of (1.9) in* $L^\infty(0,T; L^1 \cap L^2(\Omega)) \times L^\infty(0,T; W^{1,\frac{7}{5}}(I\!R^3_x)$ *for any* $T \in I\!R^+$*. Moreover this solution is bounded in* $L^\infty(0,T; L^1 \cap L^2(\Omega)) \times L^\infty(0,T; W^{1,\frac{7}{5}}(I\!R^3_x))$ *independently on* ε*.*

If we consider a sequence $(f^\varepsilon, \mathbf{E}^\varepsilon)$ *of such solutions, extracting a subsequence, we have*

$$f^\varepsilon \rightharpoonup f \text{ in } L^\infty(0,T; L^2(\Omega)) \text{ weak} - *,$$

$$\mathbf{E}^\varepsilon \to \mathbf{E} \text{ in } L^\infty(0,T; L^2_{loc}(I\!R^3_x)) \text{ strong},$$

and, denoting for any vector $\mathbf{v}$, $\mathbf{v}_{\|} = (\mathbf{v}\cdot\mathcal{M})\mathcal{M}$, *the limit* $(f, \mathbf{E})$ *satisfies*

$$\left\{\begin{array}{l} \dfrac{\partial f}{\partial t} + \mathbf{v}_{\|}\cdot\nabla_x f + \mathbf{E}_{\|}\cdot\nabla_v f = 0, \\ f_{|t=0} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f_0(\mathbf{x}, \mathbf{u}(\mathbf{v},\tau))\, d\tau, \\ \mathbf{E} = -\nabla u, \quad -\Delta u = \rho, \end{array}\right. \tag{1.11}$$

with

$$\rho = \int_{\mathbb{R}^3_v} f\, d\mathbf{v}, \tag{1.12}$$

where $\mathbf{u}(\mathbf{v},\tau)$ *is the rotation of angle* τ *around* $\mathcal{M}$ *applied to* $\mathbf{v}$ *(see (2.7) for more details).*

This Theorem is a consequence of the following result.

THEOREM 1.5 *Under assumption (1.10), extracting a subsequence, the* $2\pi-$*periodic two scale limit* $F \in L^\infty(0,T; L^\infty_{2\pi}(\mathbb{R}_\tau; L^1\cap L^2(\Omega)))$ *of* f^ε *is solution of*

$$\left\{\begin{array}{l} \dfrac{\partial F}{\partial \tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_v F = 0, \\ \dfrac{\partial F}{\partial t} + \mathbf{v}_{\|}\cdot\nabla_x F + \mathbf{E}_{\|}\cdot\nabla_v F = 0, \\ F_{|t=0} = \dfrac{1}{2\pi} f_0(\mathbf{x}, \mathbf{u}(\mathbf{v},\tau)) \\ \mathbf{E} = -\nabla u, \quad -\Delta u = \overline{\rho}, \end{array}\right. \tag{1.13}$$

with $\overline{\rho}$ *given by*

$$\overline{\rho} = \int_{\mathbb{R}^3_v} F\, d\mathbf{v}. \tag{1.14}$$

Moreover, $\overline{\rho}$ *does not depend on* τ *and*

$$\overline{\rho} = \rho. \tag{1.15}$$

This Theorem provides a rigorous justification of the procedures called subcycling and orbit averaging that are often used in Particle-In-Cell simulations of the Vlasov-Poisson equations in order to reduce the cost of the simulation. This procedure consists in advancing the particles, which provide an approximate solution of the Vlasov equation, on a smaller time step than the one used to advance the solution of the Poisson equations, see [4] for an overview.

REMARK 1.3 We may add in (1.9) an external magnetic field $\mathbf{B}^\varepsilon$ strongly converging to $\mathbf{B}$. The Theorems 1.3 and 1.1 will also apply in the same way leading to equation (1.13) with (1.13.b) replaced by

$$\frac{\partial F}{\partial t} + \mathbf{v}_{\|}\cdot\nabla_x F + (\mathbf{E}_{\|} + \mathbf{v}\times\mathbf{B}_{\|})\cdot\nabla_v F = 0,$$

and to (1.11) with (1.11.a) replaced by

$$\frac{\partial f}{\partial t} + \mathbf{v}_{\|}\cdot\nabla_x f + (\mathbf{E}_{\|} + \mathbf{v}\times\mathbf{B}_{\|})\cdot\nabla_v f = 0.$$

REMARK 1.4 the results of this paper are connected to the ones of Grenier [9, 10] and Schochet [17] concerning the pertubation of hyperbolic systems by a $\frac{1}{\varepsilon}-$depending linear operator inducing fast oscillations in time. In those work, the fast oscillations are canceled applying a "reverse-oscillating" operator to the familly of solutions of the $\varepsilon-$depending equation whose limit is seeked.
Yet, in the present work, the oscillations are treated using an ad-hoc class of oscillating test functions.

From now on, and with no loss of generality, we set $\mathcal{M} = \mathbf{e}_1$, where $\mathbf{e}_1$ is the first vector of the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $I\!R^3$.

The paper is organized as follow: In section 2, we provide the homogenization of the Vlasov equation (1.1). For this purpose, we first deduce the equation satisfied by the two scale limit F of f^ε and then we obtain the homogenized equation of Theorem 1.1.
In section 3 we study the case when a strong electric field, orthogonal to the strong magnetic field, is added in the Vlasov equation.
The last section is devoted to the homogenization of the Vlasov-Poisson system (1.9). This is an application of the result proved in section 4 once the regularity of ρ^ε implying the strong convergence of E^ε is exhibited.

2 Homogenization of the Vlasov equation with a strong external magnetic field

2.1 Two scale limit of the Vlasov equation

Let us derive first the classical a priori estimate that are available for the Vlasov equation (1.1).

LEMMA 2.1 *Under assumption (1.2) on the initial condition f_0 and (1.3),(1.4) on the fields, there exists a constant C independent of ε such that the solution f^ε of the Vlasov equation (1.1) satisfies:*

$$\|f^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{2.1}$$

Proof. We multiply the Vlasov equation (1.1) by f^ε and easily get

$$\frac{d}{dt}\int_\Omega (f^\varepsilon)^2\, d\mathbf{x}\, d\mathbf{v} + \int_\Omega v\cdot\nabla_x (f^\varepsilon)^2\, d\mathbf{x}\, d\mathbf{v} + \int_\Omega (\mathbf{E}^\varepsilon + \mathbf{v}\times(\mathbf{B}^\varepsilon + \frac{\mathcal{M}}{\varepsilon}))\cdot\nabla_v (f^\varepsilon)^2\, d\mathbf{x}\, d\mathbf{v} = 0.$$

Integrating the second and third terms by parts, they vanish. Hence

$$\frac{d}{dt}\int_\Omega (f^\varepsilon)^2\, d\mathbf{x}\, d\mathbf{v} = 0,$$

which means that the L^2 norm of f^ε is conserved and gives us the estimate on f^ε thanks to the bound on the initial condition f_0. ■

From the a priori estimate obtained in lemma 2.1 we deduce that up to a subsequence, still denoted by ε:

$$f^\varepsilon \rightharpoonup f \text{ in } L^\infty(0,T;L^2(\Omega)), \text{ weak} - *.$$

Applying the result of N'Guetseng [15] and Allaire [2] exposed in the Introduction (see Theorem 1.2) we may deduce that for every period θ, there exists a θ-periodic profile $F_\theta(t,\tau,\mathbf{x},\mathbf{v}) \in L^\infty(0,T;L^\infty_\theta(I\!R_\tau;L^2(\Omega)))$ such that for all $\psi_\theta(t,\tau,\mathbf{x},\mathbf{v})$ regular, with compact support with respect to $(t,\mathbf{x},\mathbf{v})$ and θ-periodic with respect to τ we have, up to a subsequence,

$$\int_{\mathcal{Q}} f^\varepsilon \psi_\theta^\varepsilon \, dt\, d\mathbf{x}\, d\mathbf{v} \to \int_{\mathcal{Q}} \int_0^\theta F_\theta \psi_\theta \, d\tau\, dt\, d\mathbf{x}\, d\mathbf{v}. \tag{2.2}$$

The two scale limit of equation (1.1) is led in three steps. In the first one, using a weak formulation of (1.1) with oscillating test functions (see Tartar [18] and Bensoussan, Lions and Papanicolaou [3]), and using the convergence result (2.2), we obtain a constraint equation for the θ-periodic profile F_θ, for every θ. The second step is devoted to the resolution of this constraint equation. This will lead to the natural period $\theta = 2\pi$ for the profile. In the third one, using ad hoc oscillating test functions we deduce the equation satisfied by F.

Step 1. Deduction of the constraint equation: Multiplying the Vlasov equation (1.1) by $\psi_\theta^\varepsilon \equiv \psi_\theta(t,\frac{t}{\varepsilon},\mathbf{x},\mathbf{v})$ with $\psi_\theta(t,\tau,\mathbf{x},\mathbf{v})$ regular, with compact support in $(t,\mathbf{x},\mathbf{v}) \in \mathcal{Q}$ and θ-periodic in $\tau \in I\!R_\tau$, and integrating by parts using that $div_v(\mathbf{E}^\varepsilon + \mathbf{v} \times (\mathbf{B}^\varepsilon + \frac{\mathcal{M}}{\varepsilon})) = 0$, we get

$$\int_{\mathcal{Q}} f^\varepsilon [(\frac{\partial \psi_\theta}{\partial t})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial \psi_\theta}{\partial \tau})^\varepsilon + \mathbf{v} \cdot (\nabla_x \psi_\theta)^\varepsilon + (\mathbf{E}^\varepsilon + \mathbf{v} \times (\mathbf{B}^\varepsilon + \frac{\mathcal{M}}{\varepsilon})) \cdot (\nabla_v \psi_\theta)^\varepsilon]\, dt\, d\mathbf{x}\, d\mathbf{v} = - \int_\Omega f_0 (\psi_\theta)^\varepsilon_{|t=0}\, d\mathbf{x}\, d\mathbf{v}. \tag{2.3}$$

Notice that $\mathcal{Q} = [0,T) \times \Omega$ is not an open set. Consequently, the contribution of the initial data stand in the left hand side of (2.3).
Multiplying (2.3) by ε and passing to the limit, applying (2.2), we deduce that the θ-periodic profile F_θ associated with f^ε satisfies

$$\left\{ \begin{array}{l} \displaystyle\int_{\mathcal{Q}} \int_0^\theta F_\theta (\frac{\partial \psi_\theta}{\partial \tau} + (\mathbf{v} \times \mathcal{M}) \cdot \nabla_v \psi_\theta)\, d\tau\, dt\, d\mathbf{x}\, d\mathbf{v} = 0, \\ F_\theta \text{ is } \theta\text{-periodic in } \tau. \end{array} \right. \tag{2.4}$$

As this is realized for every ψ_θ regular, compactly supported in $(t,\mathbf{x},\mathbf{v}) \in \mathcal{Q}$ and θ-periodic in $\tau \in I\!R_\tau$, it is straightforward to show that (2.4) is equivalent to

$$\left\{ \begin{array}{l} \displaystyle\int_{I\!R^3_v} \int_0^\theta F_\theta (\frac{\partial \psi_\theta}{\partial \tau} + (\mathbf{v} \times \mathcal{M}) \cdot \nabla_v \psi_\theta)\, d\tau\, dt\, d\mathbf{v} = 0, \\ F_\theta \text{ is } \theta\text{-periodic in } \tau, \end{array} \right. \tag{2.5}$$

for every $\psi_\theta \equiv \psi_\theta(\tau, \mathbf{v})$ regular, with compact support in $\mathbf{v}$ and θ-periodic in τ, for almost every $(\mathbf{x}, t)$, and also equivalent to

$$\left\{ \begin{array}{l} \dfrac{\partial F_\theta}{\partial \tau} + (\mathbf{v} \times \mathcal{M}) \cdot \nabla_v F_\theta = 0 \text{ in } \mathcal{D}'(I\!R_\tau \times I\!R^3_v), \\ F_\theta \text{ is } \theta\text{−periodic in } \tau, \end{array} \right. \tag{2.6}$$

for almost every $(\mathbf{x}, t)$. The two equivalent forms (2.5) and (2.6) will be used in the sequel. Since the solution of this equation is not unique, we call it "constraint equation", and the goal is now to derive the form this equation imposes to F_θ.

Step 2. Consequences of the constraint equation: Intuitively, the constraint equation means that F_θ is constant along the characteristics of the dynamical system $\dot{V} = V \times \mathcal{M}$. As we shall see soon, those characteristics are helicoids around the magnetic vector $\mathcal{M}$. Hence we first introduce $\mathbf{u}(\mathbf{v}, \tau)$ the transformation $\mathbf{v} \in I\!R^3 \to \mathbf{u}(\mathbf{v}, \tau) \in I\!R^3$ letting invariant the projection of $\mathbf{v}$ onto $\mathcal{M}$ and rotating of an angle τ its projection onto the plane orthogonal to $\mathcal{M}$. We have:

$$\mathbf{u}(\mathbf{v}, \tau) = v_1 \mathbf{e}_1 + [v_2 \cos\tau - v_3 \sin\tau]\mathbf{e}_2 + [v_2 \sin\tau + v_3 \cos\tau]\mathbf{e}_3. \tag{2.7}$$

Forgetting for the time being the periodicity ondition let us see what the constraint equation yields. Consider $F(\tau, \mathbf{v})$ solution of the following equation

$$\frac{\partial F}{\partial \tau} + (\mathbf{v} \times \mathcal{M}) \cdot \nabla_v F = 0 \text{ in } \mathcal{D}'(I\!R_\tau \times I\!R^3_v). \tag{2.8}$$

Lemma 2.2 *If $F(\tau, \mathbf{v}) \in L^\infty(I\!R_\tau, L^2(I\!R^3_v))$ satisfies (2.8), then there exists a function $G \in L^2(I\!R^3_u)$ such that*

$$F(\tau, \mathbf{v}) = G(\mathbf{u}(\mathbf{v}, \tau)). \tag{2.9}$$

Proof. Following Raviart [16], where weak solutions of first order hyperbolic equations are derived using their characteristic equations, F satisfies (2.8) if and only if it is the translation along the characteristics $V(\tau; \mathbf{v}, s)$ solution of

$$\left\{ \begin{array}{l} \dfrac{dV}{d\tau} = V \times \mathcal{M}, \\ V(s; \mathbf{v}, s) = \mathbf{v}, \end{array} \right. \tag{2.10}$$

of a function G. This means

$$F(\tau, \mathbf{v}) = G(V(0; \mathbf{v}, \tau)).$$

Remark 2.1 In (2.10) $V(\tau; \mathbf{v}, s)$ means the solution at time $t = \tau$ taking the value $\mathbf{v}$ at time $t = s$.

As $\mathcal{M}$ is the first basis vector $\mathbf{e}_1$ of the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $I\!R^3_v$ an easy computation yields the solution of (2.10):

$$V(\tau; \mathbf{v}, s) = v_1 \mathbf{e}_1 + [v_2 \cos(\tau - s) + v_3 \sin(\tau - s)]\mathbf{e}_2 + [-v_2 \sin(\tau - s) + v_3 \cos(\tau - s)]\mathbf{e}_3. \tag{2.11}$$

So we have

$$V(0;\mathbf{v},\tau) = \mathbf{u}(\mathbf{v},\tau),$$

where $\mathbf{u}(\mathbf{v},\tau)$ is defined by (2.7). And $F(\tau,\mathbf{v}) = G(\mathbf{u}(\mathbf{v},\tau))$. ■

Now we take the periodicity condition under consideration. First, because of the previous Lemma and since $\mathbf{u}(\mathbf{v},\tau)$ is 2π-periodic in τ, if the period θ is incommensurable with 2π a fonction $F(\tau,\mathbf{v})$ solution of (2.8) and $\theta-$periodic in τ is constant. As a consequence, the $\theta-$periodic profile F_θ contains no information on the oscillations of (f^ε).
Beside this, if θ is (a multiple of) 2π, a fonction $F(\tau,\mathbf{v})$ satisfying (2.8) naturally satisfy the periodicity condition. More precisely we have the

Lemma 2.3 *A function $F(\tau,\mathbf{v}) \in L^\infty_{2\pi}(I\!R_\tau, L^2(I\!R^3_v))$ satisfies*

$$\frac{\partial F}{\partial \tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_v F = 0 \text{ in } \mathcal{D}'(I\!R_\tau \times I\!R^3_v),$$

if and only if there exists a function $G \in L^2(I\!R^3_u)$ such that

$$F(\tau,\mathbf{v}) = G(\mathbf{u}(\mathbf{v},\tau)). \tag{2.12}$$

Hence, among every possible profile, we are incited to select the $2\pi-$ periodic one.

As a conclusion of this step, applying Lemma 2.3, the 2π-periodic profile $F(t,\tau,\mathbf{x},\mathbf{v}) = F_{2\pi}(t,\tau,\mathbf{x},\mathbf{v})$ associated with the sequence f^ε, solutions of (1.1), writes

$$F(t,\tau,\mathbf{x},\mathbf{v}) = G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)), \tag{2.13}$$

for a function $G \in L^\infty(0,T; L^2(I\!R^3_x \times I\!R^3_u))$.

Step 3. Equation satisfied by G: We now look for the equation satisfied by G linked to F by (2.13). We have

Lemma 2.4 *The function $G(t,\mathbf{x},\mathbf{u})$ linked to the 2π-periodic profile F by (2.13) is the unique solution of:*

$$\begin{aligned} &\frac{\partial G}{\partial t} + \mathbf{u}_\parallel\cdot\nabla_x G + (\mathbf{E}_\parallel + \mathbf{u}\times\mathbf{B}_\parallel)\cdot\nabla_u G = 0,\\ &G_{|t=0} = \frac{1}{2\pi} f_0, \end{aligned} \tag{2.14}$$

where for any vector field $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$, we define $\mathbf{u}_\parallel = u_1\mathbf{e}_1$ its projection onto the direction of the external magnetic field $\mathcal{M}$.

Proof. For any regular function φ, let us consider the regular and 2π-periodic in τ function $\psi(t,\tau,\mathbf{x},\mathbf{v}) = \varphi(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))$ which satisfies

$$\frac{\partial \psi}{\partial \tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_v\psi = 0. \tag{2.15}$$

Now take $\psi^\varepsilon \equiv \psi(t, \frac{t}{\varepsilon}, \mathbf{x}, \mathbf{v})$ as a test function in the weak formulation (2.3) of the Vlasov equation (1.1). Then due to (2.15), we have

$$\int_{\mathcal{Q}} f^\varepsilon[(\frac{\partial \psi}{\partial t})^\varepsilon + \mathbf{v}\cdot(\nabla_x\psi)^\varepsilon + (\mathbf{E}^\varepsilon + \mathbf{v}\times\mathbf{B}^\varepsilon)\cdot(\nabla_v\psi)^\varepsilon]\, dt\, d\mathbf{x}\, d\mathbf{v} = -\int_\Omega f_0(\psi)^\varepsilon_{|t=0}\, d\mathbf{x}\, d\mathbf{v}. \quad (2.16)$$

Passing to the two scale limit, using that $\mathbf{E}^\varepsilon$ and $\mathbf{B}^\varepsilon$ converge strongly, yields

$$\int_{\mathcal{Q}}\int_0^{2\pi} F[\frac{\partial\psi}{\partial t} + \mathbf{v}\cdot\nabla_x\psi + (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot(\nabla_v\psi)]\, d\tau\, dt\, d\mathbf{x}\, d\mathbf{v} = -\int_\Omega f_0\psi(0,0,\mathbf{x},\mathbf{v})\, d\mathbf{x}\, d\mathbf{v}, \quad (2.17)$$

which is equivalent to

$$\begin{aligned}\int_{\mathcal{Q}}\int_0^{2\pi} G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))[\frac{\partial\varphi}{\partial t}(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)) + \mathbf{v}\cdot\nabla_x\varphi(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))\\ +(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_v\{\varphi(t,\mathbf{x},\mathbf{u}(\tau,\mathbf{v}))\}]\, d\tau\, dt\, d\mathbf{x}\, d\mathbf{v} = -\int_\Omega f_0\varphi(0,\mathbf{x},\mathbf{u}(\mathbf{v},0))\, d\mathbf{x}\, d\mathbf{v}. \quad (2.18)\end{aligned}$$

Making the change of variables $\mathbf{u} = \mathbf{u}(\mathbf{v},\tau)$, noticing that $\mathbf{v} = V(\tau;\mathbf{u},0)$ with the notations of (2.10) and that $d\mathbf{u} = d\mathbf{v}$, we get on the left-hand-side of (2.18)

$$\begin{aligned}\int_{\mathcal{Q}}\int_0^{2\pi} G(t,\mathbf{x},\mathbf{u})[\frac{\partial\varphi}{\partial t}(t,\mathbf{x},\mathbf{u}) + [u_1\mathbf{e}_1 + [u_2\cos\tau + u_3\sin\tau]\mathbf{e}_2\\ +[-u_2\sin\tau + u_3\cos\tau]\mathbf{e}_3]\cdot\nabla_x\varphi(t,\mathbf{x},\mathbf{u}) + (\mathbf{E} + [u_1\mathbf{e}_1 + [u_2\cos\tau + u_3\sin\tau]\mathbf{e}_2\\ +[-u_2\sin\tau + u_3\cos\tau]\mathbf{e}_3]\times\mathbf{B})\cdot[\frac{\partial\varphi}{\partial u_1}\mathbf{e}_1 + [\cos\tau\frac{\partial\varphi}{\partial u_2} + \sin\tau\frac{\partial\varphi}{\partial u_3}]\mathbf{e}_2\\ +[-\sin\tau\frac{\partial\varphi}{\partial u_2} + \cos\tau\frac{\partial\varphi}{\partial u_3}]\mathbf{e}_3]\, d\tau\, dt\, d\mathbf{x}\, d\mathbf{u}. \quad (2.19)\end{aligned}$$

Now as neither G nor φ depend on τ, let us perform the integration with respect to τ. This yields, dividing by 2π,

$$\begin{aligned}\int_{\mathcal{Q}} G[\frac{\partial\varphi}{\partial t} + u_1\mathbf{e}_1\cdot\nabla_x\varphi + (E_1\frac{\partial\varphi}{\partial u_1} + B_1(u_3\frac{\partial\varphi}{\partial u_2} - u_2\frac{\partial\varphi}{\partial u_3})]\, dt\, d\mathbf{x}\, d\mathbf{u}\\ = -\frac{1}{2\pi}\int_\Omega f_0\varphi(0,\mathbf{x},\mathbf{u}))\, d\mathbf{x}\, d\mathbf{u}.\end{aligned}$$

as the terms in $\cos\tau$, $\sin\tau$ and $\cos\tau\sin\tau$ vanish when integrated between 0 and 2π. This gives us the equation verified by G in the sense of distributions. ■

The uniqueness of the solution of (2.14) enables to deduce that the whole sequence f^ε two scale converges to F and, because of the link (1.7) between F and f, weakly$-*$ converges to f.

2.2 Proof of Theorems 1.1 and 1.3

Replacing first $\mathbf{u}$ by $\mathbf{u}(\mathbf{v},\tau)$ in (2.14) we have

$$\begin{aligned}&\frac{\partial G}{\partial t}(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)) + \mathbf{u}_\|(\mathbf{v},\tau)\cdot\nabla_x G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))\\ &\qquad + (\mathbf{E}_\| + \mathbf{u}(\mathbf{v},\tau)\times\mathbf{B}_\|)\cdot(\nabla_u G)(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)) = 0,\\ &G(0,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)) = \frac{1}{2\pi}f_0(\mathbf{x},\mathbf{u}(\mathbf{v},\tau)).\end{aligned} \quad (2.20)$$

Secondly, we get

$$(\mathbf{E}_\parallel + \mathbf{u}(\mathbf{v},\tau)\times\mathbf{B}_\parallel)\cdot(\nabla_u G)(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)) = (\mathbf{E}_\parallel + \mathbf{v}\times\mathbf{B}_\parallel)\cdot\nabla_v(G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))). \tag{2.21}$$

Indeed, straightforward computations give

$$\begin{aligned}
&E_1\frac{\partial}{\partial v_1}(G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))) = E_1\frac{\partial G}{\partial u_1}(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)),\\
&B_1 u_3\frac{\partial}{\partial v_2}(G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))) = B_1(v_2\sin\tau\cos\tau\frac{\partial G}{\partial u_2} + v_3\cos^2\tau\frac{\partial G}{\partial u_2} +\\
&\qquad\qquad\qquad\qquad v_2\sin^2\tau\frac{\partial G}{\partial u_3} + v_3\sin\tau\cos\tau\frac{\partial G}{\partial u_3}),\\
&B_1 u_2\frac{\partial}{\partial v_2}(G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))) = -B_1(v_2\sin\tau\cos\tau\frac{\partial G}{\partial u_2} + v_3\sin^2\tau\frac{\partial G}{\partial u_2} +\\
&\qquad\qquad\qquad\qquad v_2\cos^2\tau\frac{\partial G}{\partial u_3} - v_3\sin\tau\cos\tau\frac{\partial G}{\partial u_3}),
\end{aligned}$$

and summing up these three relations yields (2.21). Hence (2.20) writes

$$\begin{aligned}
&\frac{\partial G}{\partial t}(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)) + \mathbf{u}_\parallel(\mathbf{v},\tau)\cdot\nabla_x G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))\\
&\qquad\qquad + (\mathbf{E}_\parallel + \mathbf{v}\times\mathbf{B}_\parallel)\cdot\nabla_v(G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))) = 0,\\
&G(0,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)) = \frac{1}{2\pi}f_0(\mathbf{x},\mathbf{u}(\mathbf{v},\tau)),
\end{aligned} \tag{2.22}$$

and since

$$F(t,\tau,\mathbf{x},\mathbf{v}) = G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)),$$

we obtain equation (1.8.b,c). As (1.8.a) is only the constraint, Theorem 1.3 is proved. ■

Then, in order to achieve the proof of Theorem 1.3, we shall deduce the equation satisfied by f using the integral relation (1.7) linking F and f.

As

$$f(t,\mathbf{x},\mathbf{v}) = \int_0^{2\pi} F(t,\tau,\mathbf{x},\mathbf{v})\,d\tau = \int_0^{2\pi} G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))\,d\tau,$$

integrating (2.22) in τ, gives

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_0^{2\pi} G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))\,d\tau\right) + \mathbf{v}_\parallel\cdot\nabla_x\left(\int_0^{2\pi} G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))\,d\tau\right)\\
&\qquad\qquad +(\mathbf{E}_\parallel + \mathbf{v}\times\mathbf{B}_\parallel)\cdot\nabla_v\left(\int_0^{2\pi} G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))\,d\tau\right) = 0,\\
&\int_0^{2\pi} G(0,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))\,d\tau = \frac{1}{2\pi}\int_0^{2\pi} f_0(\mathbf{x},\mathbf{u}(\mathbf{v},\tau))\,d\tau,
\end{aligned} \tag{2.23}$$

giving the homogenized Vlasov equation (1.5). Hence Theorem 1.1 is proved ■

Remark 2.2 The goal of this remark is to show how we may adapt our method to deduce the equation for the two scale limit in the case of a non uniform strong magnetic field. We assume that $\mathcal{M}(t,\mathbf{x})$ is a smooth function from $\mathcal{O}$ to the set of vectors with norme 1. That implies

$$\mathbf{e}_1 = \mathcal{R}(t,\mathbf{x})\mathcal{M}(t,\mathbf{x}), \tag{2.24}$$

where $\mathcal{R}(t,\mathbf{x})$ is a smooth map from $\mathcal{O}$ to the set of orthogonal matrices. In this framework, the way to deduce the constraint equation

$$\frac{\partial F}{\partial \tau} + (\mathbf{v} \times \mathcal{M}) \cdot \nabla_v F = 0, \tag{2.25}$$

remains the same. As $\mathcal{R}(t,\mathbf{x})$ is nothing but the matrix of a change of coordinates leading $\mathcal{M}(t,\mathbf{x})$ onto $\mathbf{e}_1$ this constraint implies

$$F(t,\tau,\mathbf{x},\mathbf{v}) = G(t,\mathbf{x},\mathcal{R}^T(t,\mathbf{x})\mathbf{u}(\mathcal{R}(t,\mathbf{x})\mathbf{v},\tau)), \tag{2.26}$$

where $\mathcal{R}^T$ is the transposed (but also the reverse) matrix of $\mathcal{R}$. In order to give a more usable form to (2.26), we denote by

$$\mathbf{r}(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tau & -\sin\tau \\ 0 & \sin\tau & \cos\tau \end{pmatrix}, \tag{2.27}$$

and since $\mathcal{R}^T\mathbf{u}(\mathcal{R}\mathbf{v},\tau)) = \mathcal{R}^T\mathbf{r}(\tau)\mathcal{R}\mathbf{v}$, (2.26) also writes

$$F(t,\tau,\mathbf{x},\mathbf{v}) = G(t,\mathbf{x},\mathcal{R}^T(t,\mathbf{x})\mathbf{r}(\tau)\mathcal{R}(t,\mathbf{x})\mathbf{v}). \tag{2.28}$$

Now, in order to get the equation satisfied by G, we proceed in a similar way as in the proof of Lemma 2.4, with this difference that we use test functions

$$\psi(t,\tau,\mathbf{x},\mathbf{v}) = \varphi(t,\mathbf{x},\mathcal{R}^T(t,\mathbf{x})\mathbf{r}(\tau)\mathcal{R}(t,\mathbf{x})\mathbf{v}). \tag{2.29}$$

Those test functions naturally satisfy the constraint and the computation leading to equation (2.17) remains valid. But here,

$$\frac{\partial \psi}{\partial t} = \frac{\partial \varphi}{\partial t} + (\frac{\partial \mathcal{R}^T}{\partial t}\mathbf{r}(\tau)\mathcal{R}\mathbf{v} + \mathcal{R}^T\mathbf{r}(\tau)\frac{\partial \mathcal{R}}{\partial t}\mathbf{v}) \cdot \nabla_u \varphi, \tag{2.30}$$

$$\frac{\partial \psi}{\partial x_i} = \frac{\partial \varphi}{\partial x_i} + (\frac{\partial \mathcal{R}^T}{\partial x_i}\mathbf{r}(\tau)\mathcal{R}\mathbf{v} + \mathcal{R}^T\mathbf{r}(\tau)\frac{\partial \mathcal{R}}{\partial x_i}\mathbf{v}) \cdot \nabla_u \varphi, \tag{2.31}$$

and

$$\nabla_v \psi = (\mathcal{R}^T\mathbf{r}(\tau)\mathcal{R})^T \nabla_u \varphi = (\mathcal{R}^T\mathbf{r}(-\tau)\mathcal{R})\nabla_u \varphi. \tag{2.32}$$

Hence, in place of formula (2.19), making the change of variables $\mathbf{u} = \mathcal{R}^T\mathbf{r}(\tau)\mathcal{R}\mathbf{v}$, ($\mathbf{v} = \mathcal{R}^T\mathbf{r}(-\tau)\mathcal{R}\mathbf{u}$), we get

$$\begin{aligned} \int_{\mathcal{Q}}\int_0^{2\pi} G(t,\mathbf{x},\mathbf{u})[\frac{\partial \varphi}{\partial t} + (\frac{\partial \mathcal{R}^T}{\partial t}\mathcal{R}\mathbf{u} + \mathcal{R}^T\mathbf{r}(\tau)\frac{\partial \mathcal{R}}{\partial t}\mathcal{R}^T\mathbf{r}(-\tau)\mathcal{R}\mathbf{u}) \cdot \nabla_u \varphi \\ +\mathcal{R}^T\mathbf{r}(-\tau)\mathcal{R}\mathbf{u} \cdot (\nabla_x \varphi + \begin{pmatrix} (\frac{\partial \mathcal{R}^T}{\partial x_1}\mathcal{R}\mathbf{u} + \mathcal{R}^T\mathbf{r}(\tau)\frac{\partial \mathcal{R}}{\partial x_1}\mathcal{R}^T\mathbf{r}(-\tau)\mathcal{R}\mathbf{u}) \cdot \nabla_u \varphi \\ (\frac{\partial \mathcal{R}^T}{\partial x_2}\mathcal{R}\mathbf{u} + \mathcal{R}^T\mathbf{r}(\tau)\frac{\partial \mathcal{R}}{\partial x_2}\mathcal{R}^T\mathbf{r}(-\tau)\mathcal{R}\mathbf{u}) \cdot \nabla_u \varphi \\ (\frac{\partial \mathcal{R}^T}{\partial x_3}\mathcal{R}\mathbf{u} + \mathcal{R}^T\mathbf{r}(\tau)\frac{\partial \mathcal{R}}{\partial x_3}\mathcal{R}^T\mathbf{r}(-\tau)\mathcal{R}\mathbf{u}) \cdot \nabla_u \varphi \end{pmatrix}) \\ +\mathbf{E} \cdot \mathcal{R}^T\mathbf{r}(-\tau)\mathcal{R}\nabla_u \varphi + ((\mathcal{R}^T\mathbf{r}(-\tau)\mathcal{R}\mathbf{u}) \times \mathbf{B}) \cdot (\mathcal{R}^T\mathbf{r}(-\tau)\mathcal{R}\nabla_u \varphi)]\, d\tau\, dt\, d\mathbf{x}\, d\mathbf{u} \\ = -\int_\Omega f_0 \varphi(0,\mathbf{x},\mathbf{u}))\, d\mathbf{x}\, d\mathbf{u}. \end{aligned} \tag{2.33}$$

Since

$$\mathbf{E} \cdot \mathcal{R}^T \mathbf{r}(-\tau)\mathcal{R}\nabla_u \varphi = \mathcal{R}^T \mathbf{r}(\tau)\mathcal{R}\mathbf{E} \cdot \nabla_u \varphi,$$

and

$$((\mathcal{R}^T \mathbf{r}(-\tau)\mathcal{R}\mathbf{u}) \times \mathbf{B}) \cdot (\mathcal{R}^T \mathbf{r}(-\tau)\mathcal{R}\nabla_u \varphi) = \mathcal{R}^T \mathbf{r}(\tau)\mathcal{R}\mathcal{B}\mathcal{R}^T \mathbf{r}(-\tau)\mathcal{R}\mathbf{u} \cdot \nabla_u \varphi,$$

with

$$\mathcal{B} = \begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix},$$

we finally get, integrating (2.33) in τ and defining

$$\overline{\mathbf{r}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned}
&\frac{\partial G}{\partial t} + \mathcal{R}^T \overline{\mathbf{r}}\mathcal{R}\mathbf{u} \cdot \nabla_x G + (\mathcal{R}^T \overline{\mathbf{r}}\mathcal{R}\mathbf{E} + \frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}^T \mathbf{r}(\tau)\mathcal{R}\mathcal{B}\mathcal{R}^T \mathbf{r}(-\tau)\mathcal{R} d\tau \ \mathbf{u}) \cdot \nabla_u G \\
&\qquad + \nabla_u \cdot [(\frac{\partial \mathcal{R}^T}{\partial t}\mathcal{R}\mathbf{u} + \frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}^T \mathbf{r}(\tau)\frac{\partial \mathcal{R}}{\partial t}\mathcal{R}^T \mathbf{r}(-\tau)\mathcal{R} d\tau \ \mathbf{u})G] \\
&\qquad + \nabla_u \cdot [(\frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}^T \mathbf{r}(-\tau)\mathcal{R}\mathbf{u} \cdot \begin{pmatrix} (\frac{\partial \mathcal{R}^T}{\partial x_1}\mathcal{R}\mathbf{u} + \mathcal{R}^T \mathbf{r}(\tau)\frac{\partial \mathcal{R}}{\partial x_1}\mathcal{R}^T \mathbf{r}(-\tau)\mathcal{R}\mathbf{u})G \\ (\frac{\partial \mathcal{R}^T}{\partial x_2}\mathcal{R}\mathbf{u} + \mathcal{R}^T \mathbf{r}(\tau)\frac{\partial \mathcal{R}}{\partial x_2}\mathcal{R}^T \mathbf{r}(-\tau)\mathcal{R}\mathbf{u})G \\ (\frac{\partial \mathcal{R}^T}{\partial x_3}\mathcal{R}\mathbf{u} + \mathcal{R}^T \mathbf{r}(\tau)\frac{\partial \mathcal{R}}{\partial x_3}\mathcal{R}^T \mathbf{r}(-\tau)\mathcal{R}\mathbf{u})G \end{pmatrix})d\tau] = 0, \\
&G_{|t=0} = \frac{1}{2\pi} f_0.
\end{aligned} \tag{2.34}$$

where for 4 vectors of $I\!R^3$, A, B, C, D we denote

$$A \cdot \begin{pmatrix} B \\ C \\ D \end{pmatrix} = A_1 B + A_2 C + A_3 D.$$

This last equation is the equation for the two scale limit when the strong magnetic field is non uniform.

3 Homogenization of the Vlasov equation with strong external magnetic and electric fields

We study here a variant of the previous problem. We homogenize the following Vlasov equation with strong and constant external magnetic and electric fields:

$$\left\{ \begin{array}{l} \dfrac{\partial f^\varepsilon}{\partial t} + \mathbf{v} \cdot \nabla_x f^\varepsilon + ((\mathbf{E}^\varepsilon + \dfrac{\mathcal{N}}{\varepsilon}) + \mathbf{v} \times (\mathbf{B}^\varepsilon + \dfrac{\mathcal{M}}{\varepsilon})) \cdot \nabla_v f^\varepsilon = 0, \\ f^\varepsilon_{|t=0} = f_0, \end{array} \right. \tag{3.1}$$

for constant vectors $\mathcal{M} \in \mathbb{S}^2$, $\mathcal{N} \in \mathbb{S}^2$, $\mathcal{M} \perp \mathcal{N}$ and under the same assumptions (1.2)-(1.4) as previously. With no lost of generality, we set $\mathcal{M} = \mathbf{e}_1$ and $\mathcal{N} = \mathbf{e}_2$.

Since the a priori estimate (2.1) remains valid we always have

$$f^\varepsilon \rightharpoonup f \text{ in } L^\infty(0,T;L^2(\Omega)), \text{ weak} - *.$$

And, there exists a 2π-periodic profile $F(t,\tau,\mathbf{x},\mathbf{v}) \in L^\infty(0,T;L^\infty_{2\pi}(\mathbb{R}_\tau;L^2(\Omega)))$ such that for all $\psi(t,\tau,\mathbf{x},\mathbf{v})$ regular, with compact support with respect to $(t,\mathbf{x},\mathbf{v})$ and 2π-periodic with respect to τ we have

$$\int_{\mathcal{Q}} f^\varepsilon \psi^\varepsilon \, dt \, d\mathbf{x} \, d\mathbf{v} \to \int_{\mathcal{Q}} \int_0^{2\pi} F\psi \, d\tau \, dt \, d\mathbf{x} \, d\mathbf{v}. \tag{3.2}$$

Proceeding as in section 2.1 we obtain the following weak formulation with oscillating test functions:

$$\begin{aligned} &\int_{\mathcal{Q}} f^\varepsilon [(\frac{\partial \psi}{\partial t})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial \psi}{\partial \tau})^\varepsilon + \mathbf{v} \cdot (\nabla_x \psi)^\varepsilon \\ &\quad + ((\mathbf{E}^\varepsilon + \frac{\mathcal{N}}{\varepsilon}) + \mathbf{v} \times (\mathbf{B}^\varepsilon + \frac{\mathcal{M}}{\varepsilon})) \cdot (\nabla_v \psi)^\varepsilon] \, dt \, d\mathbf{x} \, d\mathbf{v} = - \int_\Omega f_0 (\psi)^\varepsilon_{|t=0} \, d\mathbf{x} \, d\mathbf{v}, \end{aligned} \tag{3.3}$$

which, passing to the limit in ε yields

$$\frac{\partial F}{\partial \tau} + (\mathcal{N} + \mathbf{v} \times \mathcal{M}) \cdot \nabla_v F = 0 \text{ in } \mathcal{D}'(\mathbb{R}_\tau \times \mathbb{R}^3_v). \tag{3.4}$$

This constraint equation means that

$$F(t,\tau,\mathbf{x},\mathbf{v}) = G(t,\mathbf{x},V(0;\mathbf{v},\tau)),$$

where $V(\tau;\mathbf{v},s)$ is solution of

$$\begin{cases} \dfrac{dV}{d\tau} = \mathcal{N} + V \times \mathcal{M}, \\ V(s;\mathbf{v},s) = \mathbf{v}. \end{cases} \tag{3.5}$$

We have

$$V(\tau;\mathbf{v},s) = v_1\mathbf{e}_1 + [v_2\cos(\tau-s) + (v_3+1)\sin(\tau-s)]\mathbf{e}_2 + [-v_2\sin(\tau-s) + (v_3+1)\cos(\tau-s) - 1]\mathbf{e}_3, \tag{3.6}$$

and then

$$V(0;\mathbf{v},\tau) = \mathbf{u}(\mathbf{v},\tau) = v_1\mathbf{e}_1 + [v_2\cos\tau - (v_3+1)\sin\tau]\mathbf{e}_2 + [+v_2\sin\tau + (v_3+1)\cos\tau - 1]\mathbf{e}_3. \tag{3.7}$$

Hence we conclude

$$F(t,\tau,\mathbf{x},\mathbf{v}) = G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)),$$

for a function $G \in L^\infty(0,T;L^2(\mathbb{R}^3_x \times \mathbb{R}^3_u))$.

Remark 3.1 If $\mathcal{N}$ were not orthogonal to $\mathcal{M}$, i.e. if $\mathcal{N} = n_1\mathbf{e}_1 + \mathbf{e}_2$, the consequence of equation (3.4) would be $F(t,\tau,\mathbf{x},\mathbf{v}) = G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau) + n_1\tau\mathbf{e}_1)$. Then the periodicity condition and the L^2 in $\mathbf{v}$ regularity of F would imply $G = F = 0$.

Now in order to deduce the equation G satisfies, we take any regular function φ and we consider the regular and 2π-periodic in τ function $\psi(t,\tau,\mathbf{x},\mathbf{v}) = \varphi(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))$ which satisfies

$$\frac{\partial\psi}{\partial\tau} + (\mathcal{N} + \mathbf{v}\times\mathcal{M})\cdot\nabla_v\psi = 0. \tag{3.8}$$

Hence using this test function in the weak formulation (3.3), the term containing the constraint disappears and passing then to the limit gives

$$\int_{\mathcal{Q}}\int_0^{2\pi} F[\frac{\partial\psi}{\partial t}+\mathbf{v}\cdot\nabla_x\psi+(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot(\nabla_v\psi)]\,d\tau\,dt\,d\mathbf{x}\,d\mathbf{v} = -\int_{\Omega} f_0\psi(0,0,\mathbf{x},\mathbf{v})\,d\mathbf{x}\,d\mathbf{v}. \tag{3.9}$$

Replacing F and ψ by theire expressions, (3.9) becomes

$$\begin{aligned}\int_{\mathcal{Q}}\int_0^{2\pi} & G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))[\frac{\partial\varphi}{\partial t}(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)) + \mathbf{v}\cdot\nabla_x\varphi(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)) \\ & +(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_v\{\varphi(t,\mathbf{x},\mathbf{u}(\tau,\mathbf{v}))\}]\,d\tau\,dt\,d\mathbf{x}\,d\mathbf{v} = -\int_{\Omega} f_0\varphi(0,\mathbf{x},\mathbf{u}(\mathbf{v},0))\,d\mathbf{x}\,d\mathbf{v}.\end{aligned} \tag{3.10}$$

Making the change of variables $\mathbf{u} = \mathbf{u}(\mathbf{v},\tau)$, we get on the left-hand-side of (3.10)

$$\begin{aligned}\int_{\mathcal{Q}}\int_0^{2\pi} & G(t,\mathbf{x},\mathbf{u})[\frac{\partial\varphi}{\partial t}(t,\mathbf{x},\mathbf{u}) + [u_1\mathbf{e}_1 + [u_2\cos\tau + (u_3+1)\sin\tau]\mathbf{e}_2 \\ & \qquad +[-u_2\sin\tau + (u_3+1)\cos\tau - 1]\mathbf{e}_3]\cdot\nabla_x\varphi(t,\mathbf{x},\mathbf{u}) \\ & +(\mathbf{E} + [u_1\mathbf{e}_1 + [u_2\cos\tau + (u_3+1)\sin\tau]\mathbf{e}_2 \\ & \quad + [-u_2\sin\tau + (u_3+1)\cos\tau - 1]\mathbf{e}_3]\times\mathbf{B})\cdot[\frac{\partial\varphi}{\partial u_1}\mathbf{e}_1 + [\cos\tau\frac{\partial\varphi}{\partial u_2} + \sin\tau\frac{\partial\varphi}{\partial u_3}]\mathbf{e}_2 \\ & \qquad + [-\sin\tau\frac{\partial\varphi}{\partial u_2} + \cos\tau\frac{\partial\varphi}{\partial u_3}]\mathbf{e}_3]\,d\tau\,dt\,d\mathbf{x}\,d\mathbf{u}.\end{aligned} \tag{3.11}$$

Now as neither G nor φ depend on τ, let us perform the integration with respect to τ. This yields, dividing by 2π,

$$\begin{aligned}\int_{\mathcal{Q}} G[\frac{\partial\varphi}{\partial t} + (u_1\mathbf{e}_1 - \mathbf{e}_3)\cdot\nabla_x\varphi + ((E_1 - B_2)\frac{\partial\varphi}{\partial u_1} + B_1((u_3+1)\frac{\partial\varphi}{\partial u_2} - u_2\frac{\partial\varphi}{\partial u_3})]\,dt\,d\mathbf{x}\,d\mathbf{u} \\ = -\frac{1}{2\pi}\int_{\Omega} f_0\varphi(0,\mathbf{x},\mathbf{u}))\,d\mathbf{x}\,d\mathbf{u}.\end{aligned}$$

Hence we proved the following result

Theorem 3.1 *Under assumptions (1.2)-(1.4) and with $\mathcal{M} = \mathbf{e}_1$ and $\mathcal{N} = \mathbf{e}_2$, the function $G(t,\mathbf{x},\mathbf{u})$ linked to the 2π-periodic profile F associated with the solution of (3.1) is the unique solution of:*

$$\begin{aligned}&\frac{\partial G}{\partial t} + \begin{pmatrix} u_1 \\ 0 \\ -1 \end{pmatrix}\cdot\nabla_x G + \left[\begin{pmatrix} E_1 - B_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3+1 \end{pmatrix}\times\begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}\right]\cdot\nabla_u G = 0, \\ &G_{|t=0} = \frac{1}{2\pi}f_0,\end{aligned} \tag{3.12}$$

Now we deduce the equation satisfied by the weak limit f using the relation linking F and f.

Replacing $\mathbf{u}$ by $\mathbf{u}(\mathbf{v},\tau)$ in (3.12), we have

$$
\begin{aligned}
&\frac{\partial G}{\partial t}(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)) + \begin{pmatrix} u_1(\mathbf{v},\tau) \\ 0 \\ -1 \end{pmatrix} \cdot \nabla_x G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)) \\
&\qquad + \left[\begin{pmatrix} E_1 - B_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u_1(\mathbf{v},\tau) \\ u_2(\mathbf{v},\tau) \\ u_3(\mathbf{v},\tau)+1 \end{pmatrix} \times \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}\right] \cdot \nabla_u G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)) = 0, \\
&G(0,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)) = \frac{1}{2\pi} f_0(\mathbf{x},\mathbf{u}(\mathbf{v},\tau)).
\end{aligned}
\tag{3.13}
$$

Then as

$$
\begin{aligned}
&\left[\begin{pmatrix} E_1 - B_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u_1(\mathbf{v},\tau) \\ u_2(\mathbf{v},\tau) \\ u_3(\mathbf{v},\tau)+1 \end{pmatrix} \times \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}\right] \cdot \nabla_u G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau)) = \\
&\qquad \left[\begin{pmatrix} E_1 - B_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3+1 \end{pmatrix} \times \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}\right] \cdot \nabla_v (G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))),
\end{aligned}
\tag{3.14}
$$

and as

$$
f(t,\mathbf{x},\mathbf{v}) = \int_0^{2\pi} F(t,\tau,\mathbf{x},\mathbf{v})\,d\tau = \int_0^{2\pi} G(t,\mathbf{x},\mathbf{u}(\mathbf{v},\tau))\,d\tau,
$$

we have the

Theorem 3.2 *Under assumptions (1.2)-(1.4) and with $\mathcal{M} = \mathbf{e}_1$ and $\mathcal{N} = \mathbf{e}_2$, the weak$-*$ limit f of the sequence of solutions of (3.1) is solution of:*

$$
\begin{aligned}
&\frac{\partial f}{\partial t} + \begin{pmatrix} v_1 \\ 0 \\ -1 \end{pmatrix} \cdot \nabla_x f + \left[\begin{pmatrix} E_1 - B_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3+1 \end{pmatrix} \times \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}\right] \cdot \nabla_v f = 0, \\
&f_{|t=0} = \frac{1}{2\pi} \int_0^{2\pi} f_0(\mathbf{x},\mathbf{u}(\mathbf{v},\tau))\,d\tau,
\end{aligned}
\tag{3.15}
$$

with $\mathbf{u}(\mathbf{v},\tau)$ given by (3.7)

Remark 3.2 As

$$
\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \mathcal{N} \times \mathcal{M},
$$

the presence of -1 on the third componant of the advection vector shares a lot with the additional $\mathcal{N} \times \mathcal{M}$ drift effect usually found in the "guiding center" approximation with same order electric and magnetic fields.

4 Homogenization of the Vlasov-Poisson system with a strong external magnetic field

Assume that the initial condition satisfies the hypothesis (1.10). Then for $\varepsilon > 0$ given there exists a solution in $L^\infty(0,T;L^1\cap L^2(\Omega)) \times L^\infty(0,T;W^{1,\frac{7}{5}}(I\!R^3_x))$ for any $T \in I\!R^+$ of the Vlasov-Poisson system (1.9). Indeed for a given $\varepsilon > 0$ a force field associated to a constant magnetic field has simply been added to the system studied by Horst and Hunze [12] and their result can be extended easily to our case.

Now in order to apply the results of section 2 and pass to the two-scale limit of the Vlasov equation, we need to prove that there exists a sequence of electric fields solution of the Vlasov-Poisson system (1.9) verifying

$$\mathbf{E}^\varepsilon \to \mathbf{E} \text{ in } L^\infty(0,T;L^2_{loc}(I\!R^3_x)) \text{ strong},$$

for any $T \in I\!R^+$. For this, we shall need some apriori estimates independent on ε.

Lemma 4.1 *Assume that the initial condition f_0 is such that $f_0 \in L^2(\Omega)$ and $\int_\Omega f_0|v|^2\, d\mathbf{x}d\mathbf{v}$ is bounded, then there exists a constant C independent of ε such that the solution $(\mathbf{E}^\varepsilon, f^\varepsilon)$ of the Vlasov-Poisson equations (1.9) satisfies, for any $T \in I\!R^+$,*

$$\|f^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \le C,$$

$$\|(|v|^2 f^\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \le C,$$

and moreover

$$\|\rho^\varepsilon(\mathbf{x},t)\|_{L^\infty(0,T;L^{\frac{7}{5}}(I\!R^3_x))} \le C,$$

$$\|\mathbf{J}^\varepsilon(\mathbf{x},t)\|_{L^\infty(0,T;L^{\frac{7}{6}}(I\!R^3_x))} \le C,$$

where $\rho^\varepsilon(\mathbf{x},t) = \int f^\varepsilon\, d\mathbf{v}$ and $\mathbf{J}^\varepsilon(\mathbf{x},t) = \int \mathbf{v} f^\varepsilon\, d\mathbf{v}$.

Proof. Let us proceed formally in order to simplify the presentation. All the following is rigorously verified for regularized solutions and then the bounds, that do not depend on ε, are conserved when passing to the limit with respect to the regularizing parameter.

Multiplying the Vlasov equation by $|\mathbf{v}|^2$ and integrating with respect to $\mathbf{x}$ and $\mathbf{v}$, we get

$$\frac{d}{dt}\int_\Omega f^\varepsilon |\mathbf{v}|^2\, d\mathbf{v}\, d\mathbf{x} - 2\int_{I\!R^3_x} \mathbf{J}^\varepsilon \cdot \mathbf{E}^\varepsilon\, d\mathbf{x} = 0. \tag{4.1}$$

Now, integrating the Vlasov equation with respect to $\mathbf{v}$ yields the continuity equation

$$\frac{\partial \rho^\varepsilon}{\partial t} + \nabla \cdot \mathbf{J}^\varepsilon = 0. \tag{4.2}$$

Using this, we obtain

$$\begin{aligned}\int_{I\!R^3_x} \mathbf{J}^\varepsilon \cdot \mathbf{E}^\varepsilon\, d\mathbf{x} &= -\int_{I\!R^3_x} \mathbf{J}^\varepsilon \cdot \nabla u^\varepsilon\, d\mathbf{x} = \int_{I\!R^3_x} \nabla\cdot\mathbf{J}^\varepsilon\, u^\varepsilon\, d\mathbf{x},\\ &= -\int_{I\!R^3_x} \frac{\partial \rho_\varepsilon}{\partial t} u^\varepsilon\, d\mathbf{x}.\end{aligned}$$

But the Poisson equation yields

$$\frac{1}{2}\frac{d}{dt}\int_{I\!R^3_x}(\nabla u^\varepsilon)^2\,d\mathbf{x}=\int_{I\!R^3_x}\frac{\partial\rho_\varepsilon}{\partial t}\,u^\varepsilon\,d\mathbf{x}.$$

Hence (4.1) becomes

$$\frac{d}{dt}\Big(\int_\Omega f^\varepsilon|\mathbf{v}|^2\,d\mathbf{v}\,d\mathbf{x}+\int_{I\!R^3_x}(\nabla u^\varepsilon)^2\,d\mathbf{x}\Big)=0.$$

Finally, integrating this last equation with respect to t and using the hypothesis on the initial condition f_0, we get the bound on $\int_\Omega f^\varepsilon|\mathbf{v}|^2\,d\mathbf{v}\,d\mathbf{x}$.

Let us now proceed with the L^2 estimate on f^ε. To this aim we multiply the Vlasov equation by f^ε and easily get

$$\frac{d}{dt}\int_\Omega(f^\varepsilon)^2\,d\mathbf{x}\,d\mathbf{v}+\int_\Omega v\cdot\nabla_x(f^\varepsilon)^2\,d\mathbf{x}\,d\mathbf{v}+\int_\Omega\Big(\mathbf{E}^\varepsilon+\mathbf{v}\times\frac{\mathcal{M}}{\varepsilon}\Big)\cdot\nabla_v(f^\varepsilon)^2\,d\mathbf{x}\,d\mathbf{v}=0.$$

Integrating the second and third terms by parts, they vanish. Hence

$$\frac{d}{dt}\int_\Omega(f^\varepsilon)^2\,d\mathbf{x}\,d\mathbf{v}=0,$$

which means that the L^2 norm of f^ε is conserved and gives us the estimate on f^ε thanks to the bound on the initial condition f_0.

We now come to the last estimates, namely those on ρ^ε and $\mathbf{J}^\varepsilon$. Following the idea of Horst [11], the estimates for ρ^ε and $\mathbf{J}^\varepsilon$ can be obtained by decomposition of the velocity integral:

$$\rho^\varepsilon(\mathbf{x},t)=\int_{I\!R^3_v}f^\varepsilon\,d\mathbf{v}=\int_{|v|<R}f^\varepsilon\,d\mathbf{v}+\int_{|v|>R}f^\varepsilon\,d\mathbf{v},$$

for any $R>0$. Using the Cauchy-Schwartz inequality, we have

$$\int_{|v|<R}f^\varepsilon\,d\mathbf{v}\le\Big(\int_{|v|<R}(f^\varepsilon)^2\,d\mathbf{v}\Big)^{\frac12}\Big(\int_{|v|<R}d\mathbf{v}\Big)^{\frac12}\le C_1R^{\frac32}\Big(\int_{I\!R^3_v}(f^\varepsilon)^2\,d\mathbf{v}\Big)^{\frac12},$$

and

$$\int_{|v|>R}f^\varepsilon\,d\mathbf{v}\le\Big(\int_{|v|>R}\frac{|v|^2}{R^2}f^\varepsilon\,d\mathbf{v}\le\frac{1}{R^2}\int_{I\!R^3_v}|v|^2f^\varepsilon\,d\mathbf{v}.$$

Hence, we have for any $R>0$

$$|\rho^\varepsilon(\mathbf{x},t)|\le C_1R^{\frac32}\Big(\int_{I\!R^3_v}(f^\varepsilon)^2\,d\mathbf{v}\Big)^{\frac12}+\frac{1}{R^2}\int_{I\!R^3_v}|v|^2f^\varepsilon\,d\mathbf{v}.$$

Taking the R which minimizes the right-hand-side we obtain

$$|\rho^\varepsilon(\mathbf{x},t)|\le C_2\Big(\int_{I\!R^3_v}(f^\varepsilon)^2\,d\mathbf{v}\Big)^{\frac27}\Big(\int_{I\!R^3_v}|v|^2f^\varepsilon\,d\mathbf{v}\Big)^{\frac37},$$

and finally

$$\begin{aligned}\int_{I\!R_x^3} |\rho^\varepsilon(\mathbf{x},t)|^{\frac{7}{5}}\,d\mathbf{x} &\le C_3\int_{I\!R_x^3}(\int_{I\!R_v^3}(f^\varepsilon)^2\,d\mathbf{v})^{\frac{2}{5}}(\int_{I\!R_v^3}|v|^2 f^\varepsilon\,d\mathbf{v})^{\frac{3}{5}}\,d\mathbf{x},\\ &\le C_3(\int_{I\!R_x^3\times I\!R_v^3}(f^\varepsilon)^2\,d\mathbf{x}\,d\mathbf{v})^{\frac{2}{5}}(\int_{I\!R_x^3\times I\!R_v^3}|v|^2 f^\varepsilon\,d\mathbf{x}\,d\mathbf{v})^{\frac{3}{5}},\end{aligned}$$

thanks to the Hölder inequality. Now, knowing that the terms on the right-hand-side are bounded, we have our estimate on ρ^ε.

We can proceed with $\mathbf{J}^\varepsilon$ in exactly the same way:

$$|\mathbf{J}^\varepsilon(\mathbf{x},t)| \le \int_{I\!R_v^3}|\mathbf{v}|f^\varepsilon\,d\mathbf{v} = \int_{|v|<R}|\mathbf{v}|f^\varepsilon\,d\mathbf{v} + \int_{|v|>R}|\mathbf{v}|f^\varepsilon\,d\mathbf{v},$$

for any $R>0$. Using the Cauchy-Schwartz inequality, we have

$$\int_{|v|<R}|\mathbf{v}|f^\varepsilon\,d\mathbf{v} \le (\int_{|v|<R}(f^\varepsilon)^2\,d\mathbf{v})^{\frac{1}{2}}(\int_{|v|<R}R^2 d\mathbf{v})^{\frac{1}{2}},$$

and

$$\int_{|v|>R}|\mathbf{v}|f^\varepsilon\,d\mathbf{v} \le \int_{|v|>R}\frac{|v|^2}{R}f^\varepsilon\,d\mathbf{v}.$$

Here again we can find the minimizing R and use Hölder's inequality to obtain

$$\int_{I\!R_x^3}|\mathbf{J}^\varepsilon(\mathbf{x},t)|^{\frac{7}{6}}\,d\mathbf{x} \le C_4(\int_{I\!R_x^3\times I\!R_v^3}(f^\varepsilon)^2\,d\mathbf{x}\,d\mathbf{v})^{\frac{1}{6}}(\int_{I\!R_x^3\times I\!R_v^3}|v|^2 f^\varepsilon\,d\mathbf{x}\,d\mathbf{v})^{\frac{5}{6}},$$

which gives us the estimate on $\mathbf{J}^\varepsilon$. ■

Now, on the one hand, thanks to the classical regularizing properties of the Laplacian, $\rho^\varepsilon(\mathbf{x},t)$ bounded in $L^\infty(0,T;L^{\frac{7}{5}}(I\!R_x^3))$ implies that u^ε such that $-\Delta u^\varepsilon = \rho^\varepsilon$ is bounded in $L^\infty(0,T;W^{2,\frac{7}{5}}(I\!R_x^3))$ and hence $\mathbf{E}^\varepsilon = -\nabla u^\varepsilon$ is bounded in $L^\infty(0,T;W^{1,\frac{7}{5}}(I\!R_x^3))$. On the other hand, integrating the Vlasov equation with respect to $\mathbf{v}$, we get the continuity equation

$$\frac{\partial\rho^\varepsilon}{\partial t} + \nabla\cdot\mathbf{J}^\varepsilon = 0.$$

Hence, as $\mathbf{J}^\varepsilon$ is bounded in $L^\infty(0,T;L^{\frac{7}{6}}(I\!R_x^3))$, $\frac{\partial\rho^\varepsilon}{\partial t}$ is bounded in $L^\infty(0,T;W^{-1,\frac{7}{6}}(I\!R_x^3))$, and as we have

$$-\Delta\frac{\partial u^\varepsilon}{\partial t} = \frac{\partial\rho^\varepsilon}{\partial t},$$

the regularizing properties of the Laplacian now yield $\frac{\partial u^\varepsilon}{\partial t}$ bounded in $L^\infty(0,T;W^{1,\frac{7}{6}}(I\!R_x^3))$, which yields that $\frac{\partial\mathbf{E}^\varepsilon}{\partial t}$ is bounded in $L^\infty(0,T;L^{\frac{7}{6}}(I\!R_x^3))$.
Thus as

$$W^{1,\frac{7}{5}}(I\!R_x^3) \subset L^2_{loc}(I\!R_x^3) \subset L^{\frac{7}{6}}_{loc}(I\!R_x^3),$$

the first injection being compact and the second being continuous, the Aubin-Lions Lemma (see for example Lions [14]) yields that the functional space

$$\mathcal{U} = \{\mathbf{E}\in L^\infty(0,T;W^{1,\frac{7}{5}}(I\!R_x^3)), \frac{\partial\mathbf{E}}{\partial t}\in L^\infty(0,T;L^{\frac{7}{6}}(I\!R_x^3))\},$$

provided with the usual product norm, is compactly embedded in $L^\infty(0,T;L^2_{loc}(I\!R^3_x))$ and consequently, as $\mathbf{E}^\varepsilon$ is bounded in $\mathcal{U}$ there exists a subsequence of $\mathbf{E}^\varepsilon$ which converges strongly in $L^\infty(0,T;L^2_{loc}(I\!R^3_x))$.
Hence we have

LEMMA 4.2 *Under assumption (1.10), extracting a subsequence, the sequence* $(f^\varepsilon, E^\varepsilon) \in L^\infty(0,T;L^1\cap L^2(\Omega)) \times L^\infty(0,T;W^{1,\frac{7}{5}}(I\!R^3_x))$ *of solutions of (1.9) satisfies:*

$$f^\varepsilon \rightharpoonup f \textit{ in } L^\infty(0,T;L^2(\Omega)) \textit{ weak} - *,$$

$$f^\varepsilon \textit{ two scale converges to } F \in L^\infty(0,T;L^\infty_{2\pi}(I\!R_\tau;L^1\cap L^2(\Omega))),$$

$$\mathbf{E}^\varepsilon \to \mathbf{E} \textit{ in } L^\infty(0,T;L^2_{loc}(I\!R^3_x)) \textit{ strong.}$$

Moreover,

$$\rho^\varepsilon \rightharpoonup \rho \textit{ in } L^\infty(0,T;L^{\frac{7}{5}}(I\!R^3_x)) \textit{ weak} - *,$$

$$\rho^\varepsilon \textit{ two scale converges to } \overline{\rho} \in L^\infty(0,T;L^\infty_{2\pi}(I\!R_\tau;L^{\frac{7}{5}}(I\!R^3_x))),$$

$$\mathbf{J}^\varepsilon \rightharpoonup \mathbf{J} \textit{ in } L^\infty(0,T;L^{\frac{7}{6}}(I\!R^3_x))) \textit{ weak} - *,$$

$$\mathbf{J}^\varepsilon \textit{ two scale converges to } \overline{\mathbf{J}} \in L^\infty(0,T;L^\infty_{2\pi}(I\!R_\tau;L^{\frac{7}{6}}(I\!R^3_x))),$$

for any $T \in I\!R^+$.

Then, passing to the two scale limit in (1.9) applying the results of section 2 yields system (1.13)-(1.14).
In order to show that $\overline{\rho}$ does not depend on τ, we multiply the continuity equation (4.2) by $\varphi(t,\frac{t}{\varepsilon},x)$ with $\varphi(t,\tau,x)$ regular, with compact support in t,x and periodic in τ, we integrate it by part and we multiply it by ε. Passing then to the limit gives

$$\frac{\partial\overline{\rho}}{\partial\tau} = 0.$$

Now, summing up the above arguments proves Theorem 1.5. ■

Lastly, exactly as in section 2, by an integration with respect to τ we get Theorem 1.4. ■

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