

Axioms for the g -vector of general convex polytopes

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Abstract

McMullen's g -vector is important for simple convex polytopes. This paper postulates axioms for its extension to general convex polytopes. It also conjectures that, for each dimension d , a stated finite calculation gives the formula for the extended g -vector. This calculation is done by computer for $d = 5$ and the results analysed. The conjectures imply new linear inequalities on convex polytope flag vectors. Underlying the axioms is a hypothesised higher-order homology extension to middle perversity intersection homology (order-zero homology), which measures the failure of lower-order homology to have a ring structure.

How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?

Sherlock Holmes in *The Sign of the Four* by Sir Arthur Conan Doyle

1 Introduction

This section describes the purpose and key concepts of this paper. Every simple convex polytope X has a g -vector $g(X)$, a linear function of the face vector $f(X)$. McMullen [12] introduced the g -vector in 1971 to express his conjectured conditions on $f(X)$. In 1980 Stanley [14] proved the necessity of these conditions, and Billera and Lee [5] the sufficiency. The distant goal is to find and prove the extension of these conditions to general convex polytopes.

McMullen's conditions have three parts: (1) linear equations on $f(X)$, (2) linear inequalities $g_i(X) \geq 0$, and (3) pseudo-power growth limits $g_{i+1} \leq g_i^{(i)}$. They correspond to there being homology groups $H(X)$ that respectively satisfy Poincaré duality, satisfy hard Lefschetz and are a ring generated by the facets. McMullen at the time was not aware of this connection.

The main results in this paper are: (1) axioms which are conjectured to determine the extended g -vector; (2) a conjecture that reduces this determination to a finite calculation; and (3) this calculation performed by computer in dimension $d = 5$. Almost all of the calculation is using the axioms to eliminate the impossible.

Two partial extensions are already known. Bernstein, Khovanski and MacPherson independently (see [6, §4.1]) found one in around 1982. From a rational convex polytope X a projective algebraic variety \mathbb{P}_X can be constructed (via toric geometry). The middle perversity intersection homology (mpih) Betti numbers $h(X)$ of \mathbb{P}_X are by Poincaré duality palindromic. As in the simple case, $g_i(X) = h_i(X) - h_{i-1}(X)$ for $0 \leq 2i \leq \dim X$ defines g_i . This extends the simple definition. By hard Lefschetz, $g_i(X) \geq 0$.

This g -vector is a linear function not of the face vector of X (which counts the number of faces on X of each dimension) but of the flag vector (which counts chains of inclusions between faces). In 1985 Bayer and Billera [3] found another partial extension, which determines g up to an invertible matrix. Their result is:

Theorem 1 (Generalised Dehn-Sommerville). *The vector space \mathcal{F}_d spanned by the flag vectors of all d -dimensional convex polytopes has dimension the Fibonacci number F_{d+1} . The flag vectors $f(v(\text{pt}))$ are a basis for \mathcal{F}_d , where v ranges over the F_{d+1} degree d words in C and IC , as defined below.*

Definition 2 (C and I). *Suppose X is a convex polytope. The cone (or pyramid) CX on X is the convex hull of X with a point, the apex, not lying in the affine linear span of X . The cylinder (or prism) IX on X is the Cartesian product of an interval, say $[0, 1]$, with X .*

Thus in dimension 5 the flag vector has $F_6 = 8$ independent components while the g -vector has $\lfloor 5/2 \rfloor + 3 = 3$. We wish to extend g so that it encode the whole of f . Here are the basic hypotheses made regarding the components g_i of the extended g -vector, besides being a linear function of the flag vector. But first we need a definition.

Definition 3 (The D operator). *The difference $IC - CC$ is the D operator on convex polytopes.*

Hypothesis 1 (Effective). *For all convex polytopes X we have $g_i(X) \geq 0$.*

Hypothesis 2 (Zero-one on (C, D) words). *$g_i(w(\text{pt})) \in \{0, 1\}$ for all words w in C and D .*

By generalised Dehn-Sommerville the flag vectors $f(w(\text{pt}))$ are a basis for \mathcal{F}_d . The second hypothesis now implies that g_i is determined by $s = \{w | g_i(w(\text{pt})) = 1\}$. For each d there are finitely many (in fact 2^n where $n = F_{d+1}$) possibilities for s and thus g_i . But if $g_i(X) < 0$ then g_i fails the first hypothesis and so must be eliminated. The next hypothesis makes this elimination an explicit finite, but perhaps large, calculation.

Hypothesis 3. *The set $P_{01,d}$ of all d -dimensional 01-polytopes and their polars eliminate all impossible g_i .*

Obtaining the g -vector from what whatever remains requires additional steps that depend on precise concepts. The rest of this paper is organised as follows. After providing background and notation we define and motivate the (C, D) basis and the zero-one on (C, D) hypothesis. In particular we discuss the already known intersection homology of toric varieties, and a possible extension that encodes the remainder of the flag vector. Next we introduce axioms to give a precise statement of our assumptions and conjectures, and report on our calculations. Finally we summarise and discuss some open questions.

2 Background

This section establishes notation, provides some definitions and provides other background. Some important matters not relevant to the core of this paper, such as the decomposition theorem, are left unexplained. Concepts used in only one section are generally not placed here.

Throughout *polytope* will mean convex polytope, X will be a polytope of dimension d , w a word of degree d in C and D (as defined in the Introduction) and s will be a set of words w (also known as a *word-set*). Usually, v is a word in (C, IC) or in (C, I) .

We will use $f(X)$ to denote the *flag vector* of X and $g(X)$ is our goal, the g -vector of X . It has *components* g_i . Suppose Y is a weighted formal sum $\sum \alpha_i X_i$ of polytopes. By $f(Y)$ we will mean $\sum \alpha_i f(X_i)$, and similarly for other vector- and number-valued functions, such as g and g_i .

We use $\lambda_w = \lambda_w(X)$ to denote coefficients in the (C, D) -basis. (This should not be confused with the cd -index of convex polytopes.) We use λ_s for $\sum_{w \in s} \lambda_w$. We use $\deg_C w$ and $\deg_D w$ to denote the number of C 's and D 's in w , and write $\deg w = \deg_C w + 2 \deg_D w$. The *order* of a word is the number of occurrences of CD in it.

Every polytope X has a *polar* X^\vee . This reverses inclusions on faces and so $f(X)$ and $f(X^\vee)$ are linear functions of each other. A polytope is *simple* if its facets (codimension 1 faces) are in general position, and *simplicial* if its vertices are in general position. Simple and simplicial correspond under polarisation.

Combinatorists usually start with simplicial polytopes but for homology simple polytopes are a better starting point. Here is an example. The polar of the I operator is the bipyramid B , and [3] used B not I . The IC equation reduces $IC - CC$ to a Cartesian product operator. The polar of a Cartesian product not easy to describe. This paper takes the simple/homology point of view. For us, McMullen's conditions apply to simple polytopes, even though he formulated them for simplicial polytopes.

From every rational polytope X a toric variety \mathbb{P}_X can be constructed. It has homology groups, which we denote by $H(X)$. The definition of $H(X)$ extends to all polytopes, rational or not. We will index homology groups by complex dimension (as the others are zero). Homology will always mean middle perversity intersection homology, which for simple polytopes is the same as ordinary homology. In §5 we hypothesise the existence of *higher-order* homology groups, and call the already known part of $H(X)$ the *order-zero* homology.

We use $[a, b, c]$ to denote the polynomial $a + bx + cx^2$, and think of the order-zero part of $h(X)$ and $g(X)$ as polynomials in x . For example, $h(X) = \sum x^i h_i(X)$. This allows them to be multiplied. Thus, $h(\mathbb{P}_2) = [1, 1, 1]$ and $h(\mathbb{P}_2 \times \mathbb{P}_3) = [1, 1, 1][1, 1, 1, 1]$ by the Künneth formula.

The *hard Lefschetz theorem* says that the hyperplane class $\omega \in H_{d-1}(X)$ induces an isomorphism $\omega^{j-i} : H_i(X) \rightarrow H_j(X)$ when $i \geq j$ and $i + j = d$. A class η is *primitive* if $\omega^{j-1+1}(\eta) = 0$.

If \mathcal{B} is a Boolean expression we let (\mathcal{B}) denote 1 if \mathcal{B} is true and 0 otherwise. Thus, $\delta_{ij} = (i = j)$ is the Kronecker delta and $(w \in s)$ is the characteristic function of a set s .

Finally, some miscellaneous definitions and notations. A d -cube, or *cube* for short, will mean a d -dimensional hypercube. We do not need in this paper the definition of the *pseudo-power* $n^{(i)}$. The interested reader can find it in [12] or [4]. We use $\lfloor x \rfloor$ to denote the largest integer not greater than x . Hypotheses, axioms and conjectures are numbered separately. Everything else is numbered together.

3 The (C, D) basis

According to generalised Dehn-Sommerville every polytope flag vector has a unique expression as a linear combination of the flag vectors of $v(\text{pt})$ where v is a word in C and IC . This applies in particular to $v(\text{pt})$ where v is any word in C and I that is *not* a word in C and IC . This suggests that there is a relationship between I and C . In fact [7]:

Theorem 4 (The IC equation). *As operators on flag vectors, I and C satisfy the equation*

$$(IC - CC)I \equiv I(IC - CC).$$

Proof. Because it is central to the properties of D , we summarise the proof given in [7]. Let X be a convex polytope. By definition CCX has two apexes, but there is no geometric difference between the first and the second. Put another way, CCX is the join of X with an interval and that interval is the *apex edge* of CCX . Similarly, ICX has an apex edge. Along their apex edges CCX and ICX have the same combinatorial structure and so

$$ICX - CCX \equiv Y - Z$$

where Y and Z are respectively the truncation of ICX and CCX along their apex edges. However, Y and Z are respectively the Cartesian product of a square and a triangle with X .

Thus, as an operator on flag vectors, $IC - CC$ is a difference of Cartesian products. But I is also a Cartesian product operator, and such products commute. The IC equation follows. In fact, the IC equation together with $I(\text{pt}) = C(\text{pt})$ (which follows from the IC equation applied to the empty set) allows any word in I and C applied to a point to be reduced to an equivalent combination of C and IC words applied to a point. \square

This is one motivation for $D = IC - CC$. The proof of the IC equation establishes:

Corollary 5. *For any X*

$$DX \equiv (IC - CC)X \equiv (Y - Z) \times X$$

where Y and Z are respectively a square and a triangle.

Corollary 6 (The (C, D) basis). *Every convex polytope X has a unique representation*

$$X \equiv \sum_w \lambda_w w(\text{pt})$$

where the sum is over all degree d words w in C and D and the coefficients $\lambda_w = \lambda_w(X)$ are linear functions of the flag vector $f(X)$.

Definition 7 (CD -vector). *The mapping $w \mapsto \lambda_w$ is called the CD vector of X . If s is a set of (C, D) words we will use λ_s to denote $\sum_{w \in s} \lambda_w$.*

The zero-one hypothesis implies that each g_i is equal to λ_s for some set of words s .

4 Homology

In this section we look at the already known homology. In particular we state formulas for Betti numbers in terms of the (C, D) basis. First simple polytopes. The product of two simple polytopes is also simple. Thus if X is simple then so is DX (by which we mean that its flag vector can be written as a formal sum of the flag vectors of simple polytopes). This follows because DX is equivalent to $(Y - Z) \times X$ for simple Y and Z .

Clearly, $h(\text{pt}) = [1]$, $h(C(\text{pt})) = [1, 1]$ and similarly $h(C^j(\text{pt})) = [1, \dots, 1]$. Similarly $h(Y) = [1, 1, 1]$, $h(Z) = [1, 2, 1]$, and $h(DX) = [0, 1, 0]h(X)$. Thus $h(D^i C^j(\text{pt})) = [0, 1, 0]^i [1, \dots, 1]$. Also, any simple-polytope h -vector is a weighted sum of $D^i C^j(\text{pt})$ h -vectors and so the formal sums $D^i C^j(\text{pt})$ provide a basis for the flag vectors of simple polytopes. This proves:

Proposition 8. *Suppose X is a simple polytope. Then $X \equiv \sum_{i=0}^{\lfloor d/2 \rfloor} \lambda_i D^i C^{d-2i}(\text{pt})$ where $\lambda_i = g_i(X)$.*

For general polytopes it follows from the known formula for intersection homology Betti numbers [15] that $h(IX) = [1, 1]h(X)$. It also follows that for example if $h(X) = [a, b, c, b, a]$ then $h(CX) = [a, b, c, c, b, a]$. The general rule for C is to repeat the middle term (for d even) or the next to middle term (for d odd). If d is odd then the two next-to-middle terms are equal. Thus, if $h(X) = [a, b, b, a]$ then $h(CX) = [a, b, b, b, a]$.

From this $h(DX) = [0, 1, 0]h(X)$ follows easily. (Alternatively, the result follows from $D = Y - Z$ and the Künneth formula.) We now turn to the g -vector. The rule for $h(CX)$ then becomes $g(CX) = g(X)$ while the rule for D is $g(DX) = [0, 1, 0]g(X)$. Hence we have the following extension of the previous proposition:

Proposition 9. *Let w be a word in C and D . Then*

$$g_i(w(\text{pt})) = (\deg_D w = i)$$

and by generalised Dehn-Sommerville this equation determines g_i on all convex polytope flag vectors.

Corollary 10. *The known components g_i of g satisfy the zero-one on (C, D) hypothesis.*

The proof that $g_i(X) \geq 0$ is deep, and relies on the decomposition theorem [6].

5 Higher-order homology

This section assumes some knowledge of intersection homology and the topology of toric varieties. It provides both justification for the axioms and an interpretation of the outcome of calculations that follow.

For simple polytopes the homology $H(X)$ is a ring generated by the facets and from this the pseudo-power inequalities follow. For general polytopes $H(X)$ no longer has a ring structure. Our distant goal requires additional components in $H(X)$, which we will call the *higher-order homology*. The already known part we call the *order-zero homology*.

The main properties of higher-order homology are: (1) it should vanish when X is simple, (2) when it vanishes there is a ring structure on $H(X)$, and (3) its Betti numbers should be a linear function of the flag vector.

The order-zero homology can be constructed by taking cycles and relations whose intersection with the strata satisfy what are called perversity conditions. This allows Poincaré duality and hard Lefschetz to hold (provided we use middle perversity). It also allows non-trivial local cycles to exist. For example there is a local cycle at each vertex of the octohedron (the polar of a 3-cube).

The author expects the cycles and relations for higher-order homology to be subsets of those for order-zero homology, obtained by imposing locality conditions. In other words, fewer generators and fewer relations. For example, on the octohedron there are 8 local 1-cycles which under local equivalence are independent. However, there are 4 independent global relations among these local 1-cycles.

If $X = v(\text{pt})$ for v a word in (C, I) then the stratification is given by a single flag on X . This has consequences. Suppose η and ψ are cycles close to a stratum X_i . If η and ψ are globally equivalent then it seems that this relation can be deformed so that it too is close to X_i . In other words, on $X = v(\text{pt})$ local and global equivalence are the same. This, of course, is not true in general.

Further, if $IC = D + CC$ is used to rewrite v as a weighted sum of words in (C, D) then the weights count the primitive cycles on X with certain locality properties. Thus, each locality property gives rise to a zero-one function on the (C, D) basis. (We will see, however, that for some plausible locality properties the resulting linear function will be negative on some polytope.) To summarise what this paper needs from this section:

Hypothesis 4. *The higher-order homology vanishes on simple polytopes.*

Hypothesis 5. *Each higher-order homology group is defined by using a locality property.*

Hypothesis 6. *Each locality property induces a zero-one function on the (C, D) basis.*

6 Axioms for the g -vector

The previously stated hypotheses have consequences for the g -vector, which we will call axioms. The calculation is based on the axioms. Recall that X is any convex polytope, that d is its dimension, that w is a word of degree d in (C, D) , and that s is a set of words w . Here are the axioms.

Axiom 1 (Linearity). *$g(X)$ is a linear function of the flag vector $f(X)$.*

Axiom 2 (Components). *$g(X)$ is a map $i \mapsto g_i(X)$ from I to \mathbb{Z} , for i in some index set $I = I_d$.*

Each g_i is called a *component* of g .

Axiom 3 (Non-negative). *$g_i(X) \geq 0$ for all $i \in I$ and all X .*

The next axiom is crucial. It implies that g is given by a choice from a known and finite set. Its motivation comes from the three previous sections. It depends upon the generalised Dehn-Sommerville equations.

Axiom 4 (Zero-one on (C, D) words). *For each i in I we have $g_i = \lambda_s$ for some set of words s .*

Thus, we can take the index set I to be a subset of the power set of all degree d words in C and D . For example, for $d = 5$ we have $F_{5+1} = 8$ and so the power set has $2^8 = 256$ elements. Thus, *a priori* there are 2^{256} possible values for I . From now on we will write $s \in S$ instead of $i \in I$. The next tasks are to remove first the impossible and then the redundant. Axiom 3 provides some conditions.

Definition 11. *Say that s is effective if $\lambda_s(X) \geq 0$ for all X .*

Let $E = E_d$ denote the set of all effective s (of degree d). Now for the redundant. Clearly, if nonempty disjoint word-sets s and s' are both effective then so is their union. Such unions provide no new conditions on $f(X)$ and so are excluded from g . More generally:

Definition 12 (Extremal). *Say that s in E_d is extremal if it cannot be expressed as a weighted sum $s = \sum_{t \neq s} \alpha_t t$ of other elements of E_d with non-negative weights α_t . (The empty set is not extremal.)*

Axiom 5 (Extremal). *Each element of s of S are extremal in E .*

The g -vector should embrace both zero-order and all possible higher-order homology.

Axiom 6 (Order-zero homology). *The g_i corresponding to order-zero homology are components of g .*

Axiom 7 (Higher-order homology). *If s is effective, extremal, and λ_s vanishes on simple polytopes, then $s \in S$.*

Finally, we want g to be complete and without redundancy.

Axiom 8 (Basis). *The components of g are a basis for all linear functions of the flag vector f .*

7 Conjectures

In this section we state two conjectures. If both are true then for each dimension d a finite calculation will give a formula for the g -vector in that dimension, and hence linear inequalities $g_s(X) \geq 0$. The first conjecture is, of course:

Conjecture 1. *There is a g -vector that satisfies the axioms stated in the previous section.*

The set $E = E_d$ of effective word-sets s of degree d is central to this conjecture. The g -vector follows once we have E . To show that s is *not* in E it is enough to produce an X such that $g_s(X) < 0$. Therefore, a finite set P of test polytopes is by elimination enough to determine E . We will make this more formal.

Definition 13 (Effective word-sets). *Let P be a set of dimension d polytopes. Define $E(P)$, the word-sets effective on P , to be the set of s such that $\lambda_s(X) \geq 0$ for every X in P .*

Definition 14 (Broad set of polytopes). *Say that a set P of polytopes is broad if $E(P) = E$, or in other words that $\lambda_s(X) \geq 0$ for $X \in P$ implies $\lambda_s(X) \geq 0$ for all X .*

Proposition 15. *For each dimension d there is a broad and finite set P of d -polytopes.*

In dimension 5 there is a broad P with at most $2^8 = 256$ polytopes. However, we don't know what it is. The second conjecture, if true, provides an explicit broad set P . There is at present little evidence for it, other than the satisfactory output produced by the $d = 5$ calculation, which is the subject of the next two sections. The set P is produced from subsets of the vertices of a d -cube, together with polars. A cube can be represented so that each vertex component is either 0 or 1. We repeat the standard definition:

Definition 16. *A 01-polytope is the convex hull of a subset of the vertices of a cube.*

A cube has 2^d vertices and $2d$ facets. The polar of a cube is the *cross-polytope*, the convex hull of the d basis vectors e_i and their negatives $-e_i$. The cross-polytope has 2^d facets (namely a choice of which e_i are to have a negative sign). Every convex polytope has a *polar* X^\vee (which depends on the choice of a point p in the interior of X). Each i -face on X corresponds to a $d - i - 1$ face of X^\vee and vice versa. This bijection reverses inclusions and so the flag vector $f(X^\vee)$ of the polar is a linear function of $f(X)$ (and does not depend on the choice of p).

Definition 17. *Let $P_{01,d}$ denote the set of all d -dimensional 01-polytopes, together with their polars.*

A 01-polytope need not have the same dimension as its cube. But if of smaller dimension then there is a projection onto a face that gives an affinely equivalent polytope (which thus has the same flag vector). So without loss of generality we need only consider X and X^\vee obtained from the d -cube.

Conjecture 2 ($P_{01,d}$ is broad). *The set $P_{01,d}$ of d -dimensional 01-polytopes and their polars is broad.*

8 The $d = 5$ calculation

In the previous section we conjectured that certain calculations would give a formula for the g -vector. In this section and the next we report on some such calculations. We applied `polymake` [11] to a list of 01-polytopes for $d = 5$ supplied by Aichholzer [1]. This gave us a list of flag vectors. There are 1,226,525 such polytopes (up to cube symmetries) and the calculation took about 15 GHz days. Together with their polars this gives 688,298 distinct flag vectors.

We also wrote software [10] to compute the (C, D) vector for each of these flag vectors, and also to determine for each of the $256 = 2^8$ subsets of the $F_{5+1} = 8$ degree 5 words s in (C, D) whether or not the sum of the corresponding coefficients in the (C, D) vector was non-negative for all these flag vectors. This gives us $E(P_{01,d})$, as defined in the previous section.

Finally, we used `polymake` again to determine the extremal s in $E(P_{01,d})$. Altogether there were 13 such, of which we are expecting 8 to be components of the g -vector. The meaning of the remaining 5 is less clear. The next two results give the conjectured components of the g -vector.

Proposition 18. *For $P_{01,5}$ the order-zero g_i (in the form 'index = deg : word-set')*

$$\begin{aligned} 11111 &= 0 : CCCCC \\ 2111 &= 1 : DCCC, CDCC, CCDC, CCDD \\ 221 &= 2 : DDC, DCD, CDD \end{aligned}$$

are effective (this is already known) and extremal.

Proposition 19. For $P_{01,5}$ the higher-order g_s (in the form ‘index : word-set’) are

$$\begin{aligned} 1211 &: CDCC, CCDC, CCCD, CDD \\ 1121 &: CCDC, CCCD, DCD \\ 1112 &: CCCD \\ 122 &: CDD \\ 212 &: DCD \end{aligned}$$

A word about the notation above, which is also used later. To each s we associate an index, which is a word in the symbols 1 and 2. Thus, $g_{221} = g_2 = g_s$ where $s = \{DDC, DCD, CDD\}$. The index i , with C and D replacing 1 and 2, is the first listed element of the set s . This notation is concise and, with care, the context resolves any ambiguities.

We will now show that the above are a basis for all linear functions of the flag vector. Let w be a word. It is enough to show that λ_w is a linear combination of the g_i above. But this is easy, because the components are listed in an upper-triangular order. In other words, every word appears first in one of the sets, and it does not appear in any subsequent set. This proves:

Proposition 20. The above components (of the conjectured g -vector) are a basis for all linear functions of the flag vector.

We now turn to the remaining effective and extremal s . They do not vanish on simple polytopes, and they are not order-zero g_i numbers. This is why they are not part of the g -vector.

Proposition 21. For $P_{01,5}$ the remaining effective and extremal s (in the form ‘inequality : word-set’) are

$$\begin{aligned} g_{122} \leq g_{221} + g_{2111} &: DDC, DCD, DCCC, CDCC, CCDC, CCCD \\ g_{122} \leq g_{221} + g_{1211} &: DDC, DCD, CDD, CDCC, CCDC, CCCD \\ g_{212} \leq g_{221} + g_{1112} &: DDC, CDD, CCCD \\ g_{212} \leq g_{221} + g_{1121} &: DDC, DCD, CDD, CCDC, CCCD \\ g_{212} \leq g_{221} + g_{2111} &: DDC, CDD, DCCC, CDCC, CCDC, CCCD \end{aligned}$$

where the left column is $\lambda_s \geq 0$ written in terms of the components of the g -vector.

These inequalities, an unanticipated by-product of the calculation, are more concisely be written as

$$\begin{aligned} g_{122} - g_{221} &\leq \min(g_{2111}, g_{1211}) \\ g_{212} - g_{221} &\leq \min(g_{2111}, g_{1121}, g_{1112}) \end{aligned}$$

and in this form they are similiar to McMullen’s pseudo-power inequalities $g_{i+1} \leq g_i^{(i)}$.

9 A special test polytope

The $d = 5$ calculation gives not only a formula for g but also, implicitly, a small and broad set of test polytopes. Here is the most interesting example. The set

$$s = \{CDCC, CCDC, CCCD\}$$

has $\lambda_s = g_{1211} - g_{122}$ and is effective on the whole of $P_{01,5}$ with just one exception (up to symmetries of the cube).

In this section let X be the 01-polytope which has all the vertices of the 5-cube except the set V whose members, listed in lexicographic order, are

$$\begin{aligned} u_1 &= (0, 0, 0, 0, 0), & u_2 &= (0, 0, 0, 1, 1), & u_3 &= (0, 1, 1, 0, 0), & u_4 &= (0, 1, 1, 1, 1), \\ v_1 &= (1, 0, 1, 0, 1), & v_2 &= (1, 0, 1, 1, 0), & v_3 &= (1, 1, 0, 0, 1), & v_4 &= (1, 1, 0, 1, 0). \end{aligned}$$

The exception is the polar X^\vee of X . The polytope X^\vee has flag vector (independent components in `polymake` order)

$$f(X) = (1, 24, 112, 152, 464, 80, 400, 696)$$

while the (C, D) vector is given by

$$\begin{aligned} (CCCC, 1), & \quad (CCCD, 8), & \quad (CCDC, 56), & \quad (CDCC, -66), \\ (CDD, 20), & \quad (DCCC, 20) & \quad (DCD, 0), & \quad (DDC, -5) \end{aligned}$$

and so

$$\lambda_s(X^\vee) = \lambda_{CDCC} + \lambda_{CCDC} + \lambda_{CCCD} = -66 + 56 + 8 = -2$$

which is, as claimed, negative. (This calculation, repeated many times, gave us $E(P_{01,5})$.)

The g -vector of X^\vee , however, is

$$\begin{aligned} g_0 = 1, & \quad g_1 = 8 + 56 - 66 + 20 = 18, & \quad g_2 = 20 + 0 - 5 = 15, \\ g_{1211} = 8 + 56 - 66 + 20 = 18, & \quad g_{1121} = 8 + 56 + 0 = 64, & \quad g_{1112} = 8, & \quad g_{122} = 20, & \quad g_{212} = 0 \end{aligned}$$

whose components are (for this polytope by construction of g) non-negative.

The set V of missing vertices has a structure. The u_i lie on a 4-face, and the v_i on the opposite 4-face. The vertices u_1 and u_4 give a diagonal of the 4-face, as do u_2 and u_3 . The vertex pairs $\{u_1, u_2\}$, $\{u_1, u_3\}$, $\{u_2, u_4\}$ and $\{u_3, u_4\}$ all differ in two coordinates. The same is true for the v_i . Finally, each u_i differs from each v_j in three coordinates. To state this more formally, let d_1 denote the Hamming (or Manhattan) metric on the vertices of the cube; d_1 counts how many components differ. The following definition and proposition summarise these facts.

Definition 22 (Distance count). *Let V' be a subset of the vertices of a cube. For each vertex v of the cube the i -th component of the distance count of v from V' is the number of $v' \in V'$ with $d_1(v, v') = i$.*

Proposition 23. *Let v be any member of the missing vertices set V . Then the distance count of v from V is $(1, 0, 2, 4, 1, 0)$.*

This result has a converse. It tells us that, at least in this case, distance count can determine a vertex set. As construction of polytopes with special properties may be useful later, we state the converse and give the somewhat pedestrian proof.

Proposition 24. *Up to symmetry there is exactly one subset V of the 5-cube that has distance count $(1, 0, 2, 4, 1, 0)$.*

Proof. Let V be the subset above and let V' be another one. Pick a vertex in V' . By cube symmetry without loss of generality (wlog) it is u_1 . There's only one vertex at distance 4 from u_1 and wlog it is u_4 . Pick a point at distance 2 from u_1 . Its distance from u_4 a priori is 2 or 4. But $d_1(u_1, u_4) = 4$ and so by distance count it must be 2 and wlog the point is u_2 .

Now pick the other point at distance 2 from u_1 . As before, its first coordinate is zero. If not u_3 then it has distance 2 from u_1 , u_2 and u_4 . This violates the distance count. So it is u_3 .

Now choose a vertex v at distance 3 from all the u_i . We have $d_1(u_1, v) = 3$ and so it contains 3 ones. If the first component is 0 then $d_1(v, u_4) = 1$. So the first component is 1. It cannot be $(1, 0, 0, 1, 1)$ or $(1, 1, 1, 0, 0)$ because of u_2 and u_3 respectively. Therefore the v_i are the only vertices at distance 3 from all the u_i . \square

Finally, we restate these results putting the focus on V .

Proposition 25. *The distance count $(1, 0, 2, 4, 1, 0)$ determines, by complement and polarisation, the unique up to symmetry polytope in $P_{01,5}$ that excludes $\{CDCC, CCDC, CCCD\}$.*

10 Summary

The section reviews the rest of the paper and sketches possibilities for future work.

The key new concepts are (1) that the components of the g -vector are zero-one on the (C, D) basis, and (2) the set $E = E_d$ of effective word-sets given by $\{s | \lambda_s(X) \geq 0 \text{ for all } X\}$. They arose as follows. The author computed $\lambda_s(X)$ for all X in $P_{01,5}$ and for the s in [9]. This was to test the s for being effective. However, as we have seen, some came out negative. The author then inverted the process. He used $P_{01,5}$ to compute $E(P_{01,5})$ and hence obtained a conjecture. The structure of E , even if not as described by the axioms, is most likely worth studying. The conjectures make definite statements about $E(P_{01,d})$ that can be tested by finite calculation.

Two of the components of g for $d = 5$, namely

$$\begin{aligned} 1211 &: CDCC, CCDC, CCCD, CDD \\ 1121 &: CCDC, CCCD, DCD \end{aligned}$$

have mixed degree in D . The meaning of the $\deg_D = 2$ terms, CDD and DCD respectively, is unclear. Although they may look improbable, they are an unavoidable consequence of our axioms and conjectures. The author expects that in general the lowest degree terms will be given as in [9].

The meaning of the conjectured inequalities

$$\begin{aligned} g_{122} - g_{221} &\leq \min(g_{2111}, g_{1211}) \\ g_{212} - g_{221} &\leq \min(g_{2111}, g_{1121}, g_{1112}) \end{aligned}$$

is far from clear. The analogous growth conditions in the simple case follow from $H(X)$ being generated by the facets of X . It is possible that [8] will help here.

Further calculations would be helpful. However, the size of $P_{01,d}$ grows very rapidly with d . It might be practical to compute for the whole of $P_{01,6}$ but for $P_{01,7}$ only a subset can be used. Analysis of the $d = 4$ and 5 results may lead to smaller broad subsets of $P_{01,d}$. This is of course related to the construction of polytopes and thus the proof of sufficiency. Results in dimension 4 should be compared to known results, conjectures and constructions [4, 2, 13].

Simplicial polytopes are a useful special case. Their polars are simple, and so McMullen's conditions characterise their flag vectors. It may be instructive, say for $d = 5$, to write the already known conditions on simplicial polytopes in terms of the conjectured g -vector.

The extension of McMullen's conditions to general polytopes will have at least three components, namely: (1) linear equations on the flag vector, (2) linear equalities $g_s(X) \geq 0$, and (3) growth conditions on the g_s . The generalised Dehn-Sommerville equations of Bayer and Billera provide (1). Our first conjecture, if true, provides (2) abstractly while our second conjecture will then provide a finite calculation that gives (2) in a concrete form. It may also provide part of (3).

Here are three more distant goals: (1) produce, with supporting evidence, a conjectured formula for g in all dimensions, (2) systematically produce test polytopes, particularly in higher dimensions, (3) translate the formulas and conjectures in this paper to intersection homology. This last goal is probably the only way to prove the conjectures, probably via the decomposition theorem [6].

References

- [1] Oswin Aichholzer, personal communication, 2010.
- [2] Margaret Bayer, *The extended f -vectors of 4-polytopes*, Journal of Combinatorial Theory, Series A **44** (1987), no. 1, 141–151.
- [3] Margaret M. Bayer and Louis J. Billera, *Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets*, Invent. Math. **79** (1985), no. 1, 143–157 (English).
- [4] Margaret M. Bayer and Carl W. Lee, *Combinatorial aspects of convex polytopes*, Handbook on Convex Geometry, North Holland, 1994, pp. 485–534.

- [5] Louis J. Billera and Carl W. Lee, *Sufficiency of McMullen's conditions for f -vectors of simplicial polytopes*, Bull. Amer. Math. Soc. (N.S.) **2** (1980), no. 1, 181–185.
- [6] Mark A. de Cataldo and Luca Migliorini, *The Decomposition Theorem and the topology of algebraic maps*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 4, 535–633.
- [7] Jonathan Fine, *The Mayer-Vietoris and IC equations for convex polytopes*, Discrete & Computational Geometry **13** (1995), 177–188.
- [8] ———, *Ring structure, uniform expressions and intersection homology*, ArXiv Mathematics e-prints (1998).
- [9] ———, *A complete g -vector for convex polytopes*, ArXiv e-prints (2010).
- [10] ———, *Software*, <http://bitbucket.org/jfine/python-hvector>, 2010.
- [11] Ewgenij Gawrilow and Michael Joswig, *polymake: a framework for analyzing convex polytopes*, Polytopes — Combinatorics and Computation (Gil Kalai and Günter M. Ziegler, eds.), Birkhäuser, 2000, pp. 43–74.
- [12] Peter McMullen, *The numbers of faces of simplicial polytopes*, Israel J. Math. **9** (1971), 559–570 (English).
- [13] Andreas Paffenholz and Axel Werner, *Constructions for 4-Polytopes and the Cone of Flag Vectors*, Contemporary Mathematics **423** (2006), 283–303.
- [14] Richard P. Stanley, *The number of faces of a simplicial convex polytope*, Adv. in Math. **35** (1980), no. 3, 236–238.
- [15] ———, *Generalized H -vectors, intersection cohomology of toric varieties, and related results*, Commutative algebra and combinatorics (Kyoto, 1985) (Amsterdam-New York), Adv. Stud. Pure Math., vol. 11, North-Holland, 1987, pp. 187–213 (English).

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