A group has a planar Cayley complex if and only if it has a VAP-free Cayley graph

Agelos Georgakopoulos*

Technische Universität Graz
Steyrergasse 30, 8010
Graz, Austria

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Abstract

We prove that a group has a planar Cayley complex if and only if it has a Cayley graph that can be embedded in the euclidean plane without accumulation points of vertices.

1 Introduction

The study of groups that have Cayley graphs embeddable in the euclidean plane $\mathbb{R}^2$, called *planar groups*, has a tradition starting in 1896 with Maschke's characterization of the finite ones. Among the infinite planar groups, those that admit a *planar Cayley complex*, i.e. a Cayley complex embeddable in $\mathbb{R}^2$, have received a lot of attention. They are important in complex analysis as they include the discontinuous groups of motions of the euclidean and hyperbolic plane. Moreover, they are closely related to the surface groups [17, Section 4.10]. These groups are now well understood due to the work of Macbeath [13], Wilkie [16], and others; see [17] for a survey. Planar groups that have no planar Cayley complex are harder to analyse, and they are the subject of on-going research [4, 5, 6, 7, 8].

The aim of this paper is to show the equivalence of the property of possessing a planar Cayley complex with a well-known graph-theoretical property:

Theorem 1.1. *A planar Cayley graph of a group* Γ *is VAP-free if and only if it is the 1-skeleton of a planar Cayley complex of* Γ.

A planar graph is said to be *Vertex-Accumulation-Point-free*, or *VAP-free* for short, if it has an embedding in $\mathbb{R}^2$ such that the images of its vertices have no accumulation point. The study of a planar graph is often simplified if one knows that the graph is VAP-free; examples range from structural graph-theory [2] to percolation theory [12] and the study of spectral properties [11]. A further example is Thomassen's Theorem 2.3 below, which becomes false in the non-VAP-free case. VAP-free graphs can be characterized by a condition similar

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to that of Kuratowski's; see [9]. *VAP-free* embeddings also appear with other names in the literature, most notably "*locally finite*".

We will follow the terminology of [3] for graph-theoretical terms and that of [1] for group-theoretical ones.

2 Proof of main result

One of our main tools will be the (finitary) *cycle space* $\mathcal{C}_f(G)$ of a graph $G = (V, E)$, which is defined as the vector space over $\mathbb{Z}_2$ consisting of those subsets of E such that can be written as a sum (modulo 2) of a finite set of circuits, where a set of edges $D \subseteq E$ is called a *circuit* if it is the edge set of a cycle of G. Thus $\mathcal{C}_f(G)$ is isomorphic to the first simplicial homology group of G over $\mathbb{Z}_2$. The *circuit* of a closed walk W is the set of edges traversed by W an even number of times. Note that the direction of the edges is ignored when defining circuits and $\mathcal{C}_f(G)$.

In this section we will be assuming that our graphs have no parallel edges. In a Cayley graph this can be achieved by drawing, for every involution in the generating set, a single undirected edge rather than a pair of parallel edges with opposite directions. This convention does not affect the properties involved in Theorem 1.1, which means that our assumption comes without loss of generality for the proof of that theorem.

Our first lemma, a well-known fact which is easy to prove, relates $\mathcal{C}_f(G)$ to group presentations.

We will say that a closed walk W in G is *induced* by a relator R, if W can be obtained by starting at some vertex g and following the edges corresponding to the letters in R in order; note that for a given R there are several walks in G induced by R, one for each starting vertex $g \in V(G)$.

Lemma 2.1. *Let $G = Cay \langle S \mid R \rangle$ be a Cayley graph of the group Γ. Then the set of edge-sets of walks in G induced by relators in R generates $\mathcal{C}_f(G)$.*

Conversely, if R' is a set of relations of Γ with letters in a generating set S such that the set of cycles of $Cay(\Gamma, S)$ induced by R' generates $\mathcal{C}_f(G)$, then $\langle S \mid R' \rangle$ is a presentation of Γ.

Combined with the next easy fact, this allows one to deduce group presentations from VAP-free embeddings of a Cayley graph.

Lemma 2.2. *Let G be a VAP-free plane graph. Then the set of finite face boundaries of G generates $\mathcal{C}_f(G)$.*

Proof. It suffices to show that the edge-set of every cycle C of G is a sum of edge-sets of finite face boundaries. This is indeed the case, for as G is VAP-free there must be a side A of C containing only finitely many vertices, and so $E(C)$ is the sum of the edge-sets of the face boundaries lying in A. □

The following theorem of Thomassen generalises MacLane's classical planarity criterion to infinite VAP-free planar graphs. It will easily imply the backward implication of Theorem 1.1. A *2-basis* of G is a generating set B of $\mathcal{C}_f(G)$ such that no edge of G appears in more than two elements of B.

Theorem 2.3 ([14, Section 7]). *A connected graph G has a 2-basis if and only if it is planar and has a VAP-free embedding.*

The following fact is probably well-known to experts in the study of infinite vertex transitive graphs. We include a proof sketch for the convenience of the non-expert. A *double-ray* is a 2-way infinite path (with no repetition of vertices).

Lemma 2.4. *Let G be an infinite, connected, vertex transitive graph which is not a double-ray. Then for every pair of vertices x, y of G, no component of $G - \{x, y\}$ is finite.*

Proof. To begin with, it is easy to prove that

$$\text{for every } x \in V(G), \text{ no component of } G - \{x\} \text{ is finite,} \tag{1}$$

by considering a minimal such component C and mapping x to some vertex of C.

Suppose that some component C of $G - \{x, y\}$ is finite, and choose x, y so as to minimise $|V(C)|$. We claim that the graph C has no cut-vertex. Indeed, if $z \in V(C)$ separates C, then $G - \{x, z\}$ contains a component properly contained in C, contradicting the minimality of the latter. Moreover, each of x, y has at least two neighbours in C; for if y has a single neighbour y' in C, then we could have replaced y by y' to obtain a separator $\{x, y'\}$ cutting off a smaller component, and if y has no neighbour in C then (1) is contradicted. These two observations, combined with Menger's theorem [3, Theorem 3.3.1], imply that there are two independent x–y paths P, Q through C. Moreover, (at least) one of x, y, say x, is contained in an infinite subgraph X that does not meet P, Q except at x.

Let $z \in V(C)$, and consider an automorphism g mapping x to z. Then, there is a vertex $w = gy$ such that $\{z, w\}$ separates G. We consider three cases.

If w lies in a component $C' \neq C$ of $G - \{x, y\}$, then each of gP, gQ, gX meets both C' and C. But this is impossible since C is separated from C' by x, y and the only vertices meeting more than one of gP, gQ, gX are z and w.

If w lies in C, then some component of $G - \{z, w\}$ is properly contained in C contradicting its minimality.

Finally, if $w = y$, then as the component gC of $G - \{z, w\}$ cannot be smaller than C, it must contain the vertex x. Note that x is incident with an infinite component C' of $G - \{x, y\}$ because otherwise y contradicts (1). But then gC contains C', contradicting the fact that its translate C is finite.

Thus, in all three cases we obtained a contradiction. This proves Lemma 2.4. □

We can now prove our main result.

Proof of Theorem 1.1. For the forward implication, let $G = Cay\langle \mathcal{S} \mid \mathcal{R} \rangle$ be a planar Cayley graph of the group Γ with a VAP-free embedding σ. By Lemma 2.5 below, if F is a finite face boundary in σ, then every translate of F is a face boundary. This means that if we let $\mathcal{R}'$ be the set of relations corresponding to the finite face boundaries of σ incident with the group identity e, then every finite face boundary of σ is induced by some element of $\mathcal{R}'$, and conversely any cycle induced by some element of $\mathcal{R}'$ bounds a face in σ. By Lemmas 2.1 and 2.2, $\langle \mathcal{S} \mid \mathcal{R}' \rangle$ is a presentation of Γ. The corresponding Cayley complex is planar and VAP-free since we can embed its 1-skeleton G by σ and then every 2-complex can be embedded into the face of σ bounded by the corresponding cycle.

The backward implication follows easily from Theorem 2.3: let X be a planar Cayley complex and let G be its 1-skeleton. Let B' be the set of closed walks in G bounding a 2-simplex of X —in fact, these closed walks are cycles since X is planar— and let $B := \{E(C) \mid C \in B'\}$ be the set of the corresponding edge-sets. Note that B generates $\mathcal{C}_f(G)$ by Lemma 2.1. Moreover, since X is planar, no edge of X lies in the boundary of more than two 2-simplices. Thus B is a 2-basis of G, and so Theorem 2.3 implies that G is VAP-free. □

Lemma 2.5. *Let G be an infinite vertex transitive graph with a VAP-free embedding σ. If F is a finite face boundary in σ then every image of F under any automorphism of G is a face boundary in σ.*

Proof. Suppose to the contrary that some image $F' = gF$ of F under an automorphism g is not a face boundary. Then, as σ is VAP-free, one of the sides of F' contains at least one finite component C. Let $N(C)$ be the set of vertices of F' sending an edge to C. Then $F' - N(C)$ consists of a set of disjoint paths, which we call the *intervals*. Note that there are at least three intervals, for if $|N(C)| \leq 2$ then Lemma 2.4 is contradicted.

We claim that

$$\text{no component of } G - F' \text{ sends edges to more than one interval.} \tag{2}$$

Indeed, if such a component C' existed, then, by a topological argument, it would be impossible to embed G is such a way that both C and C' lie in the same side of F' (Figure 1), but such an embedding must be possible since F is a face boundary.

Next, we claim that at most one of the intervals sends an edge to an infinite component of $G - F'$. For if there are intervals $I \neq J$ adjacent with infinite components C_I, C_J of $G - F'$, then replacing I in F' by a path through C we would obtain a cycle D that separates C_I from C_J by (2) (Figure 1). But then $g^{-1}I, g^{-1}J$ must lie in distinct sides of $g^{-1}D$ since $F = g^{-1}F'$ and F is a face boundary, contradicting the fact that σ is VAP-free.

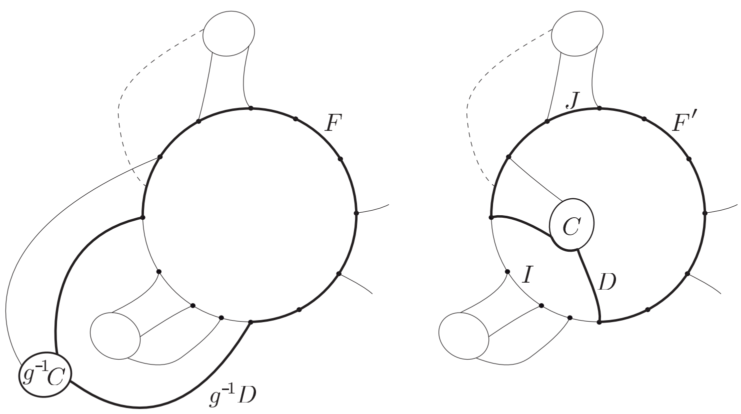

Figure 1: A contradictory situation in the proof of Lemma 2.5.

Thus our claim is proved, implying that there is a unique interval I adjacent with the infinite component of $G-F'$. This fact, combined with (2), implies that deleting the vertices $x, y \in N(C)$ bounding I leaves a finite component, namely the component of $G - \{x, y\}$ containing C. But this contradicts Lemma 2.4. $\square$

3 VAP-freeness as a group-theoretical invariant

A planar group can admit both VAP-free and non-VAP-free Cayley graphs. For example, the Cayley graph corresponding to the presentation $\langle a, b \mid b^2, abab \rangle$, of the infinite dihedral group, is VAP-free planar, but adding the redundant generator $c = ab$ keeps the Cayley graph planar and makes it non-VAP-free as the reader can check. Thus VAP-freeness is not group-theoretical invariant in general. However, it becomes an invariant if one only considers 3-connected Cayley graphs:

Theorem 3.1. *If a group Γ has a 3-connected VAP-free planar Cayley graph and a group Δ has a 3-connected non-VAP-free planar Cayley graph, then Γ is not isomorphic to Δ.*

Before proving this let us see a further example showing that it is necessary that both graphs in the assertion be 3-connected. Consider the Cayley graph corresponding to the presentation $\langle a, b \mid a^4, b^4 \rangle$. This is a free product of 4-cycles, and it is easy to see that it has a VAP-free embedding and that its connectivity is 1. Now add the redundant generators $c = ab$ and $d = a^2b^2a^2$. Note that $d^2 = 1$. Figure 2 shows that the corresponding Cayley graph is still planar, and it is easy to check that it is 3-connected. The given embedding is not VAP-free. It now follows easily from the following classical result, proved by Whitney [15, Theorem 11] for finite graphs and by Imrich [10] for infinite ones, that no embedding of this graph is VAP-free.

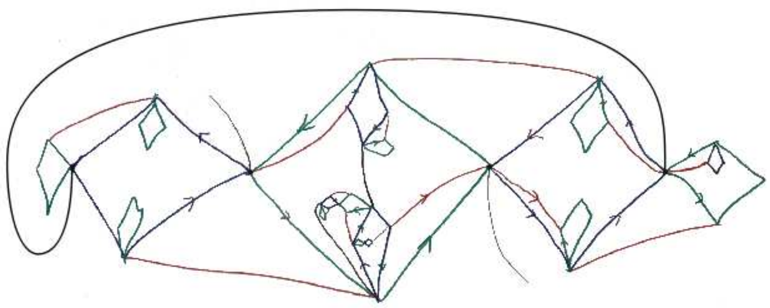

Figure 2: An embedding of the Cayley graph of $\langle a, b, c, d \mid a^4, b^4, c = ab, d = a^2b^2a^2 \rangle$.

Theorem 3.2. *Let G be a 3-connected graph embedded in the plane. Then every automorphism of G maps each facial path to a facial path.*

We will need a few lemmas for the proof of Theorem 3.1.

Lemma 3.3. *Let G be a 2-connected planar graph and let ω, ψ be distinct ends of G. Then there is a cycle C in G that* separates *ω from ψ, i.e. every double-ray with a tail in ω and a tail ψ has a vertex in C.*

Proof. Fix an embedding σ of G. Consider a finite set of vertices $S = \{s_1, s_2, \dots, s_k\}$ separating ω from ψ, and let C_1 be a cycle containing s_1, s_2; such a cycle exists since G is 2-connected. If C_1 does not separate ω from ψ then both ends lie in one of the sides of C_1, the outside say. Note that some vertex of S must also lie outside C_1, for otherwise every double-ray with a tail in ω and a tail ψ would have to go through C_1 to meet S, contradicting the fact that C_1 does not separate ω from ψ. So pick the least index j such that s_j lies outside C_1. Now consider two independent paths P_1, P_2 from s_j to C_1, and let A be the region of $\mathbb{R}^2 \backslash \{C_1 \cup P_1 \cup P_2\}$ containing rays in ω. The boundary of A is a cycle C_2 containing $P_1 \cup P_2$ and a subpath of C_1. Note that no element of $\{s_1, s_2, \dots, s_j\}$ lies in A because those points do not lie outside C_1. Repeating this argument we construct the sequence of cycles $C_1, C_2, \dots, C_m$, terminating with a cycle C_m such that the outside of C_m contains ω but none of the s_i. This cycle separates ω from ψ because every double-ray with a tail in ω and a tail ψ has to cross it to meet S. □

Using this we can prove:

Lemma 3.4. *Let G be a 2-connected graph with a VAP-free embedding σ and more than 1 end. Then at least one of the faces of σ has infinite boundary.*

Proof. By Lemma 3.3 there is a cycle C separating two ends ω, ψ of G. Since σ is VAP-free, both these ends lie in the same side of C, the outside say. Let K_ω (respectively K_ψ) be the component of $G - C$ containing rays in ω (resp. ψ). Easily, it is possible to find independent subpaths P_ω, P_ψ of C such that every vertex of C adjacent with K_ω lies in P_ω, and similarly for K_ψ and P_ψ. Let x be an endvertex of P_ω; without loss of generality, x is adjacent with K_ω.

By the choice of x we can choose an edge $e = xy$ with $y \in V(K_\omega)$ and a further edge $f = xz$ incident with x with z not in K_ω and f not in P_ω, so that e, f lie on a common face boundary F. Now if F is infinite we are done, so suppose it is finite. Consider the path $F' := F - C$. One of the endvertices of F' is x by construction, and the other endvertex x' must also lie on P_ω since F' must be contained in $K_\omega \cup C$. Now consider the cycle D contained in $F' \cup P_\omega$. By construction, K_ω, K_ψ lie in distinct sides of D. This contradicts our assumption that σ is VAP-free. □

Our last lemma is

Lemma 3.5. *There is no 3-connected vertex-transitive VAP-free planar graph with more than 1 end.*

Proof. If such a graph G exists, then by Lemma 3.4 it has an infinite face-boundary. By Theorem 3.2 this implies that every vertex of G is incident with an infinite face-boundary.

Thus we can pick two vertices x, y that lie in a common double ray R of G contained in a face-boundary. As G is 3-connected, there are three independent x–y paths P_1, P_2, P_3 by Menger's theorem [3, Theorem 3.3.1]. By an easy topological argument, there must be a pair of those paths, say P_1, P_2, whose union is a cycle C such that some side of C contains a tail of R and the other side of C contains P_3. We may assume without loss of generality that P_3 is not a single edge, for we are allowed to choose x and y far apart. Thus the side of C containing P_3 contains at least one vertex z. By our previous remarks, z is incident with an infinite face-boundary. This means that both sides of C contain infinitely many vertices, contradicting our assumption that G is VAP-free. □

We can now prove the main result of this section.

Proof of Theorem 3.1. If any of Γ, Δ is 1-ended then we are done since it is well-known, and not hard to prove, that all its planar Cayley graphs are VAP-free in this case. The result now follows immediately from Lemma 3.5. □

4 VAP-free presentations via MacLane's planarity criterion

In this section we derive a further characterisation of the groups that admit a planar Cayley complex by means of group presentations. We achieve this using Thomassen's Theorem 2.3.

Define a *VAP-free presentation* to be a group presentation $\langle \mathcal{S} \mid \mathcal{R} \rangle$ with the following two properties. First, every closed walk induced by a relator $R \in \mathcal{R}$ is a cycle; in other words, no proper subword of R is a relation. Second, for every edge e in the corresponding Cayley graph G, at most two cycles of G induced by the relators in $\mathcal{R}$ contain e. Note that the latter property can be checked by an easy algorithm once the former is guaranteed.

By Lemma 2.1 the cycles induced by the relators of a VAP-free presentation form a 2-basis. Combined with Theorem 2.3 this yields the following valuable tool, which in many cases [8, 7] allows one to deduce that certain Cayley graphs are planar by looking only at the corresponding presentations.

Corollary 4.1. *A group admits a planar Cayley complex, and a VAP-free Cayley graph, if and only if it admits a VAP-free presentation.*

Proof. Given a planar Cayley complex of the group Γ it is straightforward to derive a VAP-free presentation of Γ. By the above discussion, such a presentation yields a VAP-free Cayley graph of Γ. This in turn implies that Γ admits a planar Cayley complex by Theorem 1.1. □

The fact that a group with a planar Cayley complex admits a VAP-free presentation is implicit in [17, Theorem 4.5.6].

In fact, we can say a bit more. Thomassen [14, Theorem 7.4.] also proved that if G is 2-connected, then Theorem 2.3 can be strengthened to yield that given any 2-basis B of G, there is a VAP-free embedding σ of G such that B is the set of finite face-boundaries of σ. If G is a Cayley graph then the requirement of being 2-connected can be dropped by applying the result to the maximal 2-connected subgraphs of G, which must be either Cayley graphs too or single edges corresponding to free generators. Thus we have

Corollary 4.2. *Given a VAP-free presentation, the corresponding Cayley graph has a VAP-free embedding the finite face-boundaries of which are precisely the cycles of G induced by the relators.*

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