Minimal free resolutions for certain affine monomial curves

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This paper is dedicated to Wolmer V. Vasconcelos.

ABSTRACT. Given an arbitrary field k and an arithmetic sequence of positive integers $m_0 < \ldots < m_n$, we consider the affine monomial curve in $\mathbb{A}_k^{n+1}$ parameterized by $X_0 = t^{m_0}, \ldots, X_n = t^{m_n}$. In this paper, we conjecture that the Betti numbers of its coordinate ring are completely determined by n and the value of m_0 modulo n. We first show that the defining ideal of the monomial curve can be written as a sum of two determinantal ideals. Using this fact, we describe the minimal free resolution of the coordinate ring in the following three cases: when $m_0 \equiv 1 \mod n$ (determinantal), when $m_0 \equiv n \mod n$ (almost determinantal), and when $m_0 \equiv 2 \mod n$ and $n = 4$ (Gorenstein of codimension 4).

Introduction

Let k denote an arbitrary field and R be the polynomial ring $k[X_0, \ldots, X_n]$. Consider the k-algebra homomorphism $\varphi : R \to k[t]$ given by $\varphi(X_i) = t^{m_i}$, $i = 0, \ldots, n$. Then the ideal $\mathcal{P} := \ker \varphi \subset R$ is the defining ideal of the monomial curve in $\mathbb{A}_k^{n+1}$ given by the parametrization $X_0 = t^{m_0}, \ldots, X_n = t^{m_n}$. The k-algebra of the semigroup $\Gamma \subset \mathbb{N}$ generated by $m_0, \ldots, m_n$ is $k[\Gamma] := k[t^{m_0}, \ldots, t^{m_n}] \simeq R/\mathcal{P}$, which is one-dimensional and $\mathcal{P}$ is a perfect ideal of codimension n. It is well known that $\mathcal{P}$ is minimally generated by binomials. Moreover, $\mathcal{P}$ is a homogeneous ideal and $k[\Gamma]$ is the homogeneous coordinate ring if we give weight m_i to the variables X_i. Henceforth, homogeneous and graded would mean homogeneous and graded with respect to this weighted graduation.

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Assume now that the positive integers $m_0, \ldots, m_n$ satisfy the following properties:

(i) $\gcd(m_0, \ldots, m_n) = 1$;
(ii) $0 < m_0 < \cdots < m_n$ and $m_i = m_0 + id$ for every $i \in [1, n]$, i.e., the integers form an arithmetic progression with common difference d;
(iii) $m_0, \ldots, m_n$ generate the semigroup $\Gamma := \sum_{0 \leq i \leq n} \mathbb{N} m_i$ minimally, where $\mathbb{N} = \{0, 1, 2, \ldots\}$, i.e., $m_j \notin \sum_{0 \leq i \leq n;\, i \neq j} \mathbb{N} m_i$ for every $i \in [0, n]$.

Definition 1. A sequence of positive intergers $(\mathbf{m}) = m_0, \ldots, m_n$ is called an **arithmetic sequence** if it satisfies the conditions (i),(ii) and (iii) above.

Let us write $m_0 = an + b$ such that a, b are positive integers and $b \in [1, n]$. Note that $b \in [1, n]$ and condition (iii) on $m_0, \ldots, m_n$ ensures that $a \geq 1$; otherwise $m_0 = b$ and $m_b = m_0 + bd = (1 + d)b$ contradicts minimally condition (iii).

Let $(\mathbf{m}) = m_0, \ldots, m_n$ be an arithmetic sequence. We say that the monomial curve in $\mathbb{A}_k^{n+1}$ parameterized by $X_0 = t^{m_0}, \ldots, X_n = t^{m_n}$ is the monomial curve associated to the arithmetic sequence $(\mathbf{m})$ and denote it by $C(\mathbf{m})$. A minimal binomial generating set for the defining ideal $\mathcal{P}$ of $C(\mathbf{m})$ was given in [**P**], and it was rewritten in [**MS**] to prove that $\mathcal{P}$ is not a complete intersection if $n \geq 3$. An explicit formula for the type of $k[\Gamma]$ is given in [**PS**, Corollary 6.2] under a more general assumption of almost arithmetic sequence on the integers $(\mathbf{m}) = m_0, \ldots, m_n$ and it follows from this result that if $(\mathbf{m}) = m_0, \ldots, m_n$ is an arithmetic sequence then $k[\Gamma]$ is Gorenstein if and only if $b = 2$. In this paper, we prove in Theorem 1.1 that ideal $\mathcal{P}$ has the special structure that it can be written as a sum of two determinantal ideals, one of them being the defining ideal of the rational normal curve. We exploit this structure to construct an explicit minimal free resolution of the graded ideal $\mathcal{P}$, in the cases when $m_0 \equiv 1$ or n modulo n, and when $m_0 \equiv 2$ modulo n and $n \leq 4$. Note that a minimal free resolution for $\mathcal{P}$ had already been constructed for $n = 3$ in [**S1**] using a Gröbner basis for $\mathcal{P}$. In [**S2**] the following question was posed : do the total Betti numbers of $\mathcal{P}$ depend only on the integer m_0 modulo n ? In this article, we answer this question in affirmative for the above cases. This question will be addressed in general in our work in progress [**GSS**].

1. The defining ideal

By [**P**] and [**MS**], one knows that the number of elements in a minimal set of generators of the ideal $\mathcal{P}$ depends only on m_0 modulo n. Our first result shows that $\mathcal{P}$ has an additional structure that will be helpful in the sequel.

Theorem 1.1. *Let $(\mathbf{m}) = m_0, \ldots m_n$ be an arithmetic sequence and $\mathcal{P}$ be the defining ideal of the monomial curve $C(\mathbf{m})$ associated to $\mathbf{m}$. Then*

$$\mathcal{P} = I_2\left(\begin{pmatrix} X_0 & X_1 & \cdots & X_{n-1} \\ X_1 & X_2 & \cdots & X_n \end{pmatrix}\right) + I_2\left(\begin{pmatrix} X_n^a & X_0 & \cdots & X_{n-b} \\ X_0^{a+d} & X_b & \cdots & X_n \end{pmatrix}\right).$$

Proof. It is known from [**P**] and [**MS**], that $\mathcal{P}$ is minimally generated by the following set of binomials

$$\{\delta_{ij} \mid 0 \leq i < j \leq n-1\} \cup \{\Delta_{1,j} \mid j = 2, \ldots, n+2-b\},$$

such that

$$\delta_{i,j} := X_i X_{j+1} - X_j X_{i+1}, \quad 0 \le i < j \le n-1$$

and

$$\Delta_{1,j} := X_{b+j-2} X_n^a - X_{j-2} X_0^{a+d}, \quad j = 2, \ldots, n+2-b.$$

The binomials δ_{ij} are precisely the $\binom{n}{2}$ generators of the ideal of 2×2 minors of the matrix

$$A = \begin{pmatrix} X_0 & X_1 & \cdots & X_{n-1} \\ X_1 & X_2 & \cdots & X_n \end{pmatrix}.$$

On the other hand, the binomials $\Delta_{1,j}$ are the 2×2 minors from the first and the jth column of the matrix

$$B = \begin{pmatrix} X_n^a & X_0 & \cdots & X_{n-b} \\ X_0^{a+d} & X_b & \cdots & X_n \end{pmatrix}.$$

Since the rest of the 2×2 minors of B are already in the ideal $I_2(A)$, one gets that $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are the determinatal ideals generated by the maximal minors of the matrices A and B respectively. $\square$

The actual generators also depend only on the first term m_0, the common difference d and the length n of the arithmetic sequence $(\mathbf{m})$. This is no surprise as these three numbers do determine the arithmetic sequence. However, we also have that the number of minimal generators of the ideal $\mathcal{P}$ is $\binom{n}{2} + n - b + 1$ and hence only depends on n and b. We conjecture that the following statement is true:

Conjecture 1.2. Let $(\mathbf{m}) = m_0, \ldots, m_n$ be an arithmetic sequence. Then all the Betti numbers of the homogeneous coordinate ring $k[\Gamma]$ of the affine monomial curve $C(\mathbf{m})$ in $\mathbb{A}_k^{n+1}$ associated to $(\mathbf{m})$ are determined by n and the value of m_0 modulo n.

In the next section, we discuss the conjecture in cases $b = 1, 2$ and n and construct a minimal resolution when $b = 1$ or n using the aforesaid determinantal structures. In the first special case, $b = 1$, we have that $\mathcal{P}$ is a determinantal ideal. The second special case is $b = n$, that is when

$$B := \begin{pmatrix} X_n^a & X_0 \\ X_0^{a+d} & X_n \end{pmatrix},$$

and therefore the ideal $\mathcal{P}_2$ is the principal ideal generated by $X_0^{a+d+1} - X_n^{a+1}$. The third special case is $b = 2$, that is when $k[\Gamma]$ is Gorenstein. In this case, we wiil give the Betti numbers when the codimension n is 4.

2. The minimal resolution

2.1. First case: determinantal ($m_0 \equiv 1$ modulo n). Assume that $b = 1$. In this case,

$$B := \begin{pmatrix} X_n^a & X_0 & \cdots & X_{n-1} \\ X_0^{a+d} & X_1 & \cdots & X_n \end{pmatrix}$$

and hence $I_2(A) \subset I_2(B)$. Thus, $\mathcal{P} = \mathcal{P}_2$ is a determinantal ideal of codimension n generated by the maximal minors of the $2 \times (n+1)$ matrix B. Therefore, the homogeneous coordinate ring $k[\Gamma]$ is minimally resolved by the Eagon-Northcott complex. The resolution is given by

$$0 \to \wedge^{n+1} R^{n+1} \otimes D_{n-1}(R^2) \to \cdots \to \wedge^3 R^{n+1} \otimes D_1(R^2) \to \wedge^2 R^{n+1} \to \wedge^2 R^2 \to k[\Gamma] \to 0$$

such that $D_{s-1}(R^2) = (S_{s-1}(R^2))^*$ denotes the R-module which is the dual of the symmetric algebra of R^2 and the module $S_{s-1}(R^2)$ is free of rank s, with a basis the set of monomials of total degree $(s-1)$ in λ_0, λ_1 (symbols representing basis elements of R^2). Therefore, up to an identification, $D_{s-1}(R^2) = (S_{s-1}(R^2))^*$ is a free R-module of rank s, with a basis $\{\lambda_0^{v_0}\lambda_1^{v_1} \mid v_0, v_1 \in \mathbb{N}, v_0 + v_1 = s-1\}$. For every $s = 1, \ldots, n$, let G_s denote $\wedge^{s+1} R^{n+1} \otimes D_{s-1}(R^2)$, which is a R-free module of rank $s\binom{n+1}{s+1}$ generated by the basis elements $(e_{i_1} \wedge \cdots \wedge e_{i_{s+1}}) \otimes \lambda_0^{v_0}\lambda_1^{v_1}$, for every $1 \le i_1 < i_2 < \cdots < i_{s+1} \le n+1$ and $v_0, v_1 \in \mathbb{N}, v_0 + v_1 = s-1$. The differentials $d_s : G_s \to G_{s-1}$ are given by the following formulae:

$$d_1(e_{i_1} \wedge e_{i_2}) = X_{i_1-1}X_{i_2} - X_{i_2-1}X_{i_1}\,, \quad \forall\ 1 \le i_1 < i_2 \le n+1,$$

and for $s = 2, \ldots, n$ and $1 \le i_1 < \cdots < i_{s+1} \le n+1$,

$$\begin{aligned} d_s((e_{i_1} \wedge \cdots \wedge e_{i_{s+1}}) \otimes \lambda_0^{v_0}\lambda_1^{v_1}) \quad = \quad & \sum_{j=1}^{s+1} (-1)^{j+1} X_{i_j-1}((e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots e_{i_{s+1}}) \otimes \lambda_0^{v_0-1}\lambda_1^{v_1}) \\ & + \sum_{j=1}^{s+1} (-1)^{j+1} X_{i_j}((e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots e_{i_{s+1}}) \otimes \lambda_0^{v_0}\lambda_1^{v_1-1}) \end{aligned}$$

such that summands on the right hand side involve only non-negative powers of λ_0 and λ_1.

We have therefore proved the following:

Theorem 2.1. *The minimal free resolution of the homogeneous coordinate ring $k[\Gamma]$ of the affine monomial curve $C(\mathbf{m})$ in $\mathbb{A}_k^{n+1}$ associated to the arithmetic sequence of integers $(\mathbf{m}) = m_0, \ldots, m_n$ with the property $m_0 \equiv 1$ modulo n is*

$$0 \to R^n \xrightarrow{d_n} R^{(n-1)\binom{n+1}{n}} \xrightarrow{d_{n-1}} \cdots \longrightarrow R^{s\binom{n+1}{s+1}} \xrightarrow{d_s} \cdots \longrightarrow R^{\binom{n+1}{2}} \xrightarrow{d_1} R \xrightarrow{\varphi} k[\Gamma] \to 0.$$

In particular, the Betti numbers are $\beta_0 = 1$ and $\beta_s = s\binom{n+1}{s+1}$ for every $s \in [1, n]$.

Remark 2.2. The case $a = 1$ (and $b = 1$) corresponds to semigroups Γ of maximal embedding dimension, i.e., those semigroups such that the inequality $m(\Gamma) \ge e(\Gamma)$ is an equality, being $m(\Gamma) := m_0 = an + b$ and $e(\Gamma) := n + 1$ the multiplicity and the embedding dimension respectively.

Note that the above argument gives indeed the graded resolution (with respect to the grading given by $\deg(X_i) = m_i = m_0 + id$):

Corollary 2.3. *Under the hypothesis of Theorem 2.1, the minimal graded free resolution of $k[\Gamma]$ is given by*

$$
\begin{aligned}
0 \quad &\longrightarrow \quad \bigoplus_{k=1}^{n} R(-(a+n+d)m_0 + kd - \binom{n+1}{2}d)\\
&\xrightarrow{d_n} \quad \left(\bigoplus_{k=1}^{n-1} R(-nm_0 + kd - \binom{n+1}{2}d)\right)\\
&\qquad \oplus \left(\bigoplus_{1\leq r_1<\ldots r_{n-1}\leq n} \left(\bigoplus_{k=1}^{n-1} R(-(a+d+n-1)m_0 + kd - \sum_{i=1}^{n-1} r_i d)\right)\right)\\
&\xrightarrow{d_{n-1}} \quad \cdots\\
&\quad\vdots\\
\cdots \quad &\xrightarrow{d_s} \quad \left(\bigoplus_{1\leq r_1<\ldots<r_s\leq n} \left(\bigoplus_{k=1}^{s-1} R(-sm_0 + kd - \sum_{i=1}^{s} r_i d)\right)\right) \oplus\\
&\qquad \left(\bigoplus_{1\leq r_1<\ldots<r_{s-1}\leq n} \left(\bigoplus_{k=1}^{s-1} R(-(a+s-1+d)m_0 + kd - \sum_{i=1}^{s-1} r_i d)\right)\right)\\
&\xrightarrow{d_{s-1}} \quad \cdots\\
&\quad\vdots\\
\cdots \quad &\xrightarrow{d_2} \quad \left(\bigoplus_{1\leq r_1<r_2\leq n} R(-2m_0 - (r_1+r_2-1)d)\right) \oplus \left(\bigoplus_{k=0}^{n-1} R(-(a+1+d)m_0 - kd)\right)\\
&\xrightarrow{d_1} \quad R \quad \xrightarrow{\varphi} \quad k[\Gamma] \to 0\,.
\end{aligned}
$$

2.2. Second case: almost determinantal ($m_0 \equiv n$ modulo n). In this case, we show that $k[\Gamma]$ can be resolved minimally by a mapping cone using the resolution for $R/\mathcal{P}_1$, which is the homogeneous coordinate ring of the rational normal curve. The minimal free resolution for $R/\mathcal{P}_1$ is the Eagon-Northcott complex given by

$$\mathcal{E} : 0 \longrightarrow \wedge^n R^n \otimes (S_{n-2}(R^2))^* \longrightarrow \cdots \longrightarrow \wedge^2 R^n \otimes (S_0(R^2))^* \longrightarrow \wedge^2 R^2 \longrightarrow R/\mathcal{P}_1 \longrightarrow 0.$$

Since $E_s := \wedge^{s+1} R^n \otimes (S_{s-1}(R^2))^* = R^{s\binom{n}{s+1}}$, for every $s = 1, \ldots, n-1$, $E_0 := \wedge^2 R^2 = R$, the resolution $\mathcal{E}$ takes the form

$$0 \longrightarrow R^{n-1} \longrightarrow R^{(n-2)\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow R^{s\binom{n}{s+1}} \longrightarrow \cdots \longrightarrow R^{\binom{n}{2}} \longrightarrow R \longrightarrow R/\mathcal{P}_1 \longrightarrow 0.$$

The differentials $d_s^1 : E_s \to E_{s-1}$ are defined similarly as d_s above, for every $s = 1, \ldots, n-1$.

Now consider the short exact sequence of R-modules

$$0 \longrightarrow R/(\mathcal{P}_1 : \Delta_{1,2}) \xrightarrow{\Delta_{1,2}} R/\mathcal{P}_1 \longrightarrow R/\mathcal{P}_1 + (\Delta_{1,2}) \longrightarrow 0$$

where the map $R/(\mathcal{P}_1 : \Delta_{1,2}) \xrightarrow{\Delta_{1,2}} R/\mathcal{P}_1$ is the multiplication by $\Delta_{1,2}$. The colon ideal $(\mathcal{P}_1 : \Delta_{1,2})$ is exactly equal to $\mathcal{P}_1$, since $\mathcal{P}_1$ is a prime ideal and $\Delta_{1,2} \notin \mathcal{P}_1$, being a part of a minimal generating set for the ideal $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$. Therefore, the above short exact sequence becomes,

$$0 \longrightarrow R/\mathcal{P}_1(-(a+d+1)m_0) \xrightarrow{\Delta_{1,2}} R/\mathcal{P}_1 \longrightarrow R/\mathcal{P}_1 + (\Delta_{1,2}),$$

taking gradation into account. We define the graded complex homomorphism

$$\psi : \mathcal{E}(-(a+d+1)m_0) \longrightarrow \mathcal{E}$$

as the multiplication by $\Delta_{1,\,2}$, which is a lift of the map

$$R/\mathcal{P}_1(-(a+d+1)m_0) \xrightarrow{\Delta_{1,\,2}} R/\mathcal{P}_1.$$

In fact, the complex homomorphism ψ is simply multiplication by $\Delta_{1,\,2}$. The mapping cone $\mathcal{F}(\psi)$ is given by the free modules $F_s = E_{s-1} \oplus E_s = R^{(s-1)\binom{n}{s}} \oplus R^{s\binom{n}{s+1}}$, for every $s = 0, \ldots, n$ (with $E_{-1} = E_n = 0$) and the differentials $d_s^2 : F_s \to F_{s-1}$, defined as $d_s^2(x, y) = (\psi_{s-1}(y) + d_s^1(x),\, -d_{s-1}^1(y))$, for every $s = 1, \ldots, n$.

$R/\mathcal{P}_1 + (\Delta_{1,\,2})$ is resolved by the mapping cone $\mathcal{F}(\psi)$. It is minimal because each of the maps in ψ is of positive degree and the resolution $\mathcal{E}$ is minimal. Hence, the mapping cone $\mathcal{F}(\psi)$ is the minimal free resolution of the homogeneous coordinate ring $k[\Gamma] = R/\mathcal{P}$. We have therefore proved the following:

Theorem 2.4. *The minimal free resolution of the homogeneous coordinate ring $k[\Gamma]$ of the affine monomial curve $C(\mathbf{m})$ in $\mathbb{A}_k^{n+1}$ associated to the arithmetic sequence of integers $(\mathbf{m}) = m_0, \ldots, m_n$ with the property $m_0 \equiv n$ modulo n is*

$$0 \to R^{n-1} \xrightarrow{d_n^2} R^{(n-2)\binom{n}{n-1}+(n-1)} \xrightarrow{d_{n-1}^2} \cdots \longrightarrow R^{1+\binom{n}{2}} \xrightarrow{d_1^2} R \xrightarrow{\varphi} k[\Gamma] \to 0.$$

In particular, the Betti numbers are $\beta_0 = 1$, $\beta_1 = 1+\binom{n}{2}$ and $\beta_s = (s-1)\binom{n}{s}+s\binom{n}{s+1}$ for every $s \in [2, n]$.

As in Section 2.1, one can be more precise and give the graded resolution. First note that the graded minimal resolution of $\mathcal{P}_1$ is given by

$$\begin{array}{llll} 0 \longrightarrow & \displaystyle\bigoplus_{k=1}^{n-1} R(-nm_0 + kd - \binom{n+1}{2} d) & \xrightarrow{d_{n-1}^1} & \cdots \\ \cdots \xrightarrow{d_s^1} & \displaystyle\bigoplus_{1 \le r_1 < \ldots < r_s \le n} \left(\bigoplus_{k=1}^{s-1} R(-sm_0 + kd - \sum r_i d) \right) & \xrightarrow{d_{s-1}^1} & \cdots \\ \cdots \xrightarrow{d_2^1} & \displaystyle\bigoplus_{1 \le r_1 < r_2 \le n} R(-2m_0 - (r_1 + r_2 - 1)d) & \xrightarrow{d_1^1} & R \longrightarrow R/\mathcal{P}_1 \to 0\,. \end{array}$$

Corollary 2.5. *Under the hypothesis of Theorem 2.4, the minimal graded free resolution of $k[\Gamma]$ is given by*

$$
\begin{array}{cl}
0 \longrightarrow & \bigoplus_{k=1}^{n-1} R(-(a+n+d+1)m_0 + kd - \binom{n+1}{2} d) \\
\xrightarrow{d_n^2} & \left(\bigoplus_{k=1}^{n-1} R(-nm_0 + kd - \binom{n+1}{2} d) \right) \oplus \\
& \left(\bigoplus_{1 \le r_1 < \ldots < r_{n-1} \le n} \left(\bigoplus_{k=1}^{n-2} R(-(a+n+d)m_0 + kd - \sum r_i d) \right) \right) \\
\xrightarrow{d_{n-1}^2} & \ldots \\
\vdots & \\
\cdots \xrightarrow{d_s^2} & \left(\bigoplus_{1 \le r_1 < \ldots < r_s \le n} \left(\bigoplus_{k=1}^{s-1} R(-sm_0 + kd - \sum r_i d) \right) \right) \oplus \\
& \left(\bigoplus_{1 \le r_1 < \ldots < r_{s-1} \le n} \left(\bigoplus_{k=1}^{s-2} R(-(a+s+d)m_0 + kd - \sum r_i d) \right) \right) \\
\xrightarrow{d_{s-1}^2} & \ldots \\
\vdots & \\
\cdots \xrightarrow{d_2^2} & \left(\bigoplus_{1 \le r_1 < r_2 \le n} R(-2m_0 - (r_1 + r_2 - 1)d) \right) \oplus R(-(a+1+d)m_0) \\
\xrightarrow{d_1^2} & R \xrightarrow{\varphi} k[\Gamma] \to 0.
\end{array}
$$

2.3. Third case: Gorenstein of codimension 4 ($m_0 \equiv 2$ modulo n and $n = 4$). When $b = 2$, the ideal $\mathcal{P}$ is Gorenstein of height n and hence the first and the last Betti numbers are known. This provides a very interesting family of Gorenstein ideals of height n generated by $(n-1)(n+2)/2$ binomials.

When $n = 2$, $\mathcal{P}$ is a complete intersection and when $n = 3$, $\mathcal{P}$ is the ideal of 4×4 pfaffians of a 5×5 skew symmetric matrix. For $n = 4$, the ideal $\mathcal{P}$ is a height 4 Gorenstein ideal minimally generated by 9 elements and therefore has Betti numbers 9, 16, 9 and 1.

In fact, we can show that with this weighted grading, the shifts in a graded resolution of $R/\mathcal{P}$ are as follows:

Theorem 2.6. *Given two non-negative integers a and d, consider the monomial curve $C(\mathbf{m})$ in $\mathbb{A}_k^5$ associated to the arithmetic sequence $(\mathbf{m}) = 4a+2, 4a+2+d, 4a+2+2d, 4a+2+3d, 4a+2+4d$. Set $q := 2a+1$. Then the defining ideal $\mathcal{P}$ of $C(\mathbf{m})$ is a height four Gorenstein ideal whose minimal graded free resolution is*

$$
\begin{aligned}
0 \quad \to \quad & R(-q(q+2d+9)-9d) \\
\to \quad & \left(\bigoplus_{k=7}^{9} R(-8q-kd)\right) \oplus \left(\bigoplus_{k=3}^{7} R(-q(q+2d+5)-kd)\right) \oplus R(-q(q+2d+5)-5d) \\
\to \quad & R(-6q-4d) \oplus \left(\bigoplus_{k=5}^{7} R^2(-6q-kd)\right) \oplus R(-6q-8d) \ \oplus R(-q(q+2d+3)-d) \\
& \oplus \left(\bigoplus_{k=2}^{4} R^2(-q(q+2d+3)-kd)\right) \oplus R(-q(q+2d+3)-5d) \\
\to \quad & \left(\bigoplus_{k=2}^{6} R(-4q-kd)\right) \oplus R(-4q-4d) \oplus \left(\bigoplus_{k=0}^{2} R(-q(q+2d+1)-kd)\right) \ \to \mathcal{P} \to 0.
\end{aligned}
$$

Proof. We know that $\mathcal{P}$ is Gorenstein of height 4 and by Theorem 1.1, it is generated by the 2×2 minors of A and B, where

$$
A = \begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ X_1 & X_2 & X3 & X_4 \end{pmatrix}
$$

and

$$
B = \begin{pmatrix} X_4^a & X_0 & X_1 & X_2 \\ X_0^{a+d} & X_2 & X_3 & X_4 \end{pmatrix}.
$$

Moreover, the 9 minimal generators of $\mathcal{P}$ are the six 2×2 minors of A whose degrees are $2m_0 + kd$, $k = 2, 3, 4, 4, 5, 6$ for $m_0 := 4a+2 = 2(2a+1) = 2q$, and the 3 minors of B involving the first column which are of degrees $m_0(a+d+1)+kd$, $k = 0, 1, 2$. Note that for both determinantal ideals $I_2(A)$ and $I_2(B)$, the Eagon-Northcott complex provides a minimal resolution. The 8 determinantal relations from the Eagon-Northcott resolution of $I_2(A)$ and the 6 determinantal relations from the Eagon-Northcott resolution of $I_2(B)$ involving the first column will necessarily be among the 16 minimal syzygies of $\mathcal{P}$. The degrees of the other two relations are determined by the symmetry of the resolution. Thus the remaining two relations must be of degrees $q(q+3)+(2q+2)d$ and $q(q+3)+(2q+4)d$. This determines the rest of the degrees in the resolution. $\square$

We note that when $d = 1$, this ideal can never be determinantal even though it is height 4 and generated by 9 elements. That is, this ideal is not the ideal of 2×2 minors of a 3×3 matrix for sum of the degrees of such an ideal must be even. For in this case, the ideal is generated by 9 elements whose degrees add up to $27d + km_0$. Since m_0 is even, this number is odd whenever d is odd. Hence if d is odd, it cannot be a determinantal ideal. However, it is not clear if it can be determinantal for even values of d. Of course, the conjecture only says that the Betti numbers are determined by b.

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