# AN EXPLICIT EXAMPLE OF FROBENIUS PERIODICITY

HOLGER BRENNER AND ALMAR KAID

Abstract. In this note we show that the restriction of the cotangent bundle $\Omega_{\mathbb{P}^2}$ of the projective plane to a Fermat curve $C$ of degree $d$ in characteristic $p \equiv -1 \mod 2d$ is, up to tensoration with a certain line bundle, isomorphic to its Frobenius pull-back. This leads to a Frobenius periodicity $F^*(\mathcal{E}) \cong \mathcal{E}$ on the Fermat curve of degree $2d$, where $\mathcal{E} = \mathrm{Syz}(U^2, V^2, W^2)(3)$.

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## 1. Introduction

Let $C$ be a smooth projective curve defined over a field $K$ of characteristic $p > 0$. If $F$ denotes the absolute Frobenius morphism $F : C \to C$, then we say that a vector bundle $\mathcal{E}$ on $C$ admits an $(s,t)$-*Frobenius periodicity* if there are natural numbers $s$ and $t$, $t > s$, such that $F^{t^*}(\mathcal{E}) \cong F^{s^*}(\mathcal{E})$. Of particular interest are vector bundles which admit a $(0,t)$-Frobenius periodicity, i.e., $F^{t*}(\mathcal{E}) \cong \mathcal{E}$. By the classical theorem of H. Lange and U. Stuhler [19, Satz 1.4] such a bundle is étale trivializable, i.e., there exists an étale covering $f : D \to C$ such that $f^*(\mathcal{E}) \cong \mathcal{O}_D^r$ where $r = \mathrm{rk}(\mathcal{E})$. Hence a vector bundle $\mathcal{E}$ with a $(0,t)$-Frobenius periodicity comes from a (continuous) representation $\rho : \pi_1(C) \to GL_r(K)$ of the étale fundamental group $\pi_1(C)$ of the curve (see [ibid., Proposition 1.2]). We recall that a vector bundle which can be trivialized under an étale covering does not necessarily admit a Frobenius periodicity (see [9, Example 2.10] or [2, Example below Theorem 1.1]).

Quasicoherent modules over a scheme of positive characteristic allowing a Frobenius periodicity appear under several names ($\mathcal{F}$-finite modules, unit $\mathcal{O}_X[F]$-modules) and from several perspectives ($D$-modules, local cohomology, Cartier modules, constructible sheaves on the étale site, Riemann-Hilbert correspondence) in the literature. Beside [19] we mention work of Katz [17, Proposition 4.1.1], Lyubeznik [21], Emerton and Kisin [11], Blickle [3] and Blickle and Böckle [4].

Despite the importance of vector bundles having a Frobenius periodicity, it is not easy to write down non-trivial explicit examples. For a line bundle the condition becomes ${F^t}^*\mathcal{L} = \mathcal{L}^q = \mathcal{L}$ (with $q = p^t$), so $\mathcal{L}$ must be a torsion element in Pic $C$ of order $q-1$. For higher rank, a necessary condition is that the bundle $\mathcal{S}$ has degree 0 and is semistable. By the periodicity it follows that the bundle is in fact strongly semistable, meaning that ${F^t}^*(\mathcal{E})$ is semistable for all $t \geq 0$. On the other hand, if the curve $C$ and the bundle $\mathcal{E}$ are defined over a finite field and $\mathcal{E}$ is strongly semistable of degree 0, then there is necessarily a $(s,t)$-Frobenius periodicity due to the fact that the number of isomorphism classes of semistable vector bundles of fixed rank and degree is finite ([19, Satz 1.9]). Nevertheless, it is still hard to detect the periodicity $s$ and $t$. If we have an extension $0 \to \mathcal{O}_C \to \mathcal{S} \to \mathcal{O} \to 0$ given by $c \in H^1(C, \mathcal{O}_C)$, then its Frobenius pull-back is given by the class $F^*(c)$, and one can get (semistable, but not stable) examples by looking at the Frobenius action on $H^1(C, \mathcal{O}_C)$.

In this note we provide a down to earth example of a stable rank-2 vector bundle $\mathcal{E}$ on a suitable Fermat curve admitting a $(0,1)$-Frobenius periodicity. Moreover, this periodicity only depends on a congruence condition of the characteristic of the base field, not on its algebraic structure. Our main tools will be results of P. Monsky on the Hilbert-Kunz multiplicity of Fermat hypersurface rings and the geometric approach to Hilbert-Kunz theory developed independently by the first author in [7] and V. Trivedi in [26].

The results of this article are contained in Chapter 4 of the PhD-thesis [14] of the second author. Related results on the free resolution of Frobenius powers on a Fermat ring can be found in the preprint [18]. We thank Manuel Blickle, Aldo Conca, Neil Epstein and Andrew Kustin for useful discussions.

## 2. A lemma on global sections

To begin with we recall the notions of a *syzygy bundle*. Let $K$ be a field and let $R$ be a normal standard-graded $K$-domain of dimension $d \geq 2$. Then homogeneous $R_+$-primary elements $f_1, \ldots, f_n$ (i.e., $\sqrt{(f_1, \ldots, f_n)} = R_+$) of degrees $d_1, \ldots, d_n$ define a short exact (presenting) sequence

$$0 \longrightarrow \operatorname{Syz}(f_1, \ldots, f_n) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_X(-d_i) \stackrel{f_1,\ldots,f_n}{\longrightarrow} \mathcal{O}_X \longrightarrow 0$$

on the projective scheme $X = \operatorname{Proj} R$. The kernel $\operatorname{Syz}(f_1, \ldots, f_n)$ is locally free and is called the *syzygy bundle* for the elements $f_1, \ldots, f_n$.

In this article we only deal with restrictions of syzygy bundles of the form $\operatorname{Syz}(X^a, Y^a, Z^a)$, $a \in \mathbb{N} \setminus \{0\}$, on $\mathbb{P}^2 = \operatorname{Proj} K[X,Y,Z]$ to a plane curve $C$. Our main interest will be the case $a = 1$ which corresponds via the Euler sequence to the cotangent bundle $\Omega_{\mathbb{P}^2}|_C$ on the projective plane. Since there will be no confusion in the sequel we also denote the restricted bundle on the curve by $\operatorname{Syz}(X^a, Y^a, Z^a)$.

Let $K$ be a field and consider a smooth plane curve of the form

$$V_+(Z^d - P(X,Y)) \subset \mathbb{P}^2 = \operatorname{Proj} K[X,Y,Z],$$

where $P(X,Y) \in K[X,Y]$ denotes a homogeneous polynomial of degree $d$. In this situation we can compute global sections of a rank-2 syzygy bundle of the form $\mathrm{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})$ by the following lemma which is a slight improvement over [6, Lemma 1]. It relates the sheaves $\mathrm{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})$ with the sheaves $\mathrm{Syz}(X^{a_1}, Y^{a_2}, P(X,Y)^k)$ which come from $\mathbb{P}^1$ via the Noetherian normalization $C = V_+(Z^d - P(X,Y)) \to \mathbb{P}^1 = \mathrm{Proj}\, K[X,Y]$. We will use this result several times in the proof of our main theorem in the next section.

**Lemma 2.1.** *Let $K$ be a field and let $P(X,Y) \in K[X,Y]$ be a homogeneous polynomial of degree $d$. Suppose the plane curve*

$$C := \mathrm{Proj}(K[X,Y,Z]/(Z^d - P(X,Y)))$$

*is smooth. Further, fix $a_1, a_2, a_3 \in \mathbb{N}_+$ and write $a_3 = dk + t$ with $0 \le t < d$. Then we have for every $m \in \mathbb{Z}$ a surjective sheaf morphism*

$$\begin{aligned} \varphi_m : \mathcal{S}_k(m-t) \oplus \mathcal{S}_{k+1}(m) &\longrightarrow \mathrm{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})(m) \\ (f_1, f_2, f_3), (g_1, g_2, g_3) &\longmapsto (Z^t f_1 + g_1, Z^t f_2 + g_2, f_3 + Z^{d-t} g_3) \end{aligned}$$

*for every $m \in \mathbb{Z}$, where $\mathcal{S}_i := \mathrm{Syz}(X^{a_1}, Y^{a_2}, P(X,Y)^i)$ for $i \ge 0$. Moreover, the corresponding map on global sections*

$$\Gamma(C, \mathcal{S}_k(m-t)) \oplus \Gamma(C, \mathcal{S}_{k+1}(m)) \longrightarrow \Gamma(C, \mathrm{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})(m))$$

*is surjective for every $m \in \mathbb{Z}$.*

*Proof.* We consider the sheaf morphism

$$\begin{array}{c} \mathcal{O}_C(m-t-a_1) \oplus \mathcal{O}_C(m-t-a_2) \oplus \mathcal{O}_C(m-t-dk) \\ \oplus \\ \mathcal{O}_C(m-a_1) \oplus \mathcal{O}_C(m-a_2) \oplus \mathcal{O}_C(m-dk-d) \\ \Big\downarrow \widetilde{\varphi_m} \\ \mathcal{O}_C(m-a_1) \oplus \mathcal{O}_C(m-a_2) \oplus \mathcal{O}_C(m-a_3) \end{array}$$

which maps $(s_1, s_2, s_3, s_4, s_5, s_6) \mapsto (Z^t s_1 + s_4, Z^t s_2 + s_5, s_3 + Z^{d-t} s_6)$. Clearly, $\widetilde{\varphi_m}$ maps $\mathcal{S}_k(m-t) \oplus \mathcal{S}_{k+1}(m)$ into $\mathrm{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})(m)$. Hence, the map $\varphi_m$ is obtained from $\widetilde{\varphi_m}$ via restriction to $\mathcal{S}_k(m-t) \oplus \mathcal{S}_{k+1}(m)$ and is therefore a morphism of sheaves. It is enough to prove that $\varphi_m$ is surjective on global sections for all $m$. Let $s := (F, G, H) \in \Gamma(C, \mathrm{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})(m))$ be a non-trivial global section, i.e., $FX^{a_1} + GY^{a_2} + HZ^{a_3} = 0$ and $\deg(F) + a_1 = \deg(G) + a_2 = \deg(H) + a_3 = m$. We write

$$\begin{aligned} F &= F_0 + F_1 Z + F_2 Z^2 + \ldots + F_{d-1} Z^{d-1} \\ G &= G_0 + G_1 Z + G_2 Z^2 + \ldots + G_{d-1} Z^{d-1} \\ H &= H_0 + H_1 Z + H_2 Z^2 + \ldots + H_{d-1} Z^{d-1} \end{aligned}$$

with $F_i, G_i, H_i \in K[X,Y]$ for $i = 0, \ldots, d-1$. We have $Z^{a_3} = Z^{dk+t} = (Z^d)^k Z^t = P(X,Y)^k Z^t$. Since $s$ is a syzygy we obtain (by considering the $K[X,Y]$]-component corresponding to $Z^i$ a system of equations

$$F_i Z^i X^{a_1} + G_i Z^i Y^{a_2} + H_{j(i)} Z^{j(i)} Z^{a_3} = 0,$$

where $j(i) \equiv i - t \bmod d$. Thus $s = (F, G, H)$ is the sum of the syzygies

$$s_i := (F_i Z^i, G_i Z^i, H_{j(i)} Z^{j(i)}) \in \Gamma(C, \mathrm{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})(m)).$$

We show that each of these summands does either come from $\Gamma(C, \mathcal{S}_{k+1}(m))$ or from $\Gamma(C, \mathcal{S}_k(m-t))$. We fix one equation

$$F_{i_0} Z^{i_0} X^{a_1} + G_{i_0} Z^{i_0} Y^{a_2} + H_{j(i_0)} Z^{j(i_0)} Z^{a_3} = 0$$

with $j(i_0) \equiv i_0 - t \bmod d$. First, we treat the case where $i_0 < t$, hence $j(i_0) = i_0 - t + d$. Factoring out $Z^{i_0}$ and replacing $Z^{a_3}$ by $P(X,Y)^k Z^t$ yields

$$\begin{aligned} 0 &= Z^{i_0}(F_{i_0} X^{a_1} + G_{i_0} Y^{a_2} + H_{j(i_0)} Z^{d-t} P(X,Y)^k Z^t) \\ &= Z^{i_0}(F_{i_0} X^{a_1} + G_{i_0} Y^{a_2} + H_{j(i_0)} P(X,Y)^{k+1}). \end{aligned}$$

Hence $g_{i_0} := (Z^{i_0} F_{i_0}, Z^{i_0} G_{i_0}, Z^{i_0} H_{j(i_0)}) \in \Gamma(C, \mathcal{S}_{k+1}(m))$ and $\varphi_m(g_{i_0}) = s_{i_0}$. Next, we consider the case $i_0 \geq t$, hence $j(i_0) = i_0 - t$. We factor out $Z^t$ and replace $Z^{a_3}$. This gives

$$\begin{aligned} 0 &= F_{i_0} Z^{j(i_0)+t} X^{a_1} + G_{i_0} Z^{j(i_0)+t} Y^{a_2} + H_{j(i_0)} Z^{j(i_0)} P(X,Y)^k Z^t \\ &= Z^t (F_{i_0} Z^{j(i_0)} X^{a_1} + G_{i_0} Z^{j(i_0)} Y^{a_2} + H_{j(i_0)} Z^{j(i_0)} P(X,Y)^k). \end{aligned}$$

Hence we have $h_{i_0} := (F_{i_0} Z^{j(i_0)}, G_{i_0} Z^{j(i_0)}, H_{j(i_0)} Z^{j(i_0)}) \in \Gamma(C, \mathcal{S}_k(m-t))$ and $\varphi_m(h_{i_0}) = s_{i_0}$. □

**Remark 2.2.** It is easy to see that the morphisms $\varphi_m$, $m \in \mathbb{Z}$, are injective on both summands, i.e., the induced mappings

$$\mathcal{S}_k(m-t) \longrightarrow \mathrm{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})(m), \ (f_1, f_2, f_3) \longmapsto (Z^t f_1, Z^t f_2, f_3)$$

and

$$\mathcal{S}_{k+1}(m) \longrightarrow \mathrm{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})(m), \ (g_1, g_2, g_3) \longmapsto (g_1, g_2, Z^{d-t} g_3)$$

are both injective.

**Remark 2.3.** The sheaves $\mathcal{S}_k$ and $\mathcal{S}_{k+1}$ are the pull-backs

$$\pi^*(\mathrm{Syz}_{\mathbb{P}^1}(X^{a_1}, Y^{a_2}, P(X,Y)^k)) \text{ and } \pi^*(\mathrm{Syz}_{\mathbb{P}^1}(X^{a_1}, Y^{a_2}, P(X,Y)^{k+1}))$$

respectively under the Noetherian normalization $\pi : C \to \mathbb{P}^1 = \mathrm{Proj}\, K[X,Y]$. In particular, $\mathcal{S}_k$ and $\mathcal{S}_{k+1}$ split as a direct sum of line bundles. If $t = 0$ we have $\mathrm{Syz}_C(X^{a_1}, Y^{a_2}, Z^{a_3}) \cong \mathrm{Syz}_C(X^{a_1}, Y^{a_2}, P(X,Y)^k)$ and the bundle is therefore already defined on $\mathbb{P}^1$.

## 3. Frobenius periodicity up to a twist

Let $C$ be a smooth projective curve defined over a field of positive characteristic. It is a well-known fact that the pull-back of a semistable vector bundle under the (absolute) Frobenius morphism is in general not semistable anymore; see for instance the example of Serre in [13, Example 3.2]. Using syzygy bundles on Fermat curves one can produce fairly easy examples of this phenomenon.

**Example 3.1.** Let $C := \operatorname{Proj}(\overline{\mathbb{F}}_3[X,Y,Z]/(X^4+Y^4-Z^4)$ be the Fermat quartic in characteristic 3. The cotangent bundle $\Omega_{\mathbb{P}^2}$ is stable on the projective plane (see for instance [8, Corollary 6.4]) and so is the restriction $\Omega_{\mathbb{P}^2}|_C = \operatorname{Syz}(X,Y,Z)$ by Langer's restriction theorem [20, Theorem 2.19]. Its Frobenius pull-back is the syzygy bundle $\operatorname{Syz}(X^3,Y^3,Z^3)$. The curve equation yields the relation $X\cdot X^3+Y\cdot Y^3-Z\cdot Z^3=0$ and thus we obtain a non-trivial global section of $(F^*(\Omega_{\mathbb{P}^2}|_C))(4)$. But the degree of this bundle equals $-4$ and therefore $F^*(\Omega_{\mathbb{P}^2}|_C)$ is not semistable.

A vector bundle $\mathcal{E}$ such that $F^{e*}(\mathcal{E})$ is semistable for all $e \geq 0$ is called *strongly semistable*. This notion goes back to Miyaoka (cf. [22, Section 5]). Before we state our main theorem, we prove the following Lemma separately.

**Lemma 3.2.** *Let $d \geq 2$ be an integer and let $K$ be a field of characteristic $p \equiv -1 \bmod 2d$. Then $\Omega_{\mathbb{P}^2}|_C$ is strongly semistable on the Fermat curve $C := \operatorname{Proj}(K[X,Y,Z]/(X^d+Y^d-Z^d))$.*

*Proof.* We use Hilbert-Kunz theory and its geometric interpretation developed in [7] and [26]. The Hilbert-Kunz multiplicity $e_{HK}(R)$ of the homogeneous coordinate ring $R := K[X,Y,Z]/(X^d+Y^d-Z^d)$ of the Fermat curve equals $\frac{3d}{4}$ in characteristic $p \equiv -1 \bmod 2d$ by Monsky's result [24, Theorem 2.3]. By [7, Corollary 4.6] this is equivalent to the strong semistability of $\Omega_{\mathbb{P}^2}|_C$ in these characteristics. □

**Remark 3.3.** Note that for $d=1$ we have $C \cong \mathbb{P}^1$ and $\Omega_{\mathbb{P}^2}|_C \cong \mathcal{O}_C(-2) \oplus \mathcal{O}_C(-1)$, i.e., $\Omega_{\mathbb{P}^2}|_C$ is not even semistable. For a general characterization of strong semistability of $\Omega_{\mathbb{P}^2}|_C$ on the Fermat curve of degree $d$ depending on the characteristic of the base field see [14, Chapter 4]. The restriction of $\mathcal{S}$ to every smooth projective curve of degree $\geq 7$ is stable by Langer's restriction theorem [20, Theorem 2.19].

**Theorem 3.4.** *Let $d \geq 2$ be an integer and let $K$ be a field of characteristic $p \equiv -1 \bmod 2d$. Then $\mathcal{E} := \Omega_{\mathbb{P}^2}|_C$ is strongly semistable on the Fermat curve $C := \operatorname{Proj}(K[X,Y,Z]/(X^d+Y^d-Z^d))$ and*

$$F^*(\mathcal{E}) \cong \mathcal{E}(-\frac{3(p-1)}{2}).$$

*Proof.* The strong semistability of $\mathcal{E}$ in characteristics $p \equiv -1 \mod 2d$ has already been proved in Lemma 3.2. So we have to show that $F^*(\mathcal{E}) \cong \operatorname{Syz}(X^p, Y^p, Z^p) \cong \mathcal{E}(-\frac{3(p-1)}{2})$. Since the proof is quite long, we divide it into several steps. Note that, since semistability is preserved under base change, we may assume without loss of generality that $K$ is algebraically closed.

Step 1. We write $p = dk + (d-1)$ with $k$ odd. Accordingly, we set $t = d-1$. Further, we follow the notation of Lemma 2.1 and define the bundles

$$\mathcal{S}_k := \operatorname{Syz}(X^p, Y^p, (X^d+Y^d)^k),\ \mathcal{S}_{k+1} := \operatorname{Syz}(X^p, Y^p, (X^d+Y^d)^{k+1}).$$

We show that the surjective morphism

$$\varphi_{\frac{3p+1}{2}} : \mathcal{S}_k(\frac{3p+1}{2} - t) \oplus \mathcal{S}_{k+1}(\frac{3p+1}{2}) \longrightarrow \operatorname{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2})$$

defined in Lemma 2.1 can be identified with

$$(\mathcal{O}_C(-d+2) \oplus \mathcal{O}_C) \oplus \mathcal{O}_C^2 \longrightarrow \operatorname{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2}).$$

We consider the vector bundle $\operatorname{Syz}(U^{k+1}, V^{k+1}, (U+V)^k)(\frac{3k+1}{2})$ on the projective line $\mathbb{P}^1 = \operatorname{Proj} K[U,V]$. Since the degree of this bundle is $-1$, it has to have a non-trivial global section. Substituting $U = X^d$ and $V = Y^d$ yields a non-trivial syzygy

$$FX^{dk+d} + GY^{dk+d} + H(X^d+Y^d)^k = (FX)X^p + (GY)Y^p + H(X^d+Y^d)^k = 0$$

of total degree $\frac{3dk+d}{2}$. That is, we have a non-trivial global section of $\mathcal{S}_k(\frac{3dk+d}{2})$ on the curve $C$. We have $\Gamma(C, \mathcal{S}_k(\frac{3dk+d}{2} - 1)) = 0$ because otherwise the twisted semistable Frobenius pull-back $\operatorname{Syz}(X^p, Y^p, Z^p)(\frac{3dk+d}{2} + d - 2)$ of degree $-d$ would have a non-trivial global section too (see Remark 2.2) which is impossible by semistability. Since $\deg(\mathcal{S}_k(\frac{3dk+d}{2})) = (-d+2)d$, we obtain the splitting (rewrite $\frac{3dk+d}{2} = \frac{3p+1}{2} - (d-1) = \frac{3p+1}{2} - t$)

$$\mathcal{S}_k(\frac{3p+1}{2} - t) \cong \mathcal{O}_C(-d+2) \oplus \mathcal{O}_C.$$

The other summand $\mathcal{S}_{k+1}(\frac{3dk+d}{2} + d - 1)$ has degree 0. It follows once again from Lemma 2.1 and the semistability of $\operatorname{Syz}(X^p, Y^p, Z^p)$ that

$$\Gamma(C, \mathcal{S}_{k+1}(\frac{3dk+d}{2} + d - 2)) = 0,$$

i.e., $\mathcal{S}_{k+1}(\frac{3dk+d}{2} + d - 1)$ splits as (rewrite $\frac{3dk+d}{2} + d - 1 = \frac{3p+1}{2}$)

$$\mathcal{S}_{k+1}(\frac{3p+1}{2}) \cong \mathcal{O}_C^2.$$

Step 2. Let $(FX, GY, H)$ be the non-trivial global section of $\mathcal{S}_k(\frac{3p+1}{2} - t)$ constructed above (corresponding to the component $\mathcal{O}_C$). We show that $H(P) \neq 0$ for every point $P = (x, y, z) \in C$ satisfying $z^d = x^d + y^d = 0$.

The last component $H$ of the section $(FX, GY, H)$ is a homogeneous polynomial in $X^d$ and $Y^d$ (it stems by construction from a syzygy on $\mathbb{P}^1$ in $U$ and $V$). Let $P = (x, y, z) \in C$ be a point on the curve such that $z^d = x^d + y^d = 0$. Then $x^d = -y^d$ which implies $x = \zeta y$ where $\zeta$ is a $d$th root of $-1$. In particular, $P = (\zeta y, y, 0)$. Since $K$ is algebraically closed, $\mathrm{char}(K) \neq 2$ and $p \equiv -1 \mod 2d$, the group $\mu_{2d}(K)$ of $(2d)$th roots of unity in $K$ has order $2d$. Hence, we have

$$X^d + Y^d = \prod_{\zeta} (X - \zeta Y),$$

where $\zeta \in \mu_{2d}(K)$ runs through the elements with the property $\zeta^d = -1$ (there are exactly $d$ such roots). Now assume $H(P) = 0$. Then $H(P') = 0$ for all points $P'$ of the form $P' = (\zeta y, y, 0)$. So $X^d + Y^d$ has to divide $H$, i.e., $H = \tilde{H}(X^d + Y^d)$ with a homogeneous polynomial $\tilde{H} \in K[X, Y]$. So we have a relation

$$(FX)X^p + (GY)Y^p + \tilde{H}(X^d + Y^d)^{k+1} = 0$$

of total degree $\frac{3p+1}{2} - t$. That is, we have a non-trivial section of the bundle $\mathcal{S}_{k+1}(\frac{3p+1}{2} - t)$. This section maps by Lemma 2.1 and Remark 2.2 to a non-trivial global section of $\mathrm{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2} - t)$. But

$$\deg(\mathrm{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2} - t)) = (3p + 1 - 2t - 3p)d = (1 - 2t)d < 0.$$

Hence, the section contradicts the semistability of $\mathrm{Syz}(X^p, Y^p, Z^p)$.

<u>Step 3.</u> We show that in the surjective sheaf homomorphism

$$\varphi_{\frac{3p+1}{2}} : (\mathcal{O}_C(-d + 2) \oplus \mathcal{O}_C) \oplus \mathcal{O}_C^2 \longrightarrow \mathrm{Syz}(X^p, Y^p, Z^p)(\frac{3p + 1}{2})$$

the summand $\mathcal{O}_C(-d + 2)$ is not necessary, i.e.,

$$\varphi_{\frac{3p+1}{2}} : \mathcal{O}_C^3 = \mathcal{O}_C \oplus \mathcal{O}_C^2 \longrightarrow \mathrm{Syz}(X^p, Y^p, Z^p)(\frac{3p + 1}{2})$$

is also surjective.

Set $m := \frac{3p+1}{2}$. By the Nakayama lemma, we can check surjectivity pointwise over the residue field $K$ at every point $P = (x, y, z) \in C$. For this we have to find two linearly independent vectors in the image. First we treat the case $z \neq 0$. We show that we even have a surjective map

$$\mathcal{S}_{k+1}(m) = \mathcal{O}_C^2 \longrightarrow \mathrm{Syz}(X^p, Y^p, Z^p)(m).$$

We take basic sections

$$f = (f_1, f_2, f_3), g = (g_1, g_2, g_3) \in \Gamma(C, \mathcal{S}_{k+1}(m)) \cong \Gamma(C, \mathcal{O}_C^2).$$

Their images are $\tilde{f} = (f_1, f_2, Zf_3)$ and $\tilde{g} = (g_1, g_2, Zg_3)$. Assume there is a relation $\tilde{f}(P) + \lambda\tilde{g}(P) = 0$ with $\lambda \in K^{\times}$. Looking at each component this gives the equations

$$\begin{aligned} f_1(P) + \lambda g_1(P) &= 0, \\ f_2(P) + \lambda g_2(P) &= 0, \\ (f_3(P) + \lambda g_3(P))z &= 0. \end{aligned}$$

But since $z \neq 0$, the latter equation would mean $f_3(P) + \lambda g_3(P) = 0$ and therefore we would obtain a relation $f(P) + \lambda g(P) = 0$ which contradicts the assumption.

Now we deal with the case $z = 0$, i.e., $P = (x, y, 0)$. Let

$$f = (FX, GY, H) \in \Gamma(C, \mathcal{S}_k(m-t)) \cong \Gamma(C, \mathcal{O}_C(-d+2) \oplus \mathcal{O}_C)$$

be the section (corresponding to $\mathcal{O}_C$ which we have found in step 1. The image of $f$ in the bundle $\operatorname{Syz}(X^p, Y^p, Z^p)(m)$ is the section $(Z^tFX, Z^tGY, H)$. Evaluated at $P$ we obtain the vector $v = (0, 0, H(P))$. Since $0 = z^d = x^d + y^d$ we have $H(P) \neq 0$ by step 2. Now we take a section $0 \neq g = (g_1, g_2, g_3) \in \Gamma(C, \mathcal{S}_{k+1}(m)) \cong \Gamma(C, \mathcal{O}_C^2)$. The image of $g$ equals $(g_1, g_2, Zg_3)$. Evaluation at $P$ gives the vector $w = (g_1(P), g_2(P), 0)$, where either $g_1(P)$ or $g_2(P)$ is not 0 (otherwise $g_3(P)$ would be 0 as well). Hence we have found a vector $w$ such that $v, w$ are linearly independent over $K$.

Step 4. So far we have shown that we have a surjective morphism

$$\mathcal{O}_C^3 \longrightarrow \operatorname{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2}) \longrightarrow 0.$$

Since $\det(\operatorname{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2})) \cong \mathcal{O}_C(1)$ we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_C(-1) \longrightarrow \mathcal{O}_C^3 \longrightarrow \operatorname{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2}) \longrightarrow 0.$$

Dualizing and tensoring with $\mathcal{O}_C(-1)$ gives

$$0 \longrightarrow (\operatorname{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2}))^\vee(-1) \longrightarrow \mathcal{O}_C^3(-1) \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

where the map $\mathcal{O}_C^3(-1) \to \mathcal{O}_C$ is given by some linear forms $L_1, L_2, L_3$ in the homogeneous coordinate ring $R = K[X, Y, Z]/(X^d + Y^d - Z^d)$. In particular, we have $(\operatorname{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2}))^\vee(-1) \cong \operatorname{Syz}(L_1, L_2, L_3)$. We show that $\{L_1, L_2, L_3\}$ and $\{X, Y, Z\}$ generate the same ideal in $R$. Assume to the contrary that $L_1, L_2, L_3$ are linearly dependent. Such an equation yields a non-trivial section of $\operatorname{Syz}(L_1, L_2, L_3)(1)$. This bundle has degree $\deg(\operatorname{Syz}(L_1, L_2, L_3)(1)) = (2-3)d = -d < 0$. But since $\operatorname{Syz}(X^p, Y^p, Z^p)$ is semistable, so is $(\operatorname{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2}))^\vee$ and thus also $\operatorname{Syz}(L_1, L_2, L_3)$. So the section contradicts the semistability.

Step 5. We have already proved that

$$\mathcal{E} \cong \operatorname{Syz}(X, Y, Z) \cong (\operatorname{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2}))^\vee(-1).$$

Since $\mathrm{Syz}(X^p, Y^p, Z^p)$ is a bundle of rank 2, we have

$$\mathrm{Syz}(X^p, Y^p, Z^p) \cong \mathrm{Syz}(X^p, Y^p, Z^p)^\vee \otimes \mathcal{O}_C(-3p).$$

So finally we obtain

$$\begin{aligned}
\mathcal{E} &\cong \mathrm{Syz}(X, Y, Z) \\
&\cong (\mathrm{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2}))^\vee(-1) \\
&\cong \mathrm{Syz}(X^p, Y^p, Z^p)^\vee \otimes \mathcal{O}_C(-\frac{3p+1}{2}) \otimes \mathcal{O}_C(-1) \\
&\cong \mathrm{Syz}(X^p, Y^p, Z^p) \otimes \mathcal{O}_C(3p) \otimes \mathcal{O}_C(-\frac{3p+1}{2}) \otimes \mathcal{O}_C(-1) \\
&\cong \mathrm{Syz}(X^p, Y^p, Z^p)(\frac{3(p-1)}{2}),
\end{aligned}$$

and consequently $F^*(\mathcal{E}) \cong \mathrm{Syz}(X^p, Y^p, Z^p) \cong \mathcal{E}(-\frac{3(p-1)}{2})$ which finishes the proof. □

**Remark 3.5.** We also comment on the case $p \equiv 1 \mod 2d$. Then we write $p = dk + 1$ with $k$ even and set $t = 1$. The syzygy bundle $\mathrm{Syz}(U^k, V^k, (U+V)^{k+1})(\frac{3k}{2})$ on $\mathbb{P}^1 = \mathrm{Proj}\, K[U, V]$ has degree $-1$ and therefore has to have a non-trivial global section. Substituting $U = X^d$ and $Y^d$ and multiplying with $XY$ gives then a syzygy

$$(FY)X^p + (GX)Y^p + (HXY)(X^d + Y^d)^{k+1} = 0,$$

i.e., a global section of $\mathcal{S}_{k+1}(\frac{3dk}{2} + 2)$ on the Fermat curve. As in the proof of Theorem 3.4 we obtain the splittings (rewrite $\frac{3dk}{2} + 2 = \frac{3p+1}{2}$)

$$\mathcal{S}_{k+1}(\frac{3p+1}{2}) \cong \mathcal{O}_C(-d+2) \oplus \mathcal{O}_C \text{ and } \mathcal{S}_k(\frac{3p+1}{2} - t) \cong \mathcal{O}_C^2.$$

Unfortunately, we do not know how to prove an analog of step 2 in these characteristics, i.e., to show that $H(P) \neq 0$ for every point $P = (x, y, z) \in C$ with $z^d = x^d + y^d = 0$. Here the reasoning of the proof (step 2) of Theorem 3.4 would lead to a section of $\mathrm{Syz}(X^p, Y^p, (X^d + Y^d)^{k+2})$ which is not helpful to get a contradiction.

**Remark 3.6.** We cannot expect that Theorem 3.4 holds in every characteristic $p$ where $\Omega_{\mathbb{P}^2}|_C$ is strongly semistable. For example, consider in characteristic 2 the Fermat cubic $C = \mathrm{Proj}(K[X, Y, Z]/(X^3 + Y^3 - Z^3))$, which is an elliptic curve. It is a well-known fact that semistable vector bundles on elliptic curves are strongly semistable (see for instance [27, appendix]). Hence $\Omega_{\mathbb{P}^2}|_C \cong \mathrm{Syz}(X, Y, Z)$ is strongly semistable by [5, Proposition 6.2]. The pull-back $F^*(\Omega_{\mathbb{P}^2}|_C) \cong \mathrm{Syz}(X^2, Y^2, Z^2)$ has for the first time non-trivial global sections in total degree 3, namely the (only) syzygy $(X, Y, -Z)$ which comes from the equation of the curve. This section gives rise to the short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathrm{Syz}(X^2, Y^2, Z^2)(3) \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

i.e., $\mathrm{Syz}(X^2, Y^2, Z^2)(3)$ is the bundle $F_2$ in Atiyah's classification [1]. Since the Hasse invariant of $C$ is 0, we have $F^*(F_2) \cong \mathcal{O}_C^2$ and therefore $F^*(\Omega_{\mathbb{P}^2}|_C) \ncong \Omega_{\mathbb{P}^2}|_C(-\frac{3(p-1)}{2})$. We have $F^{2^*}(\Omega_{\mathbb{P}_2}|_C) \cong \mathcal{O}_C(-6) \oplus \mathcal{O}_C(-6)$ and we obtain (up to a twist) the periodicity $F^{3^*}(\Omega_{\mathbb{P}^2}|_C) \cong (F^{2^*}(\Omega_{\mathbb{P}^2}|_C))(-6)$.

## 4. A computation of the Hilbert-Kunz function

We recall that the *Hilbert-Kunz function* of a standard graded ring $R$ of characteristic $p > 0$ with graded maximal ideal $\mathfrak{m}$ is the function

$$e \longmapsto \varphi_R(e) := \mathrm{length}(R/\mathfrak{m}^{[p^e]}),$$

where $\mathfrak{m}^{[p^e]}$ denotes the extended ideal under the $e$-th iteration of the Frobenius endomorphism on $R$; see for instance [23] for this rather complicated function and its properties. As a consequence of Theorem 3.4 we obtain the complete Hilbert-Kunz function of the Fermat ring $R = K[X, Y, Z]/(X^d + Y^d - Z^d)$ in characteristics $p \equiv -1 \mod 2d$. The following result is implicitly contained in [12, Lemma 5.6] of P. Monsky and C. Han.

**Corollary 4.1.** *Let $d \geq 2$ be a positive integer and let $K$ be a field of characteristic $p \equiv -1 \mod 2d$. Then the Hilbert-Kunz function of the Fermat ring $R = K[X, Y, Z]/(X^d + Y^d - Z^d)$ is*

$$\varphi_R(e) = \frac{3d}{4} p^{2e} + 1 - \frac{3d}{4}.$$

*Proof.* Since the length of $R/\mathfrak{m}^{[p^e]}$, $\mathfrak{m} = (X, Y, Z)$, does not change if one enlarges the base field, we may assume that $K$ is algebraically closed. Hence, $\varphi_R(e) = \sum_{m=0}^{\infty} \dim_K(R/\mathfrak{m}^{[p^e]})_m$ (this sum is in fact finite since the algebras $R/\mathfrak{m}^{[p^e]}$ have finite length). It follows from the presenting sequence of $\mathrm{Syz}(X, Y, C)$ on the Fermat curve $C = \mathrm{Proj}\, R$ that (setting $q := p^e$)

$$(*) \qquad \dim_K(R/\mathfrak{m}^{[q]})_m = h^0(C, \mathcal{O}_C(m)) - 3h^0(C, \mathcal{O}_C(m - q)) + h^0(C, \mathrm{Syz}(X^q, Y^q, Z^q)(m)).$$

By Theorem 3.4 we have $\mathrm{Syz}(X^p, Y^p, Z^p) \cong \mathrm{Syz}(X, Y, Z)(-\frac{3(p-1)}{2})$ and consequently $\mathrm{Syz}(X^q, Y^q, Z^q) \cong \mathrm{Syz}(X, Y, Z)(-\frac{3(q-1)}{2})$ for all $q = p^e$, $e \geq 1$. The global evaluation of the presenting sequence of $\mathcal{E}(k) := \mathrm{Syz}(X, Y, Z)(k)$ gives the exact sequence

$$0 \longrightarrow \Gamma(C, \mathcal{E}(k)) \longrightarrow \Gamma(C, \mathcal{O}_C(k-1)^3) \longrightarrow \Gamma(C, \mathcal{O}_C(k)) \longrightarrow K \longrightarrow 0$$

for $k = 0$ and the short exact sequence

$$0 \longrightarrow \Gamma(C, \mathcal{E}(k)) \longrightarrow \Gamma(C, \mathcal{O}_C(k-1)^3) \longrightarrow \Gamma(C, \mathcal{O}_C(k)) \longrightarrow 0$$

for $k \geq 1$. Hence we obtain

$$h^0(C, \mathcal{E}(k)) = \begin{cases} 3h^0(C, \mathcal{O}_C(k-1)) - h^0(C, \mathcal{O}_C(k)) + 1 \text{ if } k = 0, \\ 3h^0(C, \mathcal{O}_C(k-1)) - h^0(C, \mathcal{O}_C(k)) \text{ if } k \neq 0. \end{cases}$$

For $k \geq d-2$ we have by Riemann-Roch $h^0(C, \mathcal{O}_C(k)) = dk - g + 1$, where $g$ is the genus of the curve. Since $p \equiv -1 \mod 2d$, this holds in particular for $k \geq \frac{p+1}{2}$. So the geometric formula for the Hilbert-Kunz function $(*)$ gives (in order to obtain an easier calculation we sum up to $2q$):

$$\begin{aligned} \varphi_R(e) &= \sum_{m=0}^{2q} h^0(C, \mathcal{O}_C(m)) - 3 \sum_{m=0}^{2q} h^0(C, \mathcal{O}_C(m-q)) \\ &\quad + 3 \sum_{m=0}^{2q} h^0(C, \mathcal{O}_C(m - \frac{3(q-1)}{2} - 1)) \\ &\quad - \sum_{m=0}^{2q} h^0(C, \mathcal{O}_C(m - \frac{3(q-1)}{2})) + 1 \\ &= \sum_{m=0}^{2q} h^0(C, \mathcal{O}_C(m)) - 3 \sum_{m=0}^{q} h^0(C, \mathcal{O}_C(m)) \\ &\quad + 3 \sum_{m=0}^{\frac{q+1}{2}} h^0(C, \mathcal{O}_C(m)) - \sum_{m=0}^{\frac{q+3}{2}} h^0(C, \mathcal{O}_C(m)) + 1 \\ &= \sum_{m=\frac{q+5}{2}}^{2q} h^0(C, \mathcal{O}_C(m)) - 3 \sum_{m=\frac{q+3}{2}}^{q} h^0(C, \mathcal{O}_C(m)) + 1 \\ &= \sum_{m=\frac{q+5}{2}}^{2q} (dm - g + 1) - 3 \sum_{m=\frac{q+3}{2}}^{q} (dm - g + 1) + 1 \\ &= d\left(q(2q+1) - \frac{(q+3)(q+5)}{8}\right) - \frac{3g(q-1)}{2} + \frac{3(q-1)}{2} \\ &\quad - 3d\left(\frac{q(q+1)}{2} - \frac{(q+1)(q+3)}{8}\right) + \frac{3g(q-1)}{2} - \frac{3(q-1)}{2} + 1 \\ &= \frac{3d}{4} q^2 + 1 - \frac{3d}{4}. \end{aligned}$$

Thus we have obtained the desired formula for the Hilbert-Kunz function of the ring $R$. □

**Remark 4.2.** Corollary 4.1 matches for $d = 3$ with the result [10, Theorem 4] of Buchweitz and Chen, which says that the Hilbert-Kunz function of the homogeneous coordinate ring of a plane elliptic curve defined over a field $K$ of odd characteristic $p$ equals $\frac{9}{4}p^{2e} - \frac{5}{4}$.

## 5. Examples of a (0,1)-Frobenius periodicity on Fermat curves

In this section, we show how to get via Theorem 3.4 non-trivial examples of $(0,1)$-Frobenius periodicities, i.e., we give explicit examples of vector bundles $\mathcal{E}$ on certain Fermat curves such that $\mathcal{E} \cong F^*(\mathcal{E})$.

**Example 5.1.** Let $d \geq 2$ and let $K$ be a field of characteristic $p \equiv -1 \mod 2d$. The ring homomorphism

$$K[X,Y,Z]/(X^d + Y^d - Z^d) \longrightarrow K[U,V,W]/(U^{2d} + V^{2d} - W^{2d})$$

which sends $X \mapsto U^2$, $Y \mapsto V^2$ and $Z \mapsto W^2$ induces a finite cover $f : C^{2d} \to C^d$, where $C^i$ denotes the Fermat curve of degree $i$. Since $f^*(\mathcal{O}_{C^d}(1)) \cong \mathcal{O}_{C^{2d}}(2)$, we see that $\deg(f) = 4$. The group $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ acts on $C^{2d}$ by sending $(u,v,w)$ either to $(u,v,w), (-u,v,w), (u,-v,w)$ or $(u,v,-w)$, and $C^d$ is the quotient of this action. Moreover, $f$ is a finite separable morphism and therefore $f$ preserves semistability. Theorem 3.4 gives, via pull-back under $f$, the isomorphic vector bundles

$$\begin{aligned}
\mathrm{Syz}_{C^{2d}}(U^{2p}, V^{2p}, W^{2p}) &\cong f^*(\mathrm{Syz}_{C^d}(X^p, Y^p, Z^p)) \\
&\cong f^*(\mathrm{Syz}_{C^d}(X,Y,Z)(-\frac{3(p-1)}{2})) \\
&\cong f^*(\mathrm{Syz}_{C^d}(X,Y,Z)) \otimes f^*(\mathcal{O}_{C^d}(-\frac{3(p-1)}{2})) \\
&\cong \mathrm{Syz}_{C^{2d}}(U^2, V^2, W^2)(-3(p-1))
\end{aligned}$$

on the Fermat curve $C^{2d}$. In particular, we have the periodicity

$$\mathrm{Syz}_{C^{2d}}(U^2, V^2, W^2)(3) \cong F^*(\mathrm{Syz}_{C^{2d}}(U^2, V^2, W^2)(3))$$

(note that this bundle has degree 0 and is not trivial since there are no non-trivial global sections below the degree of the curve).

**Remark 5.2.** By the classical result [19, Satz 1.4] of H. Lange and U. Stuhler the periodicity $\mathrm{Syz}_{C^{2d}}(U^2, V^2, W^2)(3) \cong F^*(\mathrm{Syz}_{C^{2d}}(U^2, V^2, W^2)(3))$ in Example 5.1 implies the existence of an étale cover

$$g : D \longrightarrow C^{2d}$$

such that

$$g^*(\mathrm{Syz}_{C^{2d}}(U^2, V^2, W^2)(3)) \cong \mathcal{O}_D^2.$$

Moreover, by [19, Proposition 1.2] the bundle $\mathrm{Syz}_{C^{2d}}(U^2, V^2, W^2)(3)$ comes from a (continuous) representation

$$\rho : \pi_1(C^{2d}) \longrightarrow GL_2(K)$$

of the algebraic fundamental group $\pi_1(C^{2d})$. It would be interesting to see how the étale trivialization $g$ and the representation $\rho$ look explicitly in this example.

**Remark 5.3.** In this remark we show that $\mathcal{E} := \mathrm{Syz}(U^2, V^2, W^2)(3)$ is not étale trivializable in characteristic 0. We consider this bundle on the smooth projective relative curve

$$\mathcal{C}^{2d} := \mathrm{Proj}(\mathbb{Z}_{2d}[U, V, W]/(U^{2d} + V^{2d} - Z^{2d})) \longrightarrow \mathrm{Spec}\,\mathbb{Z}_{2d}.$$

For a prime number $p \nmid 2d$ the special fiber $\mathcal{C}_p^{2d}$ over $(p)$ is the (smooth) Fermat curve over the finite field $\mathbb{F}_p$. The generic fiber $\mathcal{C}_0^{2d}$ over $(0)$ is the Fermat curve over $\mathbb{Q}$. To prove that $\mathcal{E}_0 := \mathcal{E}|_{\mathcal{C}_0^{2d}}$ is not étale trivializable on $\mathcal{C}_0^{2d}$ we use once again Hilbert-Kunz theory (cf. also the proof of Lemma 3.2). Note that $\mathcal{E}_0$ is semistable by [5, Proposition 2]. If $d \geq 4$ is even (the case $d = 2$ is trivial) we consider prime numbers $p \equiv d \pm 1 \mod 2d$ and if $d \geq 3$ is odd we look at prime numbers $p \equiv d \mod 2d$. In these characteristics, [24, Theorem 2.3] yields that the Hilbert-Kunz multiplicity $e_{HK}(R_p)$ of the homogeneous coordinate ring $R_p$ of the Fermat curve $C^d \to \mathrm{Spec}\,\mathbb{F}_p$ of degree $d$ equals

$$e_{HK}(R_p) = \frac{3d}{4} + \frac{(d(d-3))^2}{4dp^2} \text{ if } d \text{ is even and } e_{HK}(R_p) = \frac{3d}{4} + \frac{d^3}{4p^2} \text{ if } d \text{ is odd.}$$

Hence, $\Omega_{\mathbb{P}^2}|_{C^d}$ is not strongly semistable by [7, Corollary 4.6]. Since we can realize the fibers $\mathcal{C}_p^{2d}$, as in Example 5.1, as coverings $f : \mathcal{C}_p^{2d} \to C^d$, the bundles $\mathcal{E}_p := \mathcal{E}|_{\mathcal{C}_p^{2d}} \cong f^*(\Omega_{\mathbb{P}^2}|_{C^d})(3)$ are not strongly semistable either. Note that by the well-known theorem of Dirichlet (cf. [25, Chapitre VI, §4, Théorème and Corollaire]) there are infinitely many such fibers. Therefore, there is no étale cover $g : D \to \mathcal{C}_0^{2d}$ such that $g^*(\mathcal{E}_0) \cong \mathcal{O}_D^2$.

This observation is somehow related to the Grothendieck-Katz $p$-curvature conjecture [16, (I quat)] which states the following: Let $R$ be a $\mathbb{Z}$-domain of finite type, $\mathbb{Z} \subseteq R$, and $\mathcal{X} \to \mathrm{Spec}\,R$ a smooth projective morphism of relative dimension $d \geq 1$. If $\mathcal{E}$ is a vector bundle on $\mathcal{X} \to \mathrm{Spec}\,R$ equipped with an integrable connection $\nabla$ such that $\nabla|_{\mathcal{E}_\mathfrak{p}}$ has $p$-curvature 0 on the special fiber $\mathcal{X}_\mathfrak{p}$ for almost all closed points $\mathfrak{p} \in \mathrm{Spec}\,R$, then there exists an étale cover $g : Y \to \mathcal{X}_0$ of the generic fiber $\mathcal{X}_0$ such that $(g^*(\mathcal{E}_0), g^*(\nabla_0))$ is trivial. For a detailed account on integrable connections and $p$-curvature see [15] and [16]. In our example of the relative Fermat curve $\mathcal{C}^{2d}$, we have for infinitely many prime numbers $p \equiv -1 \mod 2d$ the Frobenius descent $F^*(\mathcal{E}_p) \cong \mathcal{E}_p$ on $\mathcal{C}_p^{2d}$. By the so-called Cartier-correspondence [15, Theorem 5.1] this is equivalent to the existence of an integrable connection $\nabla_p$ on $\mathcal{E}_p$ with vanishing $p$-curvature. If one could establish a connection on $\mathcal{E}$ (since $\mathcal{E}$ is a vector bundle over a curve, this connection would be automatically integrable) which is compatible with the connections on the special fibers $\mathcal{C}_p^{2d}$, $p \equiv -1 \mod 2d$, then our example would show that the Grothendieck-Katz conjecture does not hold if one only requires vanishing $p$-curvature for infinitely many closed points.

**Remark 5.4.** In this remark we assume that the base field is algebraically closed. We consider the *Verschiebung*

$$V : \mathcal{M}_{C^{2d}}(2, \mathcal{O}_{C^{2d}}) \dashrightarrow \mathcal{M}_{C^{2d}}(2, \mathcal{O}_{C^{2d}}),\ [\mathcal{E}] \longmapsto [F^*(\mathcal{E})]$$

induced by the Frobenius morphism on $C^{2d}$. We recall that the Verschiebung is a rational map from the moduli space $\mathcal{M}_{C^{2d}}(2, \mathcal{O}_{C^{2d}})$ parametrizing (up to $S$-equivalence) semistable vector bundles on $C^{2d}$ of rank 2 and trivial determinant to itself. The vector bundle $\mathcal{S} := \operatorname{Syz}_{\mathbb{P}^2}(U^2, V^2, W^2)$ is stable on the projective plane $\mathbb{P}^2 = \operatorname{Proj} K[U, V, W]$ by [8, Corollary 6.4]. Since the discriminant of this bundle equals $\Delta(\mathcal{S}) = 4c_2(\mathcal{S}) - c_1(\mathcal{S})^2 = 12$, the restriction of $\mathcal{S}$ to every smooth projective curve of degree $\geq 7$ remains stable by Langer's restriction theorem [20, Theorem 2.19]. In particular, $\mathcal{S}|_{C^{2d}} \cong \operatorname{Syz}_{C^{2d}}(U^2, V^2, W^2)(3)$ is stable on the Fermat curve $C^{2d}$ for $d \geq 4$. Hence, for $d \geq 4$ the bundle $\operatorname{Syz}_{C^{2d}}(U^2, V^2, W^2)(3)$ defines a closed point of $\mathcal{M}_{C^{2d}}(2, \mathcal{O}_C)$ which is fixed under the Verschiebung $V$.

**Remark 5.5.** We may pull-back the vector bundle $\mathcal{E} = \operatorname{Syz}(U^2, V^2, W^2)(3)$ along the cone mapping

$$p : T = \operatorname{Spec} K[U, V, W]/(U^{2d} + V^{2d} - W^{2d}) \setminus \{\mathfrak{m}\} \to C^{2d}$$

to obtain the bundle $\mathcal{G} = p^*(\mathcal{E})$ on the punctured spectrum with the property $F^*(\mathcal{G}) \cong \mathcal{G}$. This can however not be extended to get a Frobenius periodicity on the module level, since $F^*\Gamma(T, \mathcal{G}) \neq \Gamma(T, F^*\mathcal{G})$. A Frobenius periodicity for a coherent $R$-module $M$, where $R$ is a local noetherian domain, implies that $M$ is free. This observation follows by looking at Fitting ideals of a free resolution (we thank Manuel Blickle and Neil Epstein for this remark).

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Universität Osnabrück, Fachbereich 6: Mathematik/Informatik, Albrechtstr. 28a, 49069 Osnabrück, Germany

*E-mail address*: hbrenner@uni-osnabrueck.de and akaid@uni-osnabrueck.de