# Birkhoff strata of the Grassmannian $\mathrm{Gr}^{(2)}$: Algebraic curves

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**Abstract**

Algebraic varieties and curves arising in Birkhoff strata of the Sato Grassmannian $\mathrm{Gr}^{(2)}$ are studied. It is shown that the big cell $\Sigma_0$ contains the tower of families of the normal rational curves of all odd orders. Strata $\Sigma_{2n}$, $n = 1, 2, 3, \dots$ contain hyperelliptic curves of genus $n$ and their coordinate rings. Strata $\Sigma_{2n+1}$, $n = 0, 1, 2, 3, \dots$ contain $(2m + 1, 2m + 3)-$plane curves for $n = 2m, 2m - 1$ ($m \geq 2$) and $(3, 4)$ and $(3, 5)$ curves in $\Sigma_3$, $\Sigma_5$ respectively. Curves in the strata $\Sigma_{2n+1}$ have zero genus.

## 1 Introduction

Grassmannian $\mathrm{Gr}^{(2)}$ is a very important specialization of the universal Sato Grassmannian [1]. The most known its appearance is due to the connection with the theory of the KdV equation [2, 3]. The present paper is devoted to the study of the Grassmannian $\mathrm{Gr}^{(2)}$ within the framework proposed recently in [4]. The main idea of this approach is to analyze algebro-geometric structures arising in Sato Grassmannian, in our case in the Birkhoff strata of $\mathrm{Gr}^{(2)}$, without any *a priori* reference to any integrable system.

Recall that Sato Grassmannian Gr can be viewed as the set of closed vector subspaces in the infinite dimensional set of all formal Laurent series with coefficients in $\mathbb{C}$ with certain special properties (see e.g. [2, 3]). Each subspace W$\subset$Gr possesses an algebraic basis $(w_0(z), w_1(z), w_2(z), \dots)$ with the basis elements

$$w_n = \sum_{k=-\infty}^{n} a_k^n z^k \tag{1}$$

of finite order $n$. Grassmannian Gr is a connected Banach manifold which exhibits a stratified structure [2, 3], i.e. Gr$=\bigcup_S \Sigma_S$ where the stratum $\Sigma_S$ is a subspace in Gr formed by elements of the form (1) such that possible values $n$ are given by the infinite set $S = \{s_0, s_1, s_2, \dots\}$ of integers $s_n$ with $s_0 < s_1 < s_2 < \dots$ and $s_n = n$ for large $n$. Big cell $\Sigma_0$ corresponds to $S = \{0, 1, 2, \dots\}$. Other strata are associated with the sets $S$ different from $S_0$.

$\mathrm{Gr}^{(2)}$ is the subset of elements $W$ of Gr obeying the condition $z^2 \cdot W \subset W$ [2, 3]. This condition imposes strong constraints on the Laurent series and on the structure of the strata. Namely, Birkhoff stratum $\Sigma_S$ in $\mathrm{Gr}^{(2)}$ corresponds to the sets $S$ such that $S + 2 \subset S$, i.e. all possible $S$ having the form [2, 3]

$$S_m = \{-m, -m + 2, -m + 4, \dots, m, m + 1, m + 2, \dots\} \tag{2}$$

with $m = 0, 1, 2, \dots$. Codimension of $\Sigma_m$ is $m(m + 1)/2$. One has $\mathrm{Gr}^{(2)} = \bigcup_{m\geq 0} \Sigma_m$.

In this paper, using the properties of the Birkhoff strata $\mathrm{Gr}^{(2)}$, we show that the big cell $\Sigma_0$ contains a maximal closed subspace $W_0$ which geometrically is a tower of infinite families of rational normal (Veronese) curves of all odd orders. It is demonstrated that the strata $\Sigma_{2n}$, $n = 1, 2, \dots$ contain subspaces $W_{2n}$ closed with respect to multiplication if the coefficients of Laurent series $w_n$ obey certain associativity constraints. Geometrically the subspaces $W_{2n}$ represent infinite families of coordinate rings for the hyperelliptic curves of genus $n$. Then it is shown that the strata $\Sigma_3$ and $\Sigma_5$ contain $(3, 4)$ and $(3, 5)$

degenerate plane curves respectively. In the strata $\Sigma_{2m+1},\ m \geq 2$ one has families of $(2m+1, 2m+3)$ plane curves of zero genus.

In the second part of this work [5] the tangent cohomology of the subspaces $W_n$ and associated integrable systems of hydrodynamical type will be studied.

The paper is organized as follows. The big cell is discussed in section 2. Stratum $\Sigma_1$ is considered in section 3. Closed subspaces $W_2$ in the stratum $\Sigma_2$ and corresponding elliptic curves are studied in section 4. Stratum $\Sigma_3$ and associated $(3,4)$ curves are analysed in section 5. Section 6 is devoted to general strata $\Sigma_{2n}$, $(n = 2, 3, 4, \dots)$. Stratum $\Sigma_5$ and the generic strata $\Sigma_{2n+1}$, $(n = 3, 4, \dots)$ are discussed in section 7.

# 2 Big cell

The principal stratum $\Sigma_0$ for which $S = \{0, 1, 2, \dots\}$ (called also big cell) is a dense open set and it has codimension zero[2, 3]. It possesses a canonical basis $(p_0, p_1, p_2, \dots)$ where

$$p_i(z) = z^i + \sum_{k\geq 1} \frac{H^i_k}{z^k}, \qquad i = 0, 1, 2, \dots \tag{3}$$

with arbitrary $H^i_k$.

Accordingly to the approach proposed in [4] we first look for a subspace $W_0 \subset \Sigma_0$ closed with respect to multiplication. Similar to the big cell in the general Gr one has

**Lemma 2.1** *Laurent series (3) obey the condition $z^2W_0 \subset W_0$ and the equations*

$$p_j(z)p_k(z) = \sum_{l\geq 0} C^l_{jk} p_l(z) \tag{4}$$

*if and only if*

$$\begin{aligned} H^{2n}_i &= 0, \qquad i = 1, 2, 3, \dots, n = 0, 1, 2, \dots, \\ H^{2n+i}_{2i} &= 0, \qquad i = 1, 2, 3, \dots, n = 0, 1, 2, \dots \end{aligned} \tag{5}$$

*and*

$$\begin{aligned} &H^{2m+1}_{2(k+n)+1} - H^{2(m+n)+1}_{2k+1} - \sum_{s=0}^{n-1} H^{2m+1}_{2s+1} H^{2(n-s)-1}_{2k+1} = 0, \\ &H^{2m+1}_{2(k+n)+1} + H^{2n+1}_{2(k+m)+1} + \sum_{l=0}^{k-1} H^{2m+1}_{2l+1} H^{2n+1}_{2(k-l)-1} = 0. \end{aligned} \tag{6}$$

*The constants $C^l_{jk}$ are given by*

$$\begin{aligned} C^{2l}_{2n,2m} &= \delta^l_{m+n}, \\ C^{2l+1}_{2n,2m+1} &= \delta^l_{m+n} + H^{2m+1}_{2(n-l)-1}, \\ C^{2l}_{2n+1,2m+1} &= \delta^l_{m+n+1} + H^{2n+1}_{2(m-l)+1} + H^{2m+1}_{2(n-l)+1} \end{aligned} \tag{7}$$

*and $p_{2n} = {p_2}^2 = z^{2n}$, $n \geq 0$.*

An immediate consequence of this lemma is given by the following

**Proposition 2.2** *The subspace $W_0 \subset \Sigma_0$ with the elements of the form $w = \sum_{i=0}^{n} a_i p_i(z)$ with arbitrary $a_i$ and $n \geq 0$ and parameters $H^i_k$ obeying the constraints (6) and the condition $z^2W_0 \subset W_0$ is closed with respect to multiplication $W_0 \cdot W_0 \subset W_0$. It is a maximal closed subspace in the big cell. This subspace $W_0$ is an infinite dimensional associative commutative algebra with unity $p_0 = 1$.*

The last statement follows from the equivalence of equations (6) to the associativity conditions

$$\sum_s C^s_{ij} C^r_{ks} - C^s_{ik} C^r_{js} = 0 \tag{8}$$

for the structure constants $C^l_{jk}$.

Relations (4) written explicitly, i.e.

$$\begin{aligned} p_{2n} p_{2m} &= p_{2(m+n)}, \\ p_{2n} p_{2m+1} &= p_{2(m+n)+1} + \sum_{s=0}^{n-1} H^{2m+1}_{2s+1} p_{2(n-s)-1}, \\ p_{2n+1} p_{2m+1} &= p_{2(m+n+1)} + \sum_{s=0}^{m} H^{2n+1}_{2s+1} p_{2(m-s)} + \sum_{s=0}^{n} H^{2m+1}_{2s+1} p_{2(n-s)}, \end{aligned} \tag{9}$$

imply that

$$\begin{aligned} z^2 =& p_1^2 - 2H^1_1, \\ p_3 =& {p_1}^3 - 3H^1_1 p_1, \\ p_5 =& {p_1}^5 - 5H^1_1 {p_1}^3 + \frac{15}{2} {H^1_1}^2 p_1, \\ & \dots \end{aligned} \tag{10}$$

or equivalently

$$\lambda = {p_1}^2 - 2H^1_1, \qquad p_{2n+1} = \alpha_n(\lambda) p_1 \tag{11}$$

where $\lambda = z^2$ and $\alpha_n(\lambda) = \prod_{s=1}^n \left( \lambda - \frac{H^1_{2(n-s)+1}}{2(n-s)+1} \right)$.

Similar to [4] one can treat $\lambda, p_1, p_3, \dots$ as the affine coordinates. So one has the following geometrical interpretation of the subspace $W_0$.

**Proposition 2.3** *Big cell $\Sigma_0$ contains an infinite–dimensional algebraic variety $\Gamma_0$ with the ideal*

$$\langle \lambda - p_1^2 + 2H^1_1, l^{(2)}_1, l^{(2)}_1, \dots \rangle \tag{12}$$

*where $l^{(2)}_n = p_{2n+1} - \alpha(\lambda) p_1$ and the variables $H^j_k$ obey the constraints (6). This variety $\Gamma_0$ is an infinite tower of infinite families of rational normal (Veronese) curves of all odd orders.*

Formulas (10) represent a canonical parameterization of rational normal curves (see e.g. [6]). For instance, the curves defines by the first two equations (10) is the classical twisted cubic in the three-dimensional space with the coordinates $(\lambda, p_1, p_3)$.

There is an infinite set of independent variables among all $H^j_k$ constrained by conditions (6). A natural set of independent $H^j_k$ is given by $H^1_1, H^1_3, H^1_5, \dots$.

It is also easy to see using (11)that the ideal $I^{(2)}_0$ contains singular "hyperelliptic" curves of genus zero given by the equations

$$p^2_{2n+1} = (\lambda + 2H^1_1) \alpha_n(\lambda)^2. \tag{13}$$

Infinite family of algebraic varieties described in Proposition 2.3 is in its turn the algebraic variety in the affine space with coordinates $p_i$, $(i = 1, 2, 3, \dots)$ and $H^j_k$, $(j, k = 1, 2, 3, \dots)$ defined by the quadrics

$$f_{jk} = p_j p_k - p_{j+k} - \sum_{s=0}^{k} H^j_s p_s - \sum_{s=0}^{j} H^k_s p_s = 0 \tag{14}$$

and equations (6).

## 3 Stratum $\Sigma_1$

The stratum $\Sigma_1$ is the lowest stratum different from the big cell and it corresponds to $m = 1$ and $S = \{-1, 1, 2, \dots\}$. Due to the absence of zero order element $w_0$ the canonical basis is of the form

$$p_i(z) = z^i + H_0^i + \sum_{k\geq 1} \frac{H_k^i}{z^k}, \qquad i = 1, 2, 3, \dots \ . \tag{15}$$

**Lemma 3.1** *A set $W_1$ of Laurent series (15) obey the condition $z^2 \cdot W_1 \subset W_1$ and the equations*

$$p_j(z)p_k(z) = \sum_{l\geq 1} C_{jk}^l p_l(z), \qquad j, k = 1, 2, 3, \dots \tag{16}$$

*if and only if the parameters $H_k^j$ satisfy the constraints*

$$\begin{aligned}
&H_0^{2j+k} + H_0^{2j} H_0^{2k} = 0,\\
&H_0^{2(k+j)+1} + H_0^{2k+1} H_0^{2j+1} + \sum_{l=0}^{j-1} H_{2l+1}^{2k+1} H_0^{2(j-l)-1} = 0,\\
&H_{2(j+l)+1}^{2k+1} + H_0^{2j} H_{2l+1}^{2k+1} - H_{2l+1}^{2(k+j)+1} - H_0^{2j} H_{2l+1}^{2k+1} - H_{2(l+j)+1}^{2k+1} - \sum_{s=0}^{j-1} H_{2s+1}^{2k+1} H_{2l+1}^{2(j-s)-1} = 0,\\
&H_0^{2(k+j+1)} + H_0^{2k+1} H_0^{2j+1} + \sum_{s=0}^{k-1} H_0^{2(k-s)} H_{2s+1}^{2j+1} + \sum_{s=0}^{j-1} H_0^{2(j-s)} H_{2s+1}^{2k+1} = 0,\\
&H_{2(l+k)+1}^{2j+1} + H_{2(l+j)+1}^{2k+1} + \sum_{s=0}^{l-1} H_{2s+1}^{2j+1} H_{2(l-s)-1}^{2k+1} - \sum_{s=0}^{k-1} H_{2(l+k)+1}^{2j+1} - \sum_{s=0}^{j-1} H_{2(l+j)+1}^{2k+1} = 0
\end{aligned} \tag{17}$$

*and*

$$\begin{aligned}
C_{2j,2k}^{2l} =& \delta_{j+k}^l + H_0^{2j} \delta_k^l + H_0^{2k} \delta_j^l,\\
C_{2j,2k+1}^{2l+1} =& \delta_{j+k}^l + H_0^{2j} \delta_k^l + H_0^{2k} \delta_j^l + H_{2(j-l)+1}^{2k+1},\\
C_{2j+1,2k+1}^{2l+1} =& H_0^{2k+1} \delta_j^l + H_0^{2j+1} \delta_k^l,\\
C_{2j+1,2k+1}^{2l} =& \delta_{j+k+1}^l + H_{2(l+k)+1}^{2j+1} + H_{2(l+j)+1}^{2k+1} \ .
\end{aligned} \tag{18}$$

The analysis of the constraints (17) gives

$$H_i^{2n} = 0, \qquad n, i = 1, 2, 3, \dots \tag{19}$$

and

$$H_0^{2n} = -(-H_0^2)^n, \qquad n = 1, 2, 3, \dots, \tag{20}$$

i.e.

$$p_{2n}(z) = z^{2n} - (-H_0^2)^n, \qquad n = 1, 2, 3, \dots. \tag{21}$$

For the elements $p_{2n+1}$ one instead has

$$\begin{aligned}
p_2 =& {p_1}^2 - 2H_0^1 p_1,\\
p_3 =& {p_1}^3 - 3H_0^1 {p_1}^2 - (3H_1^1 - 3{H_0^1}^2) p_1,\\
&\dots \ .
\end{aligned} \tag{22}$$

Similar to the big cell one has a subspace $W_1$ in $\Sigma_1$ closed with respect to multiplication which algebraically is an infinite-dimensional commutative associative algebra $A_1$ with the structure constants given by (18) in the basis (15). Geometrically $W_1$ is an infinite tower of families of rational normal curves of all odd orders passing through the origin $p_1 = p_2 = p_3 = \dots = 0$.

The fact that for the stratum $\Sigma_1$ one has results which are similar to those for big cell is not that surprising. Indeed, taking into account the relations (17), namely

$$2H_1^1 - H_0^2 + {H_0^1}^2 = 0, \qquad H_0^3 + H_0^1 H_0^2 + H_0^1 H_1^1 = 0 \tag{23}$$

and the formula (21), i.e. $p_2 = z^2 + H_0^2$, one can rewrite equations (22) as

$$\begin{aligned} z^2 =&(p_1 - H_0^1)^2 - 2H_1^1, \\ p_3 - H_0^3 =&(p_1 - H_0^1)^3 - 3H_1^1(p_1 - H_0^1). \end{aligned} \tag{24}$$

In the variables

$$\tilde{p}_1 = p_1 - H_0^1, \qquad \tilde{p}_3 = p_3 - H_0^3 \tag{25}$$

the equations (24) the first two equations (11) for the big cell. It is a direct check that in the variables

$$\tilde{p}_k = p_k - H_0^k, \qquad k = 1, 2, 3, \dots \tag{26}$$

all equations (22) coincide with equations (11) for the big cell.

Thus the result for the stratum $\Sigma_1$ and big cell are connected by a simple change of variables (26). Similar situation take place for other strata $\Sigma_m$ with odd $m$.

# 4 Stratum $\Sigma_2$ and elliptic curves

For the stratum $\Sigma_2$ with $S = \{-2, 0, 2, 3, 4, \dots\}$ the positive order elements of the canonical basis are given by

$$\begin{aligned} p_0 =&1 + \sum_{k\geq 1} \frac{H_k^0}{z^k}, \\ p_j =&z^j + H_{-1}^j z + \sum_{k\geq 1} \frac{H_k^j}{z^k}, \qquad k, j = 2, 3, 4, \dots . \end{aligned} \tag{27}$$

Analogue of the Lemmas 2.1 and 3.1 is given by

**Lemma 4.1** *A set $W_2$ of Laurent series (27) obey the equations*

$$p_j(z)p_k(z) = \sum_{l=0,2,3,\dots} C_{jk}^l p_l(z) \tag{28}$$

*and the condition $z^2 W_2 \subset W_2$ is satisfied if and only if*

$$\begin{aligned} H_k^{2n} =&0, \qquad k = -1, 1, 2, 3, \dots, \quad n = 0, 1, 2, \dots, \\ H_k^{2n+1} =&0, \qquad n, k = 1, 2, 3, \dots \end{aligned} \tag{29}$$

*and*

$$\begin{aligned} &H_{2(k+n)+1}^{2m+1} - H_{2k+1}^{2(m+n)+1} - \sum_{s=-1}^{n-2} H_{2s+1}^{2m+1} H_{2k+1}^{2(n-s)+1} = 0, \\ &H_{2(n+k)+1}^{2m+1} + H_{2(m+k)+1}^{2n+1} + \sum_{s=-1}^{k} H_{2s+1}^{2m+1} H_{2(l-s)-1}^{2n+1} = 0. \end{aligned} \tag{30}$$

*Constants $C_{jk}^l$ are given by*

$$\begin{aligned} C_{2n,2m}^{2l} =&\delta_{m+n}^l, \\ C_{2n,2m+1}^{2l+1} =&\delta_{m+n}^l + H_{2(n-l)-1}^{2n+1}, \\ C_{2n+1,2m+1}^{2l} =&\delta_{m+n+1}^l + H_{2(m-l)+1}^{2n+1} + H_{2(n-l)+1}^{2m+1} + H_{-1}^{2n+1} H_{-1}^{2m+1} \delta_2^l \\ &+ \left(H_{-1}^{2n+1} H_1^{2m+1} + H_1^{2n+1} H_{-1}^{2m+1}\right) \delta_0^l. \end{aligned} \tag{31}$$

*which imply*

$$\begin{aligned}
p_{2n}p_{2m} =& p_{2(m+n)},\\
p_{2n}p_{2m+1} =& p_{2(m+n)+1} + \sum_{k=-1}^{n-2} H^{2n+1}_{2k+1} p_{2(n-k)-1},\\
p_{2n+1}p_{2m+1} =& p_{2(m+n+1)} + \sum_{k=-1}^{m} H^{2n+1}_{2k+1} p_{2(m-k)} + \sum_{k=-1}^{n} H^{2m+1}_{2k+1} p_{2(n-k)}\\
&+ H^{2n+1}_{-1} H^{2m+1}_{-1} p_2 + \left( H^{2n+1}_{-1} H^{2m+1}_{1} + H^{2n+1}_{1} H^{2m+1}_{-1} \right)
\end{aligned} \tag{32}$$

*and* $p_{2n} = z^{2n}$, $n \geq 0$.

As a consequence, one has

**Proposition 4.2** *The stratum* $\Sigma_2$ *contains a maximal closed subspace* $W_2$ *with elements of the form*

$$w = \sum_{i=0,2,3,\dots} a_i p_i(z) \tag{33}$$

*with arbitrary* $n$ *and* $a_i$ *and parameters* $H^j_k$ *obeying the constraints (29), (30) and such that* $z^2 W_2 \subset W_2$.

The relations (28) readily imply that all $p_i(z)$ are generated by two elements $z^2$ and $p_3$.

Using (32) one can show that the set of independent relations (28) is given by

$${p_3}^2 = \lambda^3 + 2H^3_{-1}\lambda^2 + \left( 2{H^3_{-1}}^2 + 2H^3_1 \right)\lambda + 2H^3_{-1}H^3_1 + 2H^3_3 \tag{34}$$

and

$$p_{2n+1} = \left( \lambda^{n-1} - \sum_{i=0}^{n-2} H^{2(n-i)-1}_{-1} \lambda^i \right) p_3 \tag{35}$$

This relation is obtained obtained using iteratively the formula

$$p_{2n+1} = \lambda p_{2n-1} + H^{2n-1}_{-1} p_3. \tag{36}$$

**Proposition 4.3** *Subspace* $W_2$ *is an infinite-dimensional commutative associative algebra with the basis* $1,\ p_2,\ p_3,\ p_4, \dots$ *isomorphic to* $\mathbb{C}[\lambda, p_3]/C_6$

where

$$C_6 = {p_3}^2 - \lambda^3 - 2H^3_{-1}\lambda^2 - \left( 2{H^3_{-1}}^2 + 2H^3_1 \right)\lambda - \left( 2H^3_{-1}H^3_1 + 2H^3_3 \right). \tag{37}$$

**Proof** Associativity follows from the fact that the conditions (29) and (30) are equivalent to the condition

$$\sum_{s=0,2,3,\dots} C^s_{jk} C^r_{sl} = \sum_{s=0,2,3,\dots} C^s_{lk} C^r_{sj} \qquad j,k,l,r = 0,2,3,\dots \tag{38}$$

for the constants $C^l_{jk}$ given by (32). $\square$

Treating now $\lambda, p_3, p_5$ and $H^j_k$ as affine coordinates one has the following geometrical interpretation of the subspace $W_2$.

**Proposition 4.4** *The subspace* $W_2$ *is an infinite dimensional algebraic variety* $\Gamma_2$ *in the affine space with coordinates* $p_j$, $(j = 2,3,4,\dots)$, $H^j_k$, $(j = 3,5,7,\dots,\ k = -1,1,3,5,\dots)$ *defined by the intersection of quadrics*

$$f_{jk} = p_j p_k - p_{j+k} - \sum_{l=0,2,3,\dots}^{j+k-1} C^l_{jk} p_l(z) = 0 \tag{39}$$

*and quadrics (30). An ideal* $I^{(2)}$ *of this variety is*

$$I^{(2)} = \langle C_6, l^{(2)}_5, l^{(2)}_7, l^{(2)}_9, \dots \rangle \tag{40}$$

*where* $l^{(2)}_{2n+1} = p_{2n+1} - \left( \lambda^{n-1} - \sum_{i=0}^{n-2} H^{2(n-i)-1}_{-1} \lambda^i \right) p_3$.

Since $W_2 \sim \mathbb{C}[\lambda, p_3]/C_6$ one can view $\Gamma_2$ as the infinite family of coordinate rings of the elliptic curve $C_6 = 0$ parameterized by the variables $H^j_k$ obeying the conditions (29) and (30). Analyzing these conditions one concludes that there is an infinite set of independent variables among all $H^j_k$, for example $H^3_{-1}, H^3_1, H^3_3, H^3_5, \dots$.

It is a direct check that the curve $C_6 = 0$ has genus one. So the stratum $\Sigma_2$ contains an infinite family of elliptic curves parameterized by $H^3_{-1}, H^3_1, H^3_3$.

The ideal $I^{(2)}$ contains singular hyperelliptic curves of all orders and of genus 1 given by

$$p^2_{2n+1} = \left(\lambda^{n-1} - \sum_{i=0}^{n-2} H_{-1}^{2(n-i)-1}\lambda^i\right)^2 \left(\lambda^2 + 2H^3_{-1}\lambda^2 + \left(2{H^3_{-1}}^2 + 2H^3_1\right)\lambda + 2H^3_{-1}H^3_1 + 2H^3_3\right) \quad (41)$$

# 5 Stratum $\Sigma_3$: (3,4) curves of zero genus

Next case corresponds to $S = \{-3, -1, 1, 3, 4, 5, \dots\}$. Due to the absence of elements of orders zero and two positive elements of the canonical basis are given by

$$\begin{aligned} p_1 &= z + H^j_0 + \sum_{k\geq 1} \frac{H^1_k}{z^k}, \\ p_j &= z^j + H^j_{-2}z^2 + H^j_0 + \sum_{k\geq 1} \frac{H^j_k}{z^k}, \qquad j = 3, 4, 5, \dots . \end{aligned} \quad (42)$$

Since ${p_1}^2$ has the order two a closed subspace can be generated only by the elements $p_3, p_4, p_5, \dots$.

**Lemma 5.1** *A set $W_3$ of Laurent series $p_j(z)$, $j = 3, 4, 5, \dots$ obey the equations*

$$p_j(z)p_k(z) = \sum_{l=3,4,5,\dots} C^l_{jk} p_l(z), \qquad j, k = 3, 4, 5, \dots \quad (43)$$

*and the condition $z^2 W_3 \subset W_3$ if and only if*

$$p_j = z^j + H^j_{-2}z^2 + H^j_0, \qquad j \geq 5, \quad (44)$$

$$\begin{aligned} &H^j_{-2} + H^{j-2}_{-2}H^4_{-2} - H^{j-2}_0 = 0, \\ &H^j_0 + H^{j-2}_{-2}H^4_0 = 0 \end{aligned} \quad (45)$$

*and*

$$\begin{aligned} &H^4_0 + 2\, H^3_0 H^3_{-2} - {H^3_{-2}}^2 H^4_{-2} - {H^4_{-2}}^2 = 0, \\ &{H^3_0}^2 - {H^3_{-2}}^2 H^4_0 - H^4_{-2}H^4_0 = 0. \end{aligned} \quad (46)$$

**Proof** Let us begin with the condition $z^{2n}W_3 \subset W_3$. One has

$$z^{2n}p_m(z) = z^{2n+m} + \dots + H^m_{2n-1}z + \dots . \quad (47)$$

In $W_3$ there is no element which contains the term of order one. Hence, with necessity $H^m_{2n-1} = 0$ for all $n = 1, 2, 3, \dots$ and $n = 3, 4, 5, \dots$, i.e.

$$p_j(z) = z^j + H^j_{-2}z^2 + H^j_0 + \sum_{n\geq 1} \frac{H^j_{2n}}{z^{2n}}, \qquad j = 3, 4, 5, \dots . \quad (48)$$

Then considering the product $p_{2k+1}p_j$ one has

$$p_{2k+1}(z)p_j(z) = z^{2k+j+1} + \dots + H^j_{2k}z + \dots . \quad (49)$$

The terms of the order $z^i$, $i \geq 3$ can be represented as a superposition of $p_3, p_4, \dots, p_{2k+j+1}$ giving the constants $C^l_{jk}$ while the coefficient in front of $z$ should vanish. Hence $H^j_{2k} = 0$ for all $k = 1, 2, 3, \dots$. So

$$p_j = z^j + H^j_{-2} z^2 + H^j_0 \qquad j \geq 3 \ . \tag{50}$$

The coefficients $H^j_{-2}$ and $H^j_0$ are not all independent. Indeed, the relations

$$\begin{aligned} z^2 p_3 =& p_5 + H^3_{-2} p_4, \\ z^2 p_4 =& p_6 + H^4_{-2} p_4, \\ z^4 p_3 =& p_7 + H^3_{-2} p_6 + H^3_0 p_4, \\ \dots \end{aligned} \tag{51}$$

imply

$$\begin{aligned} H^5_{-2} - H^3_0 + H^3_{-2} H^4_{-2} =& 0, \\ H^5_0 + H^3_{-2} H^4_0 =& 0, \\ H^6_{-2} - H^4_0 + {H^4_{-2}}^2 =& 0, \\ H^6_0 + H^4_{-2} H^4_0 =& 0, \\ H^7_{-2} + H^3_{-2} H^6_{-2} + H^3_0 H^4_{-2} =& 0, \\ H^7_0 + H^3_{-2} H^6_0 + H^3_0 H^4_0 =& 0, \\ & \dots \end{aligned} \tag{52}$$

and so on. The relations (52) are the lowest members of the relations (45). Using these relations, one can express all $H^j_{-2}$, $H^j_0$ with $j = 5, 6, 7, \dots$ in terms of $H^3_{-2}$, $H^3_0$ and $H^4_{-2}$, $H^4_0$.

Furthermore, the vanishing of the coefficients in front of $z^2$ and $z^0$ in the relation

$$p_3{}^2 - \left( p_6 + 2H^3_{-2} p_5 + {H^3_{-2}}^2 p_4 + 2H^3_0 p_3 \right) = 0 \tag{53}$$

is equivalent to the conditions

$$\begin{aligned} & H^6_{-2} - 2\, H^3_{-2} H^5_{-2} - {H^3_{-2}}^2 H^4_{-2} = 0, \\ & H^6_0 - {H^3_0}^2 - 2\, H^3_{-2} H^5_0 - {H^3_{-2}}^2 H^4_0 = 0. \end{aligned} \tag{54}$$

Finally taking into account (52), one gets the constraints (46). So there are only two independent parameters among all coefficients $H^j_{-2}$ and $H^j_0$. The simplest choice is to take $H^3_{-2}$ and $H^4_{-2}$ as independent variables. At last, the direct calculation gives

$$C^l_{jk} = \delta^l_{j+k} + H^k_{-2} \delta^l_{j+2} + H^k_0 \delta^l_j + H^j_{-2} \delta^l_{k+2} + H^j_0 \delta^l_k + H^j_{-2} H^k_{-2} \delta^l_4. \qquad \square \tag{55}$$

An immediate consequence of the Lemma 5.1 is given by

**Proposition 5.2** *The stratum $\Sigma_3$ contains the subspace $W_3$ closed with respect to multiplication $W_3 \cdot W_3 \subset W_3$. Elements of $W_3$ have the form $w = \sum_{j=3}^n a_j p_j(z)$ with arbitrary $n$ and $a_i$ and $p_i$ of the form (44) with $H^j_{-2}, H^j_0$ obeying the constraints (45) and (46). The subspace $W_3$ is an infinite-dimensional associative and commutative algebra $A_3$ with the basis (44) and structure constants (55).*

A geometrical interpretation of $W_3$ is provided by

**Proposition 5.3** *The subspace $W_3$ can be viewed as the two parametric family of algebraic varieties defined by the relations*

$$p_j p_k - \sum_l C^l_{jk} p_l = p_j p_k - \left( p_{j+k} + H^k_{-2} p_{j+2} + H^k_0 p_j + H^j_{-2} p_{k+2} + H^j_0 p_k + H^j_{-2} H^k_{-2} p_4 \right) = 0. \tag{56}$$

*The ideal of this family contains the plane* $(3,4)$ *curve (in the terminology of [7]) defined by the equation*

$$
\begin{aligned}
&{p_4}^3 - {p_3}^4 + 4\,H^3_{-2}p_3{p_4}^2 - \left(3\,H^4_{-2} - 2\,{H^3_{-2}}^2\right){p_3}^2p_4 - \left(-4\,H^3_0 + 2\,H^3_{-2}H^4_{-2}\right){p_3}^3\\
&-\left(3\,H^4_0 + 4\,H^3_0H^3_{-2} + {H^3_{-2}}^4 + {H^3_{-2}}^2H^4_{-2}\right){p_4}^2 - \left(4\,{H^3_{-2}}^2H^3_0 + 8\,H^3_{-2}H^4_0 - 2\,H^3_{-2}{H^4_{-2}}^2\right.\\
&\left.-6\,H^3_0H^4_{-2} - 2\,{H^3_{-2}}^3H^4_{-2}\right)p_3p_4 - \left(-3\,H^4_0H^4_{-2} + 6\,{H^3_0}^2 - 6\,H^3_0H^3_{-2}H^4_{-2} + 2\,{H^3_{-2}}^2H^4_0\right.\\
&\left.+{H^3_{-2}}^2{H^4_{-2}}^2 + {H^4_{-2}}^3\right){p_3}^2 - \left(-3\,{H^4_0}^2 - 2\,{H^3_0}^2{H^3_{-2}}^2 - 2\,{H^3_{-2}}^2H^4_0H^4_{-2} - 2\,{H^3_{-2}}^4H^4_0\right.\\
&\left.+3\,{H^3_0}^2H^4_{-2} + 2\,H^3_0H^3_{-2}{H^4_{-2}}^2 - 8\,H^3_0H^3_{-2}H^4_0 + 2\,H^3_0{H^3_{-2}}^3H^4_{-2}\right)p_4 - \left(-4\,{H^3_0}^3 - 4\,H^3_{-2}{H^4_0}^2\right.\\
&+6\,H^3_0H^4_0H^4_{-2} - 4\,H^3_0H^4_0{H^3_{-2}}^2 + 6\,{H^3_0}^2H^3_{-2}H^4_{-2} + 2\,H^4_0H^3_{-2}{H^4_{-2}}^2 + 2\,H^4_0{H^3_{-2}}^3H^4_{-2}\\
&\left.-2\,H^3_0{H^3_{-2}}^2{H^4_{-2}}^2 - 2\,H^3_0{H^4_{-2}}^3\right)p_3 = 0,
\end{aligned}
\tag{57}
$$

*where* $H^3_{-2}$,$H^3_0$,$H^4_{-2}$ *and* $H^4_0$ *obey the constraints (46). The (3,4) curve (57) have zero genus.*

**Proof** By direct calculation with the use of polynomial form (44) of $p_j$. □

Comparing the results of this and previous section, one observes an essential difference between the geometrical properties of the subspaces $W_2$ and $W_3$. This is due to the quite different form of the Laurent series belonging to $W_i$ which is the consequence of a different situation with elements of the first order in $z$. Namely, though in both cases $W_i$ does not contain the element $p_1(z)$, The absence in $W_3$ of the terms of order $z$ in $p_j(z)$ leads to a strong constraints leading to the polinomiality of $p_j(z)$.

We note also that due to the presence of the element $p_0 = 1$ of zero order in $W_2$ one has $z^2 \in W_2$ while $z^2 \notin W_3$. As a consequence, for instance, one can choose $p_3$, $p_4$ and $z^2p_3$ as the generators of the algebra $A_3$ instead of $p_3$, $p_4$ and $p_5$.

A way to avoid the polinomiality of all $p_j(z) \in W_3$ would be to relax the condition $z^2W_3 \subset W_3$. Since $z^2$ is not an element of $W_3$ it would be natural not to require that the product of $z^2$ and an element of $W_3$ belongs to $W_3$, but instead to require that $z^2W_3 \subset \Sigma_3$. The presence of the element $p_1(z)$ in $\Sigma_3$, allows us to avoid immediate constraints on $p_j(z)$ followed from the relations of the type (47) and (49). for instance, instead of the conditions (51) one gets the following ones

$$
\begin{aligned}
z^2p_3 - p_5 - H^3_{-2}p_4 &= H^3_1p_1,\\
z^2p_4 - p_6 - H^4_{-2}p_4 &= H^4_1p_1,
\end{aligned}
\tag{58}
$$

and so on. In virtue of the equations of this type one obtains an infinite set of relations for $H^j_k$. Computer analysis strongly indicates that these conditions again lead to the constraint $H^j_k = 0$, $k = 1,2,3,\ldots$, $j = 3,4,5,\ldots$, i.e. to the polinomiality of all $p_j(z)$.

# 6 Strata $\Sigma_{2n}$. Hyperelliptic curves of genus n

Stratum $\Sigma_{2n}$ with arbitrary $n$ is characterized by $S = \{-2n, -2n+2, -2n+4, \ldots, 0, 2, 4, \ldots, 2n, 2n+1, 2n+2, \ldots\}$. So it does not contain, in particular, $n$ elements of the order $1,3,5,\ldots,2n-1$ and the positive order elements of the canonical basis are given by

$$
\begin{aligned}
p_0 =& 1 + \sum_{k\geq 1}\frac{H^0_k}{z^k},\\
p_j =& z^j + \sum_{k=0}^{j-1} H^j_{-2k-1}z^{2k+1} + \sum_{k\geq 1}\frac{H^j_k}{z^k}, \qquad j = 2,4,6,\ldots,2n-2,\\
p_j =& z^j + \sum_{k=0}^{n-1} H^j_{-2k-1}z^{2k+1} + \sum_{k\geq 1}\frac{H^j_k}{z^k} \qquad j = 2n, 2n+1, 2n+2, 2n+3, \ldots .
\end{aligned}
\tag{59}
$$

In this case one has

**Lemma 6.1** *A set $W_{2n}$ of Laurent series (59) obey the condition $z^2 W_{2n} \subset W_{2n}$ and equations*

$$p_j(z)p_k(z) = \sum_l C^l_{jk} p_l(z) \tag{60}$$

if and only if

$$\begin{aligned} H^{2m}_k =&0, \qquad m = 0,1,2,\ldots,\ k = -2n+2, -2n+4, \ldots, -2, 0, 1, 2, 3, \ldots, \\ H^{2m+1}_{2k} =&0, \qquad m = 0,1,2,\ldots\ k = -n, -n+1, -n+2, \ldots \end{aligned} \tag{61}$$

and

$$\begin{aligned} &H^{2j+1}_{2(l+k)+1} - H^{2(j+k)+1}_{2l+1} - \sum_{s=-n}^{k-1} H^{2j+1}_{2s+1} H^{2(k-s)-1}_{2l+1} = 0, \\ &H^{2j+1}_{2(l+k)+1} + H^{2k+1}_{2(l+j)+1} + \sum_{s=-n}^{-1} H^{2j+1}_{2s+1} H^{2k+1}_{2(l-s)-1} + \sum_{r=-n}^{-1} H^{2k+1}_{2r+1} H^{2j+1}_{2(l-r)-1} + \sum_{s=0}^{l-n} H^{2j+1}_{2s+1} H^{2k+1}_{2(l-s)-1} = 0. \end{aligned} \tag{62}$$

Rewriting equation (60) separately for $p_j$ with even and odd $j$, i.e.

$$\begin{aligned} p_{2j}p_{2k} =& p_{2(j+k)}, \\ p_{2j}p_{2k+1} =& p_{2(j+k)+1} + \sum_{s=-n}^{k-1} H^{2j+1}_{2s+1} p_{2(k-s)-1}, \\ p_{2j+1}p_{2k+1} =& p_{2(j+k+1)} + \sum_{s=-n}^{j} H^{2j+1}_{2s+1} p_{2(j-s)} + \sum_{s=-n}^{k} H^{2k+1}_{2s+1} p_{2(k-s)} \\ &+ \sum_{s=-n}^{-1} \sum_{r=-n}^{-1} H^{2j+1}_{2s+1} H^{2k+1}_{2r+1} p_{-2(s+r+1)} + \sum_{s=-n}^{-1} \sum_{r=0}^{-s-1} H^{2j+1}_{2s+1} H^{2k+1}_{2r+1} p_{-2(s+r+1)} \\ &+ \sum_{r=-n}^{-1} \sum_{s=0}^{-r-1} H^{2j+1}_{2s+1} H^{2k+1}_{2r+1} p_{-2(s+r+1)}, \end{aligned} \tag{63}$$

one concludes that

$$\begin{aligned} p_{2m} =& (z^2)^m, \\ p_{2m+1} =& \alpha(\lambda) p_{2n+1}, \qquad m = n+1, n+2, \ldots,\ \lambda = z^2 \end{aligned} \tag{64}$$

for suitable $\alpha(\lambda) \in \mathbb{C}[\lambda]$. Moreover

$$p^2_{2n+1} = \lambda^{2n+1} + \sum_{k=0}^{n} u_k \lambda^k = 0 \tag{65}$$

where the coefficients $u_k$ can be obtained from

$$p^2_{2n+1} = \lambda^{2n+1} + 2\sum_{s=0}^{2n} H^{2n+1}_{2(n-s)+1} \lambda^s + \sum_{k=-n}^{n+1} \sum_{s=0}^{n-k-1} H^{2n+1}_{2k+1} H^{2n+1}_{-2(s+k)-1} \lambda^s. \tag{66}$$

Thus, one has

**Proposition 6.2** *The stratum $\Sigma_{2n}$ for $n = 2,3,4,\ldots$ contains maximal subspace $W_{2n}$ closed with respect to multiplication with elements of the form*

$$w = \sum_{k\geq 0} a_{2k} \lambda^k + \sum_{k \geq n} b_{2k+1}(\lambda) p_{2k+1} \tag{67}$$

*with parameters $H^j_k$ obeying the constraints (61) and (62).*

**Proposition 6.3** *The subspace $W_{2n}$ is the infinite-dimensional commutative associative algebra $A_{2n}$ isomorphic to $\mathbb{C}[\lambda, p_{2n+1}]/C_{2n+1}$ where $\lambda = z^2$ and*

$$C_{2n+1} = p_{2n+1}^2 - \lambda^{2n+1} - \sum_{k=0}^{n} u_k \lambda^k = 0 \tag{68}$$

*and $u_k$ are given by (66)*

**Proof** Associativity of the algebra $A_{2n}$ is the consequence of the equivalence of the constraints (61) and (62) and the associativity conditions

$$\sum_s C_{jk}^s C_{ls}^r = \sum_s C_{lk}^s C_{js}^r \tag{69}$$

for the constants $C_{jk}^l$ defined in (60) i.e.

$$\begin{aligned} C_{2j,2k}^{2l} =& \delta_{j+k}^l, \\ C_{2j+1,2k}^{2l+1} =& \delta_{j+k}^l + H_{2(k-l)-1}^{2j+1}, \\ C_{2j,2k}^{2l} =& \delta_{j+k}^l + H_{2(j-l)+1}^{2j+1} + H_{2(k-l)+1}^{2k+1} + \sum_{s=-n-1}^{-1} \sum_{r=-n-1}^{-1} H_{2s+1}^{2j+1} H_{2r+1}^{2k+1} \delta_{-2(s+r+1)}^l \\ &+ \sum_{s=-n-1}^{-1} \sum_{r=0}^{-s-1} H_{2s+1}^{2j+1} H_{2r+1}^{2k+1} \delta_{-2(s+r+1)}^l + \sum_{r=-n-1}^{-1} \sum_{s=0}^{-r-1} H_{2s+1}^{2j+1} H_{2r+1}^{2k+1} \delta_{-(s+r+1)}^l . \end{aligned} \tag{70}$$

□

Interpreting $\lambda, p_{2n+1}, p_{2n+3}, \ldots$ as the local affine coordinates we first observe that the equation

$$C_{2n+1} = 0 \tag{71}$$

defines a hyperelliptic curve of genus $n$. It is parameterized by $2n+1$ arbitrary quantities $H_{-2n+1}^{2n+1}$, $H_{-2n+1}^{2n+3}$, $\ldots$ $H_{+1}^{2n+1}$, $\ldots$, $H_{2n+1}^{2n+1}$. Their variation generates an infinite family of hyperellliptic curves. One has

**Proposition 6.4** *The subspace $W_{2n}$ in $\Sigma_{2n}$ is an infinite-dimensional algebraic variety $\Gamma_{2n}$ defined by the relations (60),(61), (62), (68). Its ideal is*

$$I_{2n+1} = \langle C_{2n+1}, l_{2n+3}^{(n)}, l_{2n+5}^{(n)}, \ldots \rangle \tag{72}$$

*where $l_{2m+1}^{(n)} = p_{2m+1} - \alpha_m(\lambda) p_{2n+1}$, $m = n+1, n+2, n+3, \ldots$.*

In other words the variety $\Gamma_{2n}$ is the intersection of the cubic $C_{2n+1} = 0$ and infinite set of algebraic curves $l_{2m+1}^{(n)}$, $m = n+1, n+2, n+3, \ldots$. One can easily see that the ideal $I_{2n}$ contains higher order hyperelliptic curves but all of them have genus $n$.

Thus stratum $\Sigma_{2n}$ is characterized by the presence of the plane hyperelliptic curves $C_{2n+1}$ of genus $n$ in the closed subspace $W_{2n}$. This is due to the presence of $n$ gaps (elements $p_1(z)$, $p_3(z), \ldots, p_{2n-1}(z)$) in the basis of $W_{2n}$. The fact that for hyperelliptic curves (Riemann surfaces) of genus $n$ one has $n$ (Weierstrass) gaps in a generic point is well known in the theory of abelian functions (see e.g. [8]). Probably not that known observation is that these gaps and consequently the properties of corresponding algebraic curves are prescribed by the structure of the Birkhoff strata $\Sigma_{2n}$ in $\mathrm{Gr}^{(2)}$. In different context an appearance of hyperelliptic curves in Birkhoff strata of $\mathrm{Gr}^{(2)}$ has been observed in [9].

# 7 Strata $\Sigma_{2n+1}$

Stratum $\Sigma_{2n+1}$, $n = 2, 3, 4, \dots$ is characterized by $S = \{-2n-1, -2n+1, \dots, -1, 1, 3, \dots, 2n+1, 2n+2, \dots\}$. So, the positive order elements of the canonical basis in $\Sigma_{2n+1}$ are of the form

$$\begin{aligned} p_j(z) =& z^j + H^j_{-j+1} z^{j-2} + H^j_{j+2} z^{j-2} + \dots + H^j_0 + \sum_{k\geq 1} \frac{H^j_k}{z^k}, \qquad j = 1, 3, \dots, 2n-1 \\ p_j(z) =& z^j + H^j_{-2n} z^{2n} + H^j_{-2n+1} z^{2n-1} + \dots + H^j_0 + \sum_{k\geq 1} \frac{H^j_k}{z^k}, \qquad j = 2n+1, 2n+2, \dots \ . \end{aligned} \tag{73}$$

Closed subspaces in $\Sigma_{2n+1}$ have different structure for different $n$. In order to see this let us begin with $\Sigma_5$. In this case the elements (73) of the canonical basis are

$$\begin{aligned} p_1 =& z + H^1_0 + \sum_{k\geq 1} \frac{H^1_k}{z^k}, \\ p_3 =& z^3 + H^3_{-2} z^2 + H^1_0 + \sum_{k\geq 1} \frac{H^1_k}{z^k}, \\ p_j =& z^j + H^j_{-4} z^4 + H^j_{-2} z^2 + H^j_0 + \sum_{k\geq 1} \frac{H^j_k}{z^k} \qquad j = 5, 6, 7, \dots \ . \end{aligned} \tag{74}$$

It is easy to see that the maximal closed subspace $W_5$ in $\Sigma_5$ is the subspace with the basis $(p_3, p_5, p_6, \dots)$.

**Lemma 7.1** *A set $W_5$ of the Laurent series $p_3, p_5, p_6, \dots$ obey the equations*

$$p_j(z) p_k(z) = \sum_{l=3,5,6,\dots} C_{jk} p_l(z) \tag{75}$$

*and the condition $z^2 W_5 \subset W_5$ if and only if $H^j_k = 0$, $j = 3, 5, 6, \dots$, $k = 1, 2, 3, \dots$, i.e. all $p_j$ are polynomials and $H^j_k$, $k = -4, -2, 0$ obey the constraints*

$$\begin{aligned} & H^5_0 = 0 \qquad H^5_{-2} = H^3_0, \qquad H^5_{-4} = H^3_{-2}, \\ & H^6_0 = -{H^3_0}^2, \qquad H^6_{-2} = -2H^3_0 H^3_{-2}, \qquad H^6_{-4} = -{H^3_{-2}}^2, \\ & \dots \end{aligned} \tag{76}$$

The proof of the polynomiality of $p_j(z)$ is exactly the same as for $W_3$ (Lemma 5.1). The constraints (76) follow from equations (75) and the condition $z^5 W_5 \subset W_5$. For instance one has $z^2 p_3 = p_5$, $z^2 p_5 = p_7 + H^5_{-4} p_6$ etc. . The constants $C^l_{jk}$ are given by

$$C^l_{jk} = \delta^l_{j+k} + \sum_{s=0}^{2} H^j_{-2s} \delta^l_{2s+k} + \sum_{s=0}^{2} H^k_{-2s} \delta^l_{2s+j} + \sum_{s,r=0}^{2} H^j_{-2s} H^k_{-2r} \delta^l_{2(r+s)}, \quad j, k \geq 3 \tag{77}$$

where, for sake of compactess, we use $H^3_{-4} = 0$. As a consequence of this lemma one has

**Proposition 7.2** *The stratum $\Sigma_5$ contains a maximal subspace $W_5$ closed with respect to multiplication $W_5 \cdot W_5 \subset W_5$. Elements of $W_5$ have the form $w = \sum_{i=3,5,6,7,\dots} a_i p_i(z)$ with arbitrary $a_i$, $n$ and $H^j_{-4}$, $H^j_{-2}$, $H^j_0$ obeying the constraints (52).*

*Algebraically $W_5$ is an infinite-dimensional commutative associative algebra $A_5$ of polynomials with the structure constants given by (77). Geometrically $W_5$ is the infinite algebraic variety $\Gamma_5$ defined by the equations (75) and (52).*

First equations of the set of equations (75) are

$$\begin{aligned} {p_3}^2 =& p_6 + 2H^3_{-2} p_5 + 2H^3_0 p_3, \\ p_3 p_5 =& p_8 + 2H^3_{-2} p_7 + {H^3_{-2}}^2 p_6 + +2H^3_0 p_5, \end{aligned} \tag{78}$$

and so on. So the algebra $A_5$ is generated by $p_3, p_5$ and $p_7$.

It is not also difficult to show that an ideal of the variety $\Gamma_5$ contains the family of plane (3,5) curve

$$p_5{}^3 - p_3{}^5 + 2\, H^3_{-2} p_3{}^3 p_5 - {H^3_{-2}}^2 p_3 p_5{}^2 + 2\, H^3_0 p_3{}^4 - 2\, H^3_0 H^3_{-2} p_3{}^2 p_5 - {H^3_0}^2 p_3{}^3 = 0 \tag{79}$$

parameterized by two variables $H^3_{-2}$ and $H^3_0$. Due to the polinomiality of $p_3$ and $p_5$ in $z$, the genus of of curve (79) is obviously equal to zero. The ideal of the varieties contains another rational plane curve given by

$$\begin{aligned} &p_6{}^5 - p_5{}^6 + 6\, H^3_{-2} p_5 p_6{}^4 + 14\, {H^3_{-2}}^2 p_5{}^2 p_6{}^3 - \left(-6\, H^3_0 - 16\, {H^3_{-2}}^3\right) p_5{}^3 p_6{}^2 - \left(-16\, H^3_0 H^3_{-2} - 9\, {H^3_{-2}}^4\right) p_5{}^4 p_6 \\ &- \left(-10\, H^3_0 {H^3_{-2}}^2 - 2\, {H^3_{-2}}^5\right) p_5{}^5 + 2\, {H^3_0}^2 p_6{}^4 + 8\, {H^3_0}^2 H^3_{-2} p_5 p_6{}^3 + 10\, {H^3_{-2}}^2 {H^3_0}^2 p_5{}^2 p_6{}^2 \\ &- \left(-14\, {H^3_0}^3 - 4\, {H^3_{-2}}^3 {H^3_0}^2\right) p_5{}^3 p_6 + 20\, {H^3_0}^3 H^3_{-2} p_5{}^4 + {H^3_0}^4 p_6{}^3 + 2\, {H^3_0}^4 H^3_{-2} p_5 p_6{}^2 + 8\, {H^3_0}^5 p_5{}^3 = 0. \end{aligned} \tag{80}$$

The stratum $\Sigma_5$ exhibits the main features of higher strata $\Sigma_{4m-1}$, $(m = 1, 2, 3, \dots)$. The maximal closed subspaces $W_{4m-1}$ have the basis $(p_{2m+1}, p_{2m+3}, \dots, p_{4m-1}, p_{4m}, p_{4m+1}, \dots)$ while the stratum $\Sigma_{4m+1}$, $(m = 1, 2, 3, \dots)$ have the basis $(p_{2m+1}, p_{2m+3}, \dots, p_{4m-1}, p_{4m+1}, p_{4m+2}, \dots)$ with the respective $p_j$. Then one can demonstrate an analog of the Lemma 7.1 for $\Sigma_{4m\pm1}$ which in particular says that all $p_j(z)$ are polynomials in $z$ obeying the equations

$$p_j(z) p_k(z) = \sum_l C^l_{jk} p_l(z), \qquad j, k, l = 2m+1, 2m+3, \dots \tag{81}$$

together with certain constraints on $H^j_{-k}$.

Consequently one has closed subspaces $W_{4m\pm1}$ in $\Sigma_{4m\pm1}$ which algebraically are commutative and associative algebras and geometrically they represent families of algebraic varieties $\Gamma_{4m\pm1}$ defined by the equation (81). Ideals of the varieties $\Gamma_{4m\pm1}$ contain plane $(2m+1, 2m+3)$ curve

$$p^{2m+3}_{2m+1} - p^{2m+1}_{2m+3} + \cdots = 0, \qquad m = 1, 2, 3, \dots \tag{82}$$

of zero genus.

Properties of these rational curves will be discussed elsewhere.

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# References

[1] Sato, M., Sato, Y.: Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold. Nonlinear partial differential equations in applied science, 259–271, North-Holland Math. Stud., **81**, North-Holland, Amsterdam, 1983.

[2] Segal, G., Wilson, G.: Loop groups and equations of KdV type, Inst. Hautes tudes Sci. Publ. Math. No. 61, 5–65 (1985)

[3] Pressley, A., Segal, G.: *Loop Groups.* Oxford University Press (1988)

[4] Konopelchenko, B. G., Ortenzi, G.: Algebraic curves in Birkhoff strata of Sato Grassmannian,. arXiv:1005.2053

[5] Konopelchenko, B. G., Ortenzi, G.: Birkhoff strata in the Grassmannian $\mathrm{Gr}^{(2)}$: tangent cohomology and integrable systems (in preparation)

[6] Harris, J.: *Algebraic geometry: a first course.* Springer-Verlag, Berlin (1992)

[7] Buchstaber, V.M., Leikin, D.V., Enolskii, V.Z.: Rational analogues of Abelian functions. Funktsional. Anal. i Prilozhen. **33**:2, 115 (1999) translated in Functional Anal. Appl. **33**, 8394 (1999)

[8] Baker, H. F.: Abelian functions, Cambridge, 1987

[9] Kodama, Y., Konopelchenko, B.G.: Singular sector of the Burgers-Hopf hierarchy and deformations of hyperelliptic curves. J. Phys. A 35, no. 31, L489-L500 (2002)