

SCHEME OF LINES ON A FAMILY OF 2-DIMENSIONAL QUADRICS: GEOMETRY AND DERIVED CATEGORY

ALEXANDER KUZNETSOV

ABSTRACT. Given a generic family Q of 2-dimensional quadrics over a smooth 3-dimensional base Y we consider the relative Fano scheme M of lines of it. The scheme M has a structure of a generically conic bundle $M \rightarrow X$ over a double covering $X \rightarrow Y$ ramified in the degeneration locus of $Q \rightarrow Y$. The double covering $X \rightarrow Y$ is singular in a finite number of points (corresponding to the points $y \in Y$ such that the quadric Q_y degenerates to a union of two planes), the fibers of M over such points are unions of two planes intersecting in a point.

The main result of the paper is a construction of a semiorthogonal decomposition for the derived category of coherent sheaves on M . This decomposition has three components, the first is the derived category of a small resolution X^+ of singularities of the double covering $X \rightarrow Y$, the second is a twisted resolution of singularities of X (given by the sheaf of even parts of Clifford algebras on Y), and the third is generated by a completely orthogonal exceptional collection.

1. INTRODUCTION

The subject of this note is a description of the structure of the derived category of coherent sheaves on the relative scheme of lines for a family of 2-dimensional quadrics. We had two motivations for investigation of this category — first of all it has an interesting structure and exhibits some interesting features. For example, it combines the minimal resolution of singularities and the twisted resolution of singularities of a certain double covering of the base of the family.

The second, and the most important motivation, comes from investigation of the derived categories of some special double covers of \mathbb{P}^3 and their relation to Enriques surfaces. See the companion paper [IK] for details.

The precise formulation of the main result of the paper is the following. Consider a family of 2-dimensional quadrics $q : Q \rightarrow Y$. This means that we are given a projectivization of a rank 4 vector bundle \mathcal{V} on Y and a divisor $Q \subset \mathbb{P}_Y(\mathcal{V})$ of relative degree 2 which is flat over Y . Such divisor is given by a line subbundle $\mathcal{L} \subset S^2\mathcal{V}^\vee$.

Given this we consider *the relative Fano scheme of lines of Q over Y* . By definition this is the zero locus on the relative Grassmannian $\text{Gr}_Y(2, \mathcal{V})$ of the global section

$$s \in \Gamma(\text{Gr}_Y(2, \mathcal{V}), \mathcal{L}^\vee \otimes S^2\mathcal{U}^\vee),$$

where $\mathcal{U} \subset \mathcal{V}$ is the tautological subbundle on the Grassmannian. We denote this relative Fano scheme by M . The fibers M_y of the projection $\rho : M \rightarrow Y$ have the following structure

- M_y is a disjoint union of two smooth conics, if the quadric Q_y is smooth;
- M_y is a single smooth conic (with a nonreduced scheme structure), if the quadric Q_y has corank 1;
- M_y is a union of two planes intersecting in a point, if the quadric Q_y has corank 2;
- M_y is a single plane (with a nonreduced scheme structure), if the quadric Q_y has corank 3.

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From now on we assume that the family $Q \rightarrow Y$ is sufficiently generic, that is generic fiber is smooth, and the codimension of the locus $D_r \subset Y$ of quadrics of corank r equals $r(r+1)/2$. In this case we have the Stein factorization for the morphism $\rho : M \rightarrow Y$:

$$\begin{array}{ccc} M & \xrightarrow{\mu} & X \\ & \searrow \rho & \swarrow f \\ & & Y \end{array}$$

where $f : X \rightarrow Y$ is the double covering ramified in the divisor D_1 , and $\mu : M \rightarrow X$ is generically a conic bundle.

The main result of this paper is a description of the derived category of M when $\dim Y = 3$. Then the above genericity assumptions imply that $D_3 = \emptyset$ and D_2 consists of a finite number N of isolated points y_1, \dots, y_N . Additionally we assume that D_1 has an ordinary double point (an ODP or a node for short) in each of y_i . The last assumption is equivalent to smoothness of M if Q is smooth.

To state the answer we need the following ingredients. First, consider the sheaf of even parts of Clifford algebras \mathcal{B}_0 on Y associated with the family $Q \rightarrow Y$ (see [K2] for details). As an \mathcal{O} -module it is given by

$$\mathcal{B}_0 = \mathcal{O}_Y \oplus \Lambda^2 \mathcal{V} \otimes \mathcal{L} \oplus \Lambda^4 \mathcal{V} \otimes \mathcal{L}^2$$

with the Clifford multiplication. If Q is smooth then the category $\mathcal{D}^b(Y, \mathcal{B}_0)$ is also smooth and can be thought of as a twisted noncommutative resolution of the double covering X .

Second, note that X has N isolated ordinary double points over D_2 , so being 3-dimensional it has 2^N small resolutions of singularities in the category of Moishezon varieties. To fix one of these resolutions we should choose for each point $y_i \in D_2$ one of the planes in the corank 2 quadric Q_{y_i} , or equivalently one of the planes in the fiber M_{y_i} of M over Y .

Let us pick one of these resolutions and denote it by $\sigma_+ : X^+ \rightarrow X$. Let us denote the planes in M_{y_i} corresponding to this choice by Σ_i^+ , and the complementary planes by Σ_i^- , so that $M_{y_i} = \Sigma_i^+ \cup \Sigma_i^-$.

The main result of this paper is the following

Theorem 1.1. *Assume that $Q \rightarrow Y$ is a family of quadrics, Y and Q are smooth, $\dim Y = 3$, and the degeneration locus D_1 has a finite number of ordinary double points $\{y_1, \dots, y_N\} = D_2$. Then the relative Fano scheme M of lines of Q over Y is smooth and there is a semiorthogonal decomposition*

$$\mathcal{D}^b(M) = \langle \mathcal{D}^b(X^+), \mathcal{D}^b(Y, \mathcal{B}_0), \{\mathcal{O}_{\Sigma_i^+}\}_{i=1}^N \rangle.$$

Here the third component is a completely orthogonal exceptional collection.

The proof goes as follows. In section 2 we prove smoothness of M and investigate the local structure of M around the planes Σ_i^\pm . In particular, we check that the sheaves $\mathcal{O}_{\Sigma_i^+}$ form a completely orthogonal exceptional collection in $\mathcal{D}^b(M)$. In section 3 we recall some facts about the sheaf of even parts of Clifford algebras \mathcal{B}_0 and construct a fully faithful embedding $\mathcal{D}^b(Y, \mathcal{B}_0) \rightarrow \mathcal{D}^b(M)$. In section 4 we show that there is a birational transformation of M , a flip in N planes Σ_i^+ , transforming it into a \mathbb{P}^1 -fibration $\mu_+ : M^+ \rightarrow X^+$ over a small resolution $X^+ \rightarrow X$. This gives an identification of the orthogonal to the collection $\{\mathcal{O}_{\Sigma_i^+}\}_{i=1}^N$ in $\mathcal{D}^b(M)$ with $\mathcal{D}^b(M^+)$. In section 5 we construct a fully faithful embedding $\mathcal{D}^b(X^+) \rightarrow \mathcal{D}^b(M^+)$ and identify the complement with $\mathcal{D}^b(Y, \mathcal{B}_0)$.

In the last section 6 we discuss another way of proving Theorem 1.1 and suggest some further directions of investigation.

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2. GEOMETRY OF M

For each quadric Q_y in the family $Q \rightarrow Y$ denote by $K_y \subset \mathcal{V}_y$ the kernel of the corresponding quadratic form (thus $\mathbb{P}(K_y)$ is the singular locus of Q_y). Note that the differential of the section $s \in \Gamma(Y, \mathcal{L}^{-1} \otimes S^2 \mathcal{V}^\vee)$ at y gives a linear map $T_y Y \rightarrow S^2 \mathcal{V}_y^\vee$. Composing it with the natural projection $S^2 \mathcal{V}_y^\vee \rightarrow S^2 K_y^\vee$ we obtain a map

$$\kappa_y : T_y Y \rightarrow S^2 K_y^\vee.$$

In term of these maps one can check the smoothness of Q and M .

Proposition 2.1. *Assume that Y is smooth. Then*

- (1) Q is smooth if and only if for any $y \in Y$ and any subspace $K \subset K_y$ with $\dim K \leq 1$, the composition $T_y Y \rightarrow S^2 K_y^\vee \rightarrow S^2 K^\vee$ is surjective;
- (2) M is smooth if and only if for any $y \in Y$ and any embedding $K \rightarrow K_y$ with $\dim K \leq 2$, the composition $T_y Y \rightarrow S^2 K_y^\vee \rightarrow S^2 K^\vee$ is surjective.

Proof: The question is local so we can assume that \mathcal{V} is a trivial bundle, $\mathcal{V} \cong V \otimes \mathcal{O}_Y$. Then the result is a simple local calculation. \square

Remark 2.2. This result generalizes to arbitrary relative isotropic Grassmannians of families of quadrics of arbitrary dimension. The smoothness of the Grassmannian of k -dimensional subspaces is equivalent to the surjectivity of the corresponding map for all K with $\dim K \leq k$.

Corollary 2.3. *Assume that Y is smooth and $D_3 = \emptyset$. Then M is smooth if and only if Q is smooth and for any $y \in D_2$ the map $\kappa_y : T_y Y \rightarrow S^2 K_y^\vee$ is surjective.*

Another consequence of smoothness of Q is smoothness of $D_1 \setminus D_2$. On the other hand, the points of D_2 are always singular on D_1 . In fact they are ordinary double points if M is smooth.

Lemma 2.4. *Assume that Q is smooth and $\dim Y = 3$. Then M is smooth if and only if D_2 is a finite number of points and any point of D_2 is an ordinary double point of D_1 .*

Proof: Take any $y \in D_2$, so that $\dim K_y = 2$. The map $T_y Y \rightarrow S^2 K_y^\vee$ can be thought of as a net of quadrics on K_y parameterized by $T_y Y$. Its degeneration locus in $\mathbb{P}(T_y Y)$ is a conic, either nondegenerate (if the map $T_y Y \rightarrow S^2 K_y^\vee$ is surjective) or singular (since the kernel of the map lies in the singular locus). But on the other hand, this degeneration locus is the base of the tangent cone to D_1 at y_i . So, if y_i is an ODP of D_1 , the conic should be nondegenerate, hence the map should be surjective. \square

Remark 2.5. Note also that if the map κ_y for $y \in D_2$ is surjective, then it is an isomorphism (since $\dim T_y Y = \dim S^2 K_y^\vee = 3$).

For each point $y_i \in D_2$ we have $Q_{y_i} = \mathbb{P}(W_i^+) \cup \mathbb{P}(W_i^-)$, both W_i^+ and W_i^- being a 3-dimensional subspaces in \mathcal{V}_{y_i} , the fiber of \mathcal{V} over y_i . These planes intersect along the line $\mathbb{P}(K_{y_i})$. We put

$$W_i^0 = K_{y_i} = W_i^+ \cap W_i^-.$$

Consequently, $M_{y_i} = \text{Gr}(2, W_i^+) \cup \text{Gr}(2, W_i^-) = \Sigma_i^+ \cup \Sigma_i^-$, both Σ_i^+ and Σ_i^- are planes. These planes intersect in a point $P_i = \text{Gr}(2, W_i^0)$. Choose arbitrary point $y = y_i \in D_2$ and one of the planes $\Sigma = \Sigma_i^\pm$.

Proposition 2.6. *If a point $y \in D_2$ is an ODP of D_1 then $\mathcal{N}_{\Sigma/M} \cong \mathcal{O}_\Sigma(-1) \oplus \mathcal{O}_\Sigma(-1)$.*

Proof: Choosing a local trivialization of the bundle \mathcal{V} we obtain an isomorphism

$$\mathcal{N}_{\Sigma/\text{Gr}(2, \mathcal{V})} \cong \mathcal{N}_{\Sigma/\text{Gr}(2, \mathcal{V}_y)} \oplus T_y Y \otimes \mathcal{O}_\Sigma \cong \mathcal{U}^\vee \oplus T_y Y \otimes \mathcal{O}_Y.$$

On the other hand, $\mathcal{N}_{M/\mathrm{Gr}_Y(2,\mathcal{V})} \cong S^2\mathcal{U}^\vee \otimes \mathcal{L}^\vee$. Hence the standard exact sequence

$$0 \rightarrow \mathcal{N}_{\Sigma/M} \rightarrow \mathcal{N}_{\Sigma/\mathrm{Gr}_Y(2,\mathcal{V})} \rightarrow (\mathcal{N}_{M/\mathrm{Gr}_Y(2,\mathcal{V})})|_{\Sigma} \rightarrow 0$$

gives

$$0 \rightarrow \mathcal{N}_{\Sigma/M} \rightarrow \mathcal{U}|_{\Sigma}^\vee \oplus T_y Y \otimes \mathcal{O}_{\Sigma} \rightarrow S^2\mathcal{U}|_{\Sigma}^\vee \rightarrow 0.$$

Since $\Sigma = \mathrm{Gr}(2, W)$, $W \subset \mathcal{V}_y$, the cohomology exact sequence looks like

$$0 \rightarrow H^0(\Sigma, \mathcal{N}_{\Sigma/M}) \rightarrow W^* \oplus T_y Y \rightarrow S^2 W^* \rightarrow H^1(\Sigma, \mathcal{N}_{\Sigma/M}) \rightarrow 0.$$

Consider the map $W^* \oplus T_y Y \rightarrow S^2 W^*$. Its first component is the multiplication by the equation of the line $K_y = W^0 \subset W$. Hence the sequence can be rewritten as

$$0 \rightarrow H^0(\Sigma, \mathcal{N}_{\Sigma/M}) \rightarrow T_y Y \rightarrow S^2 K_y^\vee \rightarrow H^1(\Sigma, \mathcal{N}_{\Sigma/M}) \rightarrow 0.$$

The middle map here is just the map κ_y , hence by Remark 2.5 it is an isomorphism. Thus the bundle $\mathcal{N}_{\Sigma/M}$ is acyclic, hence it is isomorphic to $\mathcal{O}_{\Sigma}(-1) \oplus \mathcal{O}_{\Sigma}(-1)$. \square

From now on we assume that every point $y_i \in D_2$ is an ODP of D_1 .

Corollary 2.7. *We have $(\omega_M)|_{\Sigma_i^\pm} \cong \mathcal{O}_{\Sigma_i^\pm}(-1)$.*

Proof: By adjunction formula $\mathcal{O}_{\Sigma}(-3) \cong \omega_{\Sigma} \cong \omega_{M|\Sigma} \otimes \det \mathcal{N}_{\Sigma/M} \cong \omega_{M|\Sigma} \otimes \mathcal{O}_{\Sigma}(-2)$, hence the claim. \square

The most important corollary is the following

Corollary 2.8. *The structure sheaf $\mathcal{O}_{\Sigma} \in \mathcal{D}^b(M)$ is exceptional.*

Proof: We have an isomorphism $\mathcal{E}xt^t(\mathcal{O}_{\Sigma}, \mathcal{O}_{\Sigma}) \cong \Lambda^t \mathcal{N}_{\Sigma/M} \cong \Lambda^t(\mathcal{O}_{\Sigma}(-1) \oplus \mathcal{O}_{\Sigma}(-1))$. Note that for $t = 1$, and $t = 2$ this sheaf on $\Sigma = \mathbb{P}^2$ is acyclic. Hence $\mathrm{Ext}^\bullet(\mathcal{O}_{\Sigma}, \mathcal{O}_{\Sigma}) \cong H^\bullet(\Sigma, \mathcal{H}om(\mathcal{O}_{\Sigma}, \mathcal{O}_{\Sigma})) \cong H^\bullet(\Sigma, \mathcal{O}_{\Sigma})$ implies exceptionality of \mathcal{O}_{Σ} . \square

Another simple observation is that Σ_i^\pm with different i are completely orthogonal.

Lemma 2.9. *If $i \neq j$ then $\mathrm{Ext}^\bullet(\mathcal{O}_{\Sigma_i^\pm}, \mathcal{O}_{\Sigma_j^\pm}) = 0$.*

Proof: The planes Σ_i^\pm and Σ_j^\pm are contained in fibers of M over different points $y_i, y_j \in Y$, hence there is no local Ext's between their structure sheaves. Hence global Ext's also vanish. \square

Thus choosing one plane for each y_i we obtain a completely orthogonal exceptional collection

Corollary 2.10. *The collection $\{\mathcal{O}_{\Sigma_i^\pm}\}_{i=1}^N$ is a completely orthogonal exceptional collection in $\mathcal{D}^b(M)$.*

3. THE CLIFFORD ALGEBRA

For the precise definition and basic results about the sheaves of even parts of Clifford algebras, see [K2]. Here we remind some of their properties.

Recall that besides the sheaf of algebras $\mathcal{B}_0 = \mathcal{O}_Y \oplus \Lambda^2 \mathcal{V} \otimes \mathcal{L} \oplus \Lambda^4 \mathcal{V} \otimes \mathcal{L}^2$ on Y , we also have a natural sequence of sheaves of \mathcal{B}_0 -modules, the first of them is the odd part of the sheaf of Clifford algebras \mathcal{B}_1 , which as a sheaf of \mathcal{O}_Y -modules is given by

$$\mathcal{B}_1 = \mathcal{V} \oplus \Lambda^3 \mathcal{V} \otimes \mathcal{L}$$

and with the action of \mathcal{B}_0 given by the Clifford multiplication. The other sheaves \mathcal{B}_k in the sequence are obtained from \mathcal{B}_0 and \mathcal{B}_1 by an appropriate twist

$$\mathcal{B}_{-2k} = \mathcal{B}_0 \otimes \mathcal{L}^k, \quad \mathcal{B}_{1-2k} = \mathcal{B}_1 \otimes \mathcal{L}^k.$$

This sequence can be thought of as a sequence of powers of a line bundle. In particular, the functors $-\otimes_{\mathcal{B}_0} \mathcal{B}_l$ and $\mathcal{H}om_{\mathcal{B}_0}(\mathcal{B}_l, -)$ are exact and we have

$$\mathcal{B}_k \otimes_{\mathcal{B}_0} \mathcal{B}_l \cong \mathcal{B}_{k+l}, \quad R\mathcal{H}om_{\mathcal{B}_0}(\mathcal{B}_l, \mathcal{B}_k) \cong \mathcal{B}_{k-l}. \quad (1)$$

Let α denote the embedding $M \rightarrow \mathrm{Gr}_Y(2, \mathcal{V})$. Let g denote $c_1(\mathcal{U}^\vee)$, the positive generator of the relative Picard group $\mathrm{Pic}(\mathrm{Gr}_Y(2, \mathcal{V})/Y)$. Since M is the zero locus of $s \in \Gamma(\mathrm{Gr}_Y(2, \mathcal{V}), \mathcal{L}^\vee \otimes S^2\mathcal{U}^\vee)$ we have the Koszul resolution for its structure sheaf

$$0 \rightarrow \mathcal{L}^3(-3g) \rightarrow \mathcal{L}^2 \otimes S^2\mathcal{U}(-g) \rightarrow \mathcal{L} \otimes S^2\mathcal{U} \rightarrow \mathcal{O} \rightarrow \alpha_*\mathcal{O}_M \rightarrow 0. \quad (2)$$

Now we will show that M also comes with a sequence of naturally defined \mathcal{B}_0 -modules. To unburden the notation we denote the pullbacks of the sheaves \mathcal{B}_k to $\mathrm{Gr}_Y(2, \mathcal{V})$ by the same letters. For each $k \in \mathbb{Z}$ consider the morphism $\mathcal{U} \otimes \mathcal{B}_{k-1} \rightarrow \mathcal{B}_k$ of sheaves of \mathcal{B}_0 -modules on $\mathrm{Gr}_Y(2, \mathcal{V})$ induced by the embedding $\mathcal{U} \subset \mathcal{V}$ and the Clifford multiplication $\mathcal{V} \otimes \mathcal{B}_k \rightarrow \mathcal{B}_{k+1}$.

Proposition 3.1. *There are isomorphisms $\mathrm{Coker}(\mathcal{U} \otimes \mathcal{B}_{k-1} \rightarrow \mathcal{B}_k) \cong \alpha_*\mathcal{S}_k$, where*

$$\mathcal{S}_{2k+1} = (\mathcal{V}/\mathcal{U}) \otimes \mathcal{L}^{-k}, \quad (3)$$

and there is an exact sequence

$$0 \rightarrow \mathcal{L}^{-k} \rightarrow \mathcal{S}_{2k} \rightarrow \det \mathcal{V} \otimes \mathcal{L}^{1-k}(g) \rightarrow 0. \quad (4)$$

Moreover, the sheaves \mathcal{S}_k have a structure of \mathcal{B}_0 -modules such that

$$\mathcal{S}_k \otimes_{\mathcal{B}_0} \mathcal{B}_l \cong \mathcal{S}_{k+l}, \quad R\mathcal{H}om_{\mathcal{B}_0}(\mathcal{B}_l, \mathcal{S}_k) \cong \mathcal{S}_{k-l}. \quad (5)$$

Finally, for each k there is an exact sequence

$$0 \rightarrow \mathcal{O}(-2g) \otimes \mathcal{B}_{k-4} \rightarrow \mathcal{U}(-g) \otimes \mathcal{B}_{k-3} \rightarrow \mathcal{U} \otimes \mathcal{B}_{k-1} \rightarrow \mathcal{B}_k \rightarrow \alpha_*\mathcal{S}_k \rightarrow 0. \quad (6)$$

Proof: First let us check that the cokernels are supported on M scheme-theoretically. For this we note that the composition of the maps $S^2\mathcal{U} \otimes \mathcal{B}_{k-2} \rightarrow \mathcal{U} \otimes \mathcal{B}_{k-1} \rightarrow \mathcal{B}_k$ (both of which are induced by the action of \mathcal{V} on \mathcal{B}) coincides with the map $S^2\mathcal{U} \otimes \mathcal{B}_{k-2} \cong S^2\mathcal{U} \otimes \mathcal{B}_k \otimes \mathcal{L} \rightarrow \mathcal{B}_k$ induced by the section s defining the family Q (this follows from the definition of the Clifford multiplication). It follows that the cokernel is a quotient of $\alpha_*\alpha^*\mathcal{B}_k$, hence it can be written as $\alpha_*\mathcal{S}_k$, where \mathcal{S}_k is a sheaf of \mathcal{B}_0 -modules on M . Note also that the formulas (5) follow from the definition of \mathcal{S}_k combined with equations (1) and exactness of functors $-\otimes_{\mathcal{B}_0} \mathcal{B}_l$ and $\mathcal{H}om_{\mathcal{B}_0}(\mathcal{B}_l, -)$. So, it remains to verify (3), (4) and (6).

For this consider the maps $\mathcal{B}_{k-4} \rightarrow \mathcal{B}_{k-3} \otimes \mathcal{U}^\vee$ obtained by the partial dualization from the maps $\mathcal{U} \otimes \mathcal{B}_{k-4} \rightarrow \mathcal{B}_{k-3}$. Also consider the composition $\mathcal{U} \otimes \mathcal{B}_{k-3} \rightarrow \mathcal{V} \otimes \mathcal{B}_{k-3} \rightarrow \mathcal{V}^\vee \otimes \mathcal{B}_{k-1} \rightarrow \mathcal{U}^\vee \otimes \mathcal{B}_{k-1}$, where the middle map is induced by the double action of \mathcal{V} on \mathcal{B} . Finally, after appropriate twistings and identification $\mathcal{U}^\vee(-g) \cong \mathcal{U}$, we compose a sequence

$$0 \rightarrow \mathcal{O}(-2g) \otimes \mathcal{B}_{k-4} \rightarrow \mathcal{U}(-g) \otimes \mathcal{B}_{k-3} \rightarrow \mathcal{U} \otimes \mathcal{B}_{k-1} \rightarrow \mathcal{B}_k \rightarrow 0.$$

It is easy to check that the compositions of the arrows are zero, so the constructed sequence of maps is a complex. Note that each term of the complex is naturally filtered. Consider the spectral sequence of the filtered complex in case $k = 0$. The first term looks like

$$\begin{array}{ccccccc} \mathcal{O}(-2g) \otimes \Lambda^4\mathcal{V} \otimes \mathcal{L}^4 & & & & & & \\ \mathcal{O}(-2g) \otimes \Lambda^2\mathcal{V} \otimes \mathcal{L}^3 & \longrightarrow & \mathcal{U}(-g) \otimes \Lambda^3\mathcal{V} \otimes \mathcal{L}^3 & & & & \\ \mathcal{O}(-2g) \otimes \mathcal{L}^2 & \longrightarrow & \mathcal{U}(-g) \otimes \mathcal{V} \otimes \mathcal{L}^2 & \longrightarrow & \mathcal{U} \otimes \Lambda^3\mathcal{V} \otimes \mathcal{L}^2 & \longrightarrow & \Lambda^4\mathcal{V} \otimes \mathcal{L}^2 \\ & & & & \mathcal{U} \otimes \mathcal{V} \otimes \mathcal{L} & \longrightarrow & \Lambda^2\mathcal{V} \otimes \mathcal{L} \\ & & & & & & \mathcal{O} \end{array}$$

The rows are natural complexes with maps corresponding to the wedge multiplication. Their cohomology are easy to compute, so it is not difficult to see that the second term looks like

$$\begin{array}{ccccccc}
\det \mathcal{V} \otimes \mathcal{L}^4(-2g) & & & & & & \\
& \searrow & & & & & \\
\mathcal{L}^3(-3g) & & \det \mathcal{V} \otimes \mathcal{L}^3 \otimes S^2 \mathcal{U} & & & & \\
& \searrow & \searrow & & & & \\
& & \mathcal{L}^2 \otimes S^2 \mathcal{U}(-g) & & \det \mathcal{V} \otimes \mathcal{L}^2 \otimes S^2 \mathcal{U}(g) & & \\
& & & \searrow & & \searrow & \\
& & & & \mathcal{L} \otimes S^2 \mathcal{U} & & \det \mathcal{V} \otimes \mathcal{L} \otimes \mathcal{O}(g) \\
& & & & & \searrow & \\
& & & & & & \mathcal{O}
\end{array}$$

The arrows here are induced by s . So, it is easy to see that the bottom chain is the Koszul complex of s , while the top chain is the same complex twisted by $\det \mathcal{V} \otimes \mathcal{L}(g)$. Hence the spectral sequence degenerates in the third term and shows that the cohomology of the above complex is supported in degree zero and is an extension of $\alpha_* \alpha^*(\det \mathcal{V} \otimes \mathcal{L}(g))$ by $\alpha_* \alpha^* \mathcal{O}$. Since we already know that it is supported on M , we conclude that it can be written as $\alpha_* \mathcal{S}_0$, where \mathcal{S}_0 is an extension of $\det \mathcal{V} \otimes \mathcal{L}(g)$ by \mathcal{O} on M . This gives (4) and (6) for \mathcal{S}_0 .

Analogously, consider the complex for $k = 1$. The first term of the spectral sequence looks like

$$\begin{array}{ccccccc}
\mathcal{O}(-2g) \otimes \Lambda^3 \mathcal{V} \otimes \mathcal{L}^3 & \longrightarrow & \mathcal{U}(-g) \otimes \Lambda^4 \mathcal{V} \otimes \mathcal{L}^3 & & & & \\
\mathcal{O}(-2g) \otimes \mathcal{V} \otimes \mathcal{L}^2 & \longrightarrow & \mathcal{U}(-g) \otimes \Lambda^2 \mathcal{V} \otimes \mathcal{L}^2 & \longrightarrow & \mathcal{U} \otimes \Lambda^4 \mathcal{V} \otimes \mathcal{L}^2 & & \\
& & \mathcal{U}(-g) \otimes \mathcal{L} & \longrightarrow & \mathcal{U} \otimes \Lambda^2 \mathcal{V} \otimes \mathcal{L} & \longrightarrow & \Lambda^3 \mathcal{V} \otimes \mathcal{L} \\
& & & & \mathcal{U} & \longrightarrow & \mathcal{V}
\end{array}$$

The maps are induced by the wedge multiplication, so one can check that the second term looks like

$$\begin{array}{ccccccc}
\mathcal{V}/\mathcal{U} \otimes \mathcal{L}^3(-3g) & & & & & & \\
& \searrow & & & & & \\
& & \mathcal{V}/\mathcal{U} \otimes \mathcal{L}^2 \otimes S^2 \mathcal{U}(-g) & & & & \\
& & & \searrow & & & \\
& & & & \mathcal{V}/\mathcal{U} \otimes \mathcal{L} \otimes S^2 \mathcal{U} & & \\
& & & & & \searrow & \\
& & & & & & \mathcal{V}/\mathcal{U}
\end{array}$$

The maps are induced by s , so it is the Koszul complex of s tensored with \mathcal{V}/\mathcal{U} , hence $\mathcal{S}_1 \cong \mathcal{V}/\mathcal{U}$. This gives (3) and (6) for \mathcal{S}_1 . For other \mathcal{S}_k we deduce (3), (4) and (6) by a suitable twist. \square

Applying the functor ρ_* to the resolutions (6) twisted by $\mathcal{O}(-g)$ we deduce the following

Corollary 3.2. *We have $\rho_*(\mathcal{S}_k) \cong \mathcal{B}_k$, $\rho_*(\mathcal{S}_k(-g)) = 0$.*

Corollary 3.3. *The extension in (4) is nontrivial.*

Proof: Assume that $\mathcal{S}_0 \cong \mathcal{O} \oplus \det \mathcal{V} \otimes \mathcal{L}(g)$. Then

$$\rho_*(\mathcal{S}_0(-g)) \cong \rho_*((\mathcal{O} \oplus \det \mathcal{V} \otimes \mathcal{L}(g))(-g)) \cong \rho_*(\mathcal{O}(-g) \oplus \det \mathcal{V} \otimes \mathcal{L}).$$

Using (2) it is easy to see that $\rho_* \mathcal{O}_M = \mathcal{O}_Y \oplus \det \mathcal{V} \otimes \mathcal{L}^2$ and $\rho_*(\mathcal{O}(-g)) = \det \mathcal{V} \otimes \mathcal{L} \oplus (\det \mathcal{V})^2 \otimes \mathcal{L}^3$, so the RHS is nontrivial, which contradicts 3.2. \square

Corollary 3.4. *We have $\rho_*(\mathcal{S}_k^\vee) = 0$.*

Proof: Indeed, since \mathcal{S}_k is of rank 2 and $\det \mathcal{S}_k = \det \mathcal{V} \otimes \mathcal{L}^{1-k}(g)$, we have $\mathcal{S}_k^\vee \cong \mathcal{S}_k(-g) \otimes \det \mathcal{V}^\vee \otimes \mathcal{L}^{k-1}$, hence its pushforward is a twist of $\rho_*(\mathcal{S}_k(-g))$ which is zero. \square

Another consequence is the following

Corollary 3.5. *We have $\rho_*(\mathcal{S}_l^\vee \otimes \mathcal{S}_k) \cong \mathcal{B}_{k-l}$.*

Proof: First of all consider the case $l = 0$. Then dualizing (4) we obtain an exact triple

$$0 \rightarrow \det \mathcal{V}^\vee \otimes \mathcal{L}^{-1}(-g) \rightarrow \mathcal{S}_0^\vee \rightarrow \mathcal{O}_M \rightarrow 0.$$

Tensoring it by \mathcal{S}_k , pushing forward and using 3.2, we obtain the claim. Now for arbitrary l the formula follows by tensoring with \mathcal{B}_{-l} and using (5). \square

Now we can describe the embedding $\mathcal{D}^b(Y, \mathcal{B}_0) \rightarrow \mathcal{D}^b(M)$.

Theorem 3.6. *The functor $\Phi : \mathcal{D}^b(Y, \mathcal{B}_0) \rightarrow \mathcal{D}^b(M)$, $\mathcal{F} \mapsto \mathcal{S}_0 \otimes_{\mathcal{B}_0} \rho^* \mathcal{F}$ is fully faithful. Moreover,*

$$\Phi(\mathcal{B}_k) \cong \mathcal{S}_k.$$

Proof: First, note that

$$\mathrm{Hom}(\Phi(\mathcal{F}), \mathcal{G}) = \mathrm{Hom}(\mathcal{S}_0 \otimes_{\mathcal{B}_0} \rho^* \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{B}_0}(\rho^* \mathcal{F}, \mathcal{S}_0^\vee \otimes_{\mathcal{O}_M} \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{B}_0}(\mathcal{F}, \rho_*(\mathcal{S}_0^\vee \otimes_{\mathcal{O}_M} \mathcal{G})).$$

Thus the right adjoint functor $\Phi^! : \mathcal{D}^b(M) \rightarrow \mathcal{D}^b(Y, \mathcal{B}_0)$ is given by

$$\Phi^!(\mathcal{G}) = \rho_*(\mathcal{S}_0^\vee \otimes_{\mathcal{O}_M} \mathcal{G})$$

(the structure of \mathcal{B}_0 -module is induced by that of \mathcal{S}_0^\vee). So, to check full faithfulness it suffices to compute $\Phi^! \circ \Phi$. For this we note that

$$\Phi^!(\Phi(\mathcal{F})) = \rho_*(\mathcal{S}_0^\vee \otimes_{\mathcal{O}_M} \mathcal{S}_0 \otimes_{\mathcal{B}_0} \rho^* \mathcal{F}) \cong \rho_*(\mathcal{S}_0^\vee \otimes_{\mathcal{O}_M} \mathcal{S}_0) \otimes_{\mathcal{B}_0} \mathcal{F} \cong \mathcal{B}_0 \otimes_{\mathcal{B}_0} \mathcal{F} \cong \mathcal{F}$$

(we applied 3.5). Thus $\Phi^! \circ \Phi \cong \mathrm{id}$, so Φ is fully faithful. Finally, $\Phi(\mathcal{B}_k) = \mathcal{S}_0 \otimes_{\mathcal{B}_0} \mathcal{B}_k \cong \mathcal{S}_k$ by (5). \square

We conclude the section with the following simple calculation.

Lemma 3.7. *For each i and each k we have $(\mathcal{S}_k)_{|\Sigma_i^\pm} \cong \mathcal{O}_{\Sigma_i^\pm} \oplus \mathcal{O}_{\Sigma_i^\pm}(1)$.*

Proof: Restrict (3) and (4) to $\Sigma = \Sigma_i^\pm$. Since $\mathcal{O}(g)$ restricts to Σ as $\mathcal{O}_\Sigma(1)$ we obtain the claim for even k . For odd k we have to describe the restriction of \mathcal{V}/\mathcal{U} to Σ . Since $\Sigma = \mathrm{Gr}(2, W) \subset \mathrm{Gr}(2, \mathcal{V}_y)$, we have on Σ an exact sequence

$$0 \rightarrow W/\mathcal{U} \rightarrow \mathcal{V}/\mathcal{U} \rightarrow \mathcal{V}_y/W \otimes \mathcal{O}_\Sigma \rightarrow 0.$$

The first term is $\mathcal{O}_\Sigma(1)$ and the third is \mathcal{O}_Σ . Hence $(\mathcal{V}/\mathcal{U})_{|\Sigma} \cong \mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma(1)$. \square

4. THE FLIP

From now on we choose one of the planes $\mathbb{P}(W_i^\pm) \subset Q_{y_i}$ for each point y_i , say $\mathbb{P}(W_i^+)$, and the corresponding plane $\Sigma_i^+ = \mathrm{Gr}(2, W_i^+) \subset M$. Recall that the normal bundles of Σ_i^+ in M are $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Let us apply the composition of flips in all these planes and denote by M^+ the resulting Moishezon variety. More precisely, consider the blowup $\xi : \widetilde{M} \rightarrow M$ of M in the union of all Σ_i^+ . Then each of the exceptional divisors $E_i = \xi^{-1}(\Sigma_i^+)$ is isomorphic to $\Sigma_i^+ \times \mathbb{P}^1$ and its normal bundle is $\mathcal{O}(-1, -1)$. Hence

in the category of Moishezon varieties it can be blown down onto a line $L_i \cong \mathbb{P}^1 \subset M^+$. Thus we have a diagram

$$\begin{array}{ccccc}
 & & \bigsqcup E_i & & \\
 & \swarrow & \downarrow & \searrow & \\
 \bigsqcup \Sigma_i^+ & & \widetilde{M} & & \bigsqcup L_i \\
 \downarrow & \nearrow \xi & & \searrow \xi^+ & \downarrow \\
 M & & & & M^+
 \end{array}$$

By a result of Bondal and Orlov we have the following

Proposition 4.1 ([BO]). *The functor $\xi_*(\xi^+)^* : \mathcal{D}^b(M^+) \rightarrow \mathcal{D}^b(M)$ is fully faithful. Moreover, there is a semiorthogonal decomposition*

$$\mathcal{D}^b(M) = \langle \xi_*(\xi^+)^*(\mathcal{D}^b(M^+)), \{\mathcal{O}_{\Sigma_i^+}\}_{i=1}^N \rangle.$$

Further, we will need a detailed description of the fibers of M^+ over X . Let $x_i = f^{-1}(y_i)$ be the nodal points of X and $X_{sm} = X \setminus \{y_i\}_{i=1}^N$ be the smooth locus of X .

Lemma 4.2. *There is a regular morphism $M^+ \rightarrow X$ such that the diagram*

$$\begin{array}{ccc}
 & \widetilde{M} & \\
 \xi \swarrow & & \searrow \xi_+ \\
 M & & M^+ \\
 \mu \searrow & & \swarrow \\
 & X &
 \end{array}$$

commutes. Moreover, over the smooth locus X_{sm} the maps $M \rightarrow X$ and $M^+ \rightarrow X$ coincide. Finally, the fiber $M_{x_i}^+$ of M^+ over x_i is the blowup $\widetilde{\Sigma}_i^-$ of Σ_i^- in the points P_i and the line $L_i = \xi_+(E_i)$ is the (-1) -curve on $\widetilde{\Sigma}_i^-$.

Proof: The first claim is evident — since the map $\xi_+ : \widetilde{M} \rightarrow M^+$ is a contraction of divisors E_i , and the map $\mu \circ \xi : \widetilde{M} \rightarrow X$ contract each of these divisors to a point, we conclude that $\mu \circ \xi$ factors through ξ_+ . Moreover, since ξ and ξ_+ are identities over X_{sm} , it follows that the morphisms $M \rightarrow X$ and $M^+ \rightarrow X$ coincide over X_{sm} . Finally, note that the fiber of \widetilde{M} over x_i is the union of E_i and the proper preimage of Σ_i^- . Since ξ is the blowup of Σ_i^+ , and Σ_i^+ intersect Σ_i^- transversally at P_i , the proper preimage of Σ_i^- is the blowup $\widetilde{\Sigma}_i^-$ of Σ_i^- at P_i . Note also that $E_i \cap \widetilde{\Sigma}_i^- = L_i$ is the fiber of $E_i \rightarrow \Sigma_i^+$ over P_i and simultaneously the (-1) -curve on $\widetilde{\Sigma}_i^-$. Finally, since the map ξ_+ is the contraction of $E_i = \Sigma_i^+ \times L_i$ onto L_i , hence it doesn't change $\widetilde{\Sigma}_i^-$, so the fiber $M_{x_i}^+$ coincides with $\widetilde{\Sigma}_i^-$. \square

Note that $\widetilde{\Sigma}_i^-$ being the blowup of a plane in a point is isomorphic to a Hirzebruch surface F_1 . In particular, it has a canonical contraction $\widetilde{\Sigma}_i^- \rightarrow \mathbb{P}^1$ which induces an isomorphism of the exceptional section $L_i \subset \widetilde{\Sigma}_i^-$ onto \mathbb{P}^1 . Denote (the pullback to $\widetilde{\Sigma}_i^-$ of) the generator of the Picard group of Σ_i by h and the class of the exceptional line $L_i \subset \widetilde{\Sigma}_i^-$ by l . Then the class of the fiber of the projection $\widetilde{\Sigma}_i^- \rightarrow \mathbb{P}^1$ is $h - l$.

Lemma 4.3. *We have $\omega_{M^+|\widetilde{\Sigma}_i^-} \cong \mathcal{O}_{\widetilde{\Sigma}_i^-}(-h - l)$.*

Proof: Note that $\omega_{\widetilde{M}} = \xi^*\omega_M(\sum E_i) = \xi_+^*\omega_{M^+}(2\sum E_i)$. Hence $\xi_+^*\omega_{M^+} = \xi^*\omega_M(-\sum E_i)$. Hence

$$\xi_+^*\omega_{M^+|\widetilde{\Sigma}_i^-} \cong \xi^*\omega_{M|\widetilde{\Sigma}_i^-} \otimes \mathcal{O}(-E_i)|_{\widetilde{\Sigma}_i^-} \cong \mathcal{O}_{\widetilde{\Sigma}_i^-}(-h) \otimes \mathcal{O}_{\widetilde{\Sigma}_i^-}(-l) \cong \mathcal{O}_{\widetilde{\Sigma}_i^-}(-h - l).$$

But ξ_+ is an isomorphism on $\tilde{\Sigma}_i^-$, hence the claim. \square

It turns out that M^+ has a very simple structure — it is a \mathbb{P}^1 -fibration over a small resolution of X .

Proposition 4.4. *The map $M^+ \rightarrow X$ factors through a composition $M^+ \xrightarrow{\mu_+} X^+ \xrightarrow{\sigma_+} X$, where the map $\mu_+ : M^+ \rightarrow X^+$ is a \mathbb{P}^1 -fibration and $\sigma_+ : X^+ \rightarrow X$ is a small resolution of singularities. The restriction of the map μ_+ to the fiber $M_{x_i}^+ = \tilde{\Sigma}_i^-$ coincides with the projection $\tilde{\Sigma}_i^- \rightarrow \mathbb{P}^1$. The curve $C_i = \mu_+(\tilde{\Sigma}_i^-) \cong \mathbb{P}^1$ is the exceptional locus of X^+ over $x_i \in X$.*

Proof: We apply to M^+ relative Minimal Model Program over X , see [Na]. Since the relative MMP commutes with the base change, let us first look at $M^+ \setminus \bigsqcup \tilde{\Sigma}_i^-$ which is the preimage of X_{sm} . The map $M^+ \setminus \bigsqcup \tilde{\Sigma}_i^- \rightarrow X_{sm}$ is a \mathbb{P}^1 -fibration, so its relative Picard group is \mathbb{Z} , and the relative canonical class is ample, hence the first (and the last) step of the MMP for $M^+ \setminus \bigsqcup \tilde{\Sigma}_i^-$ is the contraction $M^+ \setminus \bigsqcup \tilde{\Sigma}_i^- \rightarrow X_{sm}$.

Now consider what happens over an analytic neighborhood of singular points. Let $x = x_i$ be one of singular points. Consider an analytic neighborhood U of x in X and its preimage $M_U^+ \subset M^+$. Then the relative (over U) effective cone of M_U^+ is generated by curves in the special fiber $M_x^+ = \tilde{\Sigma}^-$, that is by the (-1) -curve L and by the fiber of the projection $\tilde{\Sigma}^- \rightarrow \mathbb{P}^1$. By Lemma 4.3 the canonical class $K_{M^+/X}$ restricts to $\tilde{\Sigma}^-$ as $-h - l$, hence L is K -positive, while the fiber is K -negative. Hence the first step in MMP is the contraction of the ray generated by the fiber of the projection $\tilde{\Sigma}^- \rightarrow \mathbb{P}^1$. By MMP this contraction should be either

- (1) a flip, or
- (2) a divisorial contraction, or
- (3) a conic bundle.

By a result of Kawamata [Ka] the case of a flip is impossible, since the center of a flip in dimension 4 is always a \mathbb{P}^2 , while in our case the only compact surface in M_U^+ is $M_x^+ = \tilde{\Sigma}^-$ which is a Hirzebruch surface F_1 . Similarly, a divisorial contraction is impossible, since then the first step of MMP on $M_U^+ \setminus \tilde{\Sigma}_i^-$ would also be a divisorial contraction, while as we have shown above it is a \mathbb{P}^1 -fibration.

Thus the first step of MMP for M_U^+ is a conic bundle $M_U^+ \rightarrow X_U^+$. Once again, over $U \setminus \{x\}$ this conic bundle should coincide with the \mathbb{P}^1 -fibration $M^+ \setminus \bigsqcup \tilde{\Sigma}_i^- \rightarrow X_{sm}$, hence gluing all these conic bundles for all singular points x_i , we obtain a global conic bundle structure on M^+ , that is a global map $\mu_+ : M^+ \rightarrow X^+$ for some Moishezon variety X^+ . Now we apply [Te] and conclude that X^+ is necessarily smooth, hence X^+ is a resolution of singularities of X . Further, the restriction of the map μ_+ to $\tilde{\Sigma}_i^-$ is a conic bundle which contracts all the fibers of the projection $\tilde{\Sigma}_i^- \rightarrow \mathbb{P}^1$, hence the fiber of X_+ over x_i is the image C_i of $\tilde{\Sigma}_i^-$. Since X^+ is smooth and the fiber of X^+ over x_i is $C_i \cong \mathbb{P}^1$, the map $\sigma_+ : X^+ \rightarrow X$ is a small resolution of singularities. So, it remains to check that $\mu_+ : M^+ \rightarrow X^+$ is a \mathbb{P}^1 -fibration.

Since we already know that μ_+ is a conic bundle, we should check that its degeneration locus is empty. But the degeneration locus of a conic bundle is a divisor, while $M^+ \rightarrow X^+$ is nondegenerate over the complement $X_{sm} = X^+ \setminus (\bigsqcup C_i)$ of a finite number of curves, hence the degeneration locus is empty. \square

Denoting $f_+ = f \circ \sigma_+$, $\rho_+ = f_+ \circ \mu_+$, we obtain a commutative diagram

$$\begin{array}{ccc}
 & \widetilde{M} & \\
 \xi \swarrow & & \searrow \xi_+ \\
 M & \overset{\text{flip}}{\dashrightarrow} & M^+ \\
 \rho \searrow & & \swarrow \rho_+ \\
 & Y & \\
 \mu \swarrow & & \searrow \mu_+ \\
 X & \xleftarrow{\sigma_+} & X^+ \\
 & f \swarrow & \nwarrow f_+
 \end{array} \tag{7}$$

Since the map $\mu_+ : \mathcal{D}^b(M^+) \rightarrow \mathcal{D}^b(X^+)$ is a \mathbb{P}^1 -fibration, the functor $\mu_+^* : \mathcal{D}^b(X^+) \rightarrow \mathcal{D}^b(M^+)$ is fully faithful. Composing with the functor given by the flip we obtain

Corollary 4.5. *The functor $\xi_* \xi_+^* \mu_+^* : \mathcal{D}^b(X^+) \rightarrow \mathcal{D}^b(M)$ is fully faithful.*

Thus we have constructed all the required components in $\mathcal{D}^b(M)$. It remains to check that they generate the whole category. This is done in the next section.

5. DERIVED CATEGORY OF M^+

As it was shown in the previous section, M^+ is a \mathbb{P}^1 -bundle over X^+ . Locally in the étale topology this can be represented as a projectivization of a rank 2 vector bundle. In general, these local bundles do not glue into a global vector bundle, however their local endomorphism algebras glue and give a sheaf of Azumaya algebras on X^+ . We denote this sheaf by \mathcal{B}^+ , it is defined by the \mathbb{P}^1 -fibration up to a Morita equivalence. The following result of Bernardara [Be] describes the derived category of M^+ .

Proposition 5.1 ([Be]). *There is a semiorthogonal decomposition*

$$\mathcal{D}^b(M^+) = \langle \mathcal{D}^b(X^+), \mathcal{D}^b(X^+, \mathcal{B}^+) \rangle.$$

So, to prove Theorem 1.1 it remains to check that the functor $\xi_* \xi_+^*$ takes $\mathcal{D}^b(X^+, \mathcal{B}^+)$ to $\Phi(\mathcal{D}^b(Y, \mathcal{B}_0))$. For this we construct analogues of the bundles \mathcal{S}_k on M^+ . Recall that $E_i \cong \Sigma_i^+ \times L_i \cong \mathbb{P}^2 \times \mathbb{P}^1$.

Lemma 5.2. *There are vector bundles \mathcal{R}_k of rank 2 on M^+ such that there is a short exact sequence*

$$0 \rightarrow \xi^* \mathcal{S}_k \rightarrow \xi_+^* \mathcal{R}_k \rightarrow \oplus \mathcal{O}_{E_i}(0, -1) \rightarrow 0. \tag{8}$$

Proof: By Lemma 3.7 we have an isomorphism $(\xi^* \mathcal{S}_k^\vee)_{|E_i} \cong \mathcal{O}_{E_i} \oplus \mathcal{O}_{E_i}(-1, 0)$. Consider the composition $\xi^* \mathcal{S}_k^\vee \rightarrow (\xi^* \mathcal{S}_k^\vee)_{|E_i} \rightarrow \mathcal{O}_{E_i}(-1, 0)$, where the second map is the unique projection. This map is clearly surjective. Denote the kernel of the sum of these maps over i by F , so that we have an exact triple

$$0 \rightarrow F \rightarrow \xi^* \mathcal{S}_k^\vee \rightarrow \oplus \mathcal{O}_{E_i}(-1, 0) \rightarrow 0.$$

Let us check that F is a pullback of a vector bundle from M^+ . Since $\xi_+ : \widetilde{M} \rightarrow M^+$ is a smooth blowup, it suffices to check that $F_{|E_i}$ is a pullback of a vector bundle from L_i . Let us restrict the above exact sequence to E_i . Since $\mathcal{N}_{E_i/\widetilde{M}} \cong \mathcal{O}_{E_i}(-1, -1)$ we obtain an exact sequence

$$0 \rightarrow \mathcal{O}_{E_i}(0, 1) \rightarrow F_{|E_i} \rightarrow \mathcal{O}_{E_i} \oplus \mathcal{O}_{E_i}(-1, 0) \rightarrow \mathcal{O}_{E_i}(-1, 0) \rightarrow 0.$$

The last map is the projection to the second summand, hence we have an exact triple

$$0 \rightarrow \mathcal{O}_{E_i}(0, 1) \rightarrow F_{|E_i} \rightarrow \mathcal{O}_{E_i} \rightarrow 0.$$

Since $\text{Ext}^1(\mathcal{O}_{E_i}, \mathcal{O}_{E_i}(0, 1)) \cong H^1(E_i, \mathcal{O}_{E_i}(0, 1)) = 0$, we see that $F|_{E_i} \cong \mathcal{O}_{E_i} \oplus \mathcal{O}_{E_i}(0, 1)$. So, $F|_{E_i}$ is a pullback of $\mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}(1)$, hence F is a pullback of a vector bundle on M^+ which restricts to L_i as $\mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}(1)$. Now we define \mathcal{R}_k as the dual of this vector bundle. So, by definition we have the following exact sequence

$$0 \rightarrow \xi_+^* \mathcal{R}_k^\vee \rightarrow \xi_+^* \mathcal{S}_k^\vee \rightarrow \oplus \mathcal{O}_{E_i}(-1, 0) \rightarrow 0. \quad (9)$$

Dualizing this sequence and taking into account that

$$\text{RHom}(\mathcal{O}_{E_i}(-1, 0), \mathcal{O}_{\widetilde{M}}) \cong \mathcal{O}_{E_i}(1, 0) \otimes \mathcal{N}_{E_i/\widetilde{M}}[-1] \cong \mathcal{O}_{E_i}(0, -1)[-1]$$

we obtain (8). \square

The bundles \mathcal{R}_k enjoy a lot of interesting properties.

Lemma 5.3. *We have $(\rho_+)_* \mathcal{R}_k^\vee = 0$.*

Proof: Indeed, using commutativity of (7) we deduce

$$(\rho_+)_* \mathcal{R}_k^\vee = (\rho_+)_*(\xi_+)_* \xi_+^* \mathcal{R}_k^\vee = \rho_* \xi_* \xi_+^* \mathcal{R}_k^\vee.$$

Applying the functor $\rho_* \xi_*$ to (9) we obtain a triangle

$$\rho_* \xi_* \xi_+^* \mathcal{R}_k^\vee \rightarrow \rho_* \xi_* \xi_+^* \mathcal{S}_k^\vee \rightarrow \oplus \rho_* \xi_* \mathcal{O}_{E_i}(-1, 0).$$

The second term equals to $\rho_* \mathcal{S}_k^\vee$ which is zero by Corollary 3.4. Since $\rho \circ \xi$ contracts E_i to the point y_i , the third term is $\oplus H^\bullet(E_i, \mathcal{O}_{E_i}(-1, 0)) \otimes \mathcal{O}_{y_i}$, so it is also zero. Hence $(\rho_+)_* \mathcal{R}_k^\vee = 0$. \square

Proposition 5.4. *The bundle \mathcal{R}_k restricts to any fiber of $\mu_+ : M^+ \rightarrow X^+$ as $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Moreover*

$$\mathcal{R}_k|_{\widetilde{\Sigma}_i^-} \cong \mathcal{O}_{\widetilde{\Sigma}_i^-}(h) \oplus \mathcal{O}_{\widetilde{\Sigma}_i^-}(l). \quad (10)$$

Proof: We are going to prove instead that \mathcal{R}_k^\vee restricts to all fibers as $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. For fibers of $\mu^+ : M^+ \setminus \bigsqcup \widetilde{\Sigma}_i^- \rightarrow X^+ \setminus \bigsqcup C_i$ this follows from Lemma 5.3. So it remains to investigate the restriction of \mathcal{R}_k^\vee to $\widetilde{\Sigma}_i^-$. For this we restrict (9):

$$0 \rightarrow (\mathcal{R}_k^\vee)|_{\widetilde{\Sigma}_i^-} \rightarrow \mathcal{O}_{\widetilde{\Sigma}_i^-} \oplus \mathcal{O}_{\widetilde{\Sigma}_i^-}(-h) \rightarrow \mathcal{O}_{L_i} \rightarrow 0.$$

It follows that either (10) holds, or $(\mathcal{R}_k^\vee)|_{\widetilde{\Sigma}_i^-} \cong \mathcal{O}_{\widetilde{\Sigma}_i^-}(-h-l) \oplus \mathcal{O}_{\widetilde{\Sigma}_i^-}$. In the former case we are done since both $\mathcal{O}_{\widetilde{\Sigma}_i^-}(h)$ and $\mathcal{O}_{\widetilde{\Sigma}_i^-}(l)$ restrict as $\mathcal{O}(1)$ to any fiber of $\widetilde{\Sigma}_i^-$ over C_i . Let us check that the case $(\mathcal{R}_k^\vee)|_{\widetilde{\Sigma}_i^-} \cong \mathcal{O}_{\widetilde{\Sigma}_i^-}(-h-l) \oplus \mathcal{O}_{\widetilde{\Sigma}_i^-}$ is impossible.

For this we note that by Lemma 5.3

$$0 = \text{Ext}^\bullet(\mathcal{O}_{y_i}, (\rho_+)_* \mathcal{R}_k^\vee) = \text{Ext}^\bullet(\rho_+^* \mathcal{O}_{y_i}, \mathcal{R}_k^\vee).$$

On the other hand, the cohomology sheaves $\mathcal{H}^l = \mathcal{H}^l(\rho_+^* \mathcal{O}_{y_i})$ are supported on $\widetilde{\Sigma}_i^-$. Moreover, $\mathcal{H}^l = 0$ for $l > 0$ (since ρ_+^* is right exact) and $\mathcal{H}^0 \cong \mathcal{O}_{\widetilde{\Sigma}_i^-}$. Consider the spectral sequence

$$\text{Ext}^q(\mathcal{H}^p, \mathcal{R}_k^\vee) \Rightarrow \text{Ext}^{q-p}(\rho_+^* \mathcal{O}_{y_i}, \mathcal{R}_k^\vee) = 0.$$

Note that by Serre duality on M^+ we have

$$\text{Ext}^q(\mathcal{H}^i, \mathcal{R}_k^\vee) \cong \text{Ext}^{4-q}(\mathcal{R}_k^\vee, \mathcal{H}^i \otimes \omega_{M^+})^\vee \cong H^{4-q}(M^+, \mathcal{H}^i \otimes \mathcal{R}_k \otimes \omega_{M^+})^\vee.$$

The RHS vanishes for $q \notin \{2, 3, 4\}$ since the sheaf \mathcal{H}^i is supported on $\tilde{\Sigma}_i^-$ and $\dim \tilde{\Sigma}_i^- = 2$. Hence the line $q = 3$ doesn't change in the spectral sequence. But if $(\mathcal{R}_k^\vee)_{|\tilde{\Sigma}_i^-} \cong \mathcal{O}_{\tilde{\Sigma}_i^-}(-h-l) \oplus \mathcal{O}_{\tilde{\Sigma}_i^-}$ then

$$\begin{aligned} \text{Ext}^3(\mathcal{H}^0, \mathcal{R}_k^\vee) &\cong H^1(M^+, \mathcal{O}_{\tilde{\Sigma}_i^-} \otimes \mathcal{R}_k \otimes \omega_{M^+})^\vee \cong \\ &\cong H^1(\tilde{\Sigma}_i^-, (\mathcal{O}_{\tilde{\Sigma}_i^-}(h+l) \oplus \mathcal{O}_{\tilde{\Sigma}_i^-}) \otimes \mathcal{O}_{\tilde{\Sigma}_i^-}(-h-l)) \cong \\ &\cong H^1(\tilde{\Sigma}_i^-, \mathcal{O}_{\tilde{\Sigma}_i^-} \oplus \mathcal{O}_{\tilde{\Sigma}_i^-}(-h-l)) \cong \mathbf{k} \end{aligned}$$

gives a nontrivial element in $\text{Ext}^3(\rho_+^* \mathcal{O}_{y_i}, \mathcal{R}_k^\vee)$, which is impossible. \square

Now let us describe the Morita equivalence class of \mathcal{B}^+ in terms of \mathcal{R}_k .

Proposition 5.5. *There is an Azumaya algebra \mathcal{B}' on X^+ which is Morita-equivalent to \mathcal{B}^+ and such that we have $\mathcal{E}nd(\mathcal{R}_0) \cong \mu_+^* \mathcal{B}'$, an isomorphism of algebras.*

Proof: Note that $\mathcal{E}nd(\mathcal{R}_0)$ restricts trivially to all fibers of the \mathbb{P}^1 -fibration $M^+ \rightarrow X^+$. Therefore $\mathcal{E}nd(\mathcal{R}_0) \cong \mu_+^* \mathcal{B}'$, where $\mathcal{B}' = (\mu_+)_* \mathcal{E}nd(\mathcal{R}_0)$. So, we have to check that \mathcal{B}' is Morita-equivalent to \mathcal{B}^+ . Consider the sheaf $\mathcal{F} = (\mu_+)_* \mathcal{R}_0$. Let us check that it has a structure of a $\mathcal{B}' - \mathcal{B}^+$ -bimodule and gives a Morita-equivalence.

Choose an étale covering $u : U \rightarrow X^+$ such that $M_U^+ = M^+ \times_{X^+} U$ is a projectivization of a vector bundle \mathcal{E} on U . Let $u : M_U^+ \rightarrow M^+$ and $\mu_U : M_U^+ \rightarrow U$ denote the projections:

$$\begin{array}{ccc} M_U^+ & \xrightarrow{u} & M^+ \\ \mu_U \downarrow & & \downarrow \mu_+ \\ U & \xrightarrow{u} & X^+ \end{array}$$

Note that the bundle $u^* \mathcal{R}_0 \otimes \mathcal{O}_{M_U^+/U}(-1)$ restricts trivially to all fibers of M_U^+ over U . Hence there exists a bundle \mathcal{R}_U on U such that

$$u^* \mathcal{R}_0 \cong \mu_U^* \mathcal{R}_U \otimes \mathcal{O}_{M_U^+/U}(1).$$

Therefore $u^*(\mu_+)_* \mathcal{R}_0 \cong (\mu_U)_* u^* \mathcal{R}_0 \cong \mathcal{R}_U \otimes \mathcal{E}^*$. On the other hand,

$$\begin{aligned} u^* \mathcal{B}' &\cong u^*(\mu_+)_* \mathcal{E}nd(\mathcal{R}_0) \cong (\mu_U)_* u^* \mathcal{E}nd(\mathcal{R}_0) \cong \\ &\cong (\mu_U)_* \mathcal{E}nd(\mu_U^* \mathcal{R}_U \otimes \mathcal{O}_{M_U^+/U}(1)) \cong (\mu_U)_* \mu_U^* \mathcal{E}nd(\mathcal{R}_U) \cong \mathcal{E}nd(\mathcal{R}_U), \end{aligned}$$

while $u^* \mathcal{B}^+ \cong \mathcal{E}nd(\mathcal{E})$ by definition of \mathcal{B}^+ . We see that $u^*(\mu_+)_* \mathcal{R}_0$ gives a Morita equivalence between $u^* \mathcal{B}'$ and $u^* \mathcal{B}^+$. Since $u : U \rightarrow X^+$ is an étale covering, we conclude that \mathcal{B}' and \mathcal{B}^+ are Morita-equivalent as well. \square

From now on we replace \mathcal{B}^+ by a Morita-equivalent Azumaya algebra \mathcal{B}' . This change does not spoil the decomposition of Theorem 5.1, it still holds after the change. On the other hand, after the change we have

$$\mathcal{E}nd(\mathcal{R}_0) \cong \mu_+^* \mathcal{B}^+. \quad (11)$$

Now we can relate (this new) \mathcal{B}^+ and \mathcal{B}_0 .

Lemma 5.6. *We have $(f_+)_* \mathcal{B}^+ \cong \mathcal{B}_0$, an isomorphism of sheaves of algebras.*

Proof: First of all

$$(f_+)_* \mathcal{B}^+ \cong (f_+)_*(\mu_+)_*(\xi_+)_* \xi_+^* \mu_+^* \mathcal{B}^+ \cong f_* \mu_* \xi_* \xi_+^* \mathcal{E}nd(\mathcal{R}_0) \cong \rho_* \xi_* \xi_+^* \mathcal{E}nd(\mathcal{R}_0).$$

On the other hand, tensoring (8) with $\xi^* \mathcal{S}_0^\vee$ and (9) with $\xi_+^* \mathcal{R}_0$ we obtain sequences

$$0 \rightarrow \xi^* \mathcal{E}nd(\mathcal{S}_0) \rightarrow \xi^* \mathcal{S}_0^\vee \otimes \xi_+^* \mathcal{R}_0 \rightarrow \oplus (\mathcal{O}_{E_i}(0, -1) \oplus \mathcal{O}_{E_i}(-1, -1)) \rightarrow 0,$$

$$0 \rightarrow \xi_+^* \mathcal{E}nd(\mathcal{R}_0) \rightarrow \xi^* \mathcal{S}_0^\vee \otimes \xi_+^* \mathcal{R}_0 \rightarrow \oplus(\mathcal{O}_{E_i}(-1, 0) \oplus \mathcal{O}_{E_i}(-1, -1)) \rightarrow 0.$$

Pushing these sequences along $\rho \circ \xi$ and noting that E_i is contracted to a point we conclude that

$$\rho_* \xi_* \xi_+^* \mathcal{E}nd(\mathcal{R}_0) \cong \rho_* \xi_* (\xi^* \mathcal{S}_0^\vee \otimes \xi_+^* \mathcal{R}_0) \cong \rho_* \xi_* \xi^* \mathcal{E}nd(\mathcal{S}_0).$$

On the other hand $\rho_* \xi_* \xi^* \mathcal{E}nd(\mathcal{S}_0) \cong \rho_* \mathcal{E}nd(\mathcal{S}_0) \cong \mathcal{B}_0$ (see 3.5 for the last isomorphism). \square

The isomorphism $(f_+)_* \mathcal{B}^+ \cong \mathcal{B}_0$ gives by adjunction a morphism $f_+^* \mathcal{B}_0 \rightarrow \mathcal{B}^+$ which equips \mathcal{B}^+ with a structure of a \mathcal{B}_0 -module. Now we define the functor $\mathcal{D}^b(Y, \mathcal{B}_0) \rightarrow \mathcal{D}^b(X^+, \mathcal{B}^+)$ by $F \mapsto F \otimes_{\mathcal{B}_0} \mathcal{B}^+$. The right adjoint functor then is $(f_+)_* : \mathcal{D}^b(X^+, \mathcal{B}^+) \rightarrow \mathcal{D}^b(Y, \mathcal{B}_0)$. Their composition takes F to

$$(f_+)_*(f_+^* F \otimes_{\mathcal{B}_0} \mathcal{B}^+) \cong F \otimes_{\mathcal{B}_0} (f_+)_* \mathcal{B}^+ = F \otimes_{\mathcal{B}_0} \mathcal{B}_0 = F.$$

This implies that this functor is fully faithful, and the orthogonal to its image consists of all objects G such that $(f_+)_* G = 0$. Since f_+ is the contraction of $(-1, -1)$ curves C_i , any object G in $\mathcal{D}^b(X^+)$ such that $(f_+)_* G = 0$ is a complex with cohomology being direct sums of $\mathcal{O}_{C_i}(-1)$. Since \mathcal{B}^+ is an Azumaya algebra, the forgetful functor $\mathcal{D}^b(X^+, \mathcal{B}^+) \rightarrow \mathcal{D}^b(X^+)$ commutes with sheaf cohomology, hence each of these direct sums of $\mathcal{O}_{C_i}(-1)$ should be a \mathcal{B}^+ -module. So, it remains to check that there is no such \mathcal{B}^+ -modules.

For this we note that (10) implies $\mathcal{E}nd(\mathcal{R}_0)|_{\tilde{\Sigma}_i^-} \cong \mathcal{O}_{\tilde{\Sigma}_i^-} \oplus \mathcal{O}_{\tilde{\Sigma}_i^-}(h-l) \oplus \mathcal{O}_{\tilde{\Sigma}_i^-}(l-h) \oplus \mathcal{O}_{\tilde{\Sigma}_i^-}$. Therefore we have $\mathcal{B}_{|C_i}^+ \cong \mathcal{O}_{C_i} \oplus \mathcal{O}_{C_i}(1) \oplus \mathcal{O}_{C_i}(-1) \oplus \mathcal{O}_{C_i} \cong \mathcal{E}nd(\mathcal{O}_{C_i} \oplus \mathcal{O}_{C_i}(-1))$, hence the category of \mathcal{B}^+ -modules supported on C_i is equivalent to the category of sheaves on C_i , the equivalence taking a sheaf F to $F \otimes (\mathcal{O}_{C_i} \oplus \mathcal{O}_{C_i}(-1)) \cong F \oplus F(-1)$. It is clear that if $F \oplus F(-1)$ is a direct sum of $\mathcal{O}(-1)$ then $F = 0$. Thus we have checked that $\mathcal{D}^b(Y, \mathcal{B}_0) \cong \mathcal{D}^b(X^+, \mathcal{B}^+)$.

Finally, we compute the composition of the equivalence $\mathcal{D}^b(Y, \mathcal{B}_0) \cong \mathcal{D}^b(X^+, \mathcal{B}^+)$, and embeddings $\mathcal{D}^b(X^+, \mathcal{B}^+) \rightarrow \mathcal{D}^b(M^+) \rightarrow \mathcal{D}^b(M)$. It acts on $F \in \mathcal{D}^b(Y, \mathcal{B}_0)$ as

$$F \mapsto \xi_* \xi_+^* (\mu_+^*(f_+^* F \otimes_{\mathcal{B}_0} \mathcal{B}^+) \otimes_{\mathcal{B}^+} \mathcal{R}_0) \cong \xi_* \xi_+^* (\rho_+^* F \otimes_{\mathcal{B}_0} \mathcal{R}_0) \cong \xi_* (\xi_+^* \rho_+^* F \otimes_{\mathcal{B}_0} \xi_+^* \mathcal{R}_0).$$

Note that $\xi_+^* \rho_+^* F \cong \xi^* \rho^* F$, so tensoring (8) by it we obtain

$$\xi^* \rho^* F \otimes_{\mathcal{B}_0} \xi^* \mathcal{S}_0 \rightarrow \xi^* \rho^* F \otimes_{\mathcal{B}_0} \xi_+^* \mathcal{R}_0 \rightarrow \oplus(\xi^* \rho^* F \otimes_{\mathcal{B}_0} \mathcal{O}_{E_i}(0, -1)).$$

Since $\xi_* \mathcal{O}_{E_i}(0, -1) = 0$, applying ξ_* we see that

$$\xi_*(\xi^* \rho^* F \otimes_{\mathcal{B}_0} \xi^* \mathcal{S}_0) \cong \rho^* F \otimes_{\mathcal{B}_0} \mathcal{S}_0,$$

which gives the required isomorphism of functors. This proves Theorem 1.1.

6. CONCLUDING REMARKS AND FURTHER QUESTIONS

Remark 6.1. There is another way of proving Theorem 1.1, avoiding use of Moishezon varieties. For this one has to perform another birational modification of M . First, consider the blowup $M' \rightarrow M$ in points $P_i = \Sigma_i^+ \cap \Sigma_i^-$. Let $E'_i \cong \mathbb{P}^3$ be the exceptional divisors of this blowup. Then the proper preimages of the planes Σ_i^+ and Σ_i^- are Hirzebruch surfaces $\tilde{\Sigma}_i^-, \tilde{\Sigma}_i^+ \subset M'$ which do not intersect. Moreover, the (-1) -curves L_i^\pm on $\tilde{\Sigma}_i^\pm$ are skew-lines in E'_i . One can check that the normal bundle to $\tilde{\Sigma}_i^\pm$ in M' restricts as $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ to any fiber of $\tilde{\Sigma}_i^\pm$ over \mathbb{P}^1 . Hence one can make a (relative over \mathbb{P}^1) flop in all $2N$ surfaces $\tilde{\Sigma}_i^\pm$ simultaneously. We will obtain an algebraic variety M'' over X . The special fibers M''_{x_i} will coincide with blowups E''_i of E'_i in the lines L_i^\pm (and each of the surfaces $\tilde{\Sigma}_i^\pm$ will be replaced by $\mathbb{P}^1 \times \mathbb{P}^1$, coinciding with the exceptional divisor of E''_i over L_i^\pm). Then by the same arguments as in Proposition 4.4 one can show that the map $M'' \rightarrow X$ factors as a \mathbb{P}^1 -fibration $M'' \rightarrow X'$, where X' is the blowup of X in all points x_i . Then a careful analysis of the relation of the categories $\mathcal{D}^b(M'')$ and $\mathcal{D}^b(M)$ allows to prove Theorem 1.1.

An interesting question for the further investigation is to describe the category $\mathcal{D}^b(M)$ without restrictions on the dimension of Y . A natural approach would be to consider first the universal family of quadrics and then to apply a base change argument (see [K1]) to obtain a decomposition in general case. Unfortunately, the approach of this paper does not work in this general setup because of the following effect — assume for simplicity that $D_3 = \emptyset$, but $\dim D_2 > 0$. Since the fibers of M over D_2 are the unions of two planes, we have an unramified double covering $\tilde{D}_2 \rightarrow D_2$. This covering in general is connected. Therefore, we cannot pick up one of the planes Σ^+ in all the fibers over D_2 and make a flip in them. However, the approach suggested in 6.1 may work, and I guess that in case $D_3 = \emptyset$ should work without big changes. As for the case of nonempty D_3 , a deeper analysis of the behavior of M over D_3 (and possibly more birational transformations) is required.

Another question which may prove interesting is investigation of the derived category of the relative scheme of lines (or other isotropic Grassmannians) of a family of quadrics of dimension bigger than 2.

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ALGEBRA SECTION, STEKLOV MATHEMATICAL INSTITUTE, 8 GUBKIN STR., MOSCOW 119991 RUSSIA

THE PONCELET LABORATORY, INDEPENDENT UNIVERSITY OF MOSCOW

E-mail address: akuznet@mi.ras.ru