Parametric Evaluations of the Rogers Ramanujan Continued Fraction

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Abstract

In this article with the help of the inverse function of the singular moduli we evaluate the Rogers Ranmanujan continued fraction and his first derivative.

1 Introductory Definitions and Formulas

For $|q| < 1$, the Rogers Ramanujan continued fraction (RRCF) (see [6]) is defined as

$$R(q) := \frac{q^{1/5}}{1+}\frac{q^1}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\cdots \tag{1}$$

We also define

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \tag{2}$$

$$f(-q) := \prod_{n=1}^{\infty}(1-q^n) = (q;q)_\infty \tag{3}$$

Ramanujan give the following relations which are very useful:

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \tag{4}$$

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)} \tag{5}$$

From the Theory of Elliptic Functions (see [6],[7],[10]),

$$K(x) = \int_0^{\pi/2}\frac{1}{\sqrt{1-x^2\sin(t)^2}}dt \tag{6}$$

is the Elliptic integral of the first kind. It is known that the inverse elliptic nome $k = k_r$, $k_r'^2 = 1 - k_r^2$ is the solution of the equation

$$\frac{K(k_r')}{K(k)} = \sqrt{r} \tag{7}$$

where $r \in \mathbf{R}_+^*$. When r is rational then the k_r are algebraic numbers.
We can also write the function f using elliptic functions. It holds (see [10]):

$$f(-q)^8 = \frac{2^{8/3}}{\pi^4} q^{-1/3} (k_r)^{2/3} (k_r')^{8/3} K(k_r)^4 \tag{8}$$

also holds

$$f(-q^2)^6 = \frac{2k_r k_r' K(k_r)^3}{\pi^3 q^{1/2}} \tag{9}$$

From [5] it is known that

$$R'(q) = 1/5 q^{-5/6} f(-q)^4 R(q) \sqrt[6]{R(q)^{-5} - 11 - R(q)^5} \tag{10}$$

Consider now for every $0 < x < 1$ the equation

$$x = k_r,$$

which, have solution

$$r = k^{(-1)}(x) \tag{11}$$

Hence for example

$$k^{(-1)}\left(\frac{1}{\sqrt{2}}\right) = 1$$

With the help of $k^{(-1)}$ function we evaluate the Rogers Ramanujan continued fraction.

2 Propositions

The relation between k_{25r} and k_r is (see [6] pg. 280):

$$k_r k_{25r} + k_r' k_{25r}' + 2 \cdot 4^{1/3} (k_r k_{25r} k_r' k_{25r}')^{1/3} = 1 \tag{12}$$

For to solve equation (12) we give the following

Proposition 1.
The solution of the equation

$$x^6 + x^3(-16 + 10x^2)w + 15x^4w^2 - 20x^3w^3 + 15x^2w^4 + x(10 - 16x^2)w^5 + w^6 = 0 \tag{13}$$

when we know the w is given by

$$\frac{y^{1/2}}{w^{1/2}} = \frac{w^{1/2}}{x^{1/2}} =$$

$$= \frac{1}{2}\sqrt{4 + \frac{2}{3}\left(\frac{L^{1/6}}{M^{1/6}} - 4\frac{M^{1/6}}{L^{1/6}}\right)^2} + \frac{1}{2}\sqrt{\frac{2}{3}}\left(\frac{L^{1/6}}{M^{1/6}} - 4\frac{M^{1/6}}{L^{1/6}}\right) \tag{14}$$

where

$$w = \sqrt{\frac{L(18+L)}{6(64+3L)}} < 1 \tag{15}$$

and

$$M = \frac{18+L}{64+3L} \tag{16}$$

If happens $x = k_r$ and $y = k_{25r}$, then $r = k^{(-1)}(x)$ and $w^2 = k_{25r}k_r$, $(w')^2 = k'_{25r}k'_r$.

Proof.

The relation (14) can be found using Mathematica. See also [6].

Proposition 2.

If $q = e^{-\pi\sqrt{r}}$ and

$$a = a_r = \left(\frac{k'_r}{k'_{25r}}\right)^2 \sqrt{\frac{k_r}{k_{25r}}} M_5(r)^{-3} \tag{17}$$

Then

$$a_r = R(q)^{-5} - 11 - R^5(q) \tag{18}$$

Where $M_5(r)$ is root of: $(5x-1)^5(1-x) = 256(k_r)^2(k'_r)^2 x$.

Proof.

Suppose that $N = n^2\mu$, where n is positive integer and μ is positive real then it holds that

$$K[n^2\mu] = M_n(\mu)K[\mu] \tag{19}$$

Where $K[\mu] = K(k_\mu)$

The following formula for $M_5(r)$ is known

$$(5M_5(r)-1)^5(1-M_5(r)) = 256(k_r)^2(k'_r)^2 M_5(r) \tag{20}$$

Thus if we use (5) and (8) and the above consequence of the Theory of Elliptic Functions, we get:

$$R^{-5}(q) - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)} = a = a_r$$

See also [4],[5].

3 The Main Theorem

From Proposition 2 and relation $w^2 = k_{25r}k_r$ we get

$$w^5 - k_r^2 w = \frac{k_r^3(k_r^2-1)}{a_r M_5(r)^3} \tag{21}$$

Combining (13) and (21), we get:

$$[-10k_r^4+26k_r^6+a_rM_5(r)^3k_r^6-16k_r^8]+[-k_r^3-6a_rM_5(r)^3k_r^3+k_r^5-6a_rM_5(r)^3k_r^5]w+$$
$$+[a_rM_5(r)^3k_r^2+15a_rM_5(r)k_r^4]w^2-20a_rM_5(r)^3k_r^3w^3+15a_rM_5(r)^3k_r^2w^4=0 \quad (22)$$

Solving with respect to $a_rM_5(r)^3$, we get

$$a_rM_5(r)^3=\frac{16k_r^6-26k_r^4-wk_r^3+10k_r^2+wk_r}{k_r^4-6k_r^3w-20k_r^3w+15w^2k_r^2-6k_rw+15w^4+w^2} \quad (23)$$

Also we have

$$\frac{K(k_{25r})}{K(k_r)}=M_5(r)=\frac{1}{m}=\left(\sqrt{\frac{k_{25r}}{k_r}}+\sqrt{\frac{k'_{25r}}{k'_r}}-\sqrt{\frac{k_{25r}k'_{25r}}{k_rk'_r}}\right)^{-1}$$
$$=\left(\frac{w}{k_r}+\frac{w'}{k'_r}-\frac{ww'}{k_rk'_r}\right)^{-1}$$

The above equalities follow from ([6] pg. 280 Entry 13-xii) and the definition of w. Note that m is the multiplier.
Hence for given $0<w<1$ we find $L\in\mathbf{R}$ and we get the following parametric evaluation for the Rogers Ramanujan continued fraction

$$R\left(e^{-\pi\sqrt{r(L)}}\right)^{-5}-11-R\left(e^{-\pi\sqrt{r(L)}}\right)^{5}=a_r=$$
$$=\frac{16k_r^6-26k_r^4-wk_r^3+10k_r^2+wk_r}{k_r^4-6k_r^3w-20k_rw^3+15w^2k_r^2-6k_rw+15w^4+w^2}\left(\frac{w}{k_r}+\frac{w'}{k'_r}-\frac{ww'}{k_rk'_r}\right)^{3} \quad (24)$$

Thus for a given w we find L and M from (15) and (16). Setting the values of M, L, w in (14) we get the values of x and y (see Proposition 1). Hence from (24) if we find $k^{(-1)}(x)=r$ we know $R(e^{-\pi\sqrt{r}})$. The clearer result is:

Main Theorem.
When w is a given real number, we can find x from equation (14). Then for the Rogers Ramanujan continued fraction holds

$$R\left(e^{-\pi\sqrt{k^{(-1)}(x)}}\right)^{-5}-11-R\left(e^{-\pi\sqrt{k^{(-1)}(x)}}\right)^{5}=a_r=$$
$$=\frac{16x^6-26x^4-wx^3+10x^2+wx}{x^4-6x^3w-20xw^3+15w^2x^2-6xw+15w^4+w^2}\times$$
$$\times\left(\frac{w}{x}+\frac{w'}{\sqrt{1-x^2}}-\frac{ww'}{x\sqrt{1-x^2}}\right)^{3} \quad (25)$$

Note. In the case of $x = k_r$, then $k^{(-1)}(x) = r$ and we have the clasical evaluation with k_{25r} (see [12]).

Theorem 1. (The first derivative)

$$R'\left(e^{-\pi\sqrt{k^{(-1)}(x)}}\right) = \frac{2^{4/3}x^{1/2}(1-x^2)}{5w^{1/6}w'^{2/3}}\left(\frac{w}{x}+\frac{w'}{\sqrt{1-x^2}}-\frac{ww'}{x\sqrt{1-x^2}}\right)^{1/2}\times$$

$$\times R\left(e^{-\pi\sqrt{k^{(-1)}(x)}}\right)\frac{K^2(x)e^{\pi\sqrt{k^{(-1)}(x)}}}{\pi^2} \tag{26}$$

Proof.
Combining (8) and (10) and Proposition 2 we get the proof.

We see how the function $k^{(-1)}(x)$ plays the same role in other continued fractions. Here we consider also the Ramanujan's Cubic fraction (see [4]), which is completely solvable using k_r.
Define the function:

$$G(x) = \frac{x}{\sqrt{2\sqrt{x}-3x+2x^{3/2}-2\sqrt{x}\sqrt{1-3\sqrt{x}+4x-3x^{3/2}+x^2}}} \tag{27}$$

Set for a given $0 < w_3 < 1$

$$x = G(w_3) \tag{28}$$

Then as in Main Theorem, for the Cubic continued fraction $V(q)$, holds (see [4]):

$$t = V\left(e^{-\pi\sqrt{k^{(-1)}(x)}}\right) = \frac{(1-x^2)^{1/3}w_3^{1/4}}{2^{1/3}x^{1/3}(1-\sqrt{w_3})} \tag{29}$$

Observe here that again we only have to know $k^{(-1)}(x)$.
If happens $x = k_r$, for a certain r, then

$$k_{9r} = \frac{w_3}{k_r} \tag{30}$$

and if we set

$$T = \sqrt{1-8V(q)^3}, \tag{31}$$

then holds

$$(k_r)^2 = x^2 = \frac{(1-T)(3+T)^3}{(1+T)(3-T)^3} \tag{32}$$

which is solvable always in radicals quartic equation. When we know w_3 we can find $k_r = x$ from $x = G(w_3)$ and hence t.
The inverse also holds: If we know $t = V(q)$ we can find T and hence $k_r = x$.

The w_3 can be find by the degree 3 modular equation which is always solvable in radicals:

$$\sqrt{k_r k_r'} + \sqrt{k_{9r} k_{9r}'} = 1$$

Let now

$$V(q) = z \Leftrightarrow q = V^{(-1)}(z) \tag{33}$$

if

$$V_i(t) := \sqrt{\frac{1-\sqrt{1-8t^3}}{1+\sqrt{1-8t^3}}\left(\frac{3+\sqrt{1-8t^3}}{3-\sqrt{1-8t^3}}\right)^3} \tag{34}$$

then

$$V_i(V(e^{-\pi\sqrt{x}})) = k_x \tag{35}$$

or

$$V(e^{-\pi\sqrt{r}}) = V_i^{(-1)}(k_r)$$

$$V(e^{-\pi\sqrt{k^{(-1)}(x)}}) = V_i^{(-1)}(x)$$

or

$$e^{-\pi\sqrt{k^{(-1)}(x)}} = V^{(-1)}(V_i^{(-1)}(x)) = (V_i \circ V)^{(-1)}(x)$$

and

$$k^{(-1)}(V_i(V(q))) = \frac{1}{\pi^2}\log(q)^2 = r \tag{36}$$

Setting now values into (36) we get values for $k^{(-1)}(.)$. The function $V_i(.)$ is an algebraic function.

4 Some Evaluations of the Rogers Ramanujan Continued Fraction

Note that if $x = k_r$, $r \in \mathbf{Q}$ then we have the classical evaluations with k_r and k_{25r}.

Evaluations.

1)

$$R(e^{-2\pi}) = \frac{-1}{2} - \frac{\sqrt{5}}{2} + \sqrt{\frac{5+\sqrt{5}}{2}}$$

$$R'(e^{-2\pi}) = 8\sqrt{\frac{2}{5}\left(9+5\sqrt{5}-2\sqrt{50+22\sqrt{5}}\right)}\frac{e^{2\pi}}{\pi^3}\Gamma\left(\frac{5}{4}\right)^4$$

2) Assume that $x = \frac{1}{\sqrt{2}}$, hence $k^{(-1)}\left(\frac{1}{\sqrt{2}}\right) = 1$. From (16) which for this x can be solved in radicals, with respect to w, we find

$$w = \frac{\sqrt{2}}{4}\left(\sqrt{5}-1\right) - \frac{1}{2}\sqrt{7\sqrt{5}-15}$$

Hence from

$$w' = \sqrt{\sqrt{1-\frac{w^4}{x^2}}\sqrt{1-x^2}}$$

we get

$$w' = \left(\frac{1+21\sqrt{-30+14\sqrt{5}}-9\sqrt{-150+70\sqrt{5}}}{\sqrt{2}}\right)^{1/4}$$

Setting these values to (25) we get the value of a_r and then $R(q)$ in radicals. The result is

$$R(e^{-\pi})^{-5}-11-R(e^{-\pi})^5 = -\frac{1}{8}\left(3+\sqrt{5}-\sqrt{-30+14\sqrt{5}}\right)[1-\sqrt{5}+\sqrt{-30+14\sqrt{5}}+$$

$$+2^{3/8}\left(-3+\sqrt{5}-\sqrt{-30+14\sqrt{5}}\right)\left(1+21\sqrt{-30+14\sqrt{5}}-9\sqrt{-150+70\sqrt{5}}\right)^{1/4}]^3\times$$

$$\times[\sqrt{-1574+704\sqrt{5}}-655\sqrt{-30+14\sqrt{5}}+293\sqrt{-150+70\sqrt{5}}]^{-1}$$

3)Set $w=1/64$ and $a=1359863889$, $b=36855$, then

$$x =$$

$$9\left(\sqrt{a}+b\right)^{5/6}[491526^{1/3}\left(\sqrt{a}+b\right)^{1/6}-960\left(\sqrt{a}+b\right)^{5/6}+2\cdot 6^{2/3}\left(\sqrt{a}+b\right)^{3/2}-$$

$$-2\cdot 6^{5/6}\left(\sqrt{a}+b\right)^{1/6}\surd[-86980957248+36855\cdot 2^{2/3}3^{1/6}\sqrt{453287963}\cdot(36855+$$

$$+\sqrt{a})^{2/3}-2358720\sqrt{a}+150958080\cdot 6^{1/3}\left(\sqrt{a}+b\right)^{1/3}+$$

$$+4096\cdot 2^{1/3}3^{5/6}\sqrt{453287963}\left(\sqrt{a}+b\right)^{1/3}+453025819\cdot 6^{2/3}\left(\sqrt{a}+b\right)^{2/3}]+$$

$$+384\cdot 2^{2/3}3^{1/6}\surd[-2358720-64\sqrt{a}+8192\cdot 6^{1/3}\left(\sqrt{a}+b\right)^{1/3}+$$

$$+12285\cdot 6^{2/3}\left(\sqrt{a}+b\right)^{2/3}+2^{2/3}3^{1/6}\sqrt{453287963}\left(\sqrt{a}+b\right)^{2/3}]^{-1}$$

4) For

$$w = \sqrt{\frac{277}{108}+\frac{13\sqrt{385}}{108}}$$

we get

$$x = \frac{\sqrt{\frac{277}{12}+\frac{13\sqrt{385}}{12}}}{4+\sqrt{7}}$$

Hence

$$R\left(\exp\left[-\pi\cdot k^{(-1)}\left(\frac{\sqrt{\frac{277}{12}+\frac{13\sqrt{385}}{12}}}{4+\sqrt{7}}\right)^{1/2}\right]\right) =$$

$$= \left(-\frac{-8071}{18} + \frac{1075\sqrt{55}}{18} + \frac{1}{18}\sqrt{5(25740148 - 3470530\sqrt{55})} \right)^{1/5}$$

5) Set $q = e^{-\pi\sqrt{r_0}}$, then from

$$V(e^{-\pi\sqrt{r_0}}) = V_i^{(-1)}(k_{r_0}) = V_0$$

and from

$$V(q^{1/3}) = \sqrt[3]{V(q)\frac{1 - V(q) + V(q)^2}{1 + 2V(q) + 4V(q)^2}}$$

We can evaluate all

$$V(q_0(n)) = b_0(n) = \text{Algebraic function of } r_0$$

where

$$q_0(n) = e^{-\pi\sqrt{r_0}/3^n}$$

and

$$V_i(V(q_0(n))) = V_i(b_0(n)) = k_{r_0/9^n}$$

hence

$$k^{(-1)}(V_i(b_0(n))) = \frac{r_0}{9^n}$$

An example for $r_0 = 2$ is

$$V(e^{-\pi\sqrt{2}}) = -1 + \sqrt{\frac{3}{2}}$$

$$V(e^{-\pi\sqrt{2}/3}) = \frac{1}{2^{1/3}}\left(-1 + \sqrt{\frac{3}{2}}\right)^{1/3}$$

$$V(e^{-\pi\sqrt{2}/9}) = \rho_3^{1/3}$$

Where ρ_3 can be evaluated in radicals but for simplicity we give the polynomial form.

$$-1 - 72x - 6408x^2 + 50048x^3 + 51264x^4 - 4608x^5 + 512x^6 = 0$$

$$\ldots$$

Then respectively we get the values

$$k^{(-1)}\left(-49 + 35\sqrt{2} + 4\sqrt{3(99 - 70\sqrt{2})}\right) = 2/9 \qquad (37)$$

$$k^{(-1)}\left(V_i(\rho_3^{1/3})\right) = 2/81 \qquad (38)$$

$$\ldots$$

Hence

$$k^{(-1)}\left(V_i(b_0(n))\right)=r_0/9^n \tag{39}$$

and

$$R\left(e^{-\pi\sqrt{r_0}/3^n}\right)^{-5}-11-R\left(e^{-\pi\sqrt{r_0}/3^n}\right)^{5}=$$

$$=\frac{16x_n^6-26x_n^4-w_nx_n^3+10x_n^2+w_nx_n}{x_n^4-6x_n^3w_n-20x_nw_n^3+15w_n^2x_n^2-6x_nw_n+15w_n^4+w_n^2}\times$$

$$\times\left(\frac{w_n}{x_n}+\frac{w_n'}{\sqrt{1-x_n^2}}-\frac{w_nw_n'}{x_n\sqrt{1-x_n^2}}\right)^3 \tag{40}$$

Where $x_n=V_i(b_0(n))=$ known. The w_n are given from (13) (in this case we don't find a way to evaluate w_n in radicals, but as a solution of (13)).
6) Set now

$$w_0=\frac{-64+a+\sqrt{4096+a(88+a)}}{6\sqrt{6}\sqrt{a}}$$

then

$$x_0=\frac{-64+a+\sqrt{4096+a(88+a)}}{\sqrt{6}\left(-4+\sqrt{-2+\frac{16}{a^{1/3}}+a^{1/3}}a^{1/6}+a^{1/3}\right)^2a^{1/6}}$$

$$R\left(e^{-\pi\sqrt{k^{(-1)}(x_0)}}\right)^{-5}-11-R\left(e^{-\pi\sqrt{k^{(-1)}(x_0)}}\right)^{5}:=A(a)$$

where the $A(a)$ is a known algebraic function of a and can calculated from the Main Theorem. Setting arbitrary real values to a we get algebraic evaluations of the RRCF as in evaluation 4.
If we set

$$g(x):=\frac{-64+x+\sqrt{4096+x(88+x)}}{\sqrt{6}\left(-4+x^{1/6}\sqrt{-2+\frac{16}{x^{1/3}}+x^{1/3}}+x^{1/3}\right)^2x^{1/6}}$$

and if we manage to write k_r in the form $g(a_r)$ for a certain a_r i.e. $V_i(V(e^{-\pi\sqrt{r}}))=k_r=g(a_r)$, then

$$R\left(e^{-\pi\sqrt{r}}\right)^{-5}-11-R\left(e^{-\pi\sqrt{r}}\right)^{5}=A(a_r)=A\left(g^{(-1)}(k_r)\right)$$

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