Striped periodic minimizers of a two-dimensional model for martensitic phase transitions

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In this paper we consider a simplified two-dimensional scalar model for the formation of mesoscopic domain patterns in martensitic shape-memory alloys at the interface between a region occupied by the parent (austenite) phase and a region occupied by the product (martensite) phase, which can occur in two variants (twins). The model, first proposed by Kohn and Müller [22], is defined by the following functional:

$$\mathcal{E}(u) = \beta ||u(0,\cdot)||^2_{H^{1/2}([0,h])} + \int_0^L dx \int_0^h dy \, (|u_x|^2 + \varepsilon |u_{yy}|)$$

where $u : [0, L] \times [0, h] \to \mathbb{R}$ is periodic in y and $u_y = \pm 1$ almost everywhere. Conti [12] proved that if $\beta \gtrsim \varepsilon L/h^2$ then the minimal specific energy scales like $\sim \min\{(\varepsilon\beta/L)^{1/2}, (\varepsilon/L)^{2/3}\}$, as $(\varepsilon/L) \to 0$. In the regime $(\varepsilon\beta/L)^{1/2} \ll (\varepsilon/L)^{2/3}$, we improve Conti's results, by computing exactly the minimal energy and by proving that minimizers are periodic one-dimensional sawtooth functions.

1. INTRODUCTION AND MAIN RESULTS

The formation of mesoscopic scale patterns in equilibrium systems is often due to a competition between interactions favoring different microscopic structures; e.g., a competition between a short range attractive interaction favoring a homogeneous ordered state and a long range repulsive interaction, which opposes such ordering on the scale of the whole sample. Mathematically, this phenomenon can be modeled by (non-convex) free-energy functionals, whose minimizers are supposed to describe the low energy states of the system. The details of the free-energy functional to be considered depend on the specific system one wants to describe: applications range from micromagnetics [10, 14, 16, 24] to diblock copolymers [2, 4, 8, 29], elasto-plasticity [5, 13], superconducting films [9, 15, 30] and martensitic phase transitions [11, 12, 22, 23], just to mention a few.

In all these cases, combinations of variational estimates and numerical simulations typically allow one to construct an approximate (and quite realistic) low temperature phase diagram, which often displays a wide range of ordering effects including formation of striped states [21, 26, 32], droplet patterns [16, 27], triangular lattices [1, 35], etc. However, a satisfactory theory of pattern formation in more than one dimension is still missing and the number of physical models for which periodicity can be rigorously proven is very small [3, 6, 7, 17–20, 25, 33, 34]. In this paper we prove periodicity of the minimizers of an anisotropic 2D free-energy functional, motivated by the theory of martensitic phase transitions. Our methods are based on a combination of reflection positivity estimates, in the spirit of [17–20], and of Poincaré-type estimates. We hope that these techniques will lead to more general examples of spontaneous pattern formation in anisotropic systems with competing interactions.

2. DEFINITION OF THE MODEL AND MAIN RESULTS

We consider a simplified two-dimensional (2D) scalar model for the formation of mesoscopic domain patterns in martensitic shape-memory alloys at the interface between a region occupied by the parent (austenite) phase and a region occupied by the product (martensite) phase, which can occur in two variants (twins). The model, first proposed by Kohn and Müller [22], is defined by the following functional:

$$\mathcal{E}(u) = \beta||u(0,\cdot)||^2_{H^{1/2}([0,h])} + \int_0^L dx \int_0^h dy \, (|u_x|^2 + \frac{\varepsilon}{2}|u_{yy}|) \tag{2.1}$$

where $u : [0,L] \times [0,h] \to \mathbb{R}$ is periodic in y and $u_y = \pm 1$ almost everywhere; therefore, the admissible functions are such that, for almost every x, the graph of $y \to u(x,y)$ looks like a (possibly irregular) sawtooth pattern. Here β and ε are nonnegative parameters. The square of the $H^{1/2}$-norm in the r.h.s. of (2.1) is defined as:

$$||u(0,\cdot)||^2_{H^{1/2}([0,h])} = 4\pi^2 \sum_{k\in\mathbb{Z}} |k||\hat{u}_0(k)|^2 \, , \tag{2.2}$$

where $\hat{u}_0(k) = h^{-1}\int_0^h dy \, u(0,y)\, e^{-2\pi iky/h}$. The equivalent x-space representation of this norm is the following:

$$||u(0,\cdot)||^2_{H^{1/2}([0,h])} = \int_0^h dy \int_{-\infty}^{+\infty} dy' \, \frac{|u(0,y) - \tilde{u}(0,y')|^2}{|y-y'|^2} \, , \tag{2.3}$$

where $\tilde{u}(0,y) : \mathbb{R} \to \mathbb{R}$ is the periodic extension of $u(0,y)$ over the whole real axis.

The problem consists in determining the minimizers of (2.1) for small values of ε; existence of the minimizer was proved in [23]. As discussed in [22], the significance of the various terms in (2.1) is the following. The rectangle $[0,L] \times [0,h]$ is the "martensite" region. The regions where $u_y = -1$ and $u_y = 1$ correspond to two distinct variants, which are separated by sharp interfaces. The term $\int |u_x|^2$ is the "strain energy"; note that it vanishes only if the interfaces between the two twin variants are precisely horizontal, i.e., if the two variants form a striped (lamellar) pattern. The term $(\varepsilon/2)\int |u_{yy}|$ is the surface energy; since u_y jumps from -1 to $+1$, $|u_{yy}|$ is like a delta function concentrated on the interfaces between the twins. It can be expressed more conventionally as

$$\frac{\varepsilon}{2}\int_0^L dx \int_0^h dy|u_{yy}| = \varepsilon \int_0^L dx\, N(x)\ , \tag{2.4}$$

where $N(x_0) = (1/2)\int_0^h dy|u_{yy}|$ is the number of twin boundaries that cross the line $x = x_0$. More precisely, $N(x)$ is defined as

$$N(x) = \frac{1}{2}\sup\Big\{ \int_0^h dy\, u_y(x,y)\varphi'(y)\ :\ \varphi \in C^\infty_{per} \text{ and } |\varphi| \le 1\Big\}\ , \tag{2.5}$$

where C^∞_{per} is the set of periodic C^∞ functions on $[0,h]$. Note that if $\int\int u_x^2 < \infty$ then $x \mapsto u(x,\cdot)$ is a continuous map from $[0,L]$ to $L^2([0,h])$ and, therefore, $N(x)$ is lower semicontinuous, being a supremum of continuous functions. Note also that $\mathcal{E}(u) < \infty$ and the fact that $u_y \in L^\infty([0,L]\times[0,h])$ imply that u has a (1/3)-Hölder continuous representative [28]; therefore, in the following, with no loss of generality, we shall assume u to be continuous in $[0,L]\times[0,h]$.

The boundary $x = 0$ represents the interface between the martensite and the austenite and the term proportional to the square of the $H^{1/2}$-norm of $u(0,y)$ is the "elastic energy in the austenite". In fact, the austenite should be imagined to occupy the region $(-\infty,0]\times[0,h]$ and to be associated with the elastic energy

$$2\pi\beta \int_{-\infty}^0 dx \int_0^h |\nabla\psi|^2\ , \tag{2.6}$$

where ψ is periodic in y, it decays to zero as $x \to -\infty$ and satisfies the boundary condition $\psi(0,y) = u(0,y)$. Since the elastic energy of the austenite is quadratic, one can perform the associated minimization explicitly. This yields

$$\psi(x,y) = \sum_{k\in\mathbb{Z}} \hat{u}_0(k) e^{2\pi i k y/h} e^{2\pi|k|x/h}\ , \tag{2.7}$$

whence

$$2\pi\beta \int_{-\infty}^0 dx \int_0^h |\nabla\psi|^2 = 4\pi^2\beta \sum_{k\in\mathbb{Z}} |k||\hat{u}_0(k)|^2\ . \tag{2.8}$$

Depending on the values of the material parameters β, ε, the minimizers of (2.1) are expected to display different qualitative features. In particular, in [22], on the basis of rigorous upper bounds and heuristic lower bounds on the ground state energy of (2.1), it was conjectured that, if $(\varepsilon/L) \ll 1$, the minimizers should display periodic striped (lamellar) order as long as

$$\Big(\frac{\varepsilon\beta}{L}\Big)^{1/2} \ll \Big(\frac{\varepsilon}{L}\Big)^{2/3} \tag{2.9}$$

and asymptotically self-similar branched patterns as long as

$$\Big(\frac{\varepsilon}{L}\Big)^{2/3} \ll \Big(\frac{\varepsilon\beta}{L}\Big)^{1/2} . \tag{2.10}$$

Recently, Conti [12] substantiated this conjecture, by proving that if $\beta \gtrsim \varepsilon L/h^2$ then E_0, the infimum of (2.1) over the admissible u's, satisfies upper and lower bounds of the following form:

$$\min\{c_s\Big(\frac{\varepsilon\beta}{L}\Big)^{1/2} , \ c_b\Big(\frac{\varepsilon}{L}\Big)^{2/3}\} \le \frac{E_0}{hL} \le \min\{C_s\Big(\frac{\varepsilon\beta}{L}\Big)^{1/2} , \ C_b\Big(\frac{\varepsilon}{L}\Big)^{2/3}\} , \tag{2.11}$$

for suitable constants c_s, c_b, C_s, C_b. The constants C_s and C_b in the r.h.s. are obtained by choosing in the variational upper bound the optimal periodic striped configurations and the optimal branched configuration, respectively.

In the present paper we improve the bounds (2.11), by proving that, if ε and β are small and such that (2.9) is satisfied, i.e., if

$$0 \le \beta \ll \Big(\frac{\varepsilon}{L}\Big)^{1/3} \ll \frac{h}{L} , \tag{2.12}$$

then the minimizers display periodic striped order. In particular, asympotically in the regime (2.12), the constant c_s in the l.h.s. of (2.11) can be chosen arbitrarily close to C_s. Our main result is summarized in the following theorem.

Theorem 1. *If $\varepsilon L^2/h^3$ and $\beta L^{1/3}\varepsilon^{-1/3}$ are positive and small enough, any minimizer $u(x,y)$ of (2.1) is a one-dimensional periodic sawtooth function, i.e.,*

$$u(x,y) = A + w_{M^*}(y - y_0) , \tag{2.13}$$

with A and y_0 two real constants, $w_M(y) := \int_0^y dz\, \mathrm{sign}(\sin \frac{\pi z M}{h})$ and

$$M^* = \mathrm{argmin}\{\mathcal{E}(w_M) \, : \, M \text{ even integer}\} . \tag{2.14}$$

Remark. An explicit computation shows that the number M^* of corner points of the periodic minimizer, as defined in (2.14), is $M^* \sim (\beta h^2/\varepsilon L)^{1/2} \gg 1$ for $\beta \gg \varepsilon L/h^2$, while it is of order 1 for $\beta \lesssim \varepsilon L/h^2$.

In order to prove Theorem 1 we proceed in several steps. First, we show that the optimal profile among the one-dimensional (1D) profiles is a sawtooth periodic function. This is proved in Section 3 and in Appendix A, by using the reflection positivity method of [17–20]. Next, we show that the minimizers of the full 2D problem are 1D in a subregime of (2.12), i.e., for

$$0 \le \beta < (2\pi^2 h)^{-1/2}\varepsilon^{1/2} . \tag{2.15}$$

The proof of this claim, which is discussed in Section 4, makes use both of the lower bound on the energy of 1D configurations of Section 3 and of a Poincaré inequality; the way in which these two bounds are combined is the key idea used in the study of the full regime, too. The proof of Theorem 1 in the full regime (2.12) requires a more sophisticated strategy: we first localize the problem in small horizontal slices, of vertical size comparable with the optimal period $2h/M^*$, and then prove that in each slice $u_x \equiv 0$, by using a combination of Poincaré-type bounds with a priori estimates on the local energy, similar to the one discussed in Section 4. This is discussed in Section 5.

3. PROOF OF THE MAIN RESULT: FIRST STEP

Let us assume that $u_x \equiv 0$ in (2.1). In this case $u(x,y) = u(0,y) \equiv u_0(y)$ and (2.1) reduces to

$$\mathcal{E}(u) = \beta \int_0^h dy \int_{-\infty}^{+\infty} dy' \, \frac{|u_0(y) - \tilde{u}_0(y')|^2}{|y-y'|^2} + \varepsilon L M_0 \,, \tag{3.1}$$

where $M_0 = N(x=0)$ is the number of jumps of $u_0'(y)$. Now, rewrite $|y-y'|^{-2}$ as

$$\frac{1}{|y-y'|^2} = \int_0^\infty d\alpha \, \alpha e^{-\alpha|y-y'|} \,, \tag{3.2}$$

so that

$$\mathcal{E}(u) = \beta \int_0^\infty d\alpha \, \alpha \int_0^h dy \int_{-\infty}^{+\infty} dy' \, |u_0(y) - \tilde{u}_0(y')|^2 e^{-\alpha|y-y'|} + \varepsilon L M_0 \,. \tag{3.3}$$

Let us denote by $0 \le y_0 < y_1 < \cdots < y_{M_0-1} < h$ the locations of the cusps of $u_0(y)$, and let us define $u_0^{(i)}$, $i = 0, \ldots, M_0 - 1$, to be the restrictions of u_0 to the intervals $[y_i, y_{i+1}]$. Given $u_0^{(i)}$ on $[y_i, y_{i+1}]$, let us extend it to the whole real axis by repeated reflections about y_i and y_{i+1}; we shall denote the extension by $\tilde{u}_0^{(i)}$. Using the chessboard estimate proved in [19] (see Appendix A for details) we find that, for any $\alpha \in (0, +\infty)$,

$$\begin{aligned} &\int_0^h dy \int_{-\infty}^{+\infty} dy' \, |u_0(y) - \tilde{u}_0(y')|^2 e^{-\alpha|y-y'|} \ge \\ &\qquad \ge \sum_{i=0}^{M_0-1} \int_{y_i}^{y_{i+1}} dy \int_{-\infty}^{+\infty} dy' \, |u_0^{(i)}(y) - \tilde{u}_0^{(i)}(y')|^2 e^{-\alpha|y-y'|} \,, \end{aligned} \tag{3.4}$$

which readily implies

$$\mathcal{E}(u) \ge \beta \sum_{i=0}^{M_0-1} \int_{y_i}^{y_{i+1}} dy \int_{-\infty}^{+\infty} dy' \, \frac{|u_0^{(i)}(y) - \tilde{u}_0^{(i)}(y')|^2}{|y-y'|^2} + \varepsilon L M_0 \,. \tag{3.5}$$

An explicit computation of the integral in (3.5) gives:

$$\int_{y_i}^{y_{i+1}} dy \int_{-\infty}^{+\infty} dy' \, \frac{|u_0^{(i)}(y) - \tilde{u}_0^{(i)}(y')|^2}{|y-y'|^2} = \frac{14\,\zeta(3)}{\pi^2}(y_{i+1}-y_i)^2 \;, \tag{3.6}$$

where we used that $\sum_{k=1}^{\infty}(2k-1)^{-3} = (7/8)\zeta(3)$. As a result:

$$\mathcal{E}(u) \geq \frac{14\,\zeta(3)}{\pi^2}\beta \sum_{i=0}^{M_0-1}(y_{i+1}-y_i)^2 + \varepsilon L M_0 = \frac{\beta c_0 h^2}{M_0} + \varepsilon L M_0 + \beta c_0 \sum_{i=0}^{M_0-1}\Big(h_i - \frac{h}{M_0}\Big)^2 \;, \tag{3.7}$$

where $c_0 = 14\,\zeta(3)/\pi^2$ and $h_i = y_{i+1} - y_i$. Defining $E_{1D}(M) = \beta c_0 h^2/M + \varepsilon L M$ and combining (3.7) with the variational bound $\mathcal{E}(u) \leq E_{1D}(M^*)$, where M^* is the even integer minimizing $E_{1D}(M)$, we find that if u is the minimizer of $\mathcal{E}(u)$ under the constraint that $u_x \equiv 0$,

$$E_{1D}(M^*) \geq \mathcal{E}(u) \geq E_{1D}(M_0) + \beta c_0 \sum_{i=0}^{M_0-1}\Big(h_i - \frac{h}{M_0}\Big)^2 \;, \tag{3.8}$$

which implies: (i) $\min\{\mathcal{E}(u) : u_x \equiv 0\} = E_{1D}(M^*)$; (ii) $M_0 = M^*$; (iii) $h_i = h/M^*$, $\forall i$. Note that even in the cases where $E_{1D}(M)$ is minimized by two distinct values of M, M_1^* and M_2^*, the only 1D minimizers are the simple periodic functions of period $2h/M_1^*$ *or* of period $2h/M_2^*$ (i.e., no function alternating bumps of size $2h/M_1^*$ and $2h/M_2^*$ can be a minimizer).

For the purpose of the forthcoming discussion, let us remark that if $\beta \gg \varepsilon L/h^2$,

$$\Big| M^* - \sqrt{\frac{\beta c_0 h^2}{\varepsilon L}} \Big| \leq 2 \tag{3.9}$$

and

$$\min\{\mathcal{E}(u) : u_x \equiv 0\} = E_{1D}(M^*) = hL\, c_s \sqrt{\frac{\beta\varepsilon}{L}} \cdot \Big(1 + O(\frac{L\varepsilon}{h^2\beta})\Big) \;, \tag{3.10}$$

with $c_s = 2\sqrt{c_0}$ the constant appearing in (2.11).

4. PROOF OF THE MAIN RESULT: SECOND STEP

The result of the previous section can be restated in the following way: if v_M is a periodic function on $[0,h]$ with $v_M' = \pm 1$ and M corners located at y_i, $i = 1, \ldots, M$, then

$$\beta ||v_M||_{H^{1/2}([0,h])} + \varepsilon L M \geq E_{1D}(M) + \beta c_0 \sum_{i=1}^{M}\Big(h_i - \frac{h}{M}\Big)^2 \;, \tag{4.1}$$

where $h_i = y_{i+1} - y_i$. In this section we make use of (4.1) and, by combining it with a Poincaré inequality, we prove that in the regime (2.15) all the minimizers are one-dimensional (and, therefore, periodic, by the results of Section 3).

Let $M = \min_{x\in[0,L]} N(x)$ and let

$$\bar{x} = \inf\{x \in [0,L] \ : \ N(x) = M\} \ . \tag{4.2}$$

Moreover, let $v_M(y) \equiv u(\bar{x}, y)$. By the lower semicontinuity of $N(x)$ (see the lines following (2.5)), $N(\bar{x}) = M$. We rewrite

$$\begin{aligned} \mathcal{E}(u) \ &= \ \Big[\beta||v_M||^2_{H^{1/2}([0,h])} + \varepsilon LM\Big] + \beta(||u_0||^2_{H^{1/2}([0,h])} - ||v_M||^2_{H^{1/2}([0,h])}) + \\ &\quad + \int_0^L dx \int_0^h dy\, |u_x|^2 + \varepsilon \int_0^L dx(N(x) - M) \geq \\ &\geq \ \Big[\beta||v_M||^2_{H^{1/2}([0,h])} + \varepsilon LM\Big] + \beta(||u_0||^2_{H^{1/2}([0,h])} - ||v_M||^2_{H^{1/2}([0,h])}) + \\ &\quad + \int_0^{\bar{x}} dx \int_0^h dy\, |u_x|^2 + \varepsilon \int_0^{\bar{x}} dx(N(x) - M) \ , \end{aligned} \tag{4.3}$$

where the right hand side of the inequality differs from the left hand side just by the upper limits of the two integrals in dx, which were set equal to $\bar{x}$ (in other words, in order to bound $\mathcal{E}(u)$ from below, we dropped the two positive integrals $\int_{\bar{x}}^L dx \int_0^h dy\, |u_x|^2$ and $\varepsilon \int_{\bar{x}}^L dx(N(x) - M)$). Now, if $\bar{x} = 0$, then u is one-dimensional, $u(x, y) = u_0(y) = v_M(y)$, and we reduce to the discussion in the previous section. Let us then suppose that $\bar{x} > 0$. In this case, the first term of the fourth line of Eq.(4.3) can be bounded from below by

$$\int_0^{\bar{x}} dx \int_0^h dy\, |u_x|^2 \geq \frac{1}{\bar{x}} \int_0^h dy |v_M(y) - u_0(y)|^2 \ . \tag{4.4}$$

The second term of the third line of Eq.(4.3) can be rewritten in the form:

$$\begin{aligned} \beta(||u_0||^2_{H^{1/2}([0,h])} - ||v_M||^2_{H^{1/2}([0,h])}) \ &= \ \beta||u_0 - v_M||^2_{H^{1/2}([0,h])} + \\ &+ \ 2\beta(v_M, u_0 - v_M)_{H^{1/2}([0,h])} \ , \end{aligned} \tag{4.5}$$

where, given two real h-periodic functions f and g,

$$\begin{aligned} (f, g)_{H^{1/2}([0,h])} \ &= \ \int_0^h dy \int_{-\infty}^{+\infty} dy' \frac{(f(y) - f(y'))(g(y) - g(y'))}{|y - y'|^2} = \\ &= \ 4\pi^2 \sum_{k\in\mathbb{Z}} |k| \hat{f}^*(k) \hat{g}(k) \ . \end{aligned} \tag{4.6}$$

Using Cauchy-Schwarz inequality we find:

$$\begin{aligned} \Big|(f, g)_{H^{1/2}([0,h])}\Big| \ &\leq \ 4\pi^2 \sum_{k\in\mathbb{Z}} |k|\, |\hat{f}(k)|\, |\hat{g}(k)| \leq \\ &\leq \ 2\pi \Big[\frac{4\pi^2}{h} \sum_{k\in\mathbb{Z}} |k|^2\, |\hat{f}(k)|^2\Big]^{1/2} \cdot \Big[h \sum_{k\in\mathbb{Z}} |\hat{g}(k)|^2\Big]^{1/2} = \\ &= \ 2\pi ||f'||_{L^2([0,h])} \cdot ||g||_{L^2([0,h])} \ . \end{aligned} \tag{4.7}$$

Using (4.5), (4.7) and the fact that $|v'_M|=1$ for a.e. y, we find that

$$\beta(||u_0||^2_{H^{1/2}([0,h])}-||v_M||^2_{H^{1/2}([0,h])}) \;\geq\; \beta||u_0-v_M||^2_{H^{1/2}([0,h])}- \\ -\; 4\pi\beta h^{1/2}||u_0-v_M||_{L^2([0,h])}\;. \qquad (4.8)$$

Combining (4.1), (4.3), (4.4) and (4.8), and neglecting the positive term $\beta||u_0-v_M||^2_{H^{1/2}([0,h])}$, we get

$$\mathcal{E}(u) \;\geq E_{1D}(M)+\beta c_0\sum_{i=1}^{M}\Big(h_i-\frac{h}{M}\Big)^2-4\pi\beta h^{1/2}||u_0-v_M||_{L^2([0,h])}+ \\ +\varepsilon\int_0^{\bar x}dx(N(x)-M)+\frac{1}{\bar x}||u_0-v_M||^2_{L^2([0,h])}\;. \qquad (4.9)$$

The term $\varepsilon\int_0^{\bar x}dx(N(x)-M)$ is bounded from below by $2\varepsilon\bar x$ (simply because, by construction, $N(x)-M\geq 2$ for all $x<\bar x$). Therefore, the last two terms in the r.h.s. of (4.9) are bounded from below by

$$\varepsilon\int_0^{\bar x}dx(N(x)-M)+\frac{1}{\bar x}||u_0-v_M||^2_{L^2([0,h])}\geq 2\varepsilon\bar x+\frac{1}{\bar x}||u_0-v_M||^2_{L^2([0,h])} \\ \geq 2\sqrt{2\varepsilon}||u_0-v_M||_{L^2([0,h])}\;, \qquad (4.10)$$

which gives us a chance to balance the error term $-4\pi\beta h^{1/2}||u_0-v_M||_{L^2([0,h])}$ in (4.9), which is linear in $||u_0-v_M||_{L^2([0,h])}$, with the sum of the interfacial and the elastic energies. In fact, by plugging (4.10) into (4.9), and neglecting a positive term, for any minimizer u we get

$$E_{1D}(M^*)\geq\mathcal{E}(u)\geq E_{1D}(M)+\beta c_0\sum_{i=1}^{M}\Big(h_i-\frac{h}{M}\Big)^2 \qquad (4.11) \\ +2(\sqrt{2\varepsilon}-2\pi\beta\sqrt{h})||u_0-v_M||_{L^2([0,h])}\;,$$

where M^* is the even integer minimizing $E_{1D}(M)$. In the regime (2.15) where $\sqrt{2\varepsilon}-2\pi\beta\sqrt{h}\geq 0$, Eq.(4.11) implies that $u_0\equiv v_M$, that is, as observed above, the minimizer is the optimal one-dimensional periodic striped state. This concludes the proof of Theorem 1 in the regime (2.15).

In the complementary regime

$$(2\pi^2h)^{-1/2}\varepsilon^{1/2}\leq\beta\ll L^{-1/3}\varepsilon^{1/3}\;, \qquad (4.12)$$

a similar strategy implies an apriori bound on M, which will be useful in the following. More precisely, by combining Eq.(4.9) with

$$\frac{1}{\bar x}||u_0-v_M||^2_{L^2([0,h])}-4\pi\beta\sqrt{h}||u_0-v_M||_{L^2([0,h])}\geq -4\pi^2\beta^2h\bar x\geq -4\pi^2\beta^2hL\;, \quad (4.13)$$

we find that for any minimizer u

$$E_{1D}(M^*)\geq\mathcal{E}(u)\geq E_{1D}(M)+\beta c_0\sum_{i=1}^{M}\Big(h_i-\frac{h}{M}\Big)^2-4\pi^2\beta^2hL\;. \qquad (4.14)$$

Recalling that $E_{1D}(M) = \beta c_0 h^2/M + \varepsilon L M$ and the fact that $|M^* - \sqrt{\beta c_0 h^2/\varepsilon L}| \leq 2$ (see (3.9)), from (4.14) we find that

$$\frac{|M - M^*|}{M^*} \leq (\text{const.}) \cdot (\beta \varepsilon^{-1/3} L^{1/3})^{3/4} \ll 1 \; . \tag{4.15}$$

5. PERIODICITY OF THE MINIMIZER: THE FULL SCALING REGIME

We are now left with proving Theorem 1 in the scaling regime (4.12). In this case the proof is much more elaborate: the rough idea is to apply the reasoning of the previous section locally in y. We localize the functional in horizontal stripes of width H_j, comparable with the optimal period $2h/M^* \sim \sqrt{\varepsilon L/\beta}$. In each strip, the combination $\sqrt{2\varepsilon} - 2\pi\beta\sqrt{h}$ appearing in the right hand side of (4.11) is replaced by $\sqrt{2\varepsilon} - C\beta\sqrt{H_j} \geq \sqrt{2\varepsilon} - C'\beta(\varepsilon L/\beta)^{1/4}$, for suitable constants C, C'; now, the latter expression is > 0 as long as $\beta \ll L^{-1/3}\varepsilon^{1/3}$, which will allow us to conclude that in every strip the minimizing configuration is 1D.

In this section, we first discuss how to localize the functional in horizontal strips, then we distinguish between "good" and "bad" localization intervals, and finally describe the lower bound on the local energy for the different intervals.

For simplicity, from now on we set $h = L = 1$. Here and below $C, C', \ldots$, and $c, c', \ldots$, denote universal constants, which might change from line to line. We assume that $u(x, y)$ is a minimizer, that $\beta \geq c\varepsilon^{1/2}$, and that ε and $\beta\varepsilon^{-1/3}$ are sufficently small.

A. A localized bound

Our purpose in this subsection is to derive a local version of the error term $-4\pi\beta h^{1/2} ||u_0 - v_M||_{L^2([0,h])}$ in Eq.(4.9). Let $u_0(y) := u(0, y)$ and $u_1(y) := u(1, y)$. Set $F(u) = \int_0^1 dx \int_0^1 dy |u_x|^2 + \varepsilon \int_0^1 (N(x) - M)$, denote by z_i, $i = 1, \ldots, M$, the locations of the corners of u_1 and by $h_i = z_{i+1} - z_i$ the distances between neighboring corners. Note that the number of corners of u_1 is equal to $M = \min_{x \in [0,1]} N(x)$, because u is a minimizer and, therefore, $u(x, y) = u(\bar{x}, y)$ for all $\bar{x} \leq x \leq 1$, with $\bar{x}$ defined as in (4.2); in fact, the choice $u(x, y) = u(\bar{x}, y)$ for all $\bar{x} \leq x \leq 1$ minimizes the two nonnegative contributions to the energy $\int_{\bar{x}}^L dx \int_0^h dy\, |u_x|^2$ and $\varepsilon \int_{\bar{x}}^L dx (N(x) - M)$, making them precisely zero, see Eq.(4.3).

Instead of v_M, we now consider a general test function $w(y)$, to be specified below, periodic on $[0, 1]$ and with a number of corners smaller or equal to M. We denote by $\bar{z}_i$, $i = 0, \ldots M_0 - 1$, the locations of the corners of w (labelled in such a way that $0 \leq \bar{z}_0 < \bar{z}_1 < \cdots < \bar{z}_{M_0 - 1} < 1$), and by $\bar{h}_i = \bar{z}_{i+1} - \bar{z}_i$ the distances between subsequent corners. In the following, it will be useful to imagine that w is associated to a sequence of exactly M corner points, even in the case that $M_0 < M$. These M corner points will be denoted by $\tilde{z}_i$, $i = 0, \dots, M-1$ and they will have the property that $0 \le \tilde{z}_0 \le \tilde{z}_1 < \cdots < \tilde{z}_{M-1} \le 1$. In the case that $M_0 = M$, the sequence of $\tilde{z}_i$'s coincide with the sequence of $\bar{z}_i$'s; otherwise, if $M_0 < M$, the sequence of the $\tilde{z}_i$'s will be formed by the original sequence of $\bar{z}_i$'s plus a set of $(M - M_0)/2$ pairs of coinciding points. We define $\tilde{h}_i = \tilde{z}_{i+1} - \tilde{z}_i$ and note that now, in general, some of the $\tilde{h}_i$'s can be equal to 0.

Proceeding as in the previous section, for any minimizer u we get

$$\begin{aligned} E_{1D}(M^*) \ge \mathcal{E}(u) \;&\ge\; \Big[\beta||w||^2_{H^{1/2}} + \varepsilon M\Big] + 2\beta(w, u_0 - w)_{H^{1/2}} + F(u) \\ &\ge\; E_{1D}(M) + \beta c_0 \sum_{i=1}^{M} (\tilde{h}_i - \frac{1}{M})^2 + 2\beta(w, u_0 - w)_{H^{1/2}} + F(u) \,. \end{aligned} \tag{5.1}$$

The first observation is that with the help of the Hilbert transform we can write (5.1) in a more local way. In fact,

$$(w, u_0 - w)_{H^{1/2}} = (\mathcal{H}w', u_0 - w)_{L^2} \,, \tag{5.2}$$

with $\mathcal{H}$ the Hilbert transform, acting on a periodic function f in the following way:

$$(\mathcal{H}f)(y) = 2\pi \sum_{k\neq 0} \frac{-ik}{|k|} \hat{f}(k) e^{2\pi i k y} = 2\pi \; P.V. \int_0^1 dy' \cot \pi(y - y') \, (f(y') - \bar{f}) \,, \tag{5.3}$$

where P.V. denotes the Cauchy principal value and $\bar{f} = \int_0^1 f(y)dy$. Combining (5.1) and (5.2) we get

$$E_{1D}(M^*) \ge \mathcal{E}(u) \ge E_{1D}(M) + \beta c_0 \sum_{i=1}^{M} (\tilde{h}_i - \frac{1}{M})^2 + 2\beta(\mathcal{H}w', u_0 - w)_{L^2} + F(u) \,. \tag{5.4}$$

We now want to bound $2\beta(\mathcal{H}w', u_0 - w)_{L^2([0,h])}$ from below by a sum of terms localized in small intervals $I_k \subset [0, h]$, which will be the local version of the error term $-4\pi\beta h^{1/2}||u_0 - v_M||_{L^2([0,h])}$ in Eq.(4.9). First of all, note that, if $\{I_k\}_{k=1,\dots,M/2}$ is a partition of the unit interval, $2\beta(\mathcal{H}w', u_0 - w)_{L^2}$ can be decomposed as

$$2\beta(\mathcal{H}w', u_0 - w)_{L^2([0,h])} = 2\beta \sum_{k=1}^{M/2} \int_{I_k} dy \, \mathcal{H}w'(y) \, (u_0(y) - w(y)) \,. \tag{5.5}$$

In the following we shall choose the partition $\{I_k\}$ in a way depending on u_1, such that each strip $[0, 1] \times I_k$ will typically (i.e., for most k) contain two or more interfaces of u (as proven by combining the definition of $\{I_k\}$ with a priori estimates on $\int_0^1 dx \int_{I_k} dy \, u_x^2$, see Lemma 1 below). Moreover, we shall choose w

in a way depending on $\{I_k\}$ and on u_0, in such a way that every I_k contains at most two corner points of w and $\int_{I_k} dy(u_0 - w) = 0$. Once that $\{I_k\}$ and w are given, every term in the r.h.s. of (5.5) can be bounded as:

$$\begin{aligned} \Big|\int_{I_k} dy\,\mathcal{H}w'\,(u_0-w)\Big| &= \Big|\int_{I_k} dy(\mathcal{H}w' - \overline{\mathcal{H}w'})(u_0-w)\Big| \\ &\le ||\mathcal{H}w' - \overline{\mathcal{H}w'}||^2_{L^2(I_k)}||u_0-w||_{L^2(I_k)} \\ &\le H_k^{1/2}||\mathcal{H}w'||^2_{BMO(I_k)}||u_0-w||_{L^2(I_k)}\;, \end{aligned} \tag{5.6}$$

where $H_k := |I_k|$, $\overline{\mathcal{H}w'} := |I_k|^{-1}\int_{I_k} dy\,\mathcal{H}w'$ and the Bounded Mean Oscillation (BMO) seminorm is defined as

$$||g||^2_{BMO(I)} = \sup_{(a,b)\subset I}\frac{1}{|b-a|}\int_a^b dy\,|g(y) - g_{(a,b)}|^2\;, \qquad g_{(a,b)} := \frac{1}{|b-a|}\int_a^b dy\,g(y)\;. \tag{5.7}$$

Now we exploit the fact that the singular kernel $\cot\pi(y-y')$ maps bounded functions into BMO functions [31]. Thus, $||\mathcal{H}w'||_{BMO(I_k)} \le C||w'||_{L^\infty(I_k)} \le C$, uniformly in w' as long as $|w'| \le 1$. Therefore, combining (5.5) with (5.6), we find that there exists a universal constant $\bar c$ such that

$$2\beta(\mathcal{H}w', u_0 - w)_{L^2} \ge -\bar c\beta\sum_{k=1}^{M/2} H_k^{1/2}||u_0-w||_{L^2(I_k)}\;, \tag{5.8}$$

which is the desired local version of the error term $-4\pi\beta h^{1/2}||u_0 - v_M||_{L^2([0,h])}$ in Eq.(4.9).

Plugging (5.8) back into (5.4) and using the fact that $E_{1D}(M^*) - E_{1D}(M) \le 0$, we find that for any periodic sawtooth function w with a number of corners $\le M$,

$$\beta c_0\sum_{i=1}^{M}(\tilde h_i - \frac{1}{M})^2 + F(u) \le \bar c\beta\sum_{k=1}^{M/2} H_k^{1/2}||u_0-w||_{L^2(I_k)}\;, \tag{5.9}$$

which is the main conclusion of this subsection.

B. The choice of the comparison function w

In this subsection we first choose the partition $\{I_k\}$ and the test function w to be used in (5.9); next, we explain how to use the latter inequality in order to prove Theorem 1.

Recall that z_i, $i = 1, \dots, M$ are the corner points of u_1'. We assume without loss of generality that $u_1' = +1$ in (z_{2k}, z_{2k+1}), $k = 1, \dots, M/2$, and we define $a_k = \frac{z_{2k}+z_{2k+1}}{2}$ and $I_k = [a_k, a_{k+1})$, $k = 1, \dots, M/2$ (since we use periodic boundary conditions, we shall use the convention that $a_0 = a_{M/2}$ and $I_0 = I_{M/2}$). Note that, by construction: (i) u_1 has exactly two jump points in every interval I_k;

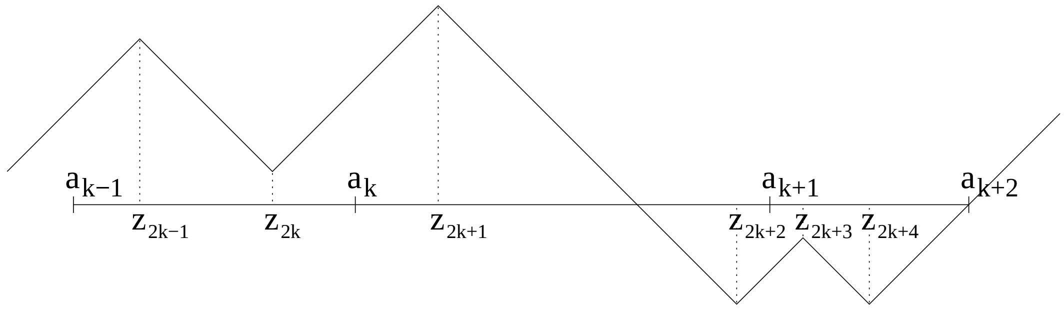

FIG. 1: The function u_1 in the intervals $I_{k-1} = [a_{k-1}, a_k)$, $I_k = [a_k, a_{k+1})$ and $I_{k+1} = [a_{k+1}, a_{k+2})$.

(ii) the jump points are "well inside" the intervals I_k; (iii) $u_1'(a_k) = +1$; (iv) $H_k = \frac{h_{2k}}{2} + h_{2k+1} + \frac{h_{2k+2}}{2}$.

Regarding the choice of the test function, we choose w to be the sawtooth function such that:

$$\text{(i) } w = u_0 \text{ on } \partial I_k\,; \qquad \text{(ii) } w' = +1 \text{ on } \partial I_k\,; \qquad \text{(iii) } \int_{I_k} (w - u_0) = 0\,, \tag{5.10}$$

for all $k = 1, \ldots, M/2$. In every interval I_k, w is uniquely specified by the two corner points $\tilde{z}_{2k+1}, \tilde{z}_{2k+2}$ chosen in such a way that: $a_k \le \tilde{z}_{2k+1} \le \tilde{z}_{2k+2} \le a_{k+1}$, $w'(y) = +1$ for $y \in (a_k, \tilde{z}_{2k+1}) \cup (\tilde{z}_{2k+2}, a_{k+1})$, $w'(y) = -1$ for $y \in (\tilde{z}_{2k+1}, \tilde{z}_{2k+2})$ and $\int_{I_k} w = \int_{I_k} u_0$ (these two corner points are uniquely defined only if $u_0' \not\equiv +1$ on I_k; if $u_0' \equiv +1$ on I_k, then we set $\tilde{z}_{2k+1} = \tilde{z}_{2k+2} = \frac{a_k + a_{k+1}}{2}$).

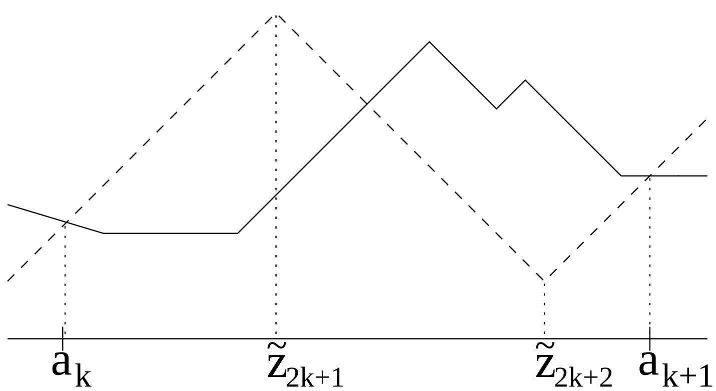

FIG. 2: The function u_0 (full line) and the test function w (dashed line) in the interval $I_k = [a_k, a_{k+1})$. The function w on I_k and, correspondingly, the locations of its corners $\tilde{z}_{2k+1}$ and $\tilde{z}_{2k+2}$, are determined by the conditions that (i) $w = u_0$ on ∂I_k, (ii) $w' = +1$ on ∂I_k, (iii) $\int_{I_k} (w - u_0) = 0$.

Note that with the definitions above, w is a sawtooth function with $M_0 \le M$ corner points, associated to which is a sequence $\tilde{z}_i$, $i = 1, \ldots, M$, satisfying the properties described before (5.1) and $\int_{I_k}(u_0 - w) = 0$; therefore, w satisfies (5.8). Let

$$I_k^L := \Big[\frac{z_{2k-1}+z_{2k}}{2}, a_k\Big)\,, \qquad I_k^R := \Big[a_{k+1}, \frac{z_{2k+3}+z_{2k+4}}{2}\Big)\,, \qquad I_k^* := I_k^L \cup I_k \cup I_k^R\,. \tag{5.11}$$

Moreover, let $I_k^{**} := I_{k-1} \cup I_k \cup I_{k+1}$. With these definitions, we can rewrite the left hand side of (5.8) as $\sum_{k=1}^{M/2} \widetilde{F}_k$, where

$$\widetilde{F}_k = \frac{\beta c_0}{7}\sum_{j=2k-2}^{2k+4}(\tilde{h}_j - \frac{1}{M})^2 + \frac{1}{3}\int_{I_k^{**}} dy \int_0^1 dx\, u_x^2 + \frac{\varepsilon}{2}\int_0^1 dx\,(N(x)|_{I_k^*} - 4)\,, \tag{5.12}$$

and $N(x)|_{I_k^*}$ is the number of corner points of $u(x,\cdot)$ in I_k^*. In the following we shall denote by $\widetilde{F}_k^{(0)}$ the first term in the r.h.s. of (5.12), by $\widetilde{F}_k^{(1)}$ the second term and by $\widetilde{F}_k^{(2)}$ the third term. Using these definitions, (5.9) can be rewritten as

$$\sum_{k=1}^{M/2}(\widetilde{F}_k - \bar{c}\beta H_k^{1/2}||u_0 - w||_{L^2(I_k)}) \le 0\,. \tag{5.13}$$

Our next goal is to derive a lower bound on the l.h.s. of (5.13) of the form

$$\sum_{k=1}^{M/2}(\widetilde{F}_k - \bar{c}\beta H_k^{1/2}||u_0 - w||_{L^2(I_k)}) \ge \frac{1}{2}\sum_{k=1}^{M/2}(\widetilde{F}_k^{(0)} + \widetilde{F}_k^{(1)}) + \Big(1 - (\beta\varepsilon^{-1/3})^\alpha\Big)\sum_{k=1}^{M/2}\widetilde{F}_k^{(2)}\,, \tag{5.14}$$

for a suitable $\alpha > 0$. Plugging (5.14) into (5.13) gives

$$\frac{\beta c_0}{2}\sum_{i=1}^{M}(\tilde{h}_i - \frac{1}{M})^2 + \frac{1}{2}\int_0^1 dy\int_0^1 dx\, u_x^2 + \varepsilon\Big(1 - (\beta\varepsilon^{-1/3})^\alpha\Big)\int_0^1 dx\,(N(x) - M) \le 0\,, \tag{5.15}$$

which implies that $u_x \equiv 0$ and $N(x) \equiv M$, and concludes the proof of Theorem 1.

The rest of the paper will be devoted to the proof of (5.14). In order to get bounds from above on $H_k^{1/2}||u_0 - w||_{L^2(I_k)}$, it will be convenient to distinguish between "good" and "bad" intervals, and to proceed in different ways, depending on the nature of the interval I_k.

C. Classification of the good and bad intervals

We shall say that

- I_k is "good" (of type 1) if $\max_{k-1\le i\le k+1} H_i \le 6/M$, $\widetilde{F}_k^{(1)} \le \eta/M^3$, and $\min_{2k-1\le j\le 2k+3} h_j \ge \kappa/M$, for suitable constants η, κ, to be conveniently fixed below.

Note that if $u(x,y)$ is the periodic sawtooth function in (2.13), as we hope to prove, then all the intervals I_k are good. Conversely, if I_k is good, then $u(x,y)|_{y\in I_k^*}$ is in some sense close to the optimal 1D configuration. More precisely, if I_k is good, then its length is of the same order as $2/M$; moreover, the corners of $u_1|_{I_k^*}$ are well separated, on the same scale, and $u_1|_{I_k^{**}}$ is close to $u_0|_{I_k^{**}}$ in L^2, on the natural scale: in fact, by the Poincaré inequality, $||u_0-u_1||^2_{L^2(I_k^{**})}\le 3\widetilde{F}_k^{(1)}\le 3\eta/M^3$.

The "bad" intervals will be further classified in three different types; we shall say that:

- I_k is of type 2 if $\max_{k-1\le i\le k+1}H_i \le 6/M$, $\widetilde{F}_k^{(1)}\le \eta/M^3$, and $\min_{2k-1\le j\le 2k+3}h_j<\kappa/M$;
- I_k is of type 3 if $\max_{k-1\le i\le k+1}H_i\le 6/M$ and $\widetilde{F}_k^{(1)}>\eta/M^3$;
- I_k is of type 4 if $\max_{k-1\le i\le k+1}H_i>6/M$.

We denote by $\mathcal{I}_q$, $q=1,\dots,4$, the set of intervals of type q; note that $\cup_q\mathcal{I}_q=\cup_k\{I_k\}$. In the following we describe how to obtain upper bounds on $\bar c\beta\sum_{k:I_k\in\mathcal{I}_q}H_k^{1/2}||u_0-w||_{L^2(I_k)}$ of the form (5.14), separately for $q=1,2,3,4$. Here and below we denote by $c,c',C,C',\dots$, universal constants independent of η,κ.

D. The lower bound: the good intervals

For intervals of type 1, the key estimates to be proven are the following.

Lemma 1 *Let I_k be an interval of type 1. If $\eta\kappa^{-3}$ is small enough, then $N(x)\Big|_{I_k}\ge 2$, $N(x)\Big|_{I_k^L}\ge 1$ and $N(x)\Big|_{I_k^R}\ge 1$, $\forall x\in[0,1]$.*

Lemma 2 *Let I_k be an interval of type 1. Let us define $\bar x_k=\inf_{x\in[0,1]}\{x : N(x)|_{I_k^*}\le 4\}$ and $\bar u(y)\equiv u(\bar x_k,y)$. If κ and $\eta\kappa^{-3}$ are small enough, then there exists a constant C independent of η,κ such that*

$$||u_0-w||_{L^2(I_k)}\le C\kappa^{-5/2}||u_0-\bar u||_{L^2(I_k^{**})}\ . \tag{5.16}$$

We first show that Lemma 1 and 2 imply the desired bound,

$$\bar c\beta H_k^{1/2}||u_0-w||_{L^2(I_k)}\le c\kappa^{-5/2}(\beta\varepsilon^{-1/3})^{3/4}\,\widetilde{F}_k\ . \tag{5.17}$$

Note that (5.17) implies (5.14) for all $\alpha<3/4$ (because $\widetilde{F}_k^{(2)}\ge 0$ for intervals of type 1). The strategy to prove (5.17) from Lemma 1 and 2 is the same followed in Section 4 to prove (4.10): we use an interpolation between the interfacial energy and the elastic energy to get a lower bound for $\widetilde{F}_k$, which is linear in $||u_0-w||_{L^2(I_k)}$. In fact, if $\bar x_k=0$, then by definition $\bar u=u_0$ and, by Lemma 2,

$u_0 \equiv w$ on I_k, in which case (5.17) is obvious. If $\bar{x}_k > 0$, then, by Lemma 1, $\widetilde{F}_k \geq \frac{1}{3}\int_0^{\bar{x}_k} dx \int_{I_k^{**}} u_x^2 + \frac{\varepsilon}{2}\int_0^{\bar{x}_k} (N(x)|_{I_k^*} - 4)$. Using the Poincaré inequality and the fact that, by definition of $\bar{x}_k$, $N(x)|_{I_k^*} - 4 \geq 1$ if $0 \leq x < \bar{x}_k$, we find:

$$\widetilde{F}_k \geq \frac{1}{3\bar{x}_k}||u_0 - \bar{u}||^2_{L^2(I_k^{**})} + \frac{\varepsilon}{2}\bar{x}_k \geq \left(\frac{2\varepsilon}{3}\right)^{1/2}||u_0 - \bar{u}||_{L^2(I_k^{**})} \geq$$
$$\geq c\varepsilon^{1/2}\kappa^{5/2}||u_0 - w||_{L^2(I_k)}\ , \qquad (5.18)$$

where in the last inequality we used Lemma 2. Using (5.18) and the fact that for type 1 intervals $H_k \leq 6/M \leq c\varepsilon^{1/2}\beta^{-1/2}$ (see (3.9) and (4.15)), we find (5.17). Let us now prove Lemma 1 and 2.

Proof of Lemma 1. Let us start by showing that $N(x)|_{I_k} \geq 2$. Let us assume by contradiction that there exists x^* such that $N(x^*)|_{I_k} < 2$. Let $v(y) \equiv u(x^*, y)$ and let us consider the intervals $J_{k,1} = (z_{2k+1} - \frac{\kappa}{4M}, z_{2k+1} + \frac{\kappa}{4M})$ and $J_{k,2} = (z_{2k+2} - \frac{\kappa}{4M}, z_{2k+2} + \frac{\kappa}{4M})$. Note that by the definition of type 1 intervals, $J_{k,1}$ and $J_{k,2}$ are disjoint and both contained in I_k. Since $v(y)$ has less than two corner points in I_k, then $v(y)$ has no corner points in at least one of the two intervals $J_{k,1}$ and $J_{k,2}$, say in $J_{k,1}$. Now, $3\widetilde{F}_k^{(1)} \geq \int_{x^*}^1 dx \int_{I_k} dy\, u_x^2 \geq ||u_1 - v||^2_{L_2(J_{k,1})}$. Using that v has no corners in $J_{k,1}$, one finds that $3\widetilde{F}_k^{(1)} \geq ||u_1 - v||^2_{L_2(J_{k,1})} \geq c\kappa^3/M^3$, a contradiction if $\eta\kappa^{-3}$ is sufficiently small. The proof that $N(x)|_{I_k^{L,R}} \geq 1$ is completely analogous. This proves Lemma 1. Moreover, it proves that $u(x,\cdot)$ has at least one corner in each of the intervals $J_{k-1,2}$, $J_{k,1}$, $J_{k,2}$, $J_{k+1,1}$, $\forall x \in [0,1]$. ∎

Proof of Lemma 2. By the definition of $\bar{x}_k$ and by the result of Lemma 1, $\bar{u}(y) = u(\bar{x}_k, y)$ has exactly 1 corner point in I_k^L (located in $J_{k-1,2}$), exactly 1 corner point in I_k^R (located in $J_{k+1,1}$) and exactly 2 corner points in I_k (one located in $J_{k,1}$ and one in $J_{k,2}$). We shall denote by z_j^*, $j = 0,1,2,3$, these corner points (with $z_0^* < z_1^* < z_2^* < z_3^*$). Moreover, $\bar{u}'(y) = +1$ if $y \in (z_0^*, z_1^*) \cup (z_2^*, z_3^*)$ and $\bar{u}'(y) = -1$ if $y \in (z_1^*, z_2^*)$. By the definition of $J_{k,1}$ and $J_{k,2}$, we have that $z_2^* - z_1^* \geq \kappa/(2M)$ and $\min\{a_k - z_0^*, z_1^* - a_k, a_{k+1} - z_2^*, z_3^* - a_{k+1}\} \geq \kappa/(4M)$.

Let $\delta := M^{3/2}||u_0 - \bar{u}||_{L^2(I_k^{**})}$. If $\delta > \delta_0$, with $\delta_0 = \bar{c}_0\kappa^{5/2}$, then (5.16) is proved; in fact, in this case, since $u_0 = w$ on ∂I_k and $|(u_0 - w)'| \leq 2$, $||u_0 - w||_{L^2(I_k)} \leq c' H_k^{3/2} \leq c'' M^{-3/2} \leq (c''/\delta_0)||u_0 - \bar{u}||_{L^2(I_k^{**})} = c'''\kappa^{-5/2}||u_0 - \bar{u}||_{L^2(I_k^{**})}$, which is the desired estimate.

Let then $\delta \leq \delta_0$ and let us note that $||w - \bar{u}||_{L^\infty(\partial I_k)} = ||u_0 - \bar{u}||_{L^\infty(\partial I_k)} \leq 4\kappa^{-1/2}\delta M^{-1}$. Indeed, if, by contradiction, $u_0(a_k) - \bar{u}(a_k) > 4\kappa^{-1/2}\delta M^{-1}$, then $u_0(x) - \bar{u}(x) = u_0(a_k) - \bar{u}(a_k) + \int_{a_k}^x (u_0' - 1)(y)dy \geq u_0(a_k) - \bar{u}(a_k) > 4\kappa^{-1/2}\delta M^{-1}$, for all $x \in (z_0^*, a_k)$ (here we used that $\bar{u}' = 1$ in (z_0^*, z_1^*) and $|u_0'| \leq 1$); similarly, if $\bar{u}(a_k) - u_0(a_k) > 4\kappa^{-1/2}\delta M^{-1}$, then $\bar{u}(x) - u_0(x) \geq \bar{u}(a_k) - u_0(a_k) > 4\kappa^{-1/2}\delta M^{-1}$, for all $x \in (a_k, z_1^*)$; in both cases, using the fact that $\min\{a_k - z_0^*, z_1^* - a_k\} \geq$

$\kappa/(4M)$, we would find $\delta M^{-3/2} \equiv ||u_0 - \bar{u}||_{L^2(I_k^{**})} > 2\delta M^{-3/2}$, a contradiction. Now, let $g = w - \bar{u}$ and let $g^* = g(y^*)$, with $y^* \in I_k$, such that $|g(y^*)| = ||g||_{L^\infty(I_k)}$. We want to prove that if κ is sufficiently small, then $|g^*| \le \kappa^{-5/2}\delta M^{-1}$; if this is the case, then $||u_0 - w||_{L^2(I_k)} \le \delta M^{-3/2} + ||g||_{L^2(I_k)} \le \delta M^{-3/2} + \sqrt{6} M^{-1/2}\kappa^{-5/2}\delta M^{-1} = (1+\sqrt{6}\kappa^{-5/2})||u_0 - \bar{u}||_{L^2(I_k^{**})}$, which is the desired bound.

Let us then assume by contradiction that $|g^*| > \kappa^{-5/2}\delta M^{-1}$. Note that by construction g has the following properties:

1. $||g||_{L^\infty(\partial I_k)} \le 4\kappa^{-1/2}\delta M^{-1}$;

2. there exist y_1, y_2, y_3, y_4 such that: (i) $a_k \le y_1 \le y_2 \le y_3 \le y_4 \le a_{k+1}$; (ii) $g'(y) = 0$ for $y \in (a_k, y_1) \cup (y_2, y_3) \cup (y_4, a_{k+1})$, $g'(y) = m$ for $y \in (y_1, y_2)$ and $g'(y) = -m$ for $y \in (y_3, y_4)$, with $|m| = 2$;

3. if we define $\Delta_1 = y_2 - y_1$, $\Delta_2 = y_3 - y_2$ and $\Delta_3 = y_4 - y_3$, then $\Delta_1 + \Delta_2 + \Delta_3 \ge z_2^* - z_1^* \ge \kappa/(2M)$.

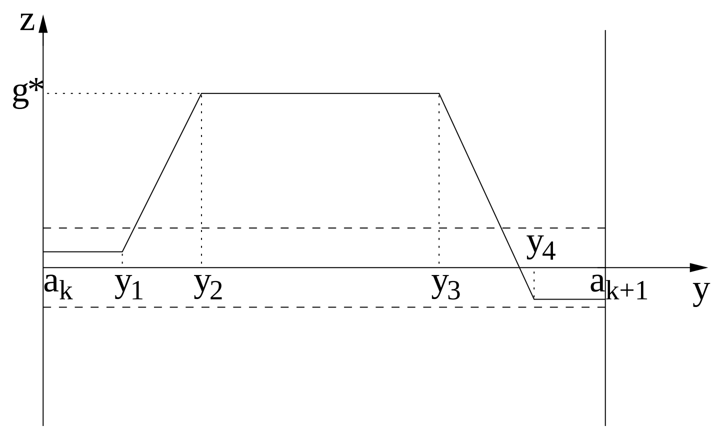

FIG. 3: The function $z = g(y)$ in the interval $I_k = [a_k, a_{k+1})$. The two horizontal dashed lines are $z = \pm 4\kappa^{-1/2}\delta M^{-1}$. Since $||g||_{L^\infty(\partial I_k)} \le 4\kappa^{-1/2}\delta M^{-1}$, the two horizontal portions of the graph of g in the intervals (a_k, y_1) and (y_4, a_{k+1}) stay inside the strip $-4\kappa^{-1/2}\delta M^{-1} \le z \le 4\kappa^{-1/2}\delta M^{-1}$.

Let us assume without loss of generality that $m = +2$, so that $g^* = \max_{I_k} g > \kappa^{-5/2}\delta M^{-1}$ and $(g^* - 4\kappa^{-1/2}\delta M^{-1})/2 \le \Delta_i \le (g^* + 4\kappa^{-1/2}\delta M^{-1})/2$, both for $i = 1$ and $i = 3$. Now, if $\Delta_2 \le \kappa/(4M)$, then $\Delta_1 + \Delta_3 \ge \kappa/(4M)$ and $\Delta_1^2 + \Delta_3^2 \ge \kappa^2/(32M^2)$. On the other hand, using that $\int_{I_k}(u_0 - \bar{u}) = \int_{I_k} g$, we find:

$$\delta M^{-3/2} = ||u_0 - \bar{u}||_{L^2(I_k^{**})} \ge H_k^{-1/2}\Big|\int_{I_k}(u_0 - \bar{u})\Big| = H_k^{-1/2}\Big|\int_{I_k} g\Big| \ge \frac{M^{1/2}}{\sqrt{6}}\Big|\int_{I_k} g\Big| . \tag{5.19}$$

Now, denoting by $\widetilde{y}_1$ and $\widetilde{y}_4$ the two points $\widetilde{y}_1 = y_2 - \frac{g^*-4\kappa^{-1/2}\delta M^{-1}}{2}$ and $\widetilde{y}_4 = y_3 + \frac{g^*-4\kappa^{-1/2}\delta M^{-1}}{2}$ such that $g(\widetilde{y}_1) = g(\widetilde{y}_4) = +4\kappa^{-1/2}\delta M^{-1}$ (see Fig.3), we can bound $|\int_{I_k} g|$ from below as $|\int_{I_k} g| \geq \int_{\widetilde{y}_1}^{y_2}[g^* - 2(y_2-y)]dy + \int_{y_3}^{\widetilde{y}_4}[g^*-2(y-y_3)]dy - 4\kappa^{-1/2}\delta M^{-1}[(\widetilde{y}_1 - a_k) + (a_{k+1} - \widetilde{y}_4)]$, which implies

$$\begin{aligned} \delta M^{-3/2} &\geq \frac{M^{1/2}}{\sqrt{6}}\Big[\frac{(g^*)^2 - 16\kappa^{-1}\delta^2 M^{-2}}{2} - C'\kappa^{-1/2}\delta M^{-2}\Big] \geq \\ &\geq \frac{M^{1/2}}{\sqrt{6}}\Big[\frac{g^* - 4\kappa^{-1/2}\delta M^{-1}}{g^* + 4\kappa^{-1/2}\delta M^{-1}}(\Delta_1^2 + \Delta_3^2) - C'\kappa^{-1/2}\delta M^{-2}\Big] > \\ &> c' M^{1/2}\Big[\frac{\kappa^2}{M^2} - C''\kappa^{-1/2}\delta M^{-2}\Big] , \end{aligned} \tag{5.20}$$

where in the last inequality we used $g^* > \kappa^{-5/2}\delta M^{-1}$ and the fact that κ is sufficiently small. Eq.(5.20) implies

$$\bar{c}_0 \kappa^{5/2} = \delta_0 \geq \delta > c''\kappa^{5/2} ,$$

a contradiction if $\bar{c}_0$ is chosen small enough. Finally, if $\Delta_2 > \kappa/(4M)$, then

$$\begin{aligned} \delta M^{-3/2} &\geq \frac{M^{1/2}}{\sqrt{6}}\Big|\int_{I_k} g\Big| \geq \frac{M^{1/2}}{\sqrt{6}}\Big[\Delta_2 g^* - C'\kappa^{-1/2}\delta M^{-2}\Big] \\ &> c'\kappa^{-3/2}\delta M^{-3/2} - C''\kappa^{-1/2}\delta M^{-3/2} , \end{aligned} \tag{5.21}$$

which leads to a contradiction if κ is sufficiently small. This concludes the proof of Lemma 2. ■

E. The lower bound: the bad intervals

For intervals of type 2, 3 and 4, the key estimate that we shall use is the following.

Lemma 3 *Let I_k be an interval of any type. There exists a constant C independent of η, κ such that*

$$||u_0 - w||_{L^2(I_k)} \leq C H_k ||u_0 - u_1||^{1/3}_{L^2(I_k)} . \tag{5.22}$$

Proof of Lemma 3. First of all, note that (5.22) is invariant under the rescaling $I_k \to \tilde{I}_k^{(\ell)} = [\ell a_k, \ell a_{k+1})$ combined with $u(y) \to \tilde{u}^{(\ell)}(y) = \ell u(y/\ell)$; therefore, we can freely assume that $H_k = 1$ and we denote by $I = [0, 1)$ the corresponding rescaled (unit) interval. Let y^* be such that $|(u_1 - u_0)(y^*)| = ||u_1 - u_0||_{L^\infty(I)}$; using that $|(u_1 - u_0)'| \leq 2$, we find that $||u_1 - u_0||_{L^\infty(I)} \leq |(u_1 - u_0)(y)| + 2|y - y^*|$, $\forall y \in I$. Without loss of generality, we can assume that y^* is in the left half of I, in which case, for any $0 \leq \delta \leq 1/2$,

$$||u_1 - u_0||_{L^\infty(I)} \leq \int_{y^*}^{y^*+\delta} \frac{dy}{\delta}|(u_1 - u_0)(y)| + \delta \leq \delta^{-1/2}||u_1 - u_0||_{L^2(I)} + \delta . \tag{5.23}$$

Now, if $||u_1-u_0||_{L^2(I)} \geq 2^{-3/2}$, then (5.22) is trivial: in fact $||u_0-w||_{L^2(I)} \leq 1/\sqrt{3}$, simply because $|(u_0-w)(y)| \leq 2\min\{y, 1-y\}$, and, therefore, $||u_0-w||_{L^2(I)} \leq \sqrt{2/3}\,||u_1-u_0||^{1/3}_{L^2(I)}$, which is the desired estimate.

Let us then suppose that $||u_1-u_0||_{L^2(I)} < 2^{-3/2}$. In this case, choosing $\delta = ||u_1-u_0||^{2/3}_{L^2(I)}$ in (5.23), we find that $\tau := ||u_1-u_0||_{L^\infty(I)} \leq 2||u_1-u_0||^{2/3}_{L^2(I)} < 1$. Let us now define, in analogy with the proof of (5.20)-(5.21), $g = w-u_1$, and let $g^* = g(y^*)$, with $y^* \in [0,1]$ such that $|g(y^*)| = ||g||_{L^\infty(I)}$. Note that by construction g has the following properties: there exist y_1, y_2, y_3, y_4 such that $0 \leq y_1 \leq y_2 \leq y_3 \leq y_4 \leq 1$ and $g'(y) = 0$ for $y \in (0, y_1) \cup (y_2, y_3) \cup (y_4, 1)$, $g'(y) = m$ for $y \in (y_1, y_2)$ and $g'(y) = -m$ for $y \in (y_3, y_4)$, with $|m| = 2$. We also define $\Delta_1 = y_2 - y_1$, $\Delta_2 = y_3 - y_2$ and $\Delta_3 = y_4 - y_3$. Let us distinguish two more subcases.

1. $|g^*| < 9\tau$. In this case, $||u_0-w||_{L^2(I)} \leq ||u_0-w||_{L^\infty(I)} \leq 10||u_1-u_0||_{L^\infty(I)} \leq 20||u_1-u_0||^{2/3}_{L^2(I)} \leq 2^{1/2}\cdot 10||u_1-u_0||^{1/3}_{L^2(I)}$, which is the desired bound.

2. $|g^*| \geq 9\tau$. In this case, proceeding as in the proof of (5.20)-(5.21), we find:

$$\begin{aligned} \tau \geq |\int_I (u_1-u_0)| \; &= \; |\int_I (u_1-w)| \geq \frac{1}{2}(\Delta_1^2+\Delta_3^2) + |g^*|\Delta_2 - \tau \\ &\geq \; \frac{1}{4}(\Delta_1+\Delta_3)^2 + |g^*|\Delta_2 - \tau\,. \end{aligned} \tag{5.24}$$

   If $\Delta_1+\Delta_3 \geq 3\sqrt{\tau}$ or $\Delta_2 \geq 1/4$, then (5.24) implies that $2\tau \geq 9\tau/4$, which is a contradiction. Therefore, $\Delta_1+\Delta_3 < 3\sqrt{\tau}$ and $\Delta_2 < 1/4$; using that $(8/9)|g^*| \leq |g^*|-\tau \leq \Delta_1+\Delta_3$, we get $|g^*| \leq (27/8)\sqrt{\tau}$. In conclusion, $||u_0-w||_{L^2(I)} \leq ||u_0-w||_{L^\infty(I)} \leq \tau + (27/8)\sqrt{\tau} < (35/8)\sqrt{\tau} \leq (35\sqrt{2}/8)||u_1-u_0||^{1/3}_{L^2(I)}$, which is the desired estimate. ∎

Let us now show how to use Lemma 3 in order to get a bound from above on $\sum_{k:I_k\in\mathcal{I}_q}(\widetilde{F}_k - \bar{c}\beta H_k^{1/2}||u_0-w||_{L^2(I_k)})$, separately for $q = 2,3,4$.

1. *Intervals of type 2*

In this case, the key remark is that, if κ and $\eta\kappa^{-3}$ are small enough, then necessarily

$$\min_{2k-1\leq j\leq 2k+3} \tilde{h}_j \leq \frac{1}{2M}\,. \tag{5.25}$$

Let us prove this fact. If $\min\{H_{k-1}, H_k, H_{k+1}\} < 1/(2M)$ the claim is obvious, so let us assume that $\min\{H_{k-1}, H_k, H_{k+1}\} \geq 1/(2M)$. Let us first consider the case that $h_{2k^*+1} := \min\{h_{2k-1}, h_{2k+1}, h_{2k+3}\} < \kappa/M$. In this case, using that $h_{2k^*+1} = H_{k^*}/2 - [u_1(a_{k^*+1}) - u_1(a_{k^*})]/2$ and $\tilde{h}_{2k^*+1} = H_{k^*}/2 - [u_0(a_{k^*+1}) - u_0(a_{k^*})]/2$, we find that $|h_{2k^*+1} - \tilde{h}_{2k^*+1}| \leq 2||u_1-u_0||_{L^\infty(I_{k^*})}$. On the other hand, if y^* is such that $|(u_1 - u_0)(y^*)| = ||u_1 - u_0||_{L^\infty(I_{k^*})}$, using that $|(u_1 - u_0)'| \le 2$, we have that $||u_1 - u_0||_{L^\infty(I_{k^*})} \le |(u_1 - u_0)(y)| + 2|y - y^*|$, $\forall y \in I_{k^*}$. Proceeding as in the proof of (5.23), we find that for any $\delta \le H_{k^*}/2$,

$$||u_1 - u_0||_{L^\infty(I_{k^*})} \le \delta^{-1/2} ||u_1 - u_0||_{L^2(I_{k^*})} + \delta \ . \tag{5.26}$$

Choosing $\delta = ||u_1 - u_0||^{2/3}_{L^2(I_{k^*})}$, which, by Poincaré inequality, is smaller than $[3\widetilde{F}^{(1)}_{k^*}]^{1/3} \le (3\eta)^{1/3} M^{-1}$ (which is in turn smaller than $H_{k^*}/2$ for η small enough), we find:

$$\begin{aligned} &|h_{2k^*+1} - \tilde{h}_{2k^*+1}| \le 2||u_1 - u_0||_{L^\infty(I_{k^*})} \le 4||u_1 - u_0||^{2/3}_{L^2(I_{k^*})} \le 4(3\eta)^{1/3} M^{-1} \quad \Rightarrow \\ &\Rightarrow \quad \tilde{h}_{2k^*+1} \le \frac{\kappa + 4(3\eta)^{1/3}}{M} \le \frac{1}{2M} \ , \end{aligned} \tag{5.27}$$

where in the last inequality we assumed that κ and η are small enough.

By definition of type 2 intervals, we are left with the case that $\min\{h_{2k}, h_{2k+2}\} < \kappa/M$. Without loss of generality, we can assume that $h_{2k} < \kappa/M$ and $\min\{h_{2k-1}, h_{2k+1}, h_{2k+3}\} \ge \kappa/M$; by contradiction, we assume that $\tilde{h}_{2k} \ge 1/(2M)$, so that $\max\{\tilde{z}_{2k+1} - a_k, a_k - \tilde{z}_{2k}\} \ge 1/(4M)$, say $\tilde{z}_{2k+1} - a_k \ge 1/(4M)$.

By (5.26), $\tau := ||u_1 - u_0||_{L^\infty(I^{**}_k)} \le \delta^{-1/2} ||u_1 - u_0||_{L^2(I^{**}_k)} + \delta$, so that, choosing $\delta = ||u_1 - u_0||^{2/3}_{L^2(I^{**}_k)}$, we get $||u_1 - u_0||_{L^\infty(I^{**}_k)} \le 2\delta \le 2(3\eta)^{1/3} M^{-1}$, by Poincarè (see the lines following (5.26)). Proceeding in a way analogous to the proof of (5.20)-(5.21), we define $g = w - u_1$, so that:

$$\frac{2(3\eta)^{1/3}}{M} \ge \tau \ge H_k^{-1} \Big| \int_{I_k} (u_0 - u_1) \Big| \ge \frac{M}{6} \Big| \int_{I_k} g \Big| \ . \tag{5.28}$$

Recall the assumptions on h_i and $\tilde{h}_{2k}$: $h_{2k} < \kappa/M$, $\min\{h_{2k-1}, h_{2k+1}, h_{2k+3}\} \ge \kappa/M$, and $\tilde{z}_{2k+1} - a_k \ge 1/(4M)$. Therefore, g has the following properties: there exist y_1, y_2, y_3, y_4 such that $a_k \le y_1 \le y_2 \le y_3 \le y_4 \le a_{k+1}$ and $g'(y) = 0$ for $y \in (0, y_1) \cup (y_2, y_3) \cup (y_4, 1)$, $g'(y) = m$ for $y \in (y_1, y_2)$ and $g'(y) = -m$ for $y \in (y_3, y_4)$, with $|m| = 2$; moreover, if $\Delta_1 := y_2 - y_1$ and $\Delta_2 = y_3 - y_2$, then $\Delta_1 \ge \kappa/M$ and $\Delta_1 + \Delta_2 \ge (1 - 2\kappa)/(4M)$. If κ and $\eta\kappa^{-3}$ are sufficiently small, by proceeding as in the proof of (5.20) and (5.21), we can bound (5.28) from below by

$$\frac{2(3\eta)^{1/3}}{M} \ge \tau \ge cM(\Delta_1^2 + \Delta_1\Delta_2 - c'\tau) \ge \frac{c''\kappa}{M} \ , \tag{5.29}$$

which is a contradiction. If $h_{2k} < \kappa/M$, $\min\{h_{2k-1}, h_{2k+1}, h_{2k+3}\} \ge \kappa/M$, and $a_k - \tilde{z}_{2k} \ge 1/(4M)$, one can proceed in a completely analogous way, by replacing I_k by I_{k-1} in (5.28). This concludes the proof of (5.25).

Once that (5.25) is proved, we find that

$$\frac{\beta c_0}{7}\sum_{j=2k-2}^{2k+4}(\tilde h_j-\frac{1}{M})^2\geq\frac{c\beta}{M^2} \tag{5.30}$$

and, as a consequence, defining $\widetilde F_k^{(01)}:=\widetilde F_k^{(0)}+\widetilde F_k^{(1)}$,

$$\widetilde F_2:=\sum_{k:I_k\in\mathcal{I}_2}\widetilde F_k^{(01)}\geq\frac{c\beta}{M^2}\mathcal{N}_2\quad\Rightarrow\quad\mathcal{N}_2\leq c^{-1}\beta^{-1}M^2\widetilde F_2\ , \tag{5.31}$$

where $\mathcal{N}_2=|\mathcal{I}_2|$ is the number of intervals of type 2. Now, by Lemma 3 and the fact that $||u_0-u_1||^2_{L^2(I_k)}\leq\int_0^1 dx\int_{I_k}dy\,u_x^2\leq 3\widetilde F_k^{(01)}$, we have that

$$\begin{aligned}\bar c\beta\sum_{k:I_k\in\mathcal{I}_2}H_k^{1/2}||u_0-w||_{L^2(I_k)}\ &\leq\ c'\beta\sum_{k:I_k\in\mathcal{I}_2}H_k^{3/2}||u_0-u_1||^{1/3}_{L^2(I_k)}\leq\\ &\leq\ c''\beta\sum_{k:I_k\in\mathcal{I}_2}M^{-3/2}[\widetilde F_k^{(01)}]^{1/6}\ .\end{aligned} \tag{5.32}$$

Using Minkowski's inequality, we find:

$$\sum_{k:I_k\in\mathcal{I}_2}[\widetilde F_k^{(01)}]^{1/6}\leq\Big[\sum_{k:I_k\in\mathcal{I}_2}\widetilde F_k^{(01)}\Big]^{1/6}\Big[\sum_{k:I_k\in\mathcal{I}_2}1\Big]^{5/6}\leq c'\widetilde F_2^{1/6}\Big[M^2\beta^{-1}\widetilde F_2\Big]^{5/6}\ . \tag{5.33}$$

Combining (5.32) and (5.33), we find that

$$\bar c\beta\sum_{k:I_k\in\mathcal{I}_2}H_k^{1/2}||u_0-w||_{L^2(I_k)}\leq c''\beta^{1/6}M^{1/6}\widetilde F_2\leq c'''(\beta\varepsilon^{-1/3})^{1/4}\widetilde F_2\ , \tag{5.34}$$

where in the last inequality we used that $M\leq c(\beta/\varepsilon)^{1/2}$, see Eq.(3.9). By using (5.34), defining $\sigma:=\beta\varepsilon^{-1/3}$ and for any $\alpha>0$, we get

$$\begin{aligned}\sum_{k:I_k\in\mathcal{I}_2}(\widetilde F_k-\bar c\beta H_k^{1/2}||u_0-w||_{L^2(I_k)})\ &\geq\ \frac12\sum_{k:I_k\in\mathcal{I}_2}\widetilde F_k^{(01)}+(1-\sigma^\alpha)\sum_{k:I_k\in\mathcal{I}_2}\widetilde F_k^{(2)}+\\ &+\ \sum_{k:I_k\in\mathcal{I}_2}\Big[(\frac12-c'''\sigma^{1/4})\widetilde F_k^{(01)}+\sigma^\alpha\widetilde F_k^{(2)}\Big]\ .\end{aligned}$$

Now, for σ small, each term in square brackets is positive, simply because $\widetilde F_k^{(01)}\geq c\varepsilon$ and $\widetilde F_k^{(2)}\geq-4\varepsilon$, so that (5.14) with the sums restricted to intervals of type 2 follows.

2. Intervals of type 3

In this case we just use the fact that $||u_0-w||^2_{L^2(I_k)}\leq\int_{I_k}dy(2y)^2=(4/3)H_k^3$, simply because $u_0=w$ on the boundary of I_k and $|(u_0-w)'|\leq 2$. Therefore, if $\sigma=\beta\varepsilon^{-1/3}$

$$\bar c\beta H_k^{1/2}||u_0-w||_{L^2(I_k)}\leq c\beta H_k^2\leq c'\beta M^{-2}\leq c''\sigma^{3/2}\widetilde F_k^{(01)}\ , \tag{5.35}$$

where in the last inequality we used that $c(\beta/\varepsilon)^{1/2} \leq M \leq c'(\beta/\varepsilon)^{1/2}$, by Eq.(3.9)-(4.15), and $M^{-3} \leq \eta^{-1}\widetilde{F}_k^{(01)}$, by the definition of type 3 interval. Using (5.35), we get

$$\sum_{k:I_k\in\mathcal{I}_3} (\widetilde{F}_k - \bar{c}\beta H_k^{1/2}||u_0 - w||_{L^2(I_k)}) \geq \frac{1}{2}\sum_{k:I_k\in\mathcal{I}_3} \widetilde{F}_k^{(01)} + (1-\sigma^\alpha)\sum_{k:I_k\in\mathcal{I}_3} \widetilde{F}_k^{(2)} + \\ + \sum_{k:I_k\in\mathcal{I}_3} \Big[(\frac{1}{2} - c''\sigma^{3/2})\widetilde{F}_k^{(01)} + \sigma^\alpha \widetilde{F}_k^{(2)}\Big] .$$

Now, for σ small, each term in square brackets is positive, simply because $\widetilde{F}_k^{(01)} \geq c\eta\varepsilon\sigma^{-3/2}$ and $\widetilde{F}_k^{(2)} \geq -4\varepsilon$, so that (5.14) with the sums restricted to intervals of type 3 follows.

3. Intervals of type 4

In this case, if $H_{k^*} := \max\{H_{k-1}, H_k, H_{k+1}\}$, we have that $\max_{2k-2\leq j\leq 2k+4} \tilde{h}_j \geq H_{k^*}/3 > 2/M$. Therefore,

$$\frac{\beta c_0}{7}\sum_{j=2k-2}^{2k+4} (\tilde{h}_j - \frac{1}{M})^2 \geq c\beta H_{k^*}^2 \geq \frac{c'\beta}{M^2} \tag{5.36}$$

and, as a consequence,

$$\widetilde{F}_4 := \sum_{k:I_k\in\mathcal{I}_4} \widetilde{F}_k^{(01)} \geq c\beta \sum_{k:I_k\in\mathcal{I}_4} H_{k^*}^2 \geq \frac{c'\beta}{M^2}\mathcal{N}_4 \quad \Rightarrow \quad \mathcal{N}_4 \leq C\beta^{-1}\widetilde{F}_4 M^2 , \tag{5.37}$$

with $\mathcal{N}_4 = |\mathcal{I}_4|$ the number of intervals of type 4. On the other hand, by Lemma 3, we have that

$$\bar{c}\beta \sum_{k:I_k\in\mathcal{I}_4} H_k^{1/2}||u_0 - w||_{L^2(I_k)} \leq c\beta \sum_{k:I_k\in\mathcal{I}_4} H_k^{3/2}||u_0 - u_1||_{L^2(I_k)}^{1/3} . \tag{5.38}$$

By Poincaré inequality, $||u_0 - u_1||_{L^2(I_k)}^2 \leq \int_0^1 dx \int_{I_k} dy\, u_x^2 \leq 3\widetilde{F}_k^{(01)}$, so that

$$\bar{c}\beta \sum_{k:I_k\in\mathcal{I}_4} H_k^{1/2}||u_0 - w||_{L^2(I_k)} \leq c\beta \sum_{k:I_k\in\mathcal{I}_4} H_k^{3/2}[\widetilde{F}_k^{(01)}]^{1/6} \leq \\ \leq c\beta\Big[\sum_{k:I_k\in\mathcal{I}_4} H_k^{9/5}\Big]^{5/6}\Big[\sum_{k:I_k\in\mathcal{I}_4} \widetilde{F}_k^{(01)}\Big]^{1/6} , \tag{5.39}$$

where the last inequality is Minkowski's. Another application of Minkowski's inequality shows that

$$\Big[\sum_{k:I_k\in\mathcal{I}_4} H_k^{9/5}\Big]^{5/6} \leq \Big[\sum_{k:I_k\in\mathcal{I}_4} H_k^2\Big]^{3/4}\Big[\sum_{k:I_k\in\mathcal{I}_4} 1\Big]^{1/12} \leq \\ \leq c\Big[\frac{\widetilde{F}_4}{\beta}\Big]^{3/4}\mathcal{N}_4^{1/12} \leq c'\Big[\frac{\widetilde{F}_4}{\beta}\Big]^{3/4}\Big[\frac{\widetilde{F}_4 M^2}{\beta}\Big]^{1/12} , \tag{5.40}$$

where in the last two inequalities we used (5.37). Substituting in (5.39) we find

$$\bar{c}\beta \sum_{k:I_k\in\mathcal{I}_4} H_k^{1/2}||u_0 - w||_{L^2(I_k)} \le c(\beta M)^{1/6}\widetilde{F}_4 \le c'\sigma^{1/4}\widetilde{F}_4\ , \qquad (5.41)$$

with $\sigma = \beta\varepsilon^{-1/3}$. Eq.(5.41) implies

$$\sum_{k:I_k\in\mathcal{I}_4} (\widetilde{F}_k - \bar{c}\beta H_k^{1/2}||u_0 - w||_{L^2(I_k)}) \ \ge\ \frac{1}{2}\sum_{k:I_k\in\mathcal{I}_4} \widetilde{F}_k^{(01)} + (1-\sigma^\alpha)\sum_{k:I_k\in\mathcal{I}_4} \widetilde{F}_k^{(2)} + \\ + \sum_{k:I_k\in\mathcal{I}_4} \Big[(\frac{1}{2} - c'\sigma^{1/4})\widetilde{F}_k^{(01)} + \sigma^\alpha \widetilde{F}_k^{(2)}\Big]\ .$$

Now, for σ small, each term in square brackets is positive, simply because $\widetilde{F}_k^{(01)} \ge c\varepsilon$ and $\widetilde{F}_k^{(2)} \ge -4\varepsilon$, so that (5.14) with the sums restricted to intervals of type 4 follows.

Combining the estimates for all different types of intervals, which are all valid for κ and $\eta\kappa^{-3}$ sufficiently small, we finally get (5.14), which implies Theorem 1, as discussed after (5.14). ∎

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Appendix A

In this appendix we prove (3.4). Without loss of generality, we can assume that $u_0(y)$ has a corner point in $y = 0$. Now, for any fixed $\alpha \in (0, +\infty)$ we rewrite:

$$\int_0^h dy \int_{-\infty}^{+\infty} dy'\, |u_0(y) - \tilde{u}_0(y')|^2 e^{-\alpha|y-y'|} = \frac{4}{\alpha}\int_0^h dy|u_0(y)|^2 - \\ -2\lim_{N\to\infty}\frac{1}{N}\int_0^{Nh} dy \int_0^{Nh} dy' \tilde{u}_0(y)\tilde{u}_0(y')e^{-\alpha|y-y'|}\ . \qquad (A.1)$$

The latter integral is in a form suitable for applying the "Chessboard estimate with Dirichlet boundary conditions" proved in [19], see (3.12) of [19]. However, in this case we want to use "ferromagnetic" reflections, rather than the "antiferromagnetic" reflections used in [19]: in other words, we want to keep reflecting u_0 around the locations of its corner points y_i, $i = 0, 1, \ldots, M_0 - 1$, *without changing sign to the reflected function*. The result, analogue to (3.12) in [19], is:

$$\frac{1}{N}\int_0^{Nh} dy \int_0^{Nh} dy' \tilde{u}_0(y)\tilde{u}_0(y')e^{-\alpha|y-y'|} \leq$$
$$\leq \sum_{i=0}^{M_0-1} \int_{y_i}^{y_{i+1}} dy \int_{-\infty}^{+\infty} dy' u_0^{(i)}(y)\tilde{u}_0^{(i)}(y')e^{-\alpha|y-y'|} , \qquad \text{(A.2)}$$

with $u_0^{(i)}$ the restriction of u_0 to the interval $[y_i, y_{i+1}]$, and $\tilde{u}_0^{(i)}$ its periodic extension to the whole real line. Eq.(A.2), combined with (A.1), gives (3.4).

For completeness, we provide here a proof of (A.2) along the lines of [19] (and using a notation as close as possible to the one of [19]). We need to introduce some definitions.

Definition 1. *Given a finite interval $[a, b]$ on the real line, let $\mathcal{E}^\alpha_{a,b} : L^2([a, b]) \to \mathbb{R}$ be the functional defined as*

$$\mathcal{E}^\alpha_{a,b}(w) := -\int_a^b dy \int_a^b dy'\, w(y)w(y')e^{-\alpha|y-y'|} . \qquad \text{(A.3)}$$

Definition 2. *Let $m, n \in \mathbb{Z}^+ \cup \{+\infty\}$ be such that $m + n \geq 1$. Let $\mathcal{F} = \{f_{-m+1}, \ldots, f_0, f_1, \ldots, f_n\}$ be a sequence of functions $f_i \in L^2([0, T_i])$ and $T_i > 0$, with $-m < i \leq n$. Let $z_{-m} = -\sum_{j=-m+1}^{0} T_j$ and $z_i = z_{-m} + \sum_{j=-m+1}^{i} T_j$, for all $-m < i \leq n$ (if $m = 0$ it is understood that $z_0 = 0$). Then we define $\varphi[\mathcal{F}] \in L^2_{\text{loc}}([z_{-m}, z_n])$ to be the function obtained by juxtaposing the functions f_i on the real line, in such a way that, if $z_{i-1} \leq y \leq z_i$, then $\varphi[\mathcal{F}](y) = f_i(y - z_{i-1})$, for all $i = -m + 1, \ldots, n$.*

Definition 3. *(i) Given $T > 0$ and $f \in L^2([0, T])$, we define $\theta f \in L^2([0, T])$ to be the reflection of f, namely $\theta f(y) = f(T - y)$, for all $y \in [0, T]$.*
(ii) If $f \in L^2([0, T])$, we define $\varphi[f] = \varphi[\mathcal{F}_\infty(f)] \in L^2_{\text{loc}}(\mathbb{R})$, where $\mathcal{F}_\infty(f) = \{\ldots, f_0, f_1, \ldots\}$ is the infinite sequence with $f_n = \theta^{n-1} f$.
(iii) Given a sequence $\mathcal{F} = \{f_{-m+1}, \ldots, f_n\}$ as in Def.2, we define $\mathcal{F}_- = \{f_{-m+1}, \ldots, f_0\}$ and $\mathcal{F}_+ = \{f_1, \ldots, f_n\}$ (if $m = 0$ or $n = 0$, it is understood that $\mathcal{F}_-$ or, respectively, $\mathcal{F}_+$ is empty) and we write $\mathcal{F} = (\mathcal{F}_-, \mathcal{F}_+)$.
(iv) The reflections of $\mathcal{F}_-$ and $\mathcal{F}_+$ are defined to be: $\theta\mathcal{F}_- = \{\theta f_0, \ldots, \theta f_{-m+1}\}$ and $\theta\mathcal{F}_+ = \{\theta f_n, \ldots, \theta f_1\}$.

Given the definitions above, the analogue of the "Chessboard estimate with Dirichlet boundary conditions" of [19] adapted to the present context is the following.

Lemma A.1 [Chessboard estimate with Dirichlet boundary conditions] *Given a finite sequence of functions $\mathcal{F} = \{f_1, \ldots, f_n\}$, $n \geq 1$, as in Definition 2, with $f_i \in L^2([0, T_1])$, we have:*

$$\mathcal{E}^{\alpha}_{0,z_n}(\varphi[\mathcal{F}]) \geq \sum_{i=1}^{n} T_i e_\infty(f_i) \;, \tag{A.4}$$

with

$$e_\infty(f_i) := \lim_{n\to\infty} \frac{\mathcal{E}^{\alpha}_{0,nT_i}(\varphi[f_i])}{nT_i} \;. \tag{A.5}$$

(Note that the limit in the r.h.s. of (A.5) exists, because $\varphi[f_i]$ is periodic and the potential $e^{-\alpha|y|}$ appearing in the definition of $\mathcal{E}^{\alpha}_{0,nT_i}$ is summable).

Lemma A.1 is the desired estimate. It immediately implies (A.2). In fact, let: (i) $n = NM_0$; (ii) $0 = y_0 < y_1 < \cdots < y_n = Nh$ be the locations of the corner points of $\tilde{u}_0$ in $[0, Nh]$; (iii) $T_i = y_i - y_{i-1}$; (iv) $f_i(y) = u_0^{(i)}(y - y_{i-1})$, $i = 1, \ldots, n$. With these definitions, $\varphi[\{f_1, \ldots, f_n\}] = \tilde{u}_0$ on $[0, Nh]$ and $e_\infty(f_i) = T_i^{-1} \int_{y_i}^{y_{i+1}} dy \int_{-\infty}^{+\infty} dy' u_0^{(i)}(y) \tilde{u}_0^{(i)}(y') e^{-\alpha|y-y'|}$; in particular, (A.4) reduces to (A.2).

We are then left with proving Lemma A.1. A basic ingredient in the proof of Lemma A.1 is the following "reflection positivity estimate" (which is the analogue of Lemma 1 of [19]).

Lemma A.2 *Given a finite sequence of functions $\mathcal{F} = \{f_{-m+1}, \ldots, f_0, f_1, \ldots, f_n\} = (\mathcal{F}_-, \mathcal{F}_+)$, as in Def.2 and 3, we have:*

$$\mathcal{E}^{\alpha}_{z_{-m},z_n}(\varphi[\mathcal{F}]) \geq \frac{1}{2}\mathcal{E}^{\alpha}_{-z_n,z_n}(\varphi[\mathcal{F}_1]) + \frac{1}{2}\mathcal{E}^{\alpha}_{z_{-m},-z_{-m}}(\varphi[\mathcal{F}_2]) \;, \tag{A.6}$$

where $\mathcal{F}_1 = (\theta\mathcal{F}_+, \mathcal{F}_+) = \{\theta f_n, \ldots, \theta f_1, f_1, \ldots, f_n\}$ and $\mathcal{F}_2 = (\mathcal{F}_-, \theta\mathcal{F}_-) = \{f_{-m+1}, \ldots, f_0, \theta f_0, \ldots, \theta f_{-m+1}\}$.

Proof of Lemma A.2. We rewrite

$$\begin{aligned}
\mathcal{E}^{\alpha}_{z_{-m},z_n}(\varphi[\mathcal{F}]) = \; & -\int_{z_{-m}}^{0} dy \int_{z_{-m}}^{0} dy' \varphi[\mathcal{F}](y)\varphi[\mathcal{F}](y') e^{-\alpha|y-y'|} \\
& -\int_{0}^{z_n} dy \int_{0}^{z_n} dy' \varphi[\mathcal{F}](y)\varphi[\mathcal{F}](y') e^{-\alpha|y-y'|} \\
& -2\int_{z_{-m}}^{0} dy \int_{0}^{z_n} dy' \varphi[\mathcal{F}](y)\varphi[\mathcal{F}](y') e^{-\alpha(y'-y)} \;.
\end{aligned} \tag{A.7}$$

Now, notice that last term on the r.h.s. of (A.7) can be rewritten and estimated as:

$$\int_{z_{-m}}^{0} dy \int_{0}^{z_n} dy' \varphi[\mathcal{F}](y)\varphi[\mathcal{F}](y') e^{-\alpha(y'-y)}$$

$$= \int_0^{-z_{-m}} dy \varphi[(\mathcal{F}_-, \theta\mathcal{F}_-)](y) e^{-\alpha y} \int_0^{z_n} dy' \varphi[(\theta\mathcal{F}_+, \mathcal{F}_+)](y') e^{-\alpha y'} \tag{A.8}$$
$$\leq \frac{1}{2}\Big[\int_0^{-z_{-m}} dy \varphi[(\mathcal{F}_-, \theta\mathcal{F}_-)](y) e^{-\alpha y}\Big]^2 + \frac{1}{2}\Big[\int_0^{z_n} dy' \varphi[(\theta\mathcal{F}_+, \mathcal{F}_+)](y') e^{-\alpha y'}\Big]^2 ,$$

which is equivalent to

$$\int_{z_{-m}}^0 dy \int_0^{z_n} dy' \varphi[\mathcal{F}](y)\varphi[\mathcal{F}](y') e^{-\alpha(y'-y)} \leq$$
$$\leq \frac{1}{2}\int_{z_{-m}}^0 dy \int_0^{-z_{-m}} dy' \varphi[\mathcal{F}_1](y)\varphi[\mathcal{F}_1](y') e^{-\alpha(y'-y)} \tag{A.9}$$
$$+\frac{1}{2}\int_{-z_n}^0 dy \int_0^{z_n} dy' \varphi[\mathcal{F}_2](y)\varphi[\mathcal{F}_2](y') e^{-\alpha(y'-y)} ,$$

with $\mathcal{F}_1 = (\mathcal{F}_-, \theta\mathcal{F}_-)$ and $\mathcal{F}_2 = (\theta\mathcal{F}_+, \mathcal{F}_+)$. Now, (A.6) follows by plugging (A.9) into (A.7) and by using that

$$-\int_{z_{-m}}^0 dy \int_{z_{-m}}^0 dy' \varphi[\mathcal{F}](y)\varphi[\mathcal{F}](y') e^{-\alpha|y-y'|} \tag{A.10}$$
$$-\frac{1}{2}\int_{z_{-m}}^0 dy \int_0^{-z_{-m}} dy' \varphi[\mathcal{F}_1](y)\varphi[\mathcal{F}_1](y') e^{-\alpha(y'-y)} = \frac{1}{2}\mathcal{E}^\alpha_{z_{-m},-z_{-m}}(\varphi[\mathcal{F}_1]) ,$$

and

$$-\int_0^{z_n} dy \int_0^{z_n} dy' \varphi[\mathcal{F}](y)\varphi[\mathcal{F}](y') e^{-\alpha|y-y'|} \tag{A.11}$$
$$-\frac{1}{2}\int_{-z_n}^0 dy \int_0^{z_n} dy' \varphi[\mathcal{F}_2](y)\varphi[\mathcal{F}_2](y') e^{-\alpha(y'-y)} = \frac{1}{2}\mathcal{E}^\alpha_{-z_n,z_n}(\varphi[\mathcal{F}_2]) .$$

■

At this point, in order to prove Lemma A.2, one needs to inductively iterate the key estimate (A.6), as explained in the following.

Proof of Lemma A.1. We proceed by induction.
(i) If $n = 1$, we first rewrite

$$\mathcal{E}^\alpha_{0,2z_1}(\varphi[\{f_1, \theta f_1\}]) = 2\mathcal{E}^\alpha_{0,z_1}(f_1) - 2\int_0^{z_1} dy \int_{z_1}^{2z_1} dy' f_1(y)\, \theta f_1(y'-z_1) e^{-\alpha(y'-y)} , \tag{A.12}$$

and we notice that, by definition of θf_1, the second term in the r.h.s. of (A.12) can be rewritten and estimated as

$$\int_0^{z_1} dy \int_{z_1}^{2z_1} dy' f_1(y) f_1(2z_1 - y') e^{-\alpha(y'-y)} = \Big[\int_0^{z_1} dy f_1(y) e^{-\alpha(z_1-y)}\Big]^2 \geq 0 \tag{A.13}$$

By combining (A.12) and (A.13) we get

$$\mathcal{E}^\alpha_{0,z_1}(f_1) \geq \frac{1}{2}\mathcal{E}^\alpha_{0,2z_1}(\varphi[\{f_1, \theta f_1\}]) . \tag{A.14}$$

Iterating the same argument, we find:

$$\mathcal{E}^{\alpha}_{0,z_1}(f_1) \geq \frac{\mathcal{E}^{\alpha}_{0,2^m z_1}(\varphi[f_1^{\otimes 2^m}])}{2^m} , \tag{A.15}$$

where, by definition,

$$f_1^{\otimes 2^m} = \{ \overbrace{f_1, \theta f_1, \ldots, f_1, \theta f_1}^{2^m \text{ times}} \} . \tag{A.16}$$

Taking the limit $m \to \infty$ in (A.15) we get the desired estimate:

$$\mathcal{E}^{\alpha}_{0,z_1}(f_1) \geq T_1 e_{\infty}(f_1) . \tag{A.17}$$

(ii) Let us now assume by induction that the bound is valid for all $1 \leq n \leq k-1$, $k \geq 2$, and let us prove it for $n = k$. There are two cases.

(a) $k = 2p$ for some $p \geq 1$. If we reflect once, by Lemma A.2 we have:

$$\begin{aligned} &\mathcal{E}^{\alpha}_{0,z_{2p}}(\varphi[\{f_1, \ldots, f_{2p}\}]) \geq \\ &\quad \geq \frac{1}{2}\mathcal{E}^{\alpha}_{0,2(z_{2p}-z_p)}(\varphi[\{\theta f_{2p}, \ldots, \theta f_{p+2}, (\theta f_{p+1})^{\otimes 2}, f_{p+2}, \ldots f_{2p}\}]) + \\ &\quad + \frac{1}{2}\mathcal{E}^{\alpha}_{0,2z_p}(\varphi[\{f_1, \ldots, f_{p-1}, f_p^{\otimes 2}, \theta f_{p-1}, \ldots, \theta f_1\}]) \end{aligned} \tag{A.18}$$

If we now regard $(\theta f_{p+1})^{\otimes 2}$ and $f_p^{\otimes 2}$ as two new functions in $L^2([0, 2T_{p+1}])$ and in $L^2([0, 2T_p])$, respectively, the two terms in the r.h.s. of (A.18) can be regarded as two terms with $n = 2p-1$ and, by the induction assumption, they satisfy the bounds:

$$\begin{aligned} &\mathcal{E}^{\alpha}_{0,2(z_{2p}-z_p)}(\varphi[\{\theta f_{2p}, \ldots, \theta f_{p+2}, (\theta f_{p+1})^{\otimes 2}, f_{p+2}, \ldots f_{2p}\}]) \geq 2 \sum_{i=p+1}^{2p} T_i e_{\infty}(f_i) , \\ &\mathcal{E}^{\alpha}_{0,2z_p}(\varphi[\{f_1, \ldots, f_{p-1}, f_p^{\otimes 2}, \theta f_{p-1}, \ldots, \theta f_1\}]) \geq 2 \sum_{i=1}^{p} T_i e_{\infty}(f_i) , \end{aligned} \tag{A.19}$$

where we used that $e_{\infty}((\theta f_{p+1})^{\otimes 2}) = e_{\infty}(f_{p+1})$ and $e_{\infty}(f_p^{\otimes 2}) = e_{\infty}(f_p)$. Therefore, the desired bound is proved.

(b) $k = 2p+1$ for some $p \geq 1$. If we reflect once, by Lemma A.2 we have:

$$\begin{aligned} &\mathcal{E}^{\alpha}_{0,z_{2p+1}}(\varphi[\{f_1, \ldots, f_{2p+1}\}]) \geq \\ &\geq \frac{1}{2}\mathcal{E}^{\alpha}_{0,2(z_{2p+1}-z_{p+1})}(\varphi[\{\theta f_{2p+1}, \ldots, \theta f_{p+3}, (\theta f_{p+2})^{\otimes 2}, f_{p+3}, \ldots, f_{2p+1}\}]) + \\ &+ \frac{1}{2}\mathcal{E}^{\alpha}_{0,2z_{p+1}}(\varphi[\{f_1, \ldots, f_p, f_{p+1}^{\otimes 2}, \theta f_p, \ldots, \theta f_1\}]) \end{aligned} \tag{A.20}$$

The first term in the r.h.s. corresponds to $n = 2p-1$ so by the induction hypothesis it is bounded below by $\sum_{i=p+2}^{2p+1} T_i e_{\infty}(f_i)$. As regards the second term, using Lemma A.2 again, we can bound it from below by

$$\frac{1}{4}\mathcal{E}^{\alpha}_{0,2z_p}(\varphi[\{f_1,\ldots,f_p,\theta f_p,\ldots,\theta f_1\}]) + \\ +\frac{1}{4}\mathcal{E}^{\alpha}_{0,2z_p+4z_{p+1}}(\varphi[\{f_1,\ldots,f_p,(f_{p+1})^{\otimes 4},\theta f_p,\ldots,\theta f_1\}]) \tag{A.21}$$

By the induction hypothesis, the first term is bounded below by $(1/2)\sum_{i=1}^{p} T_i e_\infty(f_i)$, and the second can be bounded by Lemma A.2 again. Iterating we find:

$$\mathcal{E}^{\alpha}(\varphi[\{f_1,\ldots,f_{2p+1}\}]) \geq \tag{A.22}$$
$$\geq \sum_{i=p+2}^{2p+1} T_i e_\infty(f_i) + \Big(\sum_{n\geq 1} 2^{-n}\Big)\cdot\sum_{i=1}^{p} T_i e_\infty(f_i)\Big) + \\ + \lim_{n\to\infty} 2^{-n}\mathcal{E}^{\alpha}_{0,2z_p+2^m z_{p+1}}(\varphi[\{f_1,\ldots,f_p,(f_{p+1})^{\otimes 2^m},\theta f_p,\ldots,\theta f_1\}])\ .$$

Note that the last term is equal to $T_{p+1}e_\infty(f_{p+1})$, so (A.22) is the desired bound. This concludes the proof of (A.11). ■

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