ON THE FEKETE-SZEGÖ PROBLEM FOR CONCAVE UNIVALENT FUNCTIONS

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Abstract. We consider the Fekete-Szegö problem with real parameter λ for the class $Co(\alpha)$ of concave univalent functions.

1. Introduction

Let $\mathcal{S}$ denote the class of all univalent (analytic) functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

defined on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Then the classical Fekete-Szegö inequality, presented by means of Loewner's method, for the coefficients of $f \in \mathcal{S}$ is that

$$|a_3 - \lambda a_2^2| \leq 1 + 2\exp(-2\lambda/(1-\lambda)) \quad \text{for } \lambda \in [0,1).$$

As $\lambda \to 1-$, we have the elementary inequality $|a_3 - a_2^2| \leq 1$. Moreover, the coefficient functional

$$\Lambda_\lambda(f) = a_3 - \lambda a_2^2$$

on the normalized analytic functions f in the unit disk $\mathbb{D}$ plays an important role in function theory. For example, the quantity $a_3 - a_2^2$ represents $S_f(0)/6$, where S_f denotes the Schwarzian derivative $(f''/f')' - (f''/f')^2/2$ of locally univalent functions f in $\mathbb{D}$. In the literature, there exists a large number of results about inequalities for $\Lambda_\lambda(f)$ corresponding to various subclasses of $\mathcal{S}$. The problem of maximizing the absolute value of the functional $\Lambda_\lambda(f)$ is called the Fekete-Szegö problem. In [8], Koepf solved the Fekete-Szegö problem for close-to-convex functions and the largest real number λ for which $\Lambda_\lambda(f)$ is maximized by the Koebe function $z/(1-z)^2$ is $\lambda = 1/3$, and later in [9] (see also [10]), this result was generalized for functions that are close-to-convex of order β. In [12], Pfluger employed the variational method to give another treatment of the Fekete-Szegö inequality which includes a description of the image domains under extremal functions. Later, Pfluger [13] used Jenkin's method to show that

$$|\Lambda_\lambda(f)| \leq 1 + 2|\exp(-2\lambda/(1-\lambda))|, \; f \in \mathcal{S},$$

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holds for complex λ such that $\mathrm{Re}\,(1/(1-\lambda)) \geq 1$. The inequality is sharp if and only if λ is in a certain pear shaped subregion of the disk given by

$$\lambda = 1 - (u + itv)/(u^2 + v^2),\ 1 \leq t \leq 1,$$

where $u = 1 - \log(\cos\phi)$ and $v = \tan\phi - \phi$, $0 < \phi < \pi/2$.

In this paper, we solve the Fekete-Szegö problem for functions in the class $Co(\alpha)$ of concave univalent functions, with real parameter λ.

2. Preliminaries

A function $f : \mathbb{D} \to \mathbb{C}$ is said to belong to the family $Co(\alpha)$ if f satisfies the following conditions:

(i) f is analytic in $\mathbb{D}$ with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition it satisfies $f(1) = \infty$.
(ii) f maps $\mathbb{D}$ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.
(iii) the opening angle of $f(\mathbb{D})$ at ∞ is less than or equal to $\pi\alpha$, $\alpha \in (1, 2]$.

This class has been extensively studied in the recent years and for a detailed discussion about concave functions, we refer to [1, 2, 6] and the references therein. We note that for $f \in Co(\alpha)$, $\alpha \in (1, 2]$, the closed set $\mathbb{C}\backslash f(\mathbb{D})$ is convex and unbounded. Also, we observe that $Co(2)$ contains the classes $Co(\alpha)$, $\alpha \in (1, 2]$.

We recall the analytic characterization for functions in $Co(\alpha), \alpha \in (1, 2]$: $f \in Co(\alpha)$ if and only if $\mathrm{Re}\,P_f(z) > 0$ in $\mathbb{D}$, where

$$P_f(z) = \frac{2}{\alpha - 1}\left[\frac{(\alpha + 1)}{2}\frac{1 + z}{1 - z} - 1 - z\frac{f''(z)}{f'(z)}\right].$$

In [4], we have used this characterization and proved the following theorem which will be used to prove our result.

Theorem A. *Let $\alpha \in (1, 2]$. A function $f \in Co(\alpha)$ if and only if there exists a starlike function $\phi \in \mathcal{S}^*$ such that $f(z) = \Lambda_\phi(z)$, where*

$$\Lambda_\phi(z) = \int_0^z \frac{1}{(1 - t)^{\alpha+1}}\left(\frac{t}{\phi(t)}\right)^{(\alpha-1)/2} dt.$$

We also recall a lemma due to Koepf [8, Lemma 3].

Lemma A. *Let $g(z) = z + b_2 z + b_3 z^2 + \cdots \in \mathcal{S}^*$. Then $|b_3 - \lambda b_2^2| \leq \max\{1, |3 - 4\lambda|\}$ which is sharp for the Koebe function k if $|\lambda - 3/4| \geq 1/4$ and for $(k(z^2))^{1/2} = \frac{z}{1-z^2}$ if $|\lambda - \frac{3}{4}| \leq 1/4$.*

Here $\mathcal{S}^*$ denote the family of functions $g \in \mathcal{S}$ that map $\mathbb{D}$ into domains that are starlike with respect to the origin. Each $g \in \mathcal{S}^*$ is characterized by the condition $\mathrm{Re}\,(zg'(z)/g(z)) > 0$ in $\mathbb{D}$. Ma and Minda [11] presented the Fekete-Szegö problem for more general classes through subordination, which includes the classes of starlike and convex functions, respectively. In a recent paper, the authors in [5] obtained a new method of solving the Fekete-Szegö problem for classes of close-to-convex functions defined in terms of subordination.

3. Main Result and its Proof

We recall from Theorem A that $f \in Co(\alpha)$ if and only if there exists a function $\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n \in \mathcal{S}^*$ such that

$$f'(z) = \frac{1}{(1-z)^{\alpha+1}} \left(\frac{z}{\phi(z)}\right)^{\frac{\alpha-1}{2}}, \tag{3.1}$$

where f has the form given by (1.1). Comparing the coefficients of z and z^2 on the both sides of the series expansion of (3.1), we obtain that

$$\begin{aligned} a_2 &= \frac{\alpha+1}{2} - \frac{\alpha-1}{4}\phi_2, \quad \text{and} \\ a_3 &= \frac{(\alpha+1)(\alpha+2)}{6} - \frac{\alpha^2-1}{6}\phi_2 - \frac{\alpha-1}{6}\phi_3 + \frac{\alpha^2-1}{24}\phi_2^2, \end{aligned}$$

respectively. A computation yields,

$$\begin{aligned} a_3 - \lambda a_2^2 &= \frac{(\alpha+1)^2}{4}\left(\frac{2(\alpha+2)}{3(\alpha+1)} - \lambda\right) + \frac{\alpha^2-1}{4}\left(\lambda - \frac{2}{3}\right)\phi_2 \\ &\quad -\frac{\alpha-1}{6}\left[\phi_3 - \left(\frac{2(\alpha+1) - 3\lambda(\alpha-1)}{8}\right)\phi_2^2\right]. \end{aligned} \tag{3.2}$$

Case (1): Let $\lambda \in \left(-\infty, \dfrac{2(\alpha-3)}{3(\alpha-1)}\right]$. We observe that the assumption on λ is seen to be equivalent to

$$\frac{2(\alpha+1) - 3\lambda(\alpha-1)}{8} \geq 1$$

and the first term in the last expression is nonnegative. Hence, using Lemma A for the last term in (3.2), and noting that $|\phi_2| \leq 2$, we have from the equality (3.2),

$$\begin{aligned} |a_3 - \lambda a_2^2| &\leq \frac{(\alpha+1)^2}{4}\left(\frac{2(\alpha+2)}{3(\alpha+1)} - \lambda\right) + \frac{\alpha^2-1}{4}\left(\frac{2}{3} - \lambda\right)|\phi_2| \\ &\quad + \frac{\alpha-1}{6}\left|\phi_3 - \left(\frac{2(\alpha+1) - 3\lambda(\alpha-1)}{8}\right)\phi_2^2\right| \\ &\leq \frac{(\alpha+1)^2}{4}\left(\frac{2(\alpha+2)}{3(\alpha+1)} - \lambda\right) + \frac{\alpha^2-1}{2}\left(\frac{2}{3} - \lambda\right) \\ &\quad + \frac{\alpha-1}{6}\left(\frac{2(\alpha+1) - 3\lambda(\alpha-1)}{2} - 3\right). \end{aligned}$$

Thus, simplifying the right hand expression gives

$$|a_3 - \lambda a_2^2| \leq \frac{2\alpha^2+1}{3} - \lambda\alpha^2, \quad \text{if} \quad \lambda \in \left(-\infty, \frac{2(\alpha-3)}{3(\alpha-1)}\right]. \tag{3.3}$$

Case (2): Let $\lambda \geq \dfrac{2(\alpha+2)}{3(\alpha+1)}$ so that the first term in (3.2) is nonpositive. The condition on λ in particular gives $\lambda > 2/3$ and therefore, our assumption on λ

implies that

$$\frac{2(\alpha+1)-3\lambda(\alpha-1)}{8}<\frac{1}{2}.$$

Again, it follows from Lemma A that

$$\left|\phi_3-\left(\frac{2(\alpha+1)-3\lambda(\alpha-1)}{8}\right)\phi_2^2\right|\le 3-\frac{2(\alpha+1)-3\lambda(\alpha-1)}{2}.$$

In view of these observations and an use of the inequality $|\phi_2|\le 2$, the equality (3.2) gives

$$\begin{aligned}|a_3-\lambda a_2^2| \;\le\; & -\frac{(\alpha+1)^2}{4}\left(\frac{2(\alpha+2)}{3(\alpha+1)}-\lambda\right)-\frac{\alpha^2-1}{2}\left(\frac{2}{3}-\lambda\right)\\ & +\frac{\alpha-1}{6}\left(3-\frac{2(\alpha+1)-3\lambda(\alpha-1)}{2}\right).\end{aligned}$$

Thus, simplifying the right hand expression gives

$$|a_3-\lambda a_2^2|\le \lambda\alpha^2-\frac{2\alpha^2+1}{3},\quad \text{if}\ \ \lambda\ge\frac{2(\alpha+2)}{3(\alpha+1)}. \tag{3.4}$$

The inequalities in both cases are sharp for the functions

$$f(z)=\frac{1}{2\alpha}\left[\left(\frac{1+z}{1-z}\right)^{\alpha}-1\right].$$

Case (3): To get the complete solution of the Fekete-Szegö problem, we need to consider the case

$$\lambda\in\left(\frac{2(\alpha-3)}{3(\alpha-1)},\frac{2(\alpha+2)}{3(\alpha+1)}\right). \tag{3.5}$$

Now, we deal with this case by using the formulas (3.1) and (3.2) together with the representation formula for $\phi\in\mathcal{S}^*$:

$$\frac{z\phi'(z)}{\phi(z)}=\frac{1+z\omega(z)}{1-z\omega(z)},$$

where $\omega:\mathbb{D}\to\overline{\mathbb{D}}$ is a function analytic in $\mathbb{D}$ with the Taylor series

$$\omega(z)=\sum_{n=0}^{\infty}c_nz^n.$$

Inserting the resulting formulas

$$\phi_2=2c_0\ \text{ and }\ \phi_3=c_1+3{c_0}^2$$

into (3.2) yields

$$\begin{aligned}a_3-\lambda a_2^2 \;=\; & \frac{(\alpha+1)^2}{4}\left(\frac{2(\alpha+2)}{3(\alpha+1)}-\lambda\right)+\frac{\alpha^2-1}{2}\left(\lambda-\frac{2}{3}\right)c_0\\ & -\frac{\alpha-1}{6}\left[c_1+\frac{4-2\alpha+3\lambda(\alpha-1)}{2}{c_0}^2\right]\\ =:\; & A+Bc_0+Cc_0^2+Dc_1,\end{aligned}$$

where

$$\begin{cases} A &= \dfrac{(\alpha+1)(\alpha+2)}{6} - \lambda\dfrac{(\alpha+1)^2}{4}, \\ B &= (\alpha^2-1)\left(\dfrac{\lambda}{2}-\dfrac{1}{3}\right), \\ C &= -\dfrac{(\alpha-1)\left(4-2\alpha+3\lambda(\alpha-1)\right)}{12}, \\ D &= -\dfrac{\alpha-1}{6}. \end{cases}$$

It is well-known that $|c_0| \le 1$ and $|c_1| \le 1-|c_0|^2$. Using this we obtain,

$$\begin{aligned} |a_3-\lambda a_2^2| &= |A+Bc_0+Cc_0^2+Dc_1| \\ &\le |A+Bc_0+Cc_0^2|+|D||c_1| \\ &\le |A+Bc_0+Cc_0^2|+|D|(1-|c_0|^2). \end{aligned} \tag{3.6}$$

Let $c_0 = re^{i\theta}$. First we search for the maximum of $|A+Bc_0+C{c_0}^2|$ where we fix r and vary θ. To this end, we consider the expression

$$\begin{aligned} &|A+Bc_0+C{c_0}^2|^2 \\ &\quad= |A+Bre^{i\theta}+Cr^2e^{2i\theta}|^2 \\ &\quad= (A-Cr^2)^2+B^2r^2+(2ABr+2BCr^3)\cos\theta+4ACr^2(\cos\theta)^2 \\ &\quad=: f(r,\theta). \end{aligned}$$

Afterwards, we have to find the biggest value of the maximum function, if r varies in the interval (0,1].

We need to deal with several subcases of (3.5).

<u>Case A</u>: Let $\lambda \in \left(\frac{2(\alpha-3)}{3(\alpha-1)}, \frac{2(\alpha-2)}{3(\alpha-1)}\right)$. We observe that $C>0$, $B<0$, and $A+Cr^2>0$ for $r\in[0,1]$. Hence the corresponding quadratic function

$$h(x) = (A-Cr^2)^2+B^2r^2+2Br(A+Cr^2)x+4ACr^2x^2,\ x\in[-1,1],$$

attains its maximum value for any $r\in(0,1]$ at $x=-1$. Therefore, our task is to find the maximum value of

$$g(r) = A-Br+Cr^2+\frac{\alpha-1}{6}(1-r^2).$$

The inequalities $g'(0) = -B$ and

$$g'(1) = -B+2C-\frac{\alpha-1}{3} = \frac{\alpha-1}{6}(-6\lambda+4(\alpha-1))>0$$

for $\lambda < \frac{2(\alpha-1)}{3\alpha}$ imply

$$g(r) \le g(1) = A-B+C = \frac{2\alpha^2+1}{3}-\lambda\alpha^2.$$

<u>Case B</u>: If $\lambda = \frac{2(\alpha-2)}{3(\alpha-1)}$, then $C=0$ and h is a linear function that has its maximum value at $x=-1$. The considerations of Case A apply and again we get the maximum value $g(1)$ as above.

Case C: Let $\lambda \in \left(\frac{2(\alpha-2)}{3(\alpha-1)}, \frac{2(\alpha-1)}{3\alpha}\right)$. Firstly, we prove that in this interval the quadratic function h is monotonic decreasing for $x \in [-1,1]$. Since the function $h:\mathbb{R}\to\mathbb{R}$ has its maximum at

$$x(r) = \frac{-B(A+Cr^2)}{4ACr} = \frac{-B}{4}\left(\frac{1}{Cr}+\frac{r}{A}\right),$$

it is sufficient to show that $x(r)$ is monotonic increasing and $x(1) < -1$. The first assertion is trivial and the second one is equivalent to

$$j(\lambda) = \alpha^2(3\lambda-2)^2 - 4 + 3\lambda > 0$$

for the parameters λ in question. This inequality is easily proved. Hence, we get the same upper bound as in Cases A and B. In conclusion, Cases A, B and C give,

$$|a_3 - \lambda a_2^2| \le \frac{2\alpha^2+1}{3} - \lambda\alpha^2, \quad \text{if } \lambda \in \left(\frac{2(\alpha-3)}{3(\alpha-1)}, \frac{2(\alpha-1)}{3\alpha}\right). \tag{3.7}$$

Case D: Let $\lambda \in \left[\frac{2(\alpha-1)}{3\alpha}, \frac{2}{3}\right)$ and we may factorize $j(\lambda)$ as

$$j(\lambda) = 9\alpha^2(\lambda-\lambda_1)(\lambda-\lambda_2),$$

where

$$\lambda_1 = \frac{4\alpha^2-1-\sqrt{8\alpha^2+1}}{6\alpha^2} \quad \text{and} \quad \lambda_2 = \frac{4\alpha^2-1+\sqrt{8\alpha^2+1}}{6\alpha^2}. \tag{3.8}$$

We observe that $\lambda_2 > \lambda_1$. For $\lambda \in \left[\frac{2(\alpha-1)}{3\alpha}, \lambda_1\right)$, the function h has its maximum value at $x=-1$ and the function g has its maximum value at

$$r_m = \frac{-B}{-2C + \frac{\alpha-1}{3}} \in (0,1].$$

Hence, the maximum of the Fekete-Szegö functional is

$$g(r_m) = \frac{\alpha(10-9\lambda)-(3\lambda-2)}{9(2-\lambda)+3\alpha(3\lambda-2)}.$$

For $\lambda \in \left[\lambda_1, \frac{2}{3}\right)$, the number

$$r_0 = \frac{B}{2C\left(1+\sqrt{1-\frac{B^2}{4AC}}\right)} \in (0,1]$$

is the unique solution of $x(r) = -1$ in the interval $(0,1]$. It is easily seen that $r_m < r_0$ for $\lambda < \frac{2}{3}$. Further,

$$k(r) = \sqrt{h(x(r))} + \frac{\alpha-1}{6}(1-r^2) = (A-Cr^2)\sqrt{1-\frac{B^2}{4AC}} + \frac{\alpha-1}{6}(1-r^2)$$

is monotonic decreasing for $r \ge r_0$. Hence, the maximum value of $|a_3 - \lambda a_2^2|$ is $g(r_m)$ in this part of the interval in question, too. The extremal function maps $\mathbb{D}$ onto a wedge shaped region with an opening angle at infinity less than $\pi\alpha$ and one finite vertex as in Example 3.12 in [3].

Case E: For $\lambda = \frac{2}{3}$, we have $B = 0$ and $C = -\frac{\alpha-1}{6}$. Hence the maximum is attained for $\cos\theta = 0$ and any $r \in (0, 1]$. In all these cases, we get

$$|a_3 - \lambda a_2^2| = \frac{\alpha}{3}$$

as the sharp upper bound. The extremal functions map $\mathbb{D}$ onto a region with an opening angle at infinity equal to $\pi\alpha$ and two finite vertices as in Example 3.13 in [3].

In conclusion, Cases D and E give,

$$|a_3 - \lambda a_2^2| \le \frac{\alpha(10 - 9\lambda) - (3\lambda - 2)}{9(2 - \lambda) + 3\alpha(3\lambda - 2)}, \quad \text{if} \quad \lambda \in \left(\frac{2(\alpha - 1)}{3\alpha}, \frac{2}{3}\right]. \tag{3.9}$$

Case F: Let $\lambda \in (\frac{2}{3}, \lambda_2]$. Since $B > 0$, the function $x(r)$ is monotonic decreasing now. The number

$$r_1 = \frac{B}{-2C\left(1 + \sqrt{1 - \frac{B^2}{4AC}}\right)} \in (0, 1]$$

is the unique solution of the equation $x(r) = 1$ lying in (0,1]. For $r < r_1$, we have $h(x) \le h(1)$. We consider the function

$$l(r) = A + Br + Cr^2 + \frac{\alpha - 1}{6}(1 - r^2)$$

and determine the maximum of this function to be attained at

$$r_n = \frac{B}{-2C + \frac{\alpha-1}{3}}.$$

It is easily proved that $r_n > r_1$. Since $k(r)$ is monotonic increasing, we get the maximum value of the Fekete-Szegö functional in this case as

$$k(1) = (A - C)\sqrt{1 - \frac{B^2}{4AC}} = \alpha(1 - \lambda)\sqrt{\frac{12(1 - \lambda)}{(4 - 3\lambda)^2 - \alpha^2(3\lambda - 2)^2}},$$

which is attained for $c_0 = e^{i\theta_0}$, where

$$\cos\theta_0 = \frac{-B(A + C)}{4AC}.$$

In this case, the extremal function f is defined by the solution of the following complex differential equation

$$f'(z) = \frac{(1 - ze^{i\theta_0})^{\alpha-1}}{(1 - z)^{\alpha+1}}.$$

In conclusion, in this case, we have,

$$|a_3 - \lambda a_2^2| \le \alpha(1 - \lambda)\sqrt{\frac{12(1 - \lambda)}{(4 - 3\lambda)^2 - \alpha^2(3\lambda - 2)^2}}, \quad \text{if} \quad \lambda \in (2/3, \lambda_2]. \tag{3.10}$$

Case G: Let $\lambda \in \left(\lambda_2, \frac{2(\alpha+2)}{3(\alpha+1)}\right)$. Since $x(1) < -1$ for these λ, the number

$$r_2 = \frac{B}{-2C\left(1 - \sqrt{1 - \frac{B^2}{4AC}}\right)}$$

satisfies $x(r_2) = -1$ and $r_2 \in (0,1)$. For $r \leq r_2$, we can make similar considerations as in the preceding case, i.e. for $r \leq r_1$ the function $l(r)$ takes the maximum value, and for $r \in (r_1, r_2]$, the function $k(r)$ plays this role. For $r > r_2$, the point $x(r)$ does not lie in the interval $[-1,1]$. Hence, the maximum in question is attained for $x = -1$ or $x = 1$. We see that $A + C < 0$ and $-A - Cr^2 > 0$ for the values of λ that we are considering now, the maximum of (3.6) is attained for $x = -1$, i.e. for $c_0 = -r$. Hence, for $r \in (r_2, 1]$ the maximum function is

$$n(r) = -A + Br - Cr^2 + \frac{\alpha - 1}{6}(1 - r^2).$$

Since

$$-C > \frac{\alpha - 1}{6} \quad \text{and} \quad B > 0,$$

we get $n(r) \leq n(1)$ in the interval in question and hence

$$|a_3 - \lambda a_2^2| \leq n(1) = -A + B - C = \lambda\alpha^2 - \frac{2\alpha^2 + 1}{3}$$

whenever $\lambda \in \left(\lambda_2, \frac{2(\alpha+2)}{3(\alpha+1)}\right)$.

Equations (3.3), (3.4), (3.7), (3.9), (3.10) and Case G give

Theorem. *For $\alpha \in (1,2]$, let $f \in Co(\alpha)$ have the expansion (1.1). Then, we have*

$$\left|a_3 - \lambda {a_2}^2\right| \leq \begin{cases} \dfrac{2\alpha^2+1}{3} - \lambda\alpha^2 & \text{for } \lambda \in \left(-\infty, \dfrac{2(\alpha-1)}{3\alpha}\right] \\ \dfrac{\alpha(10-9\lambda) - (3\lambda-2)}{9(2-\lambda) + 3\alpha(3\lambda-2)} & \text{for } \dfrac{2(\alpha-1)}{3\alpha} \leq \lambda \leq \dfrac{2}{3} \\ \alpha(1-\lambda)\sqrt{\dfrac{12(1-\lambda)}{(4-3\lambda)^2 - \alpha^2(3\lambda-2)^2}} & \text{for } \dfrac{2}{3} \leq \lambda \leq \lambda_2 \\ \lambda\alpha^2 - \dfrac{2\alpha^2+1}{3} & \text{for } \lambda \in [\lambda_2, \infty), \end{cases}$$

where λ_2 is given by (3.8). To emphasize the fact that the bound is a continuous function of λ for any α we mention two different expressions for the same bound for some values of λ. The inequalities are sharp.

The Fekete-Szegö inequalities for functions in the class $Co(\alpha)$ for complex values of λ remain an open problem.

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