DIRECT CONSTRUCTION OF GRASSMANN, CLIFFORD AND GEOMETRIC ALGEBRAS

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ABSTRACT. This is a simple way rigorously to construct Grassmann, Clifford and Geometric Algebras, allowing degenerate bilinear forms, infinite dimension, using fields or certain modules (characteristic 2 with limitation), and characterize the algebras in a coordinate free form. The construction is done in an orthogonal basis, and the algebras characterized by universality. The basic properties with short proofs provides a clear foundation for further development of the algebras.

CONTENTS

1. INTRODUCTION

Often proof of the existence of Grassmann or Clifford algebras are bypassed, or Chevalleys tensor approach is taken; but e.g. investigation of injectivity of the mapping from vector space to Clifford algebra is skipped. Pure mathematical books may present a lot of structures before coming to these algebras [2,4,9], and the complexity seems to give some problems [8].
Chapters 2, 3 with algebras over $\mathbb{R}^n$ in mind are recommended as minimal reading. Introductory material can be found in [5].

2. PRELIMINARIES

Our starting-point may be a field $R = \mathbb{R}$, a linear space $V = \mathbb{R}^n$ with basis $e_1 = (1, 0, \ldots 0), e_2 = (0, 1, \ldots 0) \ldots$, and indices $M = \{1, 2, \ldots, n\}$ usually ordered by $<$. Also used is a quadratic mapping $q(i) = B(e_i, e_i)$, where B is a bilinear form

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on V with diagonal form in the basis$(\,e_i|\,i\in M)$. The basic idea behind Grassmann and Clifford algebras is, that products may give new elements, e.g. $e_1e_2=e_{\{1,2\}}$. In $\mathbb{R}^n$ the new product should fulfill generator equations $e_1e_2=-e_2e_1$, $e_ie_i=q(i)\in\mathbb{R}$ and be associative. Then a product may be reordered and reduced to get a standard form without repetitions, as in

$$e_{\{1,2\}}e_3e_1e_2=-e_1e_2e_1e_3e_2=e_1e_1e_2e_3e_2=-e_1e_1e_2e_2e_3=-q(1)q(2)e_3$$

The product properties gives a dimension$\leq 2^n$, as there are 2^n subsets of M.
The goal of the algebra construction is to equip $W=\mathbb{R}^{2^n}$ with a Clifford product. A basis for W is $(\,e_K|\,K\subseteq M)$. Subspaces of W are the scalars $Re_\emptyset$ and V by identifying e_i with $e_{\{i\}}$. Sets as indices gives a compact construction. It can indeed be used together with multiindex: $e_4e_3e_5=e_{(4,3,5)}=-e_{\{4,3,5\}}=-e_{\{3,4,5\}}$.

Notation. The following notation and definitions will be used. An algebra A is a linear space equipped with a bilinear and associative composition having a unit 1_A. An algebra morphism is supposed to map unit to unit. All the algebras are over the same set of scalars, R. Silently x,y will be elements in a linear space V and X,Y elements in the algebra at hand.
The cardinality of the set K is denoted $|K|$. A product over an index set follow the order given in M. For a index set H we use $k<H$ in the meaning $\forall\,h\in H(k<h)$, implying $k<\emptyset$ is true.
$H\triangle J=(H\cup J)\backslash(H\cap J)$ is the symmetric set difference, which is associative.

3. Construction

To make the exposition general we may assume
1. R is a commutative ring with unit $1\neq 0$
2. $V\neq\{0\}$ is a free unitary R-module
3. $B:V\times V\to R$ is a bilinear form with diagonal form in the basis $(\,e_i\,|\,i\in M)$, $q(i)=B(e_i,e_i)$ and M is total ordered by a relation $<$.
4. $W=\bigoplus_{\mathcal{F}}R$, where $\mathcal{F}$ is the set of finite subsets of index set M.

A basis for W is $(\,e_K|\,K\in\mathcal{F})$. Subspaces of W are the scalars $R\,e_\emptyset$ and V by identifying e_i with $e_{\{i\}}$.
NB: If R is a field with characteristic different from 2 and V of finite dimension, any symmetric bilinear form on V has an orthogonal basis.

Factors α and β originate from ordering by swapping and reduction, respectively.

Lemma 1. *Define functions α and β for combining the sets $H,J\in\mathcal{F}$ by*

$$\alpha(H,J)=\Pi\,(-1)\ \textit{for}\ (i,j)\in H\times J\ \textit{and}\ j<i$$

$$\beta(H,J)=\Pi\,q(i)\ \textit{for}\ i\in H\cap J$$

Then for $\psi=\alpha,\beta$ holds

$$\psi(H,J)\,\psi(H\triangle J,K)=\psi(H,J\triangle K)\,\psi(J,K)$$

and therefore also for $\psi=\sigma=\alpha\beta$.

Proof: As $\alpha^2=1$ we get

$$\alpha(H\triangle J,K)=\alpha(H\backslash J,K)\,\alpha(J\backslash H,K)\,\alpha(J\cap H,K)^2=\alpha(H,K)\,\alpha(J,K)$$

and likewise $\alpha(H, J\triangle K) = \alpha(H,J)\,\alpha(H,K)$. Thus the equation for α is obvious. By Venn diagrams

$$\beta(H,J)\,\beta(H\triangle J,K) = \Pi\, q(i) \text{ for } i \text{ in just two of the sets } H,J,K,$$

and $\beta(H, J\triangle K)\beta(J,K)$ gives the same result.

Theorem 1. *Define a product* $(X,Y) \to XY$ *in* W *by* $e_H\, e_J = \sigma(H,J)\, e_{H\triangle J}$ *and bilinearity. Then* W *becomes an algebra with* $e_\emptyset$ *as unit, such that*

$$i \neq j \Rightarrow e_i e_j = -e_j e_i \text{ and } x \in V \Rightarrow x^2 = B(x,x)\, e_\emptyset$$

Proof: The associative law $(xy)z = x(yz)$ is multilinear and is verified for basis elements by

$$(e_H e_J)\, e_K = \sigma(H,J)\, e_{H\triangle J}\, e_K = \sigma(H,J)\,\sigma(H\triangle J,K)\, e_{(H\triangle J)\triangle K}$$
$$e_H\,(e_J e_K) = \sigma(J,K)\, e_H e_{J\triangle K} = \sigma(H, J\triangle K)\,\sigma(J,K)\, e_{H\triangle(J\triangle K)}$$

Furthermore

$$e_\emptyset e_K = \sigma(\emptyset,K) e_K = e_K = \sigma(K,\emptyset) e_K = e_K e_\emptyset \text{ and } e_i e_i = \sigma(\{i\},\{i\}) e_\emptyset = q(i)\, e_\emptyset.$$

Now $i < j \Rightarrow e_i e_j = \sigma(\{i\},\{j\})\, e_{\{i,j\}} = e_{\{i,j\}}$ and likewise $e_j e_i = -e_{\{i,j\}}$. If $x = \Sigma_i\, \lambda_i e_i$, then by separating into cases $i < j$, $i = j$ and $j < i$, we find

$$x^2 = \Sigma_{i,j} \lambda_i \lambda_j e_i e_j = \Sigma \lambda_i^2 e_i^2 e_\emptyset = B(x,x) e_\emptyset.$$

Corollary 1. $e_K = \Pi_{i\in K} e_i$

Proof: Use induction and $j < H \Rightarrow e_j e_H = \sigma(\{j\}, H) e_{\{j\}\cup H} = e_{\{j\}\cup H}$.

Corollary 2. *For any algebra* A *over* F *and any linear mapping* $f : V \to A$ *with*

$$f(x)^2 = B(x,x)\, 1_A$$

there exists a unique algebra morphism $F : W \to A$*, which extends f.*

Proof: Define a linear mapping $F : W \to A$ necessarily by

$$F(e_\emptyset) = 1_A \text{ and } F(e_K) = \Pi_{i\in K} f(e_i)$$

For $i \neq j$ and $x = e_i + e_j$, we get

$$f(x)^2 = B(x,x) 1_A \Rightarrow f(e_i) f(e_j) = -f(e_j) f(e_i).$$

Let $H = \{h_1, h_2, \ldots, h_p\}$ and $K = \{k_1, \ldots, k_q\}$ with $h_i < h_{i+1}$ and $k_j < k_{j+1}$ for relevant indices. Ordering from lower to higher indices by swapping neighbors in

$$F(e_H) F(e_J) = f(e_{h_1}) f(e_{h_2}) \ldots f(e_{h_p}) f(e_{k_1}) f(e_{k_2}) \ldots f(e_{k_q})$$

gives a sign change $\alpha(H,K)$, and reducing equals gives a further factor $\beta(H,K)$. Thus $F(e_H)F(e_J) = \sigma(H,J) F(e_{H\triangle J}) = F(e_H e_J)$.

Definition 1. *A Clifford algebra over* B*, denoted* $Cl_V(B)$ *or* $Cl(B)$*, is an algebra isomorphic to* $W_B = W$ *with an isomorphism fixing* V*. In the case* $B = 0$*, we have a Grassmann algebra,* $\Lambda(V) = Cl_V(0)$*, and a product, the outer product* $\wedge$*.*

Definition 2. *If the linear space of the Clifford algebra* W *also is given a Grassmann structure by the zero bilinear form, we get a double algebra* $W_{B,0} = W$*.*
A geometric algebra over B *, denoted* $\Lambda(V,B)$*, is a double algebra isomorphic to* $W_{B,0}$ *with an isomorphism fixing* V*.*

Corollary 3. *In $\Lambda(V)$ define the subspace of elements of grade $r \in \mathbb{Z}$ by $\Lambda_r(V) = span\{\wedge_{i=1}^r a_i | a_i \in V\}$ for $r \geq 0$ and otherwise $\Lambda_r(V) = \{0\}$. Then*
1. $\Lambda(V) = \oplus_r \Lambda_r(V)$ and $\Lambda_r(V) \wedge \Lambda_s(V) \subseteq \Lambda_{r+s}(V)$.
This allows us to define $X \to \langle X \rangle_r$ to be the projection on $\Lambda_r(V)$ along $\oplus_{i \neq r} \Lambda_i(V)$
2. $x_1 \wedge x_2 = -x_2 \wedge x_1$
3. $x_1 \wedge x_2 \wedge \ldots \wedge x_p$ is multilinear and alternating in the x-variables
4. $(x_1, x_2 \ldots, x_p)$ is linear independent $\Leftrightarrow x_1 \wedge x_2 \wedge \ldots \wedge x_p$ is linear independent.
5. For any algebra A over R and linear mapping $f : V \to A$ with $f(x)^2 = 0$, there exists a unique algebra morphism or outermorphism $f_\wedge : \Lambda(V) \to A$ extending f.

Proof: 1. Observe that 0 can be assigned any appropriate grade. In W_0 define $\Omega_r = \text{span}\{ e_K | |K| = r, K \in \mathcal{F}\}$ for $r \geq 0$ and otherwise $\Omega_r = \{0\}$. Then clearly $W_0 = \oplus_r \Omega_r$ and $\Omega_r \wedge \Omega_s \subseteq \Omega_{r+s}$, which implies $\wedge_{i=1}^r a_i \in \Omega_r$. Thus $\Omega_r = \Lambda_r$.
Now use the isomorphism $W_0 \to \Lambda(V)$ fixing V.
2. In $x \wedge x = 0$ set $x = x_1 + x_2$.
3. Obviously the expression is multilinear, and 0, if $x_i = x_{i+1}$. For $i < j$ and $x_i = x_j$ this situation can be obtained by swapping neighbors.
4. The full proof is in the appendix.
5. Consequence of corollary 2.

Corollary 4. *In $Cl_V(B)$ define the subspace of grade $r \in \mathbb{Z}_2 = \{0, 1\}$ by*

$$Cl_{V,r}(B) = span\{\Pi_{i=1}^s a_i \,|s \equiv r \pmod 2), a_i \in V\}$$

Then $Cl_V(B) = \oplus_r Cl_{V,r}(B)$ and $Cl_{V,r}(B) Cl_{V,s}(B) \subseteq Cl_{V,r+s}(B)$.

Proof: In W_B define $\Omega_r = \text{span}\{ e_K | |K| \equiv r \pmod 2), K \in \mathcal{F}\}$. Then we have $W_B = \oplus_r \Omega_r$ and $\Omega_r \Omega_s \subseteq \Omega_{r+s}$, since factor reductions for products are even in number. This implies $\Pi_{i=1}^s a_i \in \Omega_r$ for $s \equiv r \pmod 2$. Hence $\Omega_r = Cl_{V,r}$.
Now use the isomorphism $W_B \to Cl_V(B)$ fixing V.

4. Characterization of universal Clifford algebras

Theorem 2. *Let $\mathcal{A}_R(V, B)$ be the category of linear mappings f from V into an algebra A, such that $f(x)^2 = B(x, x)1_A$.*
A mapping $\rho : V \to U$ in $\mathcal{A}_R(V, B)$ is said to be universal, if for every linear mapping $f : V \to A$ in $\mathcal{A}_R(V, B)$, there is a unique algebra morphism $F : U \to A$ such that $F \circ \rho = f$.
In case $V \subset U$ this means F extends f , when ρ silently is taken as the injection.
If $f_i : V \to U_i, i = 1, 2$ are universal in $\mathcal{A}_R(V, B)$, then there exists a unique algebra isomorphism $F : U_1 \to U_2$ such that $F \circ f_1 = f_2$.

Proof: Universality gives unique algebra morphisms $F : U_1 \to U_2$, $G : U_2 \to U_1$, such that $F \circ f_1 = f_2$, $G \circ f_2 = f_1$. As $F \circ G \circ f_2 = f_2$ and $\text{id}_{U_2} \circ f_2 = f_2$, universality implies $\text{id}_{U_2} = F \circ G$, and likewise $\text{id}_{U_1} = G \circ F$.

Corollary 5. *$Cl_V(B)$ is universal in $\mathcal{A}_R(V, B)$.*

Proof: Follows from corollary 2 and definition 1.

Corollary 6. *The Clifford product in $Cl_V(B)$ is independent of the orthogonal basis in theorem 1.*

Proof: Let $\left(\ddot{e}_i|i\in\ddot{M}\right)$ be an orthogonal basis for V, and $\ddot{W}_B$ the Clifford algebra constructed as in theorem 1. By universality we now get an unique isomorphism $\ddot{F}:\ddot{W}_B\to Cl_V(B)$ fixing V. Hence $Cl_V(B)$ has the product defined from $\ddot{W}_B$.

Corollary 7. *In $Cl_V(B)$ the main automorphism $X\to\hat{X}$ is the universal extension of $f:V\to Cl_V(B)$ where $f(x)=-x$.*
We have $X\to\hat{X}=(-1)^rX$ for X of Clifford grade r.

Proof: By universality$f$ can be extended uniquely to an algebra morphism $\hat{x}:Cl_V(B)\to Cl_V(B)$. In a basis by linearity and $\hat{e_K}=\Pi_{i\in K}f\left(e_i\right)=(-1)^{|K|}e_K=(-1)^{|K|\bmod 2}e_K$ the statement is proved.

Corollary 8. *In $Cl_V(B)$ define the reversion $\widetilde{x}$ by $(XY)^{\sim}=\widetilde{Y}\widetilde{X}$, linearity and by fixing $1_{Cl(B)}$ and V. Then $X^{\sim\sim}=X$, and $(a_1a_2\ldots a_r)^{\sim}=a_r\ldots a_2a_1$.*

Proof: In the linear space U of $Cl_V(B)$ an algebra $(U,\diamond)$ is defined by the product $X\diamond Y=YX$. As $x\diamond x=B(x,x)1_U$ and

$$(XY)^{\sim}=\widetilde{Y}\widetilde{X}\Leftrightarrow(XY)^{\sim}=\widetilde{X}\diamond\widetilde{Y}$$

a reversion must be an algebra morphism $\widetilde{x}:Cl_V(B)\to(U,\diamond)$, and universality of $Cl_V(B)$ implies uniqueness and existence. Now $(XY)^{\sim\sim}=\left(\widetilde{Y}\widetilde{X}\right)^{\sim}=XY$ and by universality $X^{\sim\sim}=X$. The last formula follows from $(XY)^{\sim}=\widetilde{Y}\widetilde{X}$.

5. Characterization of Geometric algebras

Definition 3. Set $\chi(S)=1$, *if S is true, and else zero.*
In the geometric algebra $W_{B,0}$ define mappings $*,\rfloor$and$\lfloor$ by bilinearity and
$e_H*e_K=\chi(H=K)\,e_He_K$ *(the scalar product),*
$e_H\rfloor e_K=\chi(H\subseteq K)\,e_He_K$ *(the left contraction),*
$e_H\lfloor e_K=\chi(H\supseteq K)\,e_He_K$ *(the right contraction).*
Observe that in a geometric algebra the grading is taken from the Grassmann structure, and that $e_H\wedge e_K=\chi(H\cap K=\emptyset)\,e_He_K$.

Theorem 3. *In a geometric algebra $\Lambda(V,B)$ holds*
1. $1_{Cl(B)}=1_\Lambda$ *and* $xX=x\wedge X+x\rfloor X$
2. The two subalgebras have the same reversions, and

$$\widetilde{X}=(-1)^{r(r-1)/2}X \text{ for } X \text{ of grade } r$$

3a. $(X\lfloor Y)^{\sim}=\widetilde{Y}\rfloor\widetilde{X}$, $X*Y=\langle XY\rangle_0$, *and* $X\wedge Y=\langle XY\rangle_{r+s}$,
3b. If grade$(X)=r$ and grade$(Y)=s$, then

$$X\rfloor Y=\langle XY\rangle_{s-r},\ X\lfloor Y=\langle XY\rangle_{r-s} \text{ and } XY=\Sigma_{i=|r-s|,\ step\ 2}^{r+s}\langle XY\rangle_i$$

4. $(X\wedge Y)\rfloor Z=X\rfloor(Y\rfloor Z)$
5. $x\rfloor y=B(x,y)1_\Lambda$ *and* $x\rfloor(XY)=(x\rfloor X)Y+\hat{X}(x\rfloor Y)$
6. $x\rfloor y=B(x,y)1_\Lambda$ *and* $x\rfloor(X\wedge Y)=(x\rfloor X)\wedge Y+\hat{X}\wedge(x\rfloor Y)$
7. $x\rfloor(x_1x_2\ldots x_p)=\sum_{k=1}^p(-1)^{k-1}x_1x_2\ldots(x\rfloor x_k)\ldots x_p$
8. $x\rfloor(x_1\wedge x_2\wedge\ldots\wedge x_p)=\sum_{k=1}^p(-1)^{k-1}x_1\wedge x_2\ldots\wedge(x\rfloor x_k)\ldots\wedge x_p$
9. $x_1,x_2,\ldots,x_p$ *are pairwise orthogonal* $\Rightarrow$ $\Pi_{i=1}^p x_i=\wedge_{i=1}^p x_i$

Proof: By isomorphy and linearity it should be sufficient to sketch proofs of the statements for basis elements in $W_{B,0}$.
1. Now $h \in H \Rightarrow e_h \wedge e_H + e_h \rfloor e_H = 0 + e_h\, e_H$ and similar for $h \notin H$.
2. To reorder $\widetilde{e_H}$ requires $(-1)^{|H|(|H|-1)/2}$ swappings in boths structures.
3. The proofs are much alike, so we will take some examples:
The first in (3a): $(e_H \lfloor e_J)^{\sim} = \chi(J \subseteq H)\,(e_H e_J)^{\sim} = \chi(J \subseteq H)\widetilde{e_J}\widetilde{e_H} = \widetilde{e_J} \rfloor \widetilde{e_H}$.
The last in (3b) follows from $e_H e_J = \sigma(H,J)e_{H\triangle J}$ and
$|H\triangle J| = |H| + |J| - 2|H \cap J| = |H| - |J| + 2|J\backslash H| = |J| - |H| + 2|H\backslash J|$
4. Can be reduced to $\chi(H \cap J = \emptyset)\,\chi(H, J \subseteq K) = \chi(H \subseteq (K\backslash J))\,\chi(J \subseteq K)$
5. Clearly $e_i \rfloor e_j = B\,(e_i, e_j)\, e_\emptyset$ and the remaining can be reduced to

$$\chi(i \in (H\triangle J)) = \chi(i \in H) + (-\chi(i \in H)\,\chi(i \in J) + \chi(i \notin H)\,\chi(i \in J))$$

6. For grade$(X) = r$ and grade$(Y) = s$ use (3) and take grade $r + s - 1$ in (5)
7,8. Follows from (5,6), as e.g.

$$x \rfloor (x_1\,(x_2 \dots x_p)) = (x \rfloor x_1)\ \ (x_2 \dots x_p) - x_1 (x \rfloor (x_2 \dots x_p))\,,$$
$$x \rfloor (x_2 \dots x_p) = (x \rfloor x_2)(x_2 \dots x_p) - x_2 (x \rfloor (x_2 \dots x_p)) \text{ etc.}$$

9. By (1,7), as e.g.

$$x_1 x_2 \dots x_p = x_1 \wedge (x_2 \dots x_p) + x_1 \rfloor (x_2 \dots x_p) = x_1 \wedge (x_2 \dots x_p) \text{ etc.}$$

Theorem 4. *Given a double algebra U of $\Lambda(V)$ and $Cl_V(B)$ occupying the same linear space, such that $1_{Cl(B)} = 1_\Lambda$, and V is a common linear space. Then*

$$U \text{ is a geometric algebra over } B$$
$$\Leftrightarrow\ \forall a_1, \dots, a_p \in V\,(a_1, a_2, \dots, a_p \ \text{ are pairwise orthogonal} \Rightarrow \Pi_{i=1}^p a_i = \wedge_{i=1}^p a_i).$$

Proof: $\Rightarrow$: Follows from lemma 1.
$\Leftarrow$: By universality let $\phi : \Lambda(V, B) \to U$ be the unique isomorphism fixing V determined be the Grassmann structures and similar $\psi : \Lambda(V, B) \to U$ for the Clifford structures. Let $(e_i |\, i \in M)$ be an orthogonal basis for V. In $\Lambda(V, B)$ set $X = \wedge_{i\in K} e_i = \Pi_{i\in K} e_i$. Then $\phi = \psi$, as

$$\phi(X) = \wedge_{i\in K}\,\phi\,(e_i) = \Pi_{i\in K}\,\psi\,(e_i) = \psi(X).$$

Corollary 9. *If U is a geometric algebra over V, then the Clifford structure determines the Grassmann structure and B - and vice versa.*

Proof: By universality let $\phi : \Lambda(V, B) \to U$ be the unique isomorphism fixing V determined by the Clifford structures. Then ϕ determines the Grassmann structure of U. The converse is similar proven.

In [3] various Grassmann structures in a given Clifford algebra are used to describe interacting fermions.

6. Construction of Clifford algebras from tensor algebras

For the tensor algebra over V, $\mathcal{T} = \mathcal{T}(V, \otimes)$, this universality statement is valid: *For any algebra A over R and any linear mapping $g : V \to A$, there is an unique algebra morphism $G : \mathcal{T} \to A$ which extends g*

Suppose D is any bilinear form on V. We may then extend theorem 2 by replacing B with D and in this way extend the Clifford algebra concept.

Definition 4. *A Clifford algebra over D is an universal object in $\mathcal{A}_R(V, D)$*

Theorem 5. *Let $\mathcal{I} = \mathcal{I}(V, D)$ be the two-sided ideal in $\mathcal{T} = \mathcal{T}(V)$ generated by $\mathcal{S} = \{x \otimes x - D(x, x)1_{\mathcal{T}} | x \in V\}$, $Cl = \mathcal{T}/\mathcal{I}$ the quotient algebra and $\hat{\pi} : \mathcal{T} \to Cl$ the quotient mapping. Then $\pi = \hat{\pi}|_V$ is an universal object in $\mathcal{A}_R(V, D)$.*
If $D = B$, then π is injective.

$$\begin{array}{ccccc} V & \xrightarrow{\subset} & \mathcal{T} & \xrightarrow{\hat{\pi}} & \mathcal{T}/\mathcal{I} = Cl \\ {\scriptstyle id}\downarrow & & {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle F} \\ V & \xrightarrow{f} & A & \xrightarrow{id} & A \end{array}$$

Proof: In Cl elements are of the form $t + \mathcal{I}$, $t \in \mathcal{T}$, and obviously $x \otimes x + \mathcal{I} = D(x, x)1_{\mathcal{T}} + \mathcal{I}$. Therefore $\pi(x)^2 = D(x, x)1_{Cl}$ and π is an object in $\mathcal{A}_R(V, D)$.
To prove universality of π we will use universality for tensors. Therefore, to any object $f : V \to A$ in $\mathcal{A}_R(V, D)$ there is a unique algebra morphism $G : \mathcal{T} \to A$ which extends f. As $f(x)^2 = D(x, x)1_A$ implies $\mathcal{S} \subseteq G^{-1}(0)$, there exists a unique algebra morphism $F : Cl \to A$, such that $G = F \circ \hat{\pi}$. Hence $f = F \circ \pi$.
Conversely, as any algebra morphism $F : Cl \to A$, gives an algebra morphism $F \circ \hat{\pi}$ extending f, we have $G = F \circ \hat{\pi}$ and F is unique. Thus Cl is universal in $\mathcal{A}_R(V, D)$.
If $D = B$, by universality of Cl there exists an algebra morphism $F : Cl \to Cl_V(B)$, such that $F \circ \pi = \mathrm{id}_V$, and therefore π is injective.

7. Conclusion

On elementary basis we have defined and constructed the different algebras.
The universality principle has been described, and used in many ways:

- To prove in-dependency of orthogonal basis.
- To define the main automorphism and the reversion.
- In various proofs.
- To fully define Clifford algebras
- To establish connection to Chevalley's tensor based construction.

Comments on other constructions of geometric algebra can be found in [7].
A more general construction of Clifford algebras over modules is found in [2,4], and in a forthcoming paper.

8. Appendix

Theorem 6. *In $\Lambda(V)$ holds*

$$S = (x_1, x_2 \ldots, x_p) \text{ is linear dependent } \Leftrightarrow S_\wedge = \wedge_{k=1}^{p} x_k \text{ is linear dependent.}$$

Proof: $\Rightarrow$: Assume $\Sigma\lambda_i x_i = 0$ and $\lambda_j \neq 0$, then $\lambda_j x_j = -\Sigma_{i \neq j}\lambda_i x_i = 0$, which by corollary 3 (3) gives $\lambda_j S_\wedge = 0$.
$\Leftarrow$: If R is a field, assume S is linear independent and construct $\Lambda(V)$ from a basis containing S (corollary 6). As $S_\wedge$ is a basis element, $\lambda \neq 0 \Rightarrow \lambda S_\wedge \neq 0$.
$\Leftarrow$: Obvious for $p = 1$. Assume that S is linear independent, and $S_\wedge$ is linear dependent and p is the smallest number, for which such a set S can be found. Hence, if T is a strict subset of S, then $T_\wedge = \wedge_{x \in T} x$ is linear independent.
Let $(e_i | \, i \in M)$ be a basis for V and define a geometric algebra structure on $\Lambda(V)$

by letting (e_i) be an orthogonal basis and $q(e_i) = 1$. Set $X = x_1 \wedge x_2 \wedge \ldots \wedge x_{p-1}$. Then $\lambda X \wedge x_p = 0$ for some $\lambda \neq 0$. Furthermore $\lambda X \neq 0$, $p \geq 2$ and $\lambda x_p \neq 0$. From $0 \neq \lambda X = \Sigma_{K\in\mathcal{E}} \lambda_K e_K$ select $\lambda_K \neq 0$. By theorem 3 (8)

$$e_{i_{p-1}} \rfloor (\ldots (e_{i_1} \rfloor (x_1 \wedge \ldots \wedge x_p))) = \Sigma_{k=1}^{p} \mu_k x_k,$$

as $p-1$ of the x_i-elements are contracted. Hence

$$0 = \widetilde{e_K} \rfloor (\lambda X \wedge x_p) = \lambda_K x_p + \Sigma_{k=1}^{p-1} \mu_k x_k,$$

which contradicts the assumed in-dependency of S.

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