Holomorphic Morse inequalities and the Green-Griffiths-Lang conjecture

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Dedicated to the memory of Eckart Viehweg

Abstract. The goal of this work is to study the existence and properties of non constant entire curves $f:\mathbb{C}\to X$ drawn in a complex irreducible n-dimensional variety X, and more specifically to show that they must satisfy certain global algebraic or differential equations as soon as X is projective of general type. By means of holomorphic Morse inequalities and a probabilistic analysis of the cohomology of jet spaces, we are able to reach a significant step towards a generalized version of the Green-Griffiths-Lang conjecture.

Résumé. Le but de ce travail est d'étudier l'existence et les propriétés des courbes entières non constantes $f:\mathbb{C}\to X$ tracées sur une varété complexe irréductible de dimension n, et plus précisément de montrer que ces courbes doivent satisfaire à certaines équations algébriques ou différentielles globales dès que X est projective de type général. Au moyen des inégalités de Morse holomorphes et d'une analyse probabiliste de la cohomologie des espaces de jets, nous atteignons une première étape significative en direction d'une version généralisée de la conjecture de Green-Griffiths-Lang.

Key words. Chern curvature, holomorphic Morse inequality, jet bundle, cohomology group, entire curve, algebraic degeneration, weighted projective space, Green-Griffiths-Lang conjecture

Mots-clés. Courbure de Chern, inégalité de Morse holomorphe, fibré de jets, groupe de cohomologie, courbe entière, dégénérescence algébrique, espace projectif à poids, conjecture de Green-Griffiths-Lang.

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0. Introduction

Let X be a complex n-dimensional manifold ; most of the time we will assume that X is compact and even projective algebraic. If $\mu:\widetilde{X}\to X$ is a modification and $f:\mathbb{C}\to X$ is an entire curve whose image $f(\mathbb{C})$ is not contained in the image $\mu(E)$ of the exceptional locus, then f admits a unique lifting $\widetilde{f}:\mathbb{C}\to\widetilde{X}$. For this reason, the study of the algebraic degeneration of f is a birationally invariant problem, and singularities do not play an essential role at this stage. We will therefore assume that X is non singular, possibly after performing a suitable composition of blow-ups. We are interested more generally in the situation where the tangent bundle T_X is equipped with a *linear subspace* $V \subset T_X$, that is, an irreducible complex analytic subset of the total space such that

(0.1) all fibers $V_x := V \cap T_{X,x}$ are vector subspaces of $T_{X,x}$.

Then the problem is to study entire curves $f : \mathbb{C} \to X$ which are tangent to V, i.e. such that $f_* T_{\mathbb{C}} \subset V$. We will refer to a pair (X, V) as being a *directed variety* (or *directed manifold*). A morphism of directed varieties $\Phi : (X, V) \to (Y, W)$ is a holomorphic map $\Phi : X \to Y$ such that $\Phi_* V \subset W$; by the irreducibility, it is enough to check this condition over the dense open subset $X \smallsetminus V_{\rm sing}$ where V is actually a subbundle (here $V_{\rm sing}$ is the indeterminacy set of the associated meromorphic map $X \dashrightarrow G_r(T_X)$ to the Grassmannian of r-planes in T_X, $r = \operatorname{rank} V$). In that way, we get a category, and we will be mostly interested in the subcategory whose objects (X, V) are projective algebraic manifolds equipped with algebraic linear subspaces.

The case where $V = T_{X/S}$ is the relative tangent space of some fibration $X \to S$ is of special interest, and so is the case of a foliated variety (this is the situation where the sheaf of sections $\mathcal{O}(V)$ satisfies the Frobenius integrability condition $[\mathcal{O}(V), \mathcal{O}(V)] \subset \mathcal{O}(V)$); however, it is very useful to allow as well non integrable linear subspaces V. We refer to $V = T_X$ as being the *absolute case*. Our main target is the following deep conjecture concerning the algebraic degeneracy of entire curves, which generalizes similar statements made in [GG79] (see also [Lang86, Lang87]).

(0.2) Generalized Green-Griffiths-Lang conjecture. *Let (X, V) be a projective directed manifold such that the canonical sheaf K_V is big (in the absolute case $V = T_X$, this means that X is a variety of general type, and in the relative case we will say that (X, V) is of general type). Then there should exist an algebraic subvariety $Y \subsetneq X$ such that every non constant entire curve $f : \mathbb{C} \to X$ tangent to V is contained in Y.*

The precise meaning of K_V and of its bigness will be explained below. One says that (X, V) is Brody-hyperbolic when there are no entire curves tangent to V. According to (generalized versions of) conjectures of Kobayashi [Kob70, Kob76] the hyperbolicity of (X, V) should imply that K_V is big, and even possibly ample, in a suitable sense. It would then follow from conjecture (0.2) that (X, V) is hyperbolic if and only if for every irreducible variety $Y \subset X$, the linear subspace $V_{\widetilde{Y}} = \overline{T_{\widetilde{Y} \smallsetminus E} \cap \mu_*^{-1} V} \subset T_{\widetilde{Y}}$ has a big canonical sheaf whenever $\mu : \widetilde{Y} \to Y$ is a desingularization and E is the exceptional locus.

The most striking result known on the Green-Griffiths-Lang conjecture at this date is a recent recent of Diverio, Merker and Rousseau [DMR10] in the absolute case, confirming the statement when $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is a generic non singular hypersurface of large degree d, with an estimated sufficient lower bound $d \geqslant 2^{n^5}$.

Their proof is based in an essential way on a strategy developed by Siu [Siu02, Siu04], combined with techniques of [Dem95]. Notice that if the Green-Griffiths-Lang conjecture holds true, a much stronger and probably optimal result would be true, namely all smooth hypersurfaces of degree $d \geqslant n+3$ would satisfy the expected algebraic degeneracy statement. Moreover, by results of Clemens [Cle86] and Voisin [Voi96], a (very) generic hypersurface of degree $d \geqslant 2n+1$ would in fact be hyperbolic for every $n \geqslant 2$. Such a generic hyperbolicity statement has been obtained unconditionally by McQuillan [McQ98, McQ99] when $n = 2$ and $d \geqslant 35$, and by Demailly-El Goul [DEG00] when $n = 2$ and $d \geqslant 21$. Recently Diverio-Trapani [DT10] proved the same result when $n = 3$ and $d \geqslant 593$. By definition, proving the algebraic degeneracy means finding a non zero polynomial P on X such that all entire curves $f : \mathbb{C} \to X$ satisfy $P(f) = 0$. All known methods of proof are based on establishing first the existence of certain algebraic differential equations $P(f\,;\, f', f'', \ldots, f^{(k)}) = 0$ of some order k, and then trying to find enough such equations so that they cut out a proper algebraic locus $Y \subsetneq X$.

Let $J_k V$ be the space of k-jets of curves $f : (\mathbb{C}, 0) \to X$ tangent to V. One defines the sheaf $\mathcal{O}(E^{\mathrm{GG}}_{k,m} V^*)$ of jet differentials of order k and degree m to be the sheaf of holomorphic functions $P(z; \xi_1, \ldots \xi_k)$ on $J_k V$ which are homogeneous polynomials of degree m on the fibers of $J^k V \to X$ with respect to local coordinate derivatives $\xi_j = f^{(j)}(0)$. The degree m considered here is the weighted degree with respect to the natural $\mathbb{C}^*$ action on $J^k V$ defined by $\lambda \cdot f(t) := f(\lambda t)$, i.e. by reparametrizing the curve with a homothetic change of variable. Since $(\lambda \cdot f)^{(j)}(t) = \lambda^j f^{(j)}(\lambda t)$, the weighted action is given in coordinates by

$$\lambda \cdot (\xi_1, \xi_2, \ldots, \xi_k) = (\lambda \xi_1, \lambda^2 \xi_2, \ldots, \lambda^k \xi_k). \tag{0.3}$$

One of the major tool of the theory is the following result due to Green-Griffiths [GG79] (see also [Dem95, Dem97], [SY96a, SY96b], [Siu97]).

(0.4) Fundamental vanishing theorem. *Let (X, V) be a directed projective variety and $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ an entire curve tangent to V. Then for every global section $P \in H^0(X, E^{\mathrm{GG}}_{k,m} V^* \otimes \mathcal{O}(-A))$ where A is an ample divisor of X, one has $P(f\,;\, f', f'', \ldots, f^{(k)}) = 0$.*

It is expected that the global sections of $H^0(X, E^{\mathrm{GG}}_{k,m} V^* \otimes \mathcal{O}(-A))$ are precisely those which ultimately define the algebraic locus $Y \subsetneq X$ where the curve f should lie. The problem is then reduced to the question of showing that there are many non zero sections of $H^0(X, E^{\mathrm{GG}}_{k,m} V^* \otimes \mathcal{O}(-A))$, and further, understanding what is their joint base locus. The first part of this program is the main result of the present paper.

(0.5) Theorem. *Let (X, V) be a directed projective variety such that K_V is big and let A be an ample divisor. Then for $k \gg 1$ and $\delta \in \mathbb{Q}_+$ small enough, $\delta \leqslant c(\log k)/k$, the number of sections $h^0(X, E^{\mathrm{GG}}_{k,m} V^* \otimes \mathcal{O}(-m\delta A))$ has maximal growth, i.e. is larger that $c_k m^{n+kr-1}$ for some $m \geqslant m_k$, where $c, c_k > 0$, $n = \dim X$ and $r = \operatorname{rank} V$. In particular, entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ satisfy (many) algebraic differential equations.*

The statement is very elementary to check when $r = \operatorname{rank} V = 1$, and therefore when $n = \dim X = 1$. In higher dimensions $n \geqslant 2$, only very partial results were known at this point, concerning merely the absolute case $V = T_X$. In dimension 2, Theorem (0.5) is a consequence of the Riemann-Roch calculation of Green-Griffiths [GG79], combined with a vanishing theorem due to Bogomolov [Bog79] – the latter actually only applies to the top cohomology group H^n, and things become much more delicate when extimates of intermediate cohomology groups are needed. In higher dimensions, Diverio [Div09] proved the existence of sections of $H^0(X, E^{\rm GG}_{k,m}V^* \otimes \mathcal{O}(-1))$ whenever X is a hypersurface of $\mathbb{P}^{n+1}_{\mathbb{C}}$ of high degree $d \geqslant d_n$, assuming $k \geqslant n$ and $m \geqslant m_n$. More recently, Merker [Mer10] was able to treat the case of arbitrary hypersurfaces of general type, i.e. $d \geqslant n+3$, assuming this time k to be very large. The latter result is obtained through explicit algebraic calculations of the spaces of sections, and the proof is computationally very intensive. Bérczi [Ber10] also obtained related results with a different approach based on residue formulas, assuming $d \geqslant 2^{7n\log n}$.

All these approaches are algebraic in nature, and while they use some form of holomorphic Morse inequalities [Dem85], they only require a very special elementary algebraic case, namely the lower bound

$$h^0(X, L^{\otimes m}) \geqslant \frac{m^n}{n!}(A^n - n\,A^{n-1}\cdot B) - o(m^n)$$

for $L = \mathcal{O}(A - B)$ with A, B nef (cf. Trapani [Tra95]). Here, our techniques are based on more elaborate curvature estimates in the spirit of Cowen-Griffiths [CG76]. They require the stronger analytic form of holomorphic Morse inequalities.

(0.6) Holomorphic Morse inequalities ([Dem85])**.** *Let (L,h) be a holomorphic line bundle on a compact complex manifold X, equipped with a smooth hermitian metric h, and let E be a holomorphic vector bundle. Denote by $\Theta_{L,h} = -\frac{i}{2\pi}\partial\overline{\partial}\log h$ the curvature form of (L,h) and consider the open set*

$$X(L,h,q) = \big\{x \in X\,;\ \Theta_{L,h}(x) \text{ has signature } (n-q,q)\big\} \quad (q\text{-index set of } \Theta_{L,h}),$$

so that there is a partition $X = \coprod_{0\leqslant q\leqslant n} X(L,q) \cup S$ where $S = \{\det\Theta_{L,h}(x) = 0\}$ is the degeneracy set. Then, if we put $r = \operatorname{rank} E$, we have asymptotically as m tends to infinity:

(a) (*Weak Morse inequalities*)

$$h^q(X, E\otimes L^{\otimes m}) \leqslant \frac{m^n}{n!}\,r\int_{X(L,h,q)}(-1)^q\Theta^n_{L,h} + o(m^n).$$

(b) (*Strong Morse inequalities*) *If $X(L,h,\leqslant q) = \coprod_{j\leqslant q} X(L,h,j)$, then*

$$\sum_{j=0}^{q}(-1)^{q-j}h^j(X, E\otimes L^{\otimes m}) \leqslant \frac{m^n}{n!}\,r\int_{X(L,h,\leqslant q)}(-1)^q\Theta^n_{L,h} + o(m^n).$$

(c) (*Lower bound on* h^0)

$$h^0(X, E\otimes L^{\otimes m}) - h^1(X, E\otimes L^{\otimes m}) \geqslant \frac{m^n}{n!}\, r \int_{X(L,h,\leqslant 1)} \Theta^n_{L,h} - o(m^n).$$

The proof of the above is based on refined spectral estimates for the complex Laplace-Beltrami operators. Observe that (0.6 c) is just the special case of (0.6 b) when $q = 1$. It has been recently observed that these inequalities should be optimal in the sense that the asymptotic cohomology functional $\widehat{h}^q(X, L) := \limsup_{m\to+\infty} \frac{n!}{m^n} h^0(X, L^{\otimes m})$ satisfies

$$\widehat{h}^q(X, L) \leqslant \inf_{h\in C^\infty} \int_{X(L,h,q)} (-1)^q \Theta^n_{L,h}, \tag{0.7}$$

and that conjecturally the inequality should be an equality; it is proved in [Dem10a], [Dem10b] that this is indeed the case if $n \leqslant 2$ or $q = 0$, at least when X is projective algebraic.

Notice that holomorphic Morse inequalities are essentially insensitive to singularities, as we can pass to non singular models and blow-up X as much as we want: if $\mu : \widetilde{X} \to X$ is a modification then $\mu_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ and $R^q\mu_*\mathcal{O}_{\widetilde{X}}$ is supported on a codimension 1 analytic subset (even codimension 2 if X is smooth). It follows by the Leray spectral sequence that the estimates for L on X or for $\widetilde{L} = \mu^*L$ on $\widetilde{X}$ differ by negligible terms $O(m^{n-1})$. Finally, we can even work with singular hermitian metrics h which have (positive and rational) analytic singularities, that is, one can write locally $h = e^{-\varphi}$ where, possibly after blowing up, $\varphi(z) = c\log\sum_j |g_j|^2$ mod C^∞, with $c \in \mathbb{Q}_+$ and g_j holomorphic. Blowing-up the ideal $\mathcal{J} = (g_j)$ leads to divisorial singularities, and then by replacing L with $\widetilde{L} = \mu^*L\otimes\mathcal{O}(-E)$ where $E \in \mathrm{Div}_{\mathbb{Q}}(\widetilde{X})$ is the singularity divisor, we see that holomorphic Morse inequalities still hold for the sequence of groups $H^q(X, E\otimes L^{\otimes m}\otimes\mathcal{I}(h^{\otimes m}))$ where $\mathcal{I}(h^{\otimes m})$ is the multiplier ideal sheaf of $h^{\otimes m}$ (see Bonavero [Bon93] for more details). In the case of linear subspaces $V \subset T_X$, we introduce singular hermitian metrics as follows.

(0.8) Definition. *A singular hermitian metric on a linear subspace $V \subset T_X$ is a metric h on the fibers of V such that the function $\log h : \xi \mapsto \log|\xi|^2_h$ is locally integrable on the total space of V.*

Such a metric can also be viewed as a singular hermitian metric on the tautological line bundle $\mathcal{O}_{P(V)}(-1)$ on the projectivized bundle $P(V) = V \smallsetminus \{0\}/\mathbb{C}^*$, and therefore its dual metric h^{-1} defines a curvature current $\Theta_{\mathcal{O}_{P(V)}(1),h^{-1}}$ of type $(1,1)$ on $P(V) \subset P(T_X)$, such that

$$p^*\Theta_{\mathcal{O}_{P(V)}(1),h^{-1}} = \frac{i}{2\pi}\partial\overline{\partial}\log h, \qquad \text{where } p : V\smallsetminus\{0\} \to P(V).$$

If $\log h$ is quasi-plurisubharmonic (or quasi-psh, which means psh modulo addition of a smooth function) on V, then $\log h$ is indeed locally integrable, and we have moreover

$$\Theta_{\mathcal{O}_{P(V)}(1),h^*} \geqslant -C\omega \tag{0.9}$$

for some smooth positive $(1,1)$-form on $P(V)$ and some constant $C > 0$; conversely, if (0.9) holds, then $\log h$ is quasi psh.

(0.10) Definition. *We will say that a singular hermitian metric h on V is* admissible *if h^{-1} has analytic singularities when seen as a metric on $\mathcal{O}_{P(V)}(1)$ and if moreover h is a smooth hermitian metric on a Zariski open set $X' \subset X \smallsetminus V_{\rm sing}$.*

If h is an admissible metric, we define $\mathcal{O}_h(V^*)$ to be the sheaf of germs of holomorphic sections sections of $V^*_{|X'}$ which are h^{-1}-bounded near $X \smallsetminus X'$; by the assumption on the analytic singularities, this is a coherent sheaf, and actually a subsheaf of the sheaf $\mathcal{O}(V^*) := \mathcal{O}_{h_0}(V^*)$ associated with a smooth positive definite metric h_0 on T_X. If r is the generic rank of V and m a positive integer, we define similarly $K^m_{V,h}$ to be sheaf of germs of holomorphic sections of $(\det V^*_{|X'})^{\otimes m} = (\Lambda^r V^*_{|X'})^{\otimes m}$ which are $\det h^{-1}$-bounded, and $K^m_V := K^m_{V,h_0}$. With our assumptions, there always exists a modification $\mu : \widetilde{X}$ and an integer m_0 such that for all multiples $m = pm_0$ the pull-back $\mu^* K^m_{V,h}$ is an invertible sheaf on $\widetilde{X}$, and $\det h^{-1}$ induces a smooth non singular metric on it. We then think rather of $K_{V,h}$ (resp. K_V) as the "virtual" $\mathbb{Q}$-line bundle $\mu_*(\mu^* K^{m_0}_{V,h})^{1/m_0}$ (resp. $\mu_*(\mu^* K^{m_0}_V)^{1/m_0}$), and we say that $K_{V,h}$ is big if $h^0(X, K^m_{V,h}) \geqslant cm^n$ for $m \geqslant m_1$, with $c > 0$; notice that by definition we always have $K^m_{V,h} = \mu_*(\mu^* K^m_{V,h})$.

Our strategy can be described as following. We consider the Green-Griffiths bundle of k-jets $X^{\rm GG}_k = J^k V \smallsetminus \{0\}/\mathbb{C}^*$, which by (0.3) consists of a fibration in *weighted projective spaces*, and its associated tautological sheaf

$$L = \mathcal{O}_{X^{\rm GG}_k}(1),$$

viewed rather as a virtual $\mathbb{Q}$-line bundle $\mathcal{O}_{X^{\rm GG}_k}(m_0)^{1/m_0}$ with $m_0 = \mathrm{lcm}(1, 2, \ldots, k)$. Then, if $\pi_k : X^{\rm GG}_k \to X$ is the natural projection, we have

$$E^{\rm GG}_{k,m} = (\pi_k)_* \mathcal{O}_{X^{\rm GG}_k}(m) \quad \text{and} \quad R^q(\pi_k)_* \mathcal{O}_{X^{\rm GG}_k}(m) = 0 \text{ for } q \geqslant 1.$$

Hence, by the Leray spectral sequence we get for every invertible sheaf F on X the isomorphism

$$H^q(X, E^{\rm GG}_{k,m} V^* \otimes F) \simeq H^q(X^{\rm GG}_k, \mathcal{O}_{X^{\rm GG}_k}(m) \otimes \pi_k^* F). \tag{0.11}$$

The latter group can be evaluated thanks to holomorphic Morse inequalities. In fact we can associate with any admissible metric h on V a metric (or rather a natural family) of metrics on $L = \mathcal{O}_{X^{\rm GG}_k}(1)$. The space $X^{\rm GG}_k$ always possesses quotient singularities if $k \geqslant 2$ (and even some more if V is singular), but we do not really care since Morse inequalities still work in this setting. As we will see, it is then possible to get nice asymptotic formulas as $k \to +\infty$. They appear to be of a *probabilistic nature* if we take the components of the k-jet (i.e. the successive derivatives $\xi_j = f^{(j)}(0)$, $1 \leqslant j \leqslant k$) as random variables. This probabilistic behaviour was somehow already visible in the Riemann-Roch calculation of [GG79]. In this way, assuming K_V big, we produce a lot of sections $\sigma_j = H^0(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^* F)$, corresponding to certain divisors $Z_j \subset X_k^{\mathrm{GG}}$. The hard problem which is left in order to complete a proof of the generalized Green-Griffiths-Lang conjecture is to compute the base locus $Z = \bigcap Z_j$ and to show that $Y = \pi_k(Z) \subset X$ must be a proper algebraic variety. Although we cannot address this problem at present, we will indicate a few technical results and a couple of possibilities in this direction.

1. Hermitian geometry of weighted projective spaces

The goal of this section is to introduce natural Kähler metrics on weighted projective spaces, and to evaluate the corresponding volume forms. Here we put $d^c = \frac{i}{4\pi}(\overline{\partial} - \partial)$ so that $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$. The normalization of the d^c operator is chosen such that we have precisely $(dd^c \log |z|^2)^n = \delta_0$ for the Monge-Ampère operator in $\mathbb{C}^n$; also, for every holomorphic or meromorphic section σ of a hermitian line bundle (L, h) the Lelong-Poincaré can be formulated

$$(1.1) \qquad dd^c \log |\sigma|_h^2 = [Z_\sigma] - \Theta_{L,h},$$

where $\Theta_{L,h} = \frac{i}{2\pi} D_{L,h}^2$ is the $(1,1)$-curvature form of L and Z_σ the zero divisor of σ. The closed $(1,1)$-form $\Theta_{L,h}$ is a representative of the first Chern class $c_1(L)$. Given a k-tuple of "weights" $a = (a_1, \ldots, a_k)$, i.e. of relatively prime integers $a_s > 0$ we introduce the weighted projective space $P(a_1, \ldots, a_k)$ to be the quotient of $\mathbb{C}^k \smallsetminus \{0\}$ by the corresponding weighted $\mathbb{C}^*$ action:

$$(1.2) \qquad P(a_1, \ldots, a_k) = \mathbb{C}^k \smallsetminus \{0\}/\mathbb{C}^*, \qquad \lambda \cdot z = (\lambda^{a_1} z_1, \ldots, \lambda^{a_k} z_k).$$

As is well known, this defines a toric $k-1$-dimensional algebraic variety with quotient singularities. On this variety, we introduce the possibly singular (but almost everywhere smooth and non degenerate) Kähler form $\omega_{a,p}$ defined by

$$(1.3) \qquad \pi_a^* \omega_{a,p} = dd^c \varphi_{a,p}, \qquad \varphi_{a,p}(z) = \frac{1}{p} \log \sum_{1 \leqslant s \leqslant k} |z_s|^{2p/a_s},$$

where $\pi_a : \mathbb{C}^k \smallsetminus \{0\} \to P(a_1, \ldots, a_k)$ is the canonical projection and $p > 0$ is a positive constant. It is clear that $\varphi_{p,a}$ is real analytic on $\mathbb{C}^k \smallsetminus \{0\}$ if p is an integer and a common multiple of all weights a_s. It is at least C^2 is p is real and $p \geqslant \max(a_s)$, which will be more than sufficient for our purposes (but everything would still work for any $p > 0$). The resulting metric is in any case smooth and positive definite outside of the coordinate hyperplanes $z_s = 0$, and these hyperplanes will not matter here since they are of capacity zero with respect to all currents $(dd^c\varphi_{a,p})^\ell$. In order to evaluate the volume $\int_{P(a_1,\ldots,a_k)}\omega_{a,p}^{k-1}$, one can observe that

$$\begin{aligned}\int_{P(a_1,\ldots,a_k)}\omega_{a,p}^{k-1}&=\int_{z\in\mathbb{C}^k,\,\varphi_{a,p}(z)=0}\pi_a^*\omega_{a,p}^{k-1}\wedge d^c\varphi_{a,p}\\&=\int_{z\in\mathbb{C}^k,\,\varphi_{a,p}(z)=0}(dd^c\varphi_{a,p})^{k-1}\wedge d^c\varphi_{a,p}\\&=\frac{1}{p^k}\int_{z\in\mathbb{C}^k,\,\varphi_{a,p}(z)<0}(dd^ce^{p\varphi_{a,p}})^k.\end{aligned}\tag{1.4}$$

The first equality comes from the fact that $\{\varphi_{a,p}(z)=0\}$ is a circle bundle over $P(a_1,\ldots,a_k)$, by using the identities $\varphi_{a,p}(\lambda\cdot z)=\varphi_{a,p}(z)+\log|\lambda|^2$ and $\int_{|\lambda|=1}d^c\log|\lambda|^2=1$. The third equality can be seen by Stokes formula applied to the $(2k-1)$-form

$$(dd^ce^{p\varphi_{a,p}})^{k-1}\wedge d^ce^{p\varphi_{a,p}}=e^{p\varphi_{a,p}}(dd^c\varphi_{a,p})^{k-1}\wedge d^c\varphi_{a,p}$$

on the pseudoconvex open set $\{z\in\mathbb{C}^k\,;\ \varphi_{a,p}(z)<0\}$. Now, we find

$$(dd^ce^{p\varphi_{a,p}})^k=\Big(dd^c\sum_{1\leqslant s\leqslant k}|z_s|^{2p/a_s}\Big)^k=\prod_{1\leqslant s\leqslant k}\Big(\frac{p}{a_s}|z_s|^{\frac{p}{a_s}-1}\Big)(dd^c|z|^2)^k,\tag{1.5}$$

$$\int_{z\in\mathbb{C}^k,\,\varphi_{a,p}(z)<0}(dd^ce^{p\varphi_{a,p}})^k=\prod_{1\leqslant s\leqslant k}\frac{p}{a_s}=\frac{p^k}{a_1\ldots a_k}.\tag{1.6}$$

In fact, (1.5) and (1.6) are clear when $p=a_1=\ldots=a_k=1$ (this is just the standard calculation of the volume of the unit ball in $\mathbb{C}^k$); the general case follows by substituting formally $z_s\mapsto z_s^{p/a_s}$, and using rotational invariance along with the observation that the arguments of the complex numbers z_s^{p/a_s} now run in the interval $[0,2\pi p/a_s[$ instead of $[0,2\pi[$ (say). As a consequence of (1.4) and (1.6), we obtain the well known value

$$\int_{P(a_1,\ldots,a_k)}\omega_{a,p}^{k-1}=\frac{1}{a_1\ldots a_k},\tag{1.7}$$

for the volume. Notice that this is independent of p (as it is obvious by Stokes theorem, since the cohomology class of $\omega_{a,p}$ does not depend on p). When p tends to $+\infty$, we have $\varphi_{a,p}(z)\mapsto\varphi_{a,\infty}(z)=\log\max_{1\leqslant s\leqslant k}|z_s|^{2/a_s}$ and the volume form $\omega_{a,p}^{k-1}$ converges to a rotationally invariant measure supported by the image of the polycircle $\prod\{|z_s|=1\}$ in $P(a_1,\ldots,a_k)$. This is so because not all $|z_s|^{2/a_s}$ are equal outside of the image of the polycircle, thus $\varphi_{a,\infty}(z)$ locally depends only on $k-1$ complex variables, and so $\omega_{a,\infty}^{k-1}=0$ there by log homogeneity.

Our later calculations will require a slightly more general setting. Instead of looking at $\mathbb{C}^k$, we consider the weighted $\mathbb{C}^*$ action defined by

$$\mathbb{C}^{|r|}=\mathbb{C}^{r_1}\times\ldots\times\mathbb{C}^{r_k},\qquad\lambda\cdot z=(\lambda^{a_1}z_1,\ldots,\lambda^{a_k}z_k).\tag{1.8}$$

Here $z_s \in \mathbb{C}^{r_s}$ for some k-tuple $r = (r_1, \ldots, r_k)$ and $|r| = r_1 + \ldots + r_k$. This gives rise to a weighted projective space

$$(1.9)\qquad \begin{aligned} &P(a_1^{[r_1]}, \ldots, a_k^{[r_k]}) = P(a_1, \ldots, a_1, \ldots, a_k, \ldots, a_k), \\ &\pi_{a,r} : \mathbb{C}^{r_1} \times \ldots \times \mathbb{C}^{r_k} \smallsetminus \{0\} \longrightarrow P(a_1^{[r_1]}, \ldots, a_k^{[r_k]}) \end{aligned}$$

obtained by repeating r_s times each weight a_s. On this space, we introduce the degenerate Kähler metric $\omega_{a,r,p}$ such that

$$(1.10)\qquad \pi_{a,r}^* \omega_{a,r,p} = dd^c \varphi_{a,r,p}, \qquad \varphi_{a,r,p}(z) = \frac{1}{p} \log \sum_{1 \leqslant s \leqslant k} |z_s|^{2p/a_s}$$

where $|z_s|$ stands now for the standard hermitian norm $(\sum_{1 \leqslant j \leqslant r_s} |z_{s,j}|^2)^{1/2}$ on $\mathbb{C}^{r_s}$. This metric is cohomologous to the corresponding "polydisc-like" metric $\omega_{a,p}$ already defined, and therefore Stokes theorem implies

$$(1.11)\qquad \int_{P(a_1^{[r_1]}, \ldots, a_k^{[r_k]})} \omega_{a,r,p}^{|r|-1} = \frac{1}{a_1^{r_1} \ldots a_k^{r_k}}.$$

Since $(dd^c \log |z_s|^2)^{r_s} = 0$ on $\mathbb{C}^{r_s} \smallsetminus \{0\}$ by homogeneity, we conclude as before that the weak limit $\lim_{p \to +\infty} \omega_{a,r,p}^{|r|-1} = \omega_{a,r,\infty}^{|r|-1}$ associated with

$$(1.12)\qquad \varphi_{a,r,\infty}(z) = \log \max_{1 \leqslant s \leqslant k} |z_s|^{2/a_s}$$

is a measure supported by the image of the product of unit spheres $\prod S^{2r_s-1}$ in $P(a_1^{[r_1]}, \ldots, a_k^{[r_k]})$, which is invariant under the action of $U(r_1) \times \ldots \times U(r_k)$ on $\mathbb{C}^{r_1} \times \ldots \times \mathbb{C}^{r_k}$, and thus coincides with the hermitian area measure up to a constant determined by condition (1.11). In fact, outside of the product of spheres, $\varphi_{a,r,\infty}$ locally depends only on at most $k-1$ factors and thus, for dimension reasons, the top power $(dd^c \varphi_{a,r,\infty})^{|r|-1}$ must be zero there. In the next section, the following change of variable formula will be needed. For simplicity of exposition we restrict ourselves to continuous functions, but a standard density argument would easily extend the formula to all functions that are Lebesgue integrable with respect to the volume form $\omega_{a,r,p}^{|r|-1}$.

(1.13) Proposition. *Let $f(z)$ be a bounded function on $P(a_1^{[r_1]}, \ldots, a_k^{[r_k]})$ which is continuous outside of the hyperplane sections $z_s = 0$. We also view f as a $\mathbb{C}^*$-invariant continuous function on $\prod(\mathbb{C}^{r_s} \smallsetminus \{0\})$. Then*

$$\begin{aligned} &\int_{P(a_1^{[r_1]}, \ldots, a_k^{[r_k]})} f(z)\, \omega_{a,r,p}^{|r|-1} \\ &= \frac{(|r|-1)!}{\prod_s a_s^{r_s}} \int_{(x,u) \in \Delta_{k-1} \times \prod S^{2r_s-1}} f(x_1^{a_1/2p} u_1, \ldots, x_k^{a_k/2p} u_k) \prod_{1 \leqslant s \leqslant k} \frac{x_s^{r_s-1}}{(r_s-1)!}\, dx\, d\mu(u) \end{aligned}$$

where Δ_{k-1} is the $(k-1)$-simplex $\{x_s \geqslant 0, \sum x_s = 1\}$, $dx = dx_1 \wedge \ldots \wedge dx_{k-1}$ its standard measure, and where $d\mu(u) = d\mu_1(u_1) \ldots d\mu_k(u_k)$ is the rotation invariant probability measure on the product $\prod_s S^{2r_s-1}$ of unit spheres in $\mathbb{C}^{r_1} \times \ldots \times \mathbb{C}^{r_k}$. As a consequence

$$\lim_{p \to +\infty} \int_{P(a_1^{[r_1]}, \ldots, a_k^{[r_k]})} f(z)\, \omega_{a,r,p}^{|r|-1} = \frac{1}{\prod_s a_s^{r_s}} \int_{\prod S^{2r_s-1}} f(u)\, d\mu(u).$$

Proof. The area formula of the disc $\int_{|\lambda|<1} dd^c|\lambda|^2 = 1$ and a consideration of the unit disc bundle over $P(a_1^{[r_1]}, \ldots, a_k^{[r_k]})$ imply that

$$I_p := \int_{P(a_1^{[r_1]}, \ldots, a_k^{[r_k]})} f(z)\, \omega_{a,r,p}^{|r|-1} = \int_{z \in \mathbb{C}^{|r|}, \varphi_{a,r,p}(z)<0} f(z)\, (dd^c\varphi_{a,r,p})^{|r|-1} \wedge dd^c e^{\varphi_{a,r,p}}.$$

Now, a straightforward calculation on $\mathbb{C}^{|r|}$ gives

$$\begin{aligned}(dd^c e^{p\varphi_{a,r,p}})^{|r|} &= \Big(dd^c \sum_{1 \leqslant s \leqslant k} |z_s|^{2p/a_s}\Big)^{|r|} \\ &= \prod_{1 \leqslant s \leqslant k} \Big(\frac{p}{a_s}\Big)^{r_s+1} |z_s|^{2r_s(p/a_s-1)} (dd^c|z|^2)^{|r|}.\end{aligned}$$

On the other hand, we have $(dd^c|z|^2)^{|r|} = \frac{|r|!}{r_1! \ldots r_k!} \prod_{1 \leqslant s \leqslant k} (dd^c|z_s|^2)^{r_s}$ and

$$\begin{aligned}(dd^c e^{p\varphi_{a,r,p}})^{|r|} &= \big(p\, e^{p\varphi_{a,r,p}} (dd^c\varphi_{a,r,p} + p\, d\varphi_{a,r,p} \wedge d^c\varphi_{a,r,p})\big)^{|r|} \\ &= |r| p^{|r|+1} e^{|r|p\varphi_{a,r,p}} (dd^c\varphi_{a,r,p})^{|r|-1} \wedge d\varphi_{a,r,p} \wedge d^c\varphi_{a,r,p} \\ &= |r| p^{|r|+1} e^{(|r|p-1)\varphi_{a,r,p}} (dd^c\varphi_{a,r,p})^{|r|-1} \wedge dd^c e^{\varphi_{a,r,p}},\end{aligned}$$

thanks to the homogeneity relation $(dd^c\varphi_{a,r,p})^{|r|} = 0$. Putting everything together, we find

$$I_p = \int_{z \in \mathbb{C}^{|r|},\, \varphi_{a,r,p}(z)<0} \frac{(|r|-1)!\, p^{k-1} f(z)}{(\sum_s |z_s|^{2p/a_s})^{|r|-1/p}} \prod_s \frac{(dd^c|z_s|^2)^{r_s}}{r_s!\, a_s^{r_s+1} |z_s|^{2r_s(1-p/a_s)}}.$$

A standard calculation in polar coordinates with $z_s = \rho_s u_s$, $u_s \in S^{2r_s-1}$, yields

$$\frac{(dd^c|z_s|^2)^{r_s}}{|z_s|^{2r_s}} = 2r_s \frac{d\rho_s}{\rho_s}\, d\mu_s(u_s)$$

where μ_s is the $U(r_s)$-invariant probability measure on S^{2r_s-1}. Therefore

$$\begin{aligned}I_p &= \int_{\varphi_{a,r,p}(z)<0} \frac{(|r|-1)!\, p^{k-1} f(\rho_1 u_1, \ldots, \rho_k u_k)}{(\sum_{1 \leqslant s \leqslant k} \rho_s^{2p/a_s})^{|r|-1/p}} \prod_s \frac{2\rho_s^{2pr_s/a_s} \frac{d\rho_s}{\rho_s} d\mu_s(u_s)}{(r_s-1)!\, a_s^{r_s+1}} \\ &= \int_{u_s \in S^{2r_s-1}, \sum t_s<1} \frac{(|r|-1)!\, p^{-1} f(t_1^{a_1/2p} u_1, \ldots, t_k^{a_k/2p} u_k)}{(\sum_{1 \leqslant s \leqslant k} t_s)^{|r|-1/p}} \prod_s \frac{t_s^{r_s-1} dt_s\, d\mu_s(u_s)}{(r_s-1)!\, a_s^{r_s}}\end{aligned}$$

by putting $t_s=|z_s|^{2p/a_s}=\rho_s^{2p/a_s}$, i.e. $\rho_s=t_s^{a_s/2p}$, $t_s\in\,]0,1]$. We use still another change of variable $t_s=tx_s$ with $t=\sum_{1\leqslant s\leqslant k}t_s$ and $x_s\in\,]0,1]$, $\sum_{1\leqslant s\leqslant k}x_s=1$. Then

$$dt_1\wedge\ldots\wedge dt_k=t^{k-1}\,dx\,dt\qquad\text{where } dx=dx_1\wedge\ldots\wedge dx_{k-1}.$$

The $\mathbb{C}^*$ invariance of f shows that

$$\begin{aligned}I_p&=\int_{\substack{u_s\in S^{2r_s-1}\\ \Sigma x_s=1,\ t\in]0,1]}}(|r|-1)!f(x_1^{a_s/2p}u_1,\ldots,x_k^{a_k/2p}u_k)\prod_{1\leqslant s\leqslant k}\frac{x_s^{r_s-1}d\mu_s(u_s)}{(r_s-1)!\,a_s^{r_s}}\frac{dx\,dt}{p\,t^{1-1/p}}\\&=\int_{\substack{u_s\in S^{2r_s-1}\\ \Sigma x_s=1}}(|r|-1)!f(x_1^{a_s/2p}u_1,\ldots,x_k^{a_k/2p}u_k)\prod_{1\leqslant s\leqslant k}\frac{x_s^{r_s-1}d\mu_s(u_s)}{(r_s-1)!\,a_s^{r_s}}\,dx.\end{aligned}$$

This is equivalent to the formula given in Proposition 1.13. We have $x_s^{2a_s/p}\to 1$ as $p\to+\infty$, and by Lebesgue's bounded convergence theorem and Fubini's formula, we get

$$\lim_{p\to+\infty}I_p=\frac{(|r|-1)!}{\prod_s a_s^{r_s}}\int_{(x,u)\in\Delta_{k-1}\times\prod S^{2r_s-1}}f(u)\prod_{1\leqslant s\leqslant k}\frac{x_s^{r_s-1}}{(r_s-1)!}\,dx\,d\mu(u).$$

It can be checked by elementary integrations by parts and induction on $k,\ r_1,\ldots,r_k$ that

$$\int_{x\in\Delta_{k-1}}\prod_{1\leqslant s\leqslant k}x_s^{r_s-1}dx_1\ldots dx_{k-1}=\frac{1}{(|r|-1)!}\prod_{1\leqslant s\leqslant k}(r_s-1)!\,.\tag{1.14}$$

This implies that $(|r|-1)!\prod_{1\leqslant s\leqslant k}\frac{x_s^{r_s-1}}{(r_s-1)!}\,dx$ is a probability measure on Δ_{k-1} and that

$$\lim_{p\to+\infty}I_p=\frac{1}{\prod_s a_s^{r_s}}\int_{u\in\prod S^{2r_s-1}}f(u)\,d\mu(u).$$

Even without an explicit check, the evaluation (1.14) also follows from the fact that we must have equality for $f(z)\equiv 1$ in the latter equality, if we take into account the volume formula (1.11). □

2. Probabilistic estimate of the curvature of *k*-jet bundles

Let (X,V) be a compact complex directed non singular variety. To avoid any technical difficulty at this point, we first assume that V is a holomorphic vector subbundle of T_X, equipped with a smooth hermitian metric h.

According to the notation already specified in the introduction, we denote by J^kV the bundle of k-jets of holomorphic curves $f:(\mathbb{C},0)\to X$ tangent to V at each point. Let us set $n=\dim_{\mathbb{C}}X$ and $r=\operatorname{rank}_{\mathbb{C}}V$. Then $J^kV\to X$ is an algebraic fiber bundle with typical fiber $\mathbb{C}^{rk}$ (see below). It has a canonical $\mathbb{C}^*$-star action defined by $\lambda\cdot f:(\mathbb{C},0)\to X$, $(\lambda\cdot f)(t)=f(\lambda t)$. Fix a point x_0 in X an a local holomorphic coordinate system $(z_1,\ldots,z_n)$ centered at x_0 such that V_{x_0} is the vector subspace $\langle\partial/\partial z_1,\ldots,\partial/\partial z_r\rangle$ at x_0. Then, in a neighborhood U of x_0, V admits a holomorphic frame of the form

$$\frac{\partial}{\partial z_\beta}+\sum_{r+1\leqslant\alpha\leqslant n}a_{\alpha\beta}(z)\frac{\partial}{\partial z_\alpha},\qquad 1\leqslant\beta\leqslant r,\quad a_{\alpha\beta}(0)=0. \tag{2.1}$$

Let $f(t)=(f_1(t),\ldots,f_n(t))$ be a k-jet of curve tangent to V starting from a point $f(0)=x\in U$. Such a curve is entirely determined by its initial point and by the projection $\widetilde{f}(t):=(f_1(t),\ldots,f_r(t))$ to the first r-components, since the condition $f'(t)\in V_{f(t)}$ implies that the other components must satisfy the ordinary differential equation

$$f'_\alpha(t)=\sum_{1\leqslant\beta\leqslant r}a_{\alpha\beta}(f(t))f'_\beta(t).$$

This implies that the k-jet of f is entirely determined by the initial point x and the Taylor expansion

$$\widetilde{f}(t)-\widetilde{x}=\xi_1t+\xi_2t^2+\ldots+\xi_kt^k+O(t^{k+1}) \tag{2.2}$$

where $\xi_s=(\xi_{s\alpha})_{1\leqslant\alpha\leqslant r}\in\mathbb{C}^r$. The $\mathbb{C}^*$ action $(\lambda,f)\mapsto\lambda\cdot f$ is then expressed in coordinates by the weighted action

$$\lambda\cdot(\xi_1,\xi_2,\ldots,\xi_k)=(\lambda\xi_1,\lambda^2\xi_2,\ldots,\lambda^k\xi_k) \tag{2.3}$$

associated with the weight $a=(1^{[r]},2^{[r]},\ldots,k^{[r]})$. The quotient projectived k-jet bundle

$$X_k^{\rm GG}:=(J^kV\smallsetminus\{0\})/\mathbb{C}^* \tag{2.4}$$

considered by Green and Griffiths [GG79] is therefore in a natural way a $P(1^{[r]},2^{[r]},\ldots,k^{[r]})$ weighted projective bundle over X. As such, it possesses a canonical sheaf $\mathcal{O}_{X_k^{\rm GG}}(1)$ such that $\mathcal{O}_{X_k^{\rm GG}}(m)$ is invertible when m is a multiple of $\mathrm{lcm}(1,2,\ldots,k)$. Under the natural projection $\pi_k:X_k^{\rm GG}\to X$, the direct image $(\pi_k)_*\mathcal{O}_{X_k^{\rm GG}}(m)$ coincides with the sheaf of sections of the bundle $E_{k,m}^{\rm GG}V^*$ of jet differentials of order k and degree m, namely polynomials

$$P(z\,;\,\xi_1,\ldots,\xi_k)=\sum_{\alpha_\ell\in\mathbb{N}^r,\,1\leqslant\ell\leqslant k}a_{\alpha_1\ldots\alpha_k}(z)\,\xi_1^{\alpha_1}\ldots\xi_k^{\alpha_k} \tag{2.5}$$

of weighted degree $|\alpha_1|+2|\alpha_2|+\ldots+k|\alpha_k|=m$ on J^kV with holomorphic coefficients. The jet differentials operate on germs of curves as differential operators

$$P(f)(t)=\sum a_{\alpha_1\ldots\alpha_k}(f(t))\,f'(t)^{\alpha_1}\ldots f^{(k)}(t)^{\alpha_k} \tag{2.6}$$

In the sequel, we do not make any further use of coordinate frames as (2.1), because they need not be related in any way to the hermitian metric h of V. Instead, we choose a local holomorphic coordinate frame $(e_\alpha(z))_{1\leqslant\alpha\leqslant r}$ of V on a neighborhood U of x_0, such that

$$\langle e_\alpha(z), e_\beta(z)\rangle = \delta_{\alpha\beta} + \sum_{1\leqslant i,j\leqslant n,\, 1\leqslant\alpha,\beta\leqslant r} c_{ij\alpha\beta} z_i \overline{z}_j + O(|z|^3) \tag{2.7}$$

for suitable complex coefficients $(c_{ij\alpha\beta})$. It is a standard fact that such a normalized coordinate system always exists, and that the Chern curvature tensor $\frac{i}{2\pi}D^2_{V,h}$ of (V,h) at x_0 is then given by

$$\Theta_{V,h}(x_0) = -\frac{i}{2\pi}\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}\, dz_i\wedge d\overline{z}_j\otimes e^*_\alpha\otimes e_\beta. \tag{2.8}$$

Also, instead of defining the vectors $\xi_s\in\mathbb{C}^r$ as in (2.2), we consider a local holomorphic connection ∇ on $V_{|U}$ (e.g. the one which turns (e_α) into a parallel frame), and take $\xi_k = \nabla^k f(0)\in V_x$ defined inductively by $\nabla^1 f = f'$ and $\nabla^s f = \nabla_{f'}(\nabla^{s-1}f)$. This is just another way of parametrizing the fibers of J^kV over U by the vector bundle $V^k_{|U}$. Notice that this is highly dependent on ∇ (the bundle J^kV actually does not carry a vector bundle or even affine bundle structure); however, the expression of the weighted action (2.3) is unchanged in this new setting. Now, we fix a finite open covering $(U_\alpha)_{\alpha\in I}$ of X by open coordinated charts such that $V_{|U_\alpha}$ is trivial, along with holomorphic connections ∇_α on $V_{|U_\alpha}$. Let θ_α be a partition of unity of X subordinate to the covering (U_α). Let us fix $p>0$ and small parameters $1=\varepsilon_1\gg\varepsilon_2\gg\ldots\gg\varepsilon_k>0$. Then we define a global weighted exhaustion on J^kV by putting for any k-jet $f\in J^k_xV$

$$\Psi_{h,p,\varepsilon}(f) := \Big(\sum_{\alpha\in I}\theta_\alpha(x)\sum_{1\leqslant s\leqslant k}\varepsilon_s^{2p}\|\nabla^s_\alpha f(0)\|^{2p/s}_{h(x)}\Big)^{1/p} \tag{2.9}$$

where $\|\ \|_{h(x)}$ is the hermitian metric h of V evaluated on the fiber V_x, $x=f(0)$. The function $\Psi_{h,p,\varepsilon}$ satisfies the fundamental homogeneity property

$$\Psi_{h,p,\varepsilon}(\lambda\cdot f) = \Psi_{h,p,\varepsilon}(f)\,|\lambda|^2 \tag{2.10}$$

with respect to the $\mathbb{C}^*$ action on J^kV, in other words, it induces a hermitian metric on the dual L^* of the tautological $\mathbb{Q}$-line bundle $L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ over X_k^{GG}. The curvature of L_k is given by

$$\pi_k^*\Theta_{L_k,\Psi^*_{h,p,\varepsilon}} = dd^c\log\Psi_{h,p,\varepsilon} \tag{2.11}$$

where $\pi_k : J^kV\smallsetminus\{0\}\to X_k^{\mathrm{GG}}$ is the canonical projection. Our next goal is to compute precisely the curvature and to apply holomorphic Morse inequalities to $L\to X_k^{\mathrm{GG}}$ with the above metric. It might look a priori like an untractable problem, since the definition of $\Psi_{h,p,\varepsilon}$ is a rather unnatural one. However, the "miracle" is that the asymptotic behavior of $\Psi_{h,p,\varepsilon}$ as $\varepsilon_s/\varepsilon_{s-1}\to 0$ is in some sense uniquely defined and very natural. It will lead to a computable asymptotic formula, which is moreover simple enough to produce useful results.

(2.12) Lemma. *On each coordinate chart U equipped with a holomorphic connection ∇ of $V_{|U}$, let us define the components of a k-jet $f\in J^kV$ by $\xi_s=\nabla^s f(0)$, and consider the rescaling transformation*

$$\rho_\varepsilon(\xi_1,\xi_2,\ldots,\xi_k)=(\varepsilon_1^1\xi_1,\varepsilon_2^2\xi_2,\ldots,\varepsilon_k^k\xi_k)\quad on\ J^k_xV,\ x\in U_\alpha$$

(it commutes with the $\mathbb{C}^*$-action but is otherwise unrelated and not canonically defined over X as it depends on the choice of ∇). Then, if p is a multiple of $\mathrm{lcm}(1,2,\ldots,k)$ and $\varepsilon_s/\varepsilon_{s-1}\to 0$ for all $s=2,\ldots,k$, the rescaled function $\Psi_{h,p,\varepsilon}\circ\rho_\varepsilon^{-1}(\xi_1,\ldots,\xi_k)$ converges towards

$$\Big(\sum_{1\leqslant s\leqslant k}\|\xi_s\|_h^{2p/s}\Big)^{1/p}$$

uniformly in C^∞ topology, on every compact subset of $J^kV_{|U}\smallsetminus\{0\}$.

Proof. Let $U\subset X$ be an open set on which $V_{|U}$ is trivial and equipped with some holomorphic connection ∇. Let us pick another holomorphic connection $\widetilde\nabla=\nabla+\Gamma$ where $\Gamma\in H^0(U,\Omega^1_X\otimes\mathrm{Hom}(V,V)$. Then $\widetilde\nabla^2 f=\nabla^2 f+\Gamma(f)(f')\cdot f'$, and inductively we get

$$\widetilde\nabla^s f=\nabla^s f+P_s(f\,;\,\nabla^1 f,\ldots,\nabla^{s-1}f)$$

where $P(x\,;\,\xi_1,\ldots,\xi_{s-1})$ is a polynomial with holomorphic coefficients in $x\in U$ which is of weighted homogeneous degree s in $(\xi_1,\ldots,\xi_{s-1})$. In other words, the corresponding change in the parametrization of $J^kV_{|U}$ is given by a $\mathbb{C}^*$-homogeneous transformation

$$\widetilde\xi_s=\xi_s+P_s(x\,;\,\xi_1,\ldots,\xi_{s-1}).$$

Let us introduce the corresponding rescaled components

$$(\xi_{1,\varepsilon},\ldots,\xi_{k,\varepsilon})=(\varepsilon_1^1\xi_1,\ldots,\varepsilon_k^k\xi_k),\qquad(\widetilde\xi_{1,\varepsilon},\ldots,\widetilde\xi_{k,\varepsilon})=(\varepsilon_1^1\widetilde\xi_1,\ldots,\varepsilon_k^k\widetilde\xi_k).$$

Then

$$\begin{aligned}\widetilde\xi_{s,\varepsilon}&=\xi_{s,\varepsilon}+\varepsilon_s^s\,P_s(x\,;\,\varepsilon_1^{-1}\xi_{1,\varepsilon},\ldots,\varepsilon_{s-1}^{-(s-1)}\xi_{s-1,\varepsilon})\\&=\xi_{s,\varepsilon}+O(\varepsilon_s/\varepsilon_{s-1})^s\,O(\|\xi_{1,\varepsilon}\|+\ldots+\|\xi_{s-1,\varepsilon}\|^{1/(s-1)})^s\end{aligned}$$

and the error terms are thus polynomials of fixed degree with arbitrarily small coefficients as $\varepsilon_s/\varepsilon_{s-1}\to 0$. Now, the definition of $\Psi_{h,p,\varepsilon}$ consists of glueing the sums

$$\sum_{1\leqslant s\leqslant k}\varepsilon_s^{2p}\|\xi_k\|_h^{2p/s}=\sum_{1\leqslant s\leqslant k}\|\xi_{k,\varepsilon}\|_h^{2p/s}$$

corresponding to $\xi_k=\nabla^s_\alpha f(0)$ by means of the partition of unity $\sum\theta_\alpha(x)=1$. We see that by using the rescaled variables $\xi_{s,\varepsilon}$ the changes occurring when replacing a connection ∇_α by an alternative one ∇_β are arbitrary small in C^∞ topology, with error terms uniformly controlled in terms of the ratios $\varepsilon_s/\varepsilon_{s-1}$ on all compact subsets of $V^k\smallsetminus\{0\}$. This shows that in C^∞ topology, $\Psi_{h,p,\varepsilon}\circ\rho_\varepsilon^{-1}(\xi_1,\ldots,\xi_k)$ converges uniformly towards $(\sum_{1\leqslant s\leqslant k}\Vert\xi_k\Vert_h^{2p/s})^{1/p}$, whatever is the trivializing open set U and the holomorphic connection ∇ used to evaluate the components and perform the rescaling. □

Now, we fix a point $x_0\in X$ and a local holomorphic frame $(e_\alpha(z))_{1\leqslant\alpha\leqslant r}$ satisfying (2.7) on a neighborhood U of x_0. We introduce the rescaled components $\xi_s=\varepsilon_s^s\nabla^s f(0)$ on $J^kV_{|U}$ and compute the curvature of

$$\Psi_{h,p,\varepsilon}\circ\rho_\varepsilon^{-1}(z\,;\,\xi_1,\ldots,\xi_k)\simeq\Big(\sum_{1\leqslant s\leqslant k}\Vert\xi_s\Vert_h^{2p/s}\Big)^{1/p}$$

(by Lemma 2.12, the errors can be taken arbitrary small in C^∞ topology). We write $\xi_s=\sum_{1\leqslant\alpha\leqslant r}\xi_{s\alpha}e_\alpha$. By (2.7) we have

$$\Vert\xi_s\Vert_h^2=\sum_\alpha|\xi_{s\alpha}|^2+\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}z_i\overline{z}_j\xi_{s\alpha}\overline{\xi}_{s\beta}+O(|z|^3|\xi|^2).$$

The question is to evaluate the curvature of the weighted metric defined by

$$\begin{aligned}\Psi(z\,;\,\xi_1,\ldots,\xi_k)&=\Big(\sum_{1\leqslant s\leqslant k}\Vert\xi_s\Vert_h^{2p/s}\Big)^{1/p}\\&=\Big(\sum_{1\leqslant s\leqslant k}\Big(\sum_\alpha|\xi_{s\alpha}|^2+\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}z_i\overline{z}_j\xi_{s\alpha}\overline{\xi}_{s\beta}\Big)^{p/s}\Big)^{1/p}+O(|z|^3).\end{aligned}$$

We set $|\xi_s|^2=\sum_\alpha|\xi_{s\alpha}|^2$. A straightforward calculation yields

$$\begin{aligned}&\log\Psi(z\,;\,\xi_1,\ldots,\xi_k)=\\&=\frac{1}{p}\log\sum_{1\leqslant s\leqslant k}|\xi_s|^{2p/s}+\sum_{1\leqslant s\leqslant k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t|\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}z_i\overline{z}_j\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}+O(|z|^3).\end{aligned}$$

By (2.11), the curvature form of $L_k=\mathcal{O}_{X_k^{\rm GG}}(1)$ is given at the central point x_0 by the following formula.

(2.13) Proposition. *With the above choice of coordinates and in terms of the rescaled components $\xi_s=\varepsilon_s^s\nabla^s f(0)$ at $x_0\in X$, we have the approximate expression*

$$\Theta_{L_k,\Psi^\star_{h,p,\varepsilon}}(x_0,[\xi])\simeq\omega_{a,r,p}(\xi)+\frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t|\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\,dz_i\wedge d\overline{z}_j$$

where the error terms are $O(\max_{2\leqslant s\leqslant k}(\varepsilon_s/\varepsilon_{s-1})^s)$ *uniformly on the compact variety* $X_k^{\rm GG}$*. Here* $\omega_{a,r,p}$ *is the (degenerate) Kähler metric associated with the weight* $a=(1^{[r]},2^{[r]},\ldots,k^{[r]})$ *of the canonical* $\mathbb{C}^*$ *action on* J^kV*.*

Thanks to the uniform approximation, we can (and will) neglect the error terms in the calculations below. Since $\omega_{a,r,p}$ is positive definite on the fibers of $X_k^{\rm GG}\to X$ (at least outside of the axes $\xi_s=0$), the index of the $(1,1)$ curvature form $\Theta_{L_k,\Psi^\star_{h,p,\varepsilon}}(z,[\xi])$ is equal to the index of the $(1,1)$-form

$$\gamma_k(z,\xi):=\frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t|\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}(z)\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\,dz_i\wedge d\overline{z}_j \tag{2.14}$$

depending only on the differentials $(dz_j)_{1\leqslant j\leqslant n}$ on X. The q-index integral of $(L_k,\Psi^\star_{h,p,\varepsilon})$ on $X_k^{\rm GG}$ is therefore equal to

$$\begin{aligned}\int_{X_k^{\rm GG}(L_k,q)}&\Theta^{n+kr-1}_{L_k,\Psi^\star_{h,p,\varepsilon}}=\\&=\frac{(n+kr-1)!}{n!(kr-1)!}\int_{z\in X}\int_{\xi\in P(1^{[r]},\ldots,k^{[r]})}\omega^{kr-1}_{a,r,p}(\xi)\mathbb{1}_{\gamma_k,q}(z,\xi)\gamma_k(z,\xi)^n\end{aligned}$$

where $\mathbb{1}_{\gamma_k,q}(z,\xi)$ is the characteristic function of the open set of points where $\gamma_k(z,\xi)$ has signature $(n-q,q)$ in terms of the dz_j's. Notice that since $\gamma_k(z,\xi)^n$ is a determinant, the product $\mathbb{1}_{\gamma_k,q}(z,\xi)\gamma_k(z,\xi)^n$ gives rise to a continuous function on $X_k^{\rm GG}$. Formula 1.13 with $r_1=\ldots=r_k=r$ and $a_s=s$ yields the slightly more explicit integral

$$\begin{aligned}\int_{X_k^{\rm GG}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^\star_{h,p,\varepsilon}}&=\frac{(n+kr-1)!}{n!(k!)^r}\quad\times\\&\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}\mathbb{1}_{g_k,q}(z,x,u)g_k(z,x,u)^n\,\frac{(x_1\ldots x_k)^{r-1}}{(r-1)!^k}\,dx\,d\mu(u),\end{aligned}$$

where $g_k(z,x,u)=\gamma_k(z,x_1^{1/2p}u_1,\ldots,x_k^{k/2p}u_k)$ is given by

$$g_k(z,x,u)=\frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}(z)\,u_{s\alpha}\overline{u}_{s\beta}\,dz_i\wedge d\overline{z}_j \tag{2.15}$$

and $\mathbb{1}_{g_k,q}(z,x,u)$ is the characteristic function of its q-index set. Here

$$d\nu_{k,r}(x)=(kr-1)!\,\frac{(x_1\ldots x_k)^{r-1}}{(r-1)!^k}\,dx \tag{2.16}$$

is a probability measure on Δ_{k-1}, and we can rewrite

$$\begin{aligned}\int_{X_k^{\rm GG}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^\star_{h,p,\varepsilon}}&=\frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\quad\times\\&\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}\mathbb{1}_{g_k,q}(z,x,u)g_k(z,x,u)^n\,d\nu_{k,r}(x)\,d\mu(u).\end{aligned} \tag{2.17}$$

Now, formula (2.15) shows that $g_k(z,x,u)$ is a "Monte Carlo" evaluation of the curvature tensor, obtained by averaging the curvature at random points $u_s\in S^{2r-1}$ with certain positive weights x_s/s; we should then think of the k-jet f as some sort of random parameter such that the derivatives $\nabla^k f(0)$ are uniformly distributed in all directions. Let us compute the expected value of $(x,u)\mapsto g_k(z,x,u)$ with respect to the probability measure $d\nu_{k,r}(x)\,d\mu(u)$. Since $\int_{S^{2r-1}}u_{s\alpha}\overline{u}_{s\beta}d\mu(u_s)=\frac{1}{r}\delta_{\alpha\beta}$ and $\int_{\Delta_{k-1}}x_s\,d\nu_{k,r}(x)=\frac{1}{k}$, we find

$$\mathbf{E}(g_k(z,\bullet,\bullet))=\frac{1}{kr}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\cdot\frac{i}{2\pi}\sum_{i,j,\alpha}c_{ij\alpha\alpha}(z)\,dz_i\wedge d\overline{z}_j.$$

In other words, we get the normalized trace of the curvature, i.e.

$$\mathbf{E}(g_k(z,\bullet,\bullet))=\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\Theta_{\det(V^*),\det h^*},\tag{2.18}$$

where $\Theta_{\det(V^*),\det h^*}$ is the $(1,1)$-curvature form of $\det(V^*)$ with the metric induced by h. It is natural to guess that $g_k(z,x,u)$ behaves asymptotically as its expected value $\mathbf{E}(g_k(z,\bullet,\bullet))$ when k tends to infinity. If we replace brutally g_k by its expected value in (2.17), we get the integral

$$\frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\frac{1}{(kr)^n}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n\int_X \mathbb{1}_{\eta,q}\eta^n,$$

where $\eta:=\Theta_{\det(V^*),\det h^*}$ and $\mathbb{1}_{\eta,q}$ is the characteristic function of its q-index set in X. The leading constant is equivalent to $(\log k)^n/n!(k!)^r$ modulo a multiplicative factor $1+O(1/\log k)$. By working out a more precise analysis of the deviation, we will prove the following result.

(2.19) Probabilistic estimate. *Fix smooth hermitian metrics h on V and $\omega=\frac{i}{2\pi}\sum\omega_{ij}dz_i\wedge d\overline{z}_j$ on X. Denote by $\Theta_{V,h}=-\frac{i}{2\pi}\sum c_{ij\alpha\beta}dz_i\wedge d\overline{z}_j\otimes e_\alpha^*\otimes e_\beta$ the curvature tensor of V with respect to an h-orthonormal frame (e_α), and put*

$$\eta(z)=\Theta_{\det(V^*),\det h^*}=\frac{i}{2\pi}\sum_{1\leqslant i,j\leqslant n}\eta_{ij}dz_i\wedge d\overline{z}_j,\qquad \eta_{ij}=\sum_{1\leqslant\alpha\leqslant r}c_{ij\alpha\alpha}.$$

Finally consider the k-jet line bundle $L_k=\mathcal{O}_{X_k^{\rm GG}}(1)\to X_k^{\rm GG}$ equipped with the induced metric $\Psi^\star_{h,p,\varepsilon}$ (as defined above, with $1=\varepsilon_1\gg\varepsilon_2\gg\ldots\gg\varepsilon_k>0$). When k tends to infinity, the integral of the top power of the curvature of L_k on its q-index set $X_k^{\rm GG}(L_k,q)$ is given by

$$\int_{X_k^{\rm GG}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^\star_{h,p,\varepsilon}}=\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_X\mathbb{1}_{\eta,q}\eta^n+O((\log k)^{-1})\bigg)$$

for all $q=0,1,\ldots,n$, and the error term $O((\log k)^{-1})$ can be bounded explicitly in terms of Θ_V, η and ω. Moreover, the left hand side is identically zero for $q>n$.

The final statement follows from the observation that the curvature of L_k is positive along the fibers of $X_k^{\rm GG} \to X$, by the plurisubharmonicity of the weight (this is true even when the partition of unity terms are taken into account, since they depend only on the base); therefore the q-index sets are empty for $q > n$. We start with three elementary lemmas.

(2.20) Lemma. *The integral*

$$I_{k,r,n} = \int_{\Delta_{k-1}} \Big(\sum_{1\leqslant s\leqslant k} \frac{x_s}{s}\Big)^n d\nu_{k,r}(x)$$

is given by the expansion

$$\text{(a)} \qquad I_{k,r,n} = \sum_{1\leqslant s_1,s_2,\ldots,s_n\leqslant k} \frac{1}{s_1 s_2\ldots s_n}\,\frac{(kr-1)!}{(r-1)!^k}\,\frac{\prod_{1\leqslant i\leqslant k}(r-1+\beta_i)!}{(kr+n-1)!}.$$

where $\beta_i = \beta_i(s) = \mathrm{card}\{j\,;\ s_j = i\}$, $\sum \beta_i = n$, $1 \leqslant i \leqslant k$. The quotient

$$I_{k,r,n}\Big/\frac{r^n}{kr(kr+1)\ldots(kr+n-1)}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n$$

is bounded below by 1 and bounded above by

$$\text{(b)} \qquad 1 + \frac{1}{3}\sum_{m=2}^{n} \frac{2^m n!}{(n-m)!}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^{-m} = 1 + O((\log k)^{-2})$$

As a consequence

$$\text{(c)} \qquad \begin{aligned} I_{k,r,n} &= \frac{1}{k^n}\Big(\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n + O((\log k)^{n-2})\Big)\\ &= \frac{(\log k+\gamma)^n + O((\log k)^{n-2})}{k^n}\end{aligned}$$

where γ is the Euler-Mascheroni constant.

Proof. Let us expand the n-th power $\big(\sum_{1\leqslant s\leqslant k} \frac{x_s}{s}\big)^n$. This gives

$$I_{k,r,n} = \sum_{1\leqslant s_1,s_2,\ldots,s_n\leqslant k} \frac{1}{s_1 s_2\ldots s_n}\int_{\Delta_{k-1}} x_1^{\beta_1}\ldots x_k^{\beta_k}\,d\nu_{k,r}(x)$$

and by definition of the measure $\nu_{k,r}$ we have

$$\int_{\Delta_{k-1}} x_1^{\beta_1}\ldots x_k^{\beta_k}\,d\nu_{k,r}(x) = \frac{(kr-1)!}{(r-1)!^k}\int_{\Delta_{k-1}} x_1^{r+\beta_1-1}\ldots x_k^{r+\beta_k-1}\,dx_1\ldots dx_k.$$

By Formula (1.14), we find

$$\begin{aligned}\int_{\Delta_{k-1}} x_1^{\beta_1}\ldots x_k^{\beta_k}\,d\nu_{k,r}(x) &= \frac{(kr-1)!}{(r-1)!^k}\;\frac{\prod_{1\leqslant i\leqslant k}(r+\beta_i-1)!}{(kr+n-1)!}\\ &= \frac{r^n\prod_{i,\,\beta_i\geqslant 1}(1+\frac{1}{r})(1+\frac{2}{r})\ldots(1+\frac{\beta_i-1}{r})}{kr(kr+1)\ldots(kr+n-1)},\end{aligned}$$

and (2.20 a) follows from the first equality. The final product is minimal when $r=1$, thus

$$\begin{aligned}\frac{r^n}{kr(kr+1)\ldots(kr+n-1)} &\leqslant \int_{\Delta_{k-1}} x_1^{\beta_1}\ldots x_k^{\beta_k}\,d\nu_{k,r}(x)\\ &\leqslant \frac{r^n\prod_{1\leqslant i\leqslant k}\beta_i!}{kr(kr+1)\ldots(kr+n-1)}.\end{aligned}\tag{2.21}$$

Also, the integral is maximal when all β_i vanish except one, in which case one gets

$$\int_{\Delta_{k-1}} x_j^n\,d\nu_{k,r}(x) = \frac{r(r+1)\ldots(r+n-1)}{kr(kr+1)\ldots(kr+n-1)}.\tag{2.22}$$

By (2.21), we find the lower and upper bounds

$$I_{k,r,n}\geqslant \frac{r^n}{kr(kr+1)\ldots(kr+n-1)}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n,\tag{2.23}$$

$$I_{k,r,n}\leqslant \frac{r^n}{kr(kr+1)\ldots(kr+n-1)}\sum_{1\leqslant s_1,\ldots,s_n\leqslant k}\frac{\beta_1!\ldots\beta_k!}{s_1\ldots s_n}.\tag{2.24}$$

In order to make the upper bound more explicit, we reorganize the n-tuple $(s_1,\ldots,s_n)$ into those indices $t_1<\ldots<t_\ell$ which appear a certain number of times $\alpha_i=\beta_{t_i}\geqslant 2$, and those, say $t_{\ell+1}<\ldots<t_{\ell+m}$, which appear only once. We have of course $\sum\beta_i=n-m$, and each choice of the t_i's corresponds to $n!/\alpha_1!\ldots\alpha_\ell!$ possibilities for the n-tuple $(s_1,\ldots,s_n)$. Therefore we get

$$\sum_{1\leqslant s_1,\ldots,s_n\leqslant k}\frac{\beta_1!\ldots\beta_k!}{s_1\ldots s_n}\leqslant n!\sum_{m=0}^{n}\;\sum_{\ell,\,\Sigma\alpha_i=n-m}\;\sum_{(t_i)}\frac{1}{t_1^{\alpha_1}\ldots t_\ell^{\alpha_\ell}}\,\frac{1}{t_{\ell+1}\ldots t_{\ell+m}}.$$

A trivial comparison series vs. integral yields

$$\sum_{s<t<+\infty}\frac{1}{t^\alpha}\leqslant\frac{1}{\alpha-1}\,\frac{1}{s^{\alpha-1}}$$

and in this way, using successive integrations in t_ℓ, $t_{\ell-1}$, … , we get inductively

$$\sum_{1\leqslant t_1<\ldots<t_\ell<+\infty}\frac{1}{t_1^{\alpha_1}\ldots t_\ell^{\alpha_\ell}}\leqslant\frac{1}{\prod_{1\leqslant i\leqslant\ell}(\alpha_{\ell-i+1}+\ldots+\alpha_\ell-i)}\leqslant\frac{1}{\ell!}\,,$$

since $\alpha_i \geqslant 2$ implies $\alpha_{\ell-i+1}+\ldots+\alpha_\ell - i \geqslant i$. On the other hand

$$\sum_{1\leqslant t_{\ell+1}<\ldots<t_{\ell+m}\leqslant k} \frac{1}{t_{\ell+1}\ldots t_{\ell+m}} \leqslant \frac{1}{m!}\sum_{1\leqslant s_1,\ldots,s_m\leqslant k}\frac{1}{s_1\ldots s_m} = \frac{1}{m!}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^m.$$

Since partitions $\alpha_1+\ldots+\alpha_\ell = n-m$ satisfying the additional restriction $\alpha_i\geqslant 2$ correspond to $\alpha'_i=\alpha_i-2$ satisfying $\sum\alpha'_i = n-m-2\ell$, their number is equal to

$$\binom{n-m-2\ell+\ell-1}{\ell-1} = \binom{n-m-\ell-1}{\ell-1} \leqslant 2^{n-m-\ell-1}$$

and we infer from this

$$\sum_{1\leqslant s_1,\ldots,s_n\leqslant k}\frac{\beta_1!\ldots\beta_k!}{s_1\ldots s_n} \leqslant \sum_{\substack{\ell\geqslant 1\\ 2\ell+m\leqslant n}} \frac{2^{n-m-\ell-1}n!}{\ell!\,m!}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^m + \Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n$$

where the last term corresponds to the special case $\ell=0$, $m=n$. Therefore

$$\begin{aligned}\sum_{1\leqslant s_i\leqslant k}\frac{\beta_1!\ldots\beta_k!}{s_1\ldots s_n} &\leqslant \frac{e^{1/2}-1}{2}\sum_{m=0}^{n-2}\frac{2^{n-m}n!}{m!}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^m + \Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n\\ &\leqslant \frac{1}{3}\sum_{m=2}^{n}\frac{2^m n!}{(n-m)!}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^{n-m} + \Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n.\end{aligned}$$

This estimate combined with (2.23, 2.24) implies the upper bound (2.20 b) (the lower bound 1 being now obvious). The asymptotic estimate (2.20 c) follows immediately. □

(2.25) Lemma. *If A is a hermitian $n\times n$ matrix, set $\mathbb{1}_{A,q}$ to be equal to 1 if A has signature $(n-q,q)$ and 0 otherwise. Then for all $n\times n$ hermitian matrices A, B we have the estimate*

$$\big|\mathbb{1}_{A,q}\det A - \mathbb{1}_{B,q}\det B\big| \leqslant \|A-B\| \sum_{0\leqslant i\leqslant n-1}\|A\|^i\|B\|^{n-1-i},$$

where $\|A\|$, $\|B\|$ are the hermitian operator norms of the matrices.

Proof. We first check that the estimate holds true for $|\det A - \det B|$. Let $\lambda_1\leqslant\ldots\leqslant\lambda_n$ be the eigenvalues of A and $\lambda'_1\leqslant\ldots\leqslant\lambda'_n$ be the eigenvalues of B. We have $|\lambda_i|\leqslant\|A\|$, $|\lambda'_i|\leqslant\|B\|$ and the minimax principle implies that $|\lambda_i-\lambda'_i|\leqslant\|A-B\|$. We then get the desired estimate by writing

$$\det A - \det B = \lambda_1\ldots\lambda_n - \lambda'_1\ldots\lambda'_n = \sum_{1\leqslant i\leqslant n}\lambda_1\ldots\lambda_{i-1}(\lambda_i-\lambda'_i)\lambda'_{i+1}\ldots\lambda'_n.$$

This already implies (2.25) if A or B is degenerate. If A and B are non degenerate we only have to prove the result when one of them (say A) has signature $(n-q,q)$ and the other one (say B) has a different signature. If we put $M(t)=(1-t)A+tB$, the already established estimate for the determinant yields

$$\Big|\frac{d}{dt}\det M(t)\Big| \leqslant n\|A-B\|\,\|M(t)\| \leqslant n\|A-B\|\big((1-t)\|A\|+t\|B\|\big)^{n-1}.$$

However, since the signature of $M(t)$ is not the same for $t=0$ and $t=1$, there must exist $t_0\in\,]0,1[$ such that $(1-t_0)A+t_0B$ is degenerate. Our claim follows by integrating the differential estimate on the smallest such interval $[0,t_0]$, after observing that $M(0)=A$, $\det M(t_0)=0$, and that the integral of the right hand side on $[0,1]$ is the announced bound. □

(2.26) Lemma. *Let Q_A be the hermitian quadratic form associated with the hermitian operator A on $\mathbb{C}^n$. If μ is the rotation invariant probability measure on the unit sphere S^{2n-1} of $\mathbb{C}^n$ and λ_i are the eigenvalues of A, we have*

$$\int_{|\zeta|=1}|Q_A(\zeta)|^2d\mu(\zeta)=\frac{1}{n(n+1)}\Big(\sum\lambda_i^2+\Big(\sum\lambda_i\Big)^2\Big).$$

The norm $\|A\|=\max|\lambda_i|$ satisfies the estimate

$$\frac{1}{n^2}\|A\|^2\leqslant\int_{|\zeta|=1}|Q_A(\zeta)|^2d\mu(\zeta)\leqslant\|A\|^2.$$

Proof. The first identity as an easy calculation, and the inequalities follow by computing the eigenvalues of the quadratic form $\sum\lambda_i^2+\big(\sum\lambda_i\big)^2-c\lambda_{i_0}^2$, $c>0$. The lower bound is attained e.g. for $Q_A(\zeta)=|\zeta_1|^2-\frac{1}{n}(|\zeta_2|^2+\ldots+|\zeta_n|^2)$ when we take $i_0=1$ and $c=1+\frac{1}{n}$. □

Proof of Proposition 2.19. Take a vector $\zeta\in T_{X,z}$, $\zeta=\sum\zeta_i\frac{\partial}{\partial z_i}$, with $\|\zeta\|_\omega=1$, and introduce the trace free sesquilinear quadratic form

$$Q_{z,\zeta}(u)=\sum_{i,j,\alpha,\beta}\widetilde{c}_{ij\alpha\beta}(z)\,\zeta_i\overline{\zeta}_j\,u_\alpha\overline{u}_\beta,\qquad \widetilde{c}_{ij\alpha\beta}=c_{ij\alpha\beta}-\frac{1}{r}\eta_{ij}\delta_{\alpha\beta},\qquad u\in\mathbb{C}^r$$

where $\eta_{ij}=\sum_{1\leqslant\alpha\leqslant r}c_{ij\alpha\alpha}$. We consider the corresponding trace free curvature tensor

$$\widetilde{\Theta}_V=\frac{i}{2\pi}\sum_{i,j,\alpha,\beta}\widetilde{c}_{ij\alpha\beta}\,dz_i\wedge d\overline{z}_j\otimes e_\alpha^*\otimes e_\beta. \tag{2.27}$$

As a general matter of notation, we adopt here the convention that the canonical correspondence between hermitian forms and $(1,1)$-forms is normalized as $\sum a_{ij}dz_i\otimes d\overline{z}_j \;\leftrightarrow\; \frac{i}{2\pi}\sum a_{ij}dz_i\wedge d\overline{z}_j$, and we take the liberty of using the same symbols for both types of objects; we do so especially for $g_k(z,x,u)$ and $\eta(z)=\frac{i}{2\pi}\sum\eta_{ij}(z)dz_i\wedge d\overline{z}_j=\mathrm{Tr}\,\Theta_V(z)$. First observe that for all k-tuples of unit vectors $u=(u_1,\ldots,u_k)\in(S^{2r-1})^k$, $u_s=(u_{s\alpha})_{1\leqslant\alpha\leqslant r}$, we have

$$\int_{(S^{2r-1})^k}\Bigg|\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\sum_{i,j,\alpha,\beta}\widetilde{c}_{ij\alpha\beta}(z)\,\zeta_i\overline{\zeta}_j u_{s\alpha}\overline{u}_{s\beta}\Bigg|^2 d\mu(u)=\sum_{1\leqslant s\leqslant k}\frac{x_s^2}{s^2}\mathbf{V}(Q_{z,\zeta})$$

where $\mathbf{V}(Q_{z,\zeta})$ is the variance of $Q_{z,\zeta}$ on S^{2r-1}. This is so because we have a sum over s of independent random variables on $(S^{2r-1})^k$, all of which have zero mean value. (Lemma 2.26 shows that the variance $\mathbf{V}(Q)$ of a trace free hermitian quadratic form $Q(u)=\sum_{1\leqslant\alpha\leqslant r}\lambda_\alpha|u_\alpha|^2$ on the unit sphere S^{2r-1} is equal to $\frac{1}{r(r+1)}\sum\lambda_\alpha^2$, but we only give the formula to fix the ideas). Formula (2.22) yields

$$\int_{\Delta_{k-1}}x_s^2 d\nu_{k,r}(x)=\frac{r+1}{k(kr+1)}.$$

Therefore, according to notation (2.15), we obtain the partial variance formula

$$\begin{aligned}\int_{\Delta_{k-1}\times(S^{2r-1})^k}&\big|g_k(z,x,u)(\zeta)-\overline{g}_k(z,x)(\zeta)\big|^2 d\nu_{k,r}(x)d\mu(u)\\&=\frac{(r+1)}{k(kr+1)}\Bigg(\sum_{1\leqslant s\leqslant k}\frac{1}{s^2}\Bigg)\sigma_h(\widetilde{\Theta}_V(\zeta,\zeta))^2\end{aligned}$$

in which

$$\overline{g}_k(z,x)(\zeta)=\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\frac{1}{r}\sum_{ij\alpha}c_{ij\alpha\alpha}\zeta_i\overline{\zeta}_j=\Bigg(\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\Bigg)\frac{1}{r}\eta(z)(\zeta),$$

$$\sigma_h(\widetilde{\Theta}_V(\zeta,\zeta))^2=\mathbf{V}\big(u\mapsto\langle\widetilde{\Theta}_V(\zeta,\zeta)u,u\rangle_h\big)=\int_{u\in S^{2r-1}}\big|\langle\widetilde{\Theta}_V(\zeta,\zeta)u,u\rangle_h\big|^2 d\mu(u).$$

By integrating over $\zeta\in S^{2n-1}\subset\mathbb{C}^n$ and applying the left hand inequality in Lemma 2.26 we infer

$$\begin{aligned}\int_{\Delta_{k-1}\times(S^{2r-1})^k}&\big\|g_k(z,x,u)-\overline{g}_k(z,x)\big\|_\omega^2 d\nu_{k,r}(x)d\mu(u)\\&\leqslant\frac{n^2(r+1)}{k(kr+1)}\Bigg(\sum_{1\leqslant s\leqslant k}\frac{1}{s^2}\Bigg)\sigma_{\omega,h}(\widetilde{\Theta}_V)^2\end{aligned}\tag{2.28}$$

where $\sigma_{\omega,h}(\widetilde{\Theta}_V)$ is the standard deviation of $\widetilde{\Theta}_V$ on $S^{2n-1}\times S^{2r-1}$:

$$\sigma_{\omega,h}(\widetilde{\Theta}_V)^2=\int_{|\zeta|_\omega=1,\,|u|_h=1}\big|\langle\widetilde{\Theta}_V(\zeta,\zeta)u,u\rangle_h\big|^2 d\mu(\zeta)\,d\mu(u).$$

On the other hand, brutal estimates give the hermitian operator norm estimates

$$\|\overline{g}_k(z,x)\|_\omega \leqslant \Big(\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\Big)\frac{1}{r}\|\eta(z)\|_\omega, \tag{2.29}$$

$$\|g_k(z,x,u)\|_\omega \leqslant \Big(\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\Big)\|\Theta_V\|_{\omega,h} \tag{2.30}$$

where

$$\|\Theta_V\|_{\omega,h} = \sup_{|\zeta|_\omega=1,\,|u|_h=1}\big|\langle\Theta_V(\zeta,\zeta)u,u\rangle_h\big|.$$

We use these estimates to evaluate the q-index integrals. The integral associated with $\overline{g}_k(z,x)$ is much easier to deal with than $g_k(z,x,u)$ since the characteristic function of the q-index set depends only on z. By Lemma 2.25 we find

$$\begin{aligned}&\big|\mathbb{1}_{g_k,q}(z,x,u)\det g_k(z,x,u)-\mathbb{1}_{\eta,q}(z)\det\overline{g}_k(z,x)\big|\\&\qquad\leqslant\big\|g_k(z,x,u)-\overline{g}_k(z,x)\big\|_\omega\sum_{0\leqslant i\leqslant n-1}\|g_k(z,x,u)\|_\omega^i\|\overline{g}_k(z,x)\|_\omega^{n-1-i}.\end{aligned}$$

The Cauchy-Schwarz inequality combined with (2.28 – 2.30) implies

$$\begin{aligned}&\int_{\Delta_{k-1}\times(S^{2r-1})^k}\big|\mathbb{1}_{g_k,q}(z,x,u)\det g_k(z,x,u)-\mathbb{1}_{\eta,q}(z)\det\overline{g}_k(z,x)\big|\,d\nu_{k,r}(x)d\mu(u)\\
&\leqslant\Big(\int_{\Delta_{k-1}\times(S^{2r-1})^k}\big\|g_k(z,x,u)-\overline{g}_k(z,x)\big\|_\omega^2 d\nu_{k,r}(x)d\mu(u)\Big)^{1/2}\times\\
&\quad\Big(\int_{\Delta_{k-1}\times(S^{2r-1})^k}\Big(\sum_{0\leqslant i\leqslant n-1}\|g_k(z,x,u)\|_\omega^i\|\overline{g}_k(z,x)\|_\omega^{n-1-i}\Big)^2 d\nu_{k,r}(x)d\mu(u)\Big)^{1/2}\\
&\leqslant\frac{n(1+1/r)^{1/2}}{(k(k+1/r))^{1/2}}\Big(\sum_{1\leqslant s\leqslant k}\frac{1}{s^2}\Big)^{1/2}\sigma_{\omega,h}(\widetilde{\Theta}_V)\sum_{1\leqslant i\leqslant n-1}\|\Theta_V\|_{\omega,h}^i\Big(\frac{1}{r}\|\eta(z)\|_\omega\Big)^{n-1-i}\\
&\qquad\times\Big(\int_{\Delta_{k-1}}\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)^{2n-2}d\nu_{k,r}(x)\Big)^{1/2}\;=\;O\Big(\frac{(\log k)^{n-1}}{k^n}\Big)\end{aligned}$$

by Lemma 2.20 with n replaced by $2n-2$. This is the essential error estimate. As one can see, the growth of the error mainly depends on the final integral factor, since the initial multiplicative factor is uniformly bounded over X. In order to get the principal term, we compute

$$\begin{aligned}\int_{\Delta_{k-1}}\det\overline{g}_k(z,x)\,d\nu_{k,r}(x)&=\frac{1}{r^n}\det\eta(z)\int_{\Delta_{k-1}}\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)^n d\nu_{k,r}(x)\\&\sim\frac{(\log k)^n}{r^nk^n}\det\eta(z).\end{aligned}$$

From there we conclude that

$$\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k} \mathbb{1}_{g_k,q}(z,x,u)g_k(z,x,u)^n\,d\nu_{k,r}(x)d\mu(u) = \frac{(\log k)^n}{r^nk^n}\int_X \mathbb{1}_{\eta,q}\eta^n+O\Big(\frac{(\log k)^{n-1}}{k^n}\Big)$$

The probabilistic estimate 2.19 follows by (2.17). □

(2.31) Remark. If we take care of the precise bounds obtained above, the proof gives in fact the explicit estimate

$$\int_{X_k^{\rm GG}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^\star_{h,p,\varepsilon}} = \frac{(n+kr-1)!\;I_{k,r,n}}{n!(k!)^r(kr-1)!}\Big(\int_X \mathbb{1}_{\eta,q}\eta^n+\varepsilon_{k,r,n}J\Big)$$

where

$$J=n\,(1+1/r)^{1/2}\Big(\sum_{s=1}^k\frac{1}{s^2}\Big)^{1/2}\int_X\sigma_{\omega,h}(\widetilde\Theta_V)\sum_{i=1}^{n-1}r^{i+1}\|\Theta_V\|^i_{\omega,h}\|\eta(z)\|^{n-1-i}_\omega\omega^n$$

and

$$|\varepsilon_{k,r,n}|\leqslant\frac{\Big(\int_{\Delta_{k-1}}\Big(\sum_{s=1}^k\frac{x_s}{s}\Big)^{2n-2}d\nu_{k,r}(x)\Big)^{1/2}}{(k(k+1/r))^{1/2}\int_{\Delta_{k-1}}\Big(\sum_{s=1}^k\frac{x_s}{s}\Big)^n d\nu_{k,r}(x)}$$

$$\leqslant\frac{\Big(1+\frac13\sum_{m=2}^{2n-2}\frac{2^m(2n-2)!}{(2n-2-m)!}\big(1+\frac12+\ldots+\frac1k\big)^{-m}\Big)^{1/2}}{1+\frac12+\ldots+\frac1k}\sim\frac{1}{\log k}$$

by the lower and upper bounds of $I_{k,r,n}$, $I_{k,r,2n-2}$ obtained in Lemma 2.20. As $(2n-2)!/(2n-2-m)!\leqslant(2n-2)^m$, one easily shows that

$$|\varepsilon_{k,r,n}|\leqslant\frac{(31/15)^{1/2}}{\log k}\qquad\text{for } k\geqslant e^{5n-5}. \tag{2.32}$$

Also, we see that the error terms vanish if $\widetilde\Theta_V$ is identically zero, but this is of course a rather unexpected circumstance. In general, since the form $\widetilde\Theta_V$ is trace free, Lemma 2.23 applied to the quadratic form $u\mapsto\langle\widetilde\Theta_V(\zeta,\zeta)u,u\rangle$ on $\mathbb{C}^r$ implies $\sigma_{\omega,h}(\widetilde\Theta_V)\leqslant(r+1)^{-1/2}\|\widetilde\Theta_V\|_{\omega,h}$. This yields the simpler bound

$$J=n\,r^{1/2}\Big(\sum_{s=1}^k\frac{1}{s^2}\Big)^{1/2}\int_X\|\widetilde\Theta_V\|_{\omega,h}\sum_{i=1}^{n-1}r^i\|\Theta_V\|^i_{\omega,h}\|\eta(z)\|^{n-1-i}_\omega\omega^n. \tag{2.33}$$ □

It will be useful to extend the above estimates to the case of sections of

$$L_k = \mathcal{O}_{X_k^{\rm GG}}(1) \otimes \pi_k^* \mathcal{O}\Big(\frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)F\Big) \tag{2.34}$$

where $F \in \mathrm{Pic}_{\mathbb{Q}}(X)$ is an arbitrary $\mathbb{Q}$-line bundle on X and $\pi_k : X_k^{\rm GG} \to X$ is the natural projection. We assume here that F is also equipped with a smooth hermitian metric h_F. In formula (2.20), the renormalized metric $\eta_k(z,x,u)$ of L_k takes the form

$$\eta_k(z,x,u) = \frac{1}{\frac{1}{kr}(1+\frac{1}{2}+\ldots+\frac{1}{k})} g_k(z,x,u) + \Theta_{F,h_F}(z), \tag{2.35}$$

and by the same calculations its expected value is

$$\eta(z) := \mathbf{E}(\eta_k(z,\bullet,\bullet)) = \Theta_{\det V^*,\det h^*}(z) + \Theta_{F,h_F}(z). \tag{2.36}$$

Then the variance estimate for $\eta_k - \eta$ is unchanged, and the L^p bounds for η_k are still valid, since our forms are just shifted by adding the constant smooth term $\Theta_{F,h_F}(z)$. The probabilistic estimate 2.18 is therefore still true in exactly the same form, provided we use (2.34 – 2.36) instead of the previously defined L_k, η_k and η. An application of holomorphic Morse inequalities gives the desired cohomology estimates for

$$\begin{aligned} h^q\Big(X, E_{k,m}^{\rm GG}V^* \otimes \mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big) \\ = h^q(X_k^{\rm GG}, \mathcal{O}_{X_k^{\rm GG}}(m) \otimes \pi_k^*\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big),\end{aligned}$$

provided m is sufficiently divisible to give a multiple of F which is a $\mathbb{Z}$-line bundle.

(2.37) Theorem. *Let (X,V) be a directed manifold, $F \to X$ a $\mathbb{Q}$-line bundle, (V,h) and (F,h_F) smooth hermitian structure on V and F respectively. We define*

$$\begin{aligned} L_k &= \mathcal{O}_{X_k^{\rm GG}}(1) \otimes \pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big), \\ \eta &= \Theta_{\det V^*,\det h^*} + \Theta_{F,h_F}. \end{aligned}$$

Then for all $q \geqslant 0$ and all $m \gg k \gg 1$ such that m is sufficiently divisible, we have

(a) $$h^q(X_k^{\rm GG}, \mathcal{O}(L_k^{\otimes m})) \leqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,q)} (-1)^q\eta^n + O((\log k)^{-1})\Big),$$

(b) $$h^0(X_k^{\rm GG}, \mathcal{O}(L_k^{\otimes m})) \geqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,\leqslant 1)} \eta^n - O((\log k)^{-1})\Big),$$

(c) $$\chi(X_k^{\rm GG}, \mathcal{O}(L_k^{\otimes m})) = \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\big(c_1(V^*\otimes F)^n + O((\log k)^{-1})\big).$$

Green and Griffiths [GG79] already checked the Riemann-Roch calculation (2.37 c) in the special case $V = T^*_X$ and $F = \mathcal{O}_X$. Their proof is much simpler since it relies only on Chern class calculations, but it cannot provide any information on the individual cohomology groups, except in very special cases where vanishing theorems can be applied; in fact in dimension 2, the Euler characteristic satisfies $\chi = h^0 - h^1 + h^2 \leqslant h^0 + h^2$, hence it is enough to get the vanishing of the top cohomology group H^2 to infer $h^0 \geqslant \chi$; this works for surfaces by means of a well-known vanishing theorem of Bogomolov which implies in general

$$H^n\Big(X, E^{\rm GG}_{k,m}T^*_X \otimes \mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big)\Big) = 0$$

as soon as $K_X \otimes F$ is big and $m \gg 1$.

In fact, thanks to Bonavero's singular holomorphic Morse inequalities [Bon93], everything works almost unchanged in the case where $V \subset T_X$ has singularities and h is an admissible metric on V (see (0.10)). We only have to find a blow-up $\mu : \widetilde{X}_k \to X_k$ so that the resulting pull-backs $\mu^* L_k$ and $\mu^* V$ are locally free, and $\mu^* \det h^*$, $\mu^*\Psi_{h,p,\varepsilon}$ only have divisorial singularities. Then η is a $(1,1)$-current with logarithmic poles, and we have to deal with smooth metrics on $\mu^* L_k^{\otimes m} \otimes \mathcal{O}(-mE_k)$ where E_k is a certain effective divisor on X_k (which, by our assumption (0.10), does not project onto X). The cohomology groups involved are then the twisted cohomology groups

$$H^q(X^{\rm GG}_k, \mathcal{O}(L_k^{\otimes m}) \otimes \mathcal{J}_{k,m})$$

where $\mathcal{J}_{k,m} = \mu_*(\mathcal{O}(-mE_k))$ is the corresponding multiplier ideal sheaf, and the Morse integrals need only be evaluated in the complement of the poles, that is on $X(\eta, q) \smallsetminus S$ where $S = V_{\rm sing} \cup {\rm Sing}(h)$. Since

$$(\pi_k)_*\big(\mathcal{O}(L_k^{\otimes m}) \otimes \mathcal{J}_{k,m}\big) \subset E^{\rm GG}_{k,m}V^* \otimes \mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big)$$

we still get a lower bound for the H^0 of the latter sheaf (or for the H^0 of the un-twisted line bundle $\mathcal{O}(L_k^{\otimes m})$ on $X^{\rm GG}_k$). If we assume that $K_V \otimes F$ is big, these considerations also allow us to obtain a strong estimate in terms of the volume, by using an approximate Zariski decomposition on a suitable blow-up of (X, V). The following corollary implies in particular Theorem 0.5.

(2.38) Corollary. *If F is an arbitrary $\mathbb{Q}$-line bundle over X, one has*

$$\begin{aligned} h^0\Big(X^{\rm GG}_k, \mathcal{O}_{X^{\rm GG}_k}(m) \otimes \pi_k^*\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big) \\ \geqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\mathrm{Vol}(K_V\otimes F) - O((\log k)^{-1})\Big) - o(m^{n+kr-1}),\end{aligned}$$

when $m \gg k \gg 1$, in particular there are many sections of the k-jet differentials of degree m twisted by the appropriate power of F if $K_V \otimes F$ is big.

Proof. The volume is computed here as usual, i.e. after performing a suitable modification $\mu : \widetilde{X} \to X$ which converts K_V into an invertible sheaf. There is of course nothing to prove if $K_V \otimes F$ is not big, so we can assume $\mathrm{Vol}(K_V \otimes F) > 0$. Let us fix smooth hermitian metrics h_0 on T_X and h_F on F. They induce a metric $\mu^*(\det h_0^{-1} \otimes h_F)$ on $\mu^*(K_V \otimes F)$ which, by our definition of K_V, is a smooth metric. By the result of Fujita [Fuj94] on approximate Zariski decomposition, for every $\delta > 0$, one can find a modification $\mu_\delta : \widetilde{X}_\delta \to X$ dominating μ such that

$$\mu_\delta^*(K_V \otimes F) = \mathcal{O}_{\widetilde{X}_\delta}(A + E)$$

where A and E are $\mathbb{Q}$-divisors, A ample and E effective, with

$$\mathrm{Vol}(A) = A^n \geqslant \mathrm{Vol}(K_V \otimes F) - \delta.$$

If we take a smooth metric h_A with positive definite curvature form Θ_{A,h_A}, then we get a singular hermitian metric $h_A h_E$ on $\mu_\delta^*(K_V \otimes F)$ with poles along E, i.e. the quotient $h_A h_E / \mu^*(\det h_0^{-1} \otimes h_F)$ is of the form $e^{-\varphi}$ where φ is quasi-psh with log poles $\log|\sigma_E|^2$ (mod $C^\infty(\widetilde{X}_\delta)$) precisely given by the divisor E. We then only need to take the singular metric h on T_X defined by

$$h = h_0 e^{\frac{1}{r}(\mu_\delta)_* \varphi}$$

(the choice of the factor $\frac{1}{r}$ is there to correct adequately the metric on $\det V$). By construction h induces an admissible metric on V and the resulting curvature current $\eta = \Theta_{K_V, \det h^*} + \Theta_{F,h_F}$ is such that

$$\mu_\delta^* \eta = \Theta_{A,h_A} + [E], \qquad [E] = \text{current of integration on } E.$$

Then the 0-index Morse integral in the complement of the poles is given by

$$\int_{X(\eta,0) \smallsetminus S} \eta^n = \int_{\widetilde{X}_\delta} \Theta_{A,h_A}^n = A^n \geqslant \mathrm{Vol}(K_V \otimes F) - \delta$$

and (2.38) follows from the fact that δ can be taken arbitrary small. □

(2.39) Example. In some simple cases, the above estimates can lead to very explicit results. Take for instance X to be a smooth complete intersection of multidegree $(d_1, d_2, \ldots, d_s)$ in $\mathbb{P}^{n+s}_{\mathbb{C}}$ and consider the absolute case $V = T_X$. Then

$$K_X = \mathcal{O}_X(d_1 + \ldots + d_s - n - s - 1).$$

Assume that X is of general type, i.e. $\sum d_j > n + s + 1$. Let us equip $V = T_X$ with the restriction of the Fubini-Study metric $h = \Theta_{\mathcal{O}(1)}$; a better choice might be the Kähler-Einstein metric but we want to keep the calculations as elementary as possible. The standard formula for the curvature tensor of a submanifold gives

$$\Theta_{T_X,h} = (\Theta_{T_{\mathbb{P}^{n+s}},h})_{|X} + \beta^* \wedge \beta$$

where $\beta\in C^\infty\big(\Lambda^{1,0}T_X^*\otimes\mathrm{Hom}(T_X,\bigoplus\mathcal{O}(d_j))\big)$ is the second fundamental form. In other words, by the well known formula for the curvature of projective space, we have

$$\langle\Theta_{T_X,h}(\zeta,\zeta)u,u\rangle=|\zeta|^2|u|^2+|\langle\zeta,u\rangle|^2-|\beta(\zeta)\cdot u|^2.$$

The curvature ρ of $(K_X,\det h^{-1})$ (i.e. the opposite of the Ricci form $\mathrm{Tr}\,\Theta_{T_X,h}$) is given by

$$\rho=-\mathrm{Tr}\,\Theta_{TX,h}=\mathrm{Tr}(\beta\wedge\beta^*)-(n+1)h\geqslant-(n+1)h. \tag{2.40}$$

We take here $F=\mathcal{O}_X(-a)$, $a\in\mathbb{Q}_+$, and we want to determine conditions for the existence of sections

$$H^0\bigg(X,E^{\mathrm{GG}}_{k,m}T_X^*\otimes\mathcal{O}\Big(-a\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\Big)\bigg),\qquad m\gg1. \tag{2.41}$$

We have to choose $K_X\otimes\mathcal{O}_X(-a)$ ample, i.e. $\sum d_j>n+s+a+1$, and then (by an appropriate choice of the metric of $F=\mathcal{O}_X(-a)$), the form $\eta=\Theta_{K_X\otimes\mathcal{O}_X(-a)}$ can be taken to be any positive form cohomologous to $(\sum d_j-(n+s+a+1))h$. We use remark 2.31 and estimate the error terms by considering the Kähler metric

$$\omega=\rho+(n+s+2)h\equiv\Big(\sum d_j+1\Big)h.$$

Inequality (2.40) shows that $\omega\geqslant 2h$ and also that $\omega\geqslant\mathrm{Tr}(\beta\wedge\beta^*)$. From this, one easily concludes that $\|\eta\|_\omega\leqslant1$ by an appropriate choice of η, as well as $\|\Theta_{T_X,h}\|_{\omega,h}\leqslant1$ and $\|\widetilde\Theta_{T_X,h}\|_{\omega,h}\leqslant2$. By (2.33), we obtain for $n\geqslant2$

$$J\leqslant n^{3/2}\frac{\pi}{\sqrt6}\times2\,\frac{n^n-1}{n-1}\int_X\omega^n<\frac{4\pi}{\sqrt6}\,n^{n+1/2}\int_X\omega^n$$

where $\int_X\omega^n=\big(\sum d_j+1\big)^n\deg(X)$. On the other hand, the leading term $\int_X\eta^n$ equals $\big(\sum d_j-n-s-a-1\big)^n\deg(X)$ with $\deg(X)=d_1\ldots d_s$. By the bound (2.32) on the error term $\varepsilon_{k,r,n}$, we find that the leading coefficient of the growth of our spaces of sections is strictly controlled below by a multiple of

$$\Big(\sum d_j-n-s-a-1\Big)^n-4\pi\Big(\frac{31}{90}\Big)^{1/2}\frac{n^{n+1/2}}{\log k}\Big(\sum d_j+1\Big)^n$$

if $k\geqslant e^{5n-5}$. A sufficient condition for the existence of sections in (2.41) is thus

$$k\geqslant\exp\bigg(7.38\,n^{n+1/2}\Big(\frac{\sum d_j+1}{\sum d_j-n-s-a-1}\Big)^n\bigg). \tag{2.42}$$

This is good in the sense that we can cover arbitrary smooth complete intersections of general type, but on the other hand, even when the degrees d_j are very large, we still get a large lower bound $k\sim\exp(7.38\,n^{n+1/2})$ on the order of jets. This is far from being optimal, since Diverio [Div09] has shown e.g. that one can take $k=n$ for smooth hypersurfaces of high degree. It is however not completely unlikely that we could improve estimate (2.42) with better choices of the metrics h, ω. □

3. On the base locus of sections of k-jet bundles

The final step required for a complete solution of the Green-Griffiths conjecture would be to calculate the base locus $B_k \subset X_k^{\rm GG}$ of the space of sections

$$H^0(X_k^{\rm GG}, \mathcal{O}_{X_k^{\rm GG}}(m) \otimes \pi_k^*\mathcal{O}(-m\delta_k A)), \qquad A \text{ ample on } X,\ \ \delta_k \leqslant c\frac{\log k}{k},\ \ c \ll 1,$$

and to show that $Y_k = \pi_k(B_k)$ is a proper algebraic subvariety of X for k large, under the assumption that K_V is big. This does not look completely hopeless, since the statistics of curvature in the Morse inequalities do involve currents for which the sets of poles depend only on the bigness of K_V and therefore project onto a proper subvariety S of X (see the last step of the proof in section 2). It is not unreasonable to think that a further analysis of the asymptotic behavior of sections, e.g. through estimates of the Bergman kernel, might lead to such results.

Even if the required property of the base locus cannot be obtained directly, it would be enough, for a suitable irreducible analytic set $Z \subset X_k^{\rm GG}$ contained in the base locus at some stage, to construct non zero sections in

$$H^0(Z, \mathcal{O}_{X_k^{\rm GG}}(m)_{|Z} \otimes \pi_k^*\mathcal{O}(-m\delta_k A)_{|Z})$$

whenever $\pi_k(Z) = X$, and then to proceed inductively to cut-down the base locus until one reaches some $Z' \subset Z$ with $\pi_k(Z') \subsetneq X$. Hence we have to estimate the cohomology groups H^0 and H^q not just on $X_k^{\rm GG}$, but also on all irreducible subvarieties $Z \subset X_k^{\rm GG}$ such that $\pi_k(Z) = X$. We are not able to do this in such a generality, but our method does provide interesting results in this direction.

(3.1) Theorem. *Let (X, V) be a compact directed n-dimensional manifold, let $r = \operatorname{rank} V$ and F be a holomorphic line bundle on X. Fix an irreducible analytic set $Z_{k_0} \subset X_{k_0}^{\rm GG}$ or equivalently some $\mathbb{C}^*$-invariant set $Z'_{k_0} \subset J^{k_0}V$, and assume that $\pi_{k_0}(Z_{k_0}) = X$. For $k \gg k_0$, denote by $Z_k \subset X_k^{\rm GG}$ the irreducible set corresponding to the inverse image of Z'_{k_0} by the canonical morphism $J^kV \to J^{k_0}V$. Let h be an admissible metric on V, h_F a metric with analytic singularities on F and*

$$L_k = \mathcal{O}_{X_k^{\rm GG}}(1) \otimes \pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)F\Big),$$
$$\eta = \Theta_{K_V, \det h^*} + \Theta_{F, h_F}, \qquad S = \operatorname{Sing}(\eta).$$

Then for $m \gg k \gg k_0$ and $p_k = \dim Z_k = \dim Z_{k_0} + (k - k_0)r$ we have

$$h^0(Z_k, \mathcal{O}(L_k^{\otimes m})_{|Z_k}) \geqslant \frac{m^{p_k}}{p_k!}\frac{(\log k)^n}{n!}\deg_{X_k^{\rm GG}/X}(Z_k)\bigg(\int_{X(\eta,\leqslant 1)\smallsetminus S} \eta^n - O((\log k)^{-1})\bigg) - o(m^{p_k})$$

where $\deg_{X_k^{\rm GG}/X}(Z_k) = \deg_{X_{k_0}^{\rm GG}/X}(Z_{k_0})\big(\frac{k_0!}{k!}\big)^r$ is the relative degree of Z_k over X with respect to the normalized weighted "Kähler metric" $\omega_{a,r,p}$ introduced in (1.10).

We would also get similar upper and lower Morse bounds for the higher cohomology groups, provided that the sheaves $\mathcal{O}_{X_k^{\rm GG}}(m)$ are twisted by the appropriate multiplier ideal sheaves $\mathcal{J}_{k,m}$ already described. The main trouble to proceed further in the analysis of the base locus is that we have to take $k \gg k_0$ and that the $O(...)$ and $o(...)$ bounds depend on Z_{k_0}. Hence the newer sections can only be constructed for higher and higher orders k, without any indication that we can actually terminate the process somewhere, except possibly by some extremely delicate uniform estimates which seem at present beyond reach.

Proof. The technique is a minor variation of what has been done in section 2, hence we will only indicate the basic idea. Essentially the k-jet of f is no longer completely random, its projection onto the first k_0 components $(\nabla^j f(0))_{1\leqslant j\leqslant k_0}$ is assigned to belong to some given analytic set $Z'_{k_0} \subset J^{k_0}V$. This means that in the curvature formula (2.15)

$$g_k(z,x,u) = \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}(z)\,u_{s\alpha}\overline{u}_{s\beta}\,dz_i\wedge d\overline{z}_j$$

only the sum $\sum_{k_0<s\leqslant k}$ is perfectly random. The partial sum $\sum_{1\leqslant s\leqslant k_0}$ remains bounded, while the harmonic series diverges as $\log k$. This implies that the "non randomness" of the initial terms perturbs the estimates merely by bounded quantities, and in the end, the expected value is still similar to (2.18), i.e.

$$\mathbf{E}(g_k(z,\bullet,\bullet)) = \frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}+O(1)\Big)\big(\Theta_{K_V,\det h^*}+\Theta_{F,h_F}\big).$$

Once we are there, the calculation of standard deviation and the other estimates are just routine, and Theorem 3.1 follows again from Proposition 2.13 when we integrate the Morse integrals over Z_k instead of the whole k-jet space $X_k^{\rm GG}$. $\square$

Another possibility to analyze the base locus is to study the restriction maps

$$(3.2)\qquad \rho_{k,m}(x) : H^0(X, E_{k,m}^{\rm GG}V^*\otimes\mathcal{O}(-m\delta_k A)) \to \big(E_{k,m}^{\rm GG}V^*\otimes\mathcal{O}(-m\delta_k A)\big)_x$$

at generic points $x\in X$. If $\rho_{k,m}(x)$ can be shown to be surjective at a generic point, then a fortiori the projection $Y_k = \pi_k(B_k)$ of the base locus does not contain x and so Y_k is a proper algebraic subvariety of X. Now, proving the surjectivity of $\rho_{k,m}(x)$ could be done by proving the vanishing of the H^1 group of our sheaf twisted by the maximal ideal $\mathfrak{m}_{X,x}$. We cannot exactly reach such a precise vanishing result, but Morse inequalities can be used to show that the H^1 groups do not grow too fast.

In fact assume that A is an ample $\mathbb{Q}$-divisor on X which is chosen so small that $K_V\otimes\mathcal{O}(-A)$ is still big. By our estimates, we can then take $\delta_k = \frac{1}{kr}(1+\frac{1}{2}+\ldots+\frac{1}{k})$. Pick a very ample divisor G on X and n pencils of sections $\sigma_{j,t}\in H^0(X,\mathcal{O}(G))$, $1\leqslant j\leqslant n$, $t\in\mathbb{P}^1_{\mathbb{C}}$, such that the divisors $\sigma_{j,t_j}(z)=0$ intersect transversally at isolated points for generic choices of the parameters $t_j\in\mathbb{P}^1_{\mathbb{C}}$. We select an admissible metric h on V which provides a strictly positive curvature current on $K_V \otimes \mathcal{O}(-A)$ and multiply it by the additional weight factor $(e^{\varphi})^{1/rm\delta_k}$ where

$$\varphi(z) = \log \sum_{1 \leqslant j \leqslant n} \prod_{t \in T_j} |\sigma_{j,t}(z)|^{2n}_{h_G}$$

and $T_j \subset \mathbb{P}^1_{\mathbb{C}}$ are generic finite subsets of given cardinality N. The multiplier ideal sheaf of φ is precisely equal to the ideal $\mathcal{I}_E$ of germs of functions vanishing on a certain 0-dimensional set $E = \{x_1, \ldots, x_s\} \subset X$ of cardinality $s = N^n G^n$. Also the resulting curvature form

$$\eta = \Theta_{K_V, \det h^*} - \Theta_{A,h_A} + \frac{1}{m\delta_k} dd^c\varphi \geqslant \Theta_{K_V, \det h^*} - \Theta_{A,h_A} - \frac{N}{m\delta_k}\Theta_{G,h_G}$$

can be made to be strictly positive as a current provided that $N \sim cm\delta_k$ with $c \ll 1$. Then the corresponding multiplier ideal sheaf of the induced hermitian metric on

$$\mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathcal{O}(-m\delta_k A)$$

is the original multiplier sheaf $\mathcal{J}_{k,m}$ twisted by $\pi_k^*\mathcal{I}_E$ above x_j, provided that the x_j lie outside of V_{sing} and outside of the projection of the support $V(\mathcal{J}_{k,m})$. Consider the exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathcal{O}(-m\delta_k A) \otimes \mathcal{J}_{k,m} \otimes \pi_k^*\mathcal{I}_E \\ &\longrightarrow \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathcal{O}(-m\delta_k A) \otimes \mathcal{J}_{k,m} \\ &\longrightarrow \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathcal{O}(-m\delta_k A) \otimes \mathcal{J}_{k,m} \otimes \pi_k^*(\mathcal{O}_X/\mathcal{I}_E) \longrightarrow 0. \end{aligned}$$

Its cohomology exact sequence yields an "almost surjective arrow"

$$H^0\big(\mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathcal{O}(-m\delta_k A) \otimes \mathcal{J}_{k,m}\big) \to \bigoplus_{1 \leqslant j \leqslant s} \big(E^{\mathrm{GG}}_{k,m}V^* \otimes \mathcal{O}(-m\delta_k A)\big)_{x_j},$$

namely the image contains the kernel of the map

$$\bigoplus_{1 \leqslant j \leqslant s} \big(E^{\mathrm{GG}}_{k,m}V^* \otimes \mathcal{O}(-m\delta_k A)\big)_{x_j} \to H^1\big(\mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathcal{O}(-m\delta_k A) \otimes \mathcal{J}_{k,m} \otimes \pi_k^*\mathcal{I}_E\big).$$

Now, we have a Morse upper bound

$$h^1\big(\mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathcal{O}(-m\delta_k A) \otimes \mathcal{J}_{k,m} \otimes \pi_k^*\mathcal{I}_E\big) \leqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r} O\big((\log k)^{-1}\big)$$

since the 1-index integral $\int_{X(\eta,1)} h^n$ is identically zero. At the same time we have $s = N^n G^n \sim c' m^n (\log k)^n / k^n$, and it follows that

$$\dim \bigoplus_{1 \leqslant j \leqslant s} \big(E^{\mathrm{GG}}_{k,m}V^* \otimes \mathcal{O}(-m\delta_k A)\big)_{x_j} \sim s\frac{m^{kr-1}}{(kr-1)!(k!)^r} \sim \frac{c' m^{n+kr-1}}{(kr-1)!(k!)^r}\frac{(\log k)^n}{k^n}.$$

By selecting a suitable point x_j and a trivial lower semi-continuity argument we get the desired almost surjectivity.

(3.3) Corollary. *If A is an ample $\mathbb{Q}$-divisor on X such that $K_V \otimes \mathcal{O}(-A)$ is big and $\delta_k = \frac{1}{kr}(1+\frac{1}{2}+\ldots+\frac{1}{k})$, $r = \operatorname{rank} V$, the restriction map*

$$\rho_{k,m}(x) : H^0(X, E_{k,m}^{\mathrm{GG}} V^* \otimes \mathcal{O}(-m\delta_k A)) \to \big(E_{k,m}^{\mathrm{GG}} V^* \otimes \mathcal{O}(-m\delta_k A)\big)_x$$

has an image of dimension larger than $(1 - O((\log k)^{-1})) \dim E_{k,m}^{\mathrm{GG}} V^*$ at a generic point $x \in X$ for $m \gg k \gg 1$.

Such a result puts an upper bound on the vanishing order that a generic section may have on X_k^{GG} above a generic point of X. Our hope is that one can then completely "eliminate" the base locus by taking vertical derivatives along the fibers of $J^k V \to X$; those derivations will necessarily have some poles $\mathcal{O}(pA)$ which we hope to get cancelled by the negative powers $\mathcal{O}(-m\delta_k A)$. This strategy first devised by [Siu02, Siu04] has indeed been successful in some cases for the study of generic algebraic degeneracy (e.g. for hypersurfaces of very large degree in $\mathbb{P}_{\mathbb{C}}^{n+1}$). This would work rather easily if the rough error term $O((\log k)^{-1})$ could be replaced e.g. by $O(m^{-\varepsilon_k})$ in Corollary (3.3), but this is maybe too much to ask for.

We finally discuss yet another approach. For this we have to introduce invariant jet differentials along the lines of [Dem95]. In fact, to any directed manifold (X, V) one can associate its tower of Semple k-jet spaces, which is a sequence of directed pairs (X_k, V_k) starting with $(X_0, V_0) = (X, V)$, together with morhisms $\widetilde{\pi}_k : (X_k, V_k) \to (X_{k-1}, V_{k-1})$. These spaces are constructed inductively by putting $X_k = P(V_{k-1})$ and $V_k = (\widetilde{\pi}_k)_*^{-1}(\mathcal{O}_{X_k}(-1))$ where

$$\mathcal{O}_{X_k}(-1) \subset (\widetilde{\pi}_k)^* V_{k-1} \subset (\widetilde{\pi}_k)^* T_{X_{k-1}}$$

is the tautological subbundle (cf. [Dem95]). In the case where V is not a subbundle, we can first construct the absolute tower $(\overline{X}_k, \overline{V}_k)$ by starting from $\overline{V}_0 = T_X$, and then take X_k to be the closure in $\overline{X}_k$ of the k-step X'_k of the relative tower (X'_k, V'_k) constructed over the dense Zariski open set $X' = X \smallsetminus V_{\mathrm{sing}}$. In this way, the tower (X_k, V_k) is at least birationally well defined – in such a birational context we can even assume that X_k is smooth after performing a suitable modification at each stage. Even if we start with $V = T_X$ (or an integrable subbundle $V \subset T_X$), the k-jet lifting V_k will not be integrable in general, the only exception being when $\operatorname{rank} V_k = \operatorname{rank} V = 1$. Now, if

$$\pi_{k,0} = \widetilde{\pi}_k \circ \ldots \circ \widetilde{\pi}_1 : X_k \to X_0 = X,$$

it is shown in [Dem95] that the direct image sheaf

$$\pi_{k,0} \mathcal{O}_{X_k}(m) := E_{k,m} V^* \subset E_{k,m}^{\mathrm{GG}} V^*$$

consists of algebraic differential operators $P(f^{(j)}_{j\leqslant k})$ which satisfy the invariance property

$$P((f\circ\varphi)^{(j)}_{j\leqslant k}) = (\varphi')^m P(f^{(j)}_{j\leqslant k})\circ\varphi$$

when $\varphi\in\mathbb{G}_k$ is in the group of k-jets of biholomorphisms $\varphi:(\mathbb{C},0)\to(\mathbb{C},0)$. Since we already assume $\mathbb{C}^*$ invariance, it is enough to require invariance by the nilpotent subgroup $\mathbb{G}'_k\subset\mathbb{G}_k$ of k-jets tangent to identity. The group $\mathbb{G}'_k$ is a semi-direct product of additive groups $(\mathbb{C},+)$ consisting of biholomorphisms $\tau_{j,a}:t\mapsto t+at^j+O(t^{j+1})$, $2\leqslant j\leqslant k$, $a\in\mathbb{C}$. In this tower, the biholomorphisms $\tau_{k,a}$ actually generate a normal subgroup of $\mathbb{G}'_k$, and we have $\mathbb{G}'_k/\{\tau_{k,a}\}\simeq\mathbb{G}'_{k-1}$. Now, assume that we have found a section

$$P\in H^0(X,E^{\mathrm{GG}}_{k,m}V^*\otimes\mathcal{O}(-m\delta_k A))$$

for some ample $\mathbb{Q}$-divisor A on A. Then we have an expansion

$$P_a(f^{(j)}_{j\leqslant k}) := P((f\circ\tau_{k,a})^{(j)}_{j\leqslant k}) = \sum_{0\leqslant s\leqslant m/k} a^s P_s(f^{(j)}_{j\leqslant k})$$

and the highest non zero term P_s is $\{\tau_{k,a}\}$-invariant of weighted degree $m-(k-1)s$; this comes from the fact that the homothety $h_\lambda(t)=\lambda t$ satisfies

$$\tau_{k,a}\circ h_\lambda = h_\lambda\circ\tau_{k,a\lambda^{k-1}}.$$

Then it makes sense to look at the action of $\{\tau_{k-1,a}\}$ on P_s, and proceeding inductively we reach a non zero $\mathbb{G}'_k$-invariant (and thus $\mathbb{G}_k$-invariant) polynomial

$$Q\in H^0(X,E_{k,m'}V^*\otimes\mathcal{O}(-m\delta_k A))$$

of degree $m'\leqslant m$ (and possibly of order $k'\leqslant k$ but we can still consider it to be of order k). By raising Q to some power p and using the $\mathbb{Q}$-ampleness of A, we obtain a genuine integral section

$$Q^p\sigma_A^{p(m-m')\delta_k}\in H^0(X,E_{k,pm'}V^*\otimes\mathcal{O}(-pm'\delta_k A)).$$

(3.4) Corollary. *Let (X,V) be a projective directed manifold such that K_V is big, and A an ample $\mathbb{Q}$-divisor on X such that $K_V\otimes\mathcal{O}(-A)$ is still big. Then, if we put $\delta_k=\frac{1}{kr}(1+\frac{1}{2}+\ldots+\frac{1}{k})$, $r=\operatorname{rank}V$, the space of global invariant jet differentials*

$$H^0(X,E_{k,m}V^*\otimes\mathcal{O}(-m\delta_k A))$$

has (many) non zero sections for $m\gg k\gg 1$.

If we have a directed projective variety (X,V) with K_V big, we conclude that there exists $k\geqslant 1$ and a proper analytic set $Z\subset X_k$ such that all entire curves have the image of their k-jet $f_{[k]}(\mathbb{C})$ contained in Z. Let Z' be an irreducible component of Z such that $\pi_{k,0}(Z') = X$ (if $\pi_{k,0}(Z') \subsetneq X$ there is nothing more to do). Consider the linear subspace $V' = \overline{T_{Z' \smallsetminus Z''} \cap V_k}$ where $Z'' \subset Z'$ is chosen such that $Z' \smallsetminus Z''$ is non singular and the intersection $T_{Z' \smallsetminus Z''} \cap V_k$ is a subbundle of $T_{Z' \smallsetminus Z''}$. If $f_{[k]}(\mathbb{C})$ is not contained identically in Z'', then the curve $g = f_{[k]}$ is tangent to (Z', V'). On the other hand, if $f_{[k]}(\mathbb{C}) \subset Z''$ we can replace Z' by Z'' and argue inductively on $\dim Z'$. What we have gained here is that we have replaced the initial directed space (X, V) with another one (Z', V') such that $\operatorname{rank} V' < \operatorname{rank} V$, and we can try to argue by induction on $r = \operatorname{rank} V$.

Observe that the generalized Green-Griffiths conjecture is indeed trivial for $r = 1$ (assuming $K_V = \mathcal{O}(V^*)$ big): in fact we get in this case a non zero section $P \in H^0(X, V^{*\otimes k} \otimes \mathcal{O}(-A))$ for some $k \gg 1$ and so $P(f) \cdot (f')^k$ must vanish for every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$. Therefore $f(\mathbb{C}) \subset Y := \{P(z) = 0\} \subsetneq X$. The main difficulty in this inductive approach is that when we start with (X, V) with K_V big, it seems to be very hard to say anything about $K_{V'}$ on (Z', V'). Especially, the singularities of Z' and V' do not seem to be under control. The only hope would be to have enough control on the sections cutting out Z', and this requires anyway to understand much more precisely the behavior and vanishing order of generic sections $P \in H^0(X, E_{k,m}V^* \otimes \mathcal{O}(-m\delta_k A))$. One could try in this context to take A to approach the positive part in the Zariski decomposition of K_V, in such a way that the sections P do not have much space to move around statistically.

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