Algebra of differential forms with exterior differential $d^3 = 0$ in dimension one

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Abstract

In this work, we construct the algebra of differential forms with exterior differential d satisfying $d^3 = 0$ on one-dimensional space. We prove that this algebra is a graded q-differential algebra where q is a cubic root of unity. Since $d^2 \neq 0$ the algebra of differential forms is generated not only by the first order differential dx but also by the second order differential d^2x of a coordinate x. We study the bimodule generated by this second order differential, and show that its structure is similar to the structure of bimodule generated by the first order differential dx in the case of the anyonic line.

1 Introduction

It is well known that one of possible ways to generalize a classical Grassmann algebra is to increase the power of nilpotency of its generators. This means that a possible generalization can be defined as an associative unital algebra generated by $\theta_1, \theta_2, \ldots, \theta_n$ satisfying $\theta_i^N = 0, N > 2, i = 1, 2, \ldots, n$. In order to obtain an analogue of classical Grassmann algebra one should add commutation relations between generators to the above mentioned definition and this leads to different generalizations of classical Grassmann algebra in the case $n > 1$. If one imposes the commutation relations $\theta_i\theta_j = q\, \theta_j\theta_i\, (i < j)$, where q is an N-th primitive root of unity, then the corresponding structure is called a *generalized Grassmann algebra* [1],[3]. Another approach based on ternary commutation relations and the representation of group of cyclic permutations Z_3 by cubic roots of unity was developed in [2],[3] and the corresponding structure is called a *ternary Grassmann algebra*. It should be mentioned that both generalizations of classical Grassmann algebra mentioned above coincide in the trivial case of one generator θ satisfying $\theta^N = 0$ and this algebra known as anyonic line [4] is closely related to fractional supersymmetry [5].

A classical Grassmann algebra underlies an exterior calculus on a smooth manifold with exterior differential d satisfying $d^2 = 0$. Therefore the above mentioned generalizations of Grassmann algebra raise a natural question of possible generalizations of classical exterior calculus to one with exterior differential satisfying $d^N = 0,\ N > 2$. From an algebraic point of view an adequate algebraic structure underlying an exterior calculus is the notion of graded differential algebra. Hence one can generalize a classical exterior calculus with the help of an appropriate generalization of graded differential algebra. This generalization called *graded q-differential algebra* was proposed and studied by M. Dubois-Violette in the series of papers [6], where the author constructed several realizations of graded q-differential algebra.

According to the definition given by M. Dubois-Violette a graded q-differential algebra is an associative unital $\mathbf{N}$-graded algebra endowed with linear endomorphism d of degree 1 satisfying $d^N = 0$ and the graded q-Leibniz rule

$$d(\alpha\beta) = d(\alpha)\,\beta + q^a\ \alpha\, d(\beta), \tag{1}$$

where a is the grading of an element α, q is a primitive N-th root of unity.

From a point of view of differential geometry the above definition can be used to generalize the de Rham complex on a finite-dimensional smooth manifold. This question was studied in the series of papers [7], [8], where the authors used the notion of ternary Grassmann algebra (considering the first non-trivial generalization of classical exterior calculus corresponding to $N = 3$) to construct the algebra of differential forms. If one assumes $d^2 \neq 0$ replacing it by $d^3 = 0$, then in order to construct a self-consistent theory of differential forms it is necessary to add to the first order differentials of local coordinates $dx^1, dx^2, \ldots, dx^m$ a set of *second order differentials* $d^2x^1, d^2x^2, \ldots, d^2x^m$ (and higher order differentials in the case of $N > 3$). Appearance of higher order differentials, which are missing in classical exterior calculus, is a peculiar property of a proposed generalization of differential forms. This has as a consequence certain problems. It is well known that in classical exterior calculus functions commute with differentials, i.e.

$$f\,dx^i = dx^i\,f, \qquad \forall i = 1, 2, \ldots, m, \tag{2}$$

where f is a smooth function on a manifold, or, from an algebraic point of view, the space of 1-forms is a free finite bimodule over the algebra of smooth functions generated by the first order differentials and (2) shows how its left and right structures are related to each other. Now, assuming that d is no more classical exterior differential, i.e. $d^2 \neq 0$, and differentiating (2) with regard to graded q-Leibniz rule (1) one immediately obtains

$$d^2x^i\,f = f\,d^2x^i + [df, dx^i]_q, \tag{3}$$

where $[df, dx^i]_q = df\,dx^i - q\;dx^i\,df$ and we assign grade 1 to first order differentials dx^i. The above relations (3) are *not homogeneous* in the sense that the commutation relations between functions and second order differentials include first order differentials as well.

In this paper, we show that commutation relations between functions and second order differentials can be made homogeneous, i.e. they will not include first order differentials, if we take a non-commutative geometry point of view. In order to make our construction more transparent we begin with the simplest case of one-dimensional space. We construct the differential forms on this space with exterior differential satisfying $d^3 = 0$ and show that they form a graded q-differential algebra. In our construction we use a *coordinate calculi* developed in [9]. Then we study the commutation relations between functions and second order differentials and show that the requirement of homogeneity (i.e. vanishing of the second term in the right-hand side of (3)) implies that a coordinate calculi we are considering is the differential calculus on the anyonic line, which means that the commutation relations between functions and second order differentials are homogeneous, i.e they take on the form $d^2x\,f = f\,d^2x$, only in the case of the anyonic line (in dimension one).

2 Graded q-differential algebra on one-dimensional space

In this section, we construct a graded q-differential algebra of differential forms with exterior differential d satisfying $d^3 = 0$ in dimension one. We study the structure of a bimodule of second order differentials and show that it is homogeneous in the case of the anyonic line. In this section, q is a primitive cube root of unity, i.e. $q^3 = 1$.

Let $\mathcal{A}$ be a free unital associative $\mathbf{C}$-algebra generated by a variable x. If $\xi : \mathcal{A} \to \mathcal{A}$ is a homomorphism of this algebra and $\partial : \mathcal{A} \to \mathcal{A}$ is a linear map such that

$$\partial(x) = 1, \qquad \partial(f\,g) = \partial(f)\,g + \xi(f)\,\partial(g), \quad \forall f, g \in \mathcal{A}, \tag{4}$$

then according to coordinate calculi the map

$$d : f \to \partial(f)\,dx, \tag{5}$$

where dx is the first order differential of a variable x, is a coordinate differential, i.e. d is a linear map $d : \mathcal{A} \to D_\xi(A)$ satisfying the Leibniz rule

$$d(f\,g) = d(f)\,g + f\,d(g), \tag{6}$$

and $D_\xi(A)$ is a free left module over $\mathcal{A}$ generated by dx with the right module structure defined by the commutation rule

$$dx\,f = \xi(f)\,dx. \tag{7}$$

If $f = \sum_m \alpha_m\,x^m$ is an element of $\mathcal{A}$ then the derivative $\partial : \mathcal{A} \to \mathcal{A}$ can be written in explicit form

$$\partial(f) = \sum_{m\geq 1}\sum_{k=0}^{m-1} \alpha_m\,\xi^k(x)\,x^{m-k-1}. \tag{8}$$

In order to construct a generalization of exterior calculus with exterior differential d satisfying $d^2 \neq 0$ let us introduce the second order differential d^2x. Let $(dx)^k\,(d^2x)^m$ be monomials composed from first and second order differentials, where k, m nonnegative integers. As usual we assume that $(dx)^0 = (d^2x)^0 = 1$ where 1 is the unit element of $\mathcal{A}$. Let $\Omega_\xi(\mathcal{A})$ be a free left module over the algebra $\mathcal{A}$ generated by the above introduced monomials. It is easy to see that $\mathcal{A} \subset \Omega_\xi(\mathcal{A}),\ D_\xi(\mathcal{A}) \subset \Omega_\xi(\mathcal{A})$. The module $\Omega_\xi(\mathcal{A})$ becomes an unital associative algebra if we define a multiplication law on $\Omega_\xi(\mathcal{A})$ by the relations

$$\begin{aligned} dx\, x &= \xi(x)\, dx, \qquad & d^2x\, f = \xi(f)\, d^2x + [\partial, \xi]_q\,(f)\,(dx)^2, \qquad (9)\\ (dx)^3 &= 0, \qquad & d^2x\, dx = q^2\, dx\, d^2x, \qquad (10) \end{aligned}$$

where $[\partial, \xi]_q(f) = \partial\,(\xi(f)) - q\,\xi\,(\partial(f))$, and $f \in \mathcal{A}$.

Analyzing the defining commutation relations (9,10) of the algebra $\Omega_\xi(\mathcal{A})$ one can note that the bimodule $D_\xi(\mathcal{A})$ of a coordinate calculi is a submodule of $\Omega_\xi(\mathcal{A})$ and the relation (7) between its left and right structures follows from the commutation relation between the first order differential dx and a variable x. The second remark concerns the structure of the algebra $\Omega_\xi(\mathcal{A})$ with respect to the second order differential. The relations (9, 10) show that any power of the second order differential does not vanish. Hence the algebra $\Omega_\xi(\mathcal{A})$ is an infinite-dimensional vector space and an arbitrary element ω of this algebra can be written in the form

$$\omega = \sum_{m\geq 0} \sum_{k=0}^{2} f_{km}\,(dx)^k\,(d^2x)^m, \qquad f_{km} \in \mathcal{A}. \qquad (11)$$

We shall call elements of the algebra $\Omega_\xi(\mathcal{A})$ differential forms on one-dimensional space generated by a variable x. The algebra of differential forms $\Omega_\xi(\mathcal{A})$ becomes a **N**-graded algebra if we assign grading zero to each element of the algebra $\mathcal{A}$ and grading $k+2m$ to monomial $(dx)^k(d^2x)^m$, i.e. we assume that a variable x has grading zero and the gradings of the differentials dx, d^2x are respectively 1,2. Then the algebra of differential forms splits into the direct sum of its subspaces

$$\Omega_\xi(\mathcal{A}) = \bigoplus_{m=0}^{\infty} \Omega_\xi^m(\mathcal{A})$$

where

$$\begin{aligned} \Omega_\xi^0(\mathcal{A}) &= \mathcal{A},\\ \Omega_\xi^{2k}(\mathcal{A}) &= \{f\,(d^2x)^k + h\,(dx)^2\,(d^2x)^{k-1} :\ f, h \in \mathcal{A}\},\ k = 1, 2, \ldots \qquad (12)\\ \Omega_\xi^{2k+1}(\mathcal{A}) &= \{f\,dx\,(d^2x)^k : f \in \mathcal{A}\}, \quad k = 0, 1, \ldots. \qquad (13) \end{aligned}$$

We now extend the differential (5) of the coordinate calculi to the whole algebra $\Omega_\xi(\mathcal{A})$ as follows:

$$d(\omega) = (\partial f - h)\, dx\, (d^2x)^k, \qquad \omega \in \Omega_\xi^{2k}(\mathcal{A})$$

$$d(\omega) = f\, (d^2x)^{k+1} + \partial f\, (dx)^2\, (d^2x)^k, \quad \omega \in \Omega_\xi^{2k+1}(\mathcal{A}).$$

We shall call the above defined differential an exterior differential on the algebra of differential forms $\Omega_\xi(\mathcal{A})$. It follows from the definition that exterior differential is an endomorphism of degree 1 of the algebra $\Omega_\xi(\mathcal{A})$, i.e. $d : \Omega_\xi^m(\mathcal{A}) \to \Omega_\xi^{m+1}$.

Proposition 1 *The algebra of differential forms $\Omega_\xi(\mathcal{A})$ is a graded q-differential algebra with respect to exterior differential d, i.e. for any two differential forms ω, θ the exterior differential d satisfies*

$$d^3(\omega) = 0, \tag{14}$$
$$d(\omega\theta) = d(\omega)\,\theta + q^{|\omega|}\,\omega\, d(\theta), \tag{15}$$

where $|\omega|$ is the grading of a form ω.

Proof. It follows from (12,13) that any differential form ω can be decomposed into the sum of two forms ω_o, ω_e respectively of odd and even grading, where

$$\omega_e = \sum_{k\geq 1} [f_k\, (d^2x)^k + h_{k-1}\, (dx)^2\, (d^2x)^{k-1}], \tag{16}$$
$$\omega_o = \sum_{k\geq 0} g_k\, dx\, (d^2x)^k.$$

From the definition of the exterior differential it also follows that a differential form of even grading(16) is d-closed ($d\omega_e = 0$) if it satisfies the following condition

$$\partial f_k = h_{k-1}. \tag{17}$$

Now it is easy to show that any form of odd grading is d^2-closed, i.e. $d^2\omega_o = 0$. Indeed applying the exterior differential d to form ω_o, one obtains the form of even grading

$$d\omega_o = \sum_{k\geq 0} [\, g_k\, (d^2x)^{k+1} + \partial g_k\, (dx)^2\, (d^2x)^k],$$

which is d-closed according to (17). Differentiating form (16) of even grading twice, one obtains the form

$$d^2\omega_e = \sum_{k\geq 1}[\,(\partial f_k - h_{k-1})\,(d^2x)^{k+2} + (\partial^2 f_k - \partial h_{k-1})\,(dx)^2\,(d^2x)^{k+1}], \tag{18}$$

which is d-closed. Thus the cube nilpotency (14) of the exterior differential is proved. The q-Leibniz rule (15) can be verified by a direct calculation.

A homomorphism ξ plays a role of a parameter in the structure of the algebra of differential forms, and, choosing particular homomorphism, we can specify the structure of $\Omega_\xi(\mathcal{A})$. We remind that according to the definition the algebra of differential forms $\Omega_\xi(\mathcal{A})$ is a free left module over the algebra $\mathcal{A}$ generated by the monomials $(dx)^k(d^2x)^m$ with associative multiplication law determined by the relations (9, 10). Actually this algebra is generated by three generators x, dx, d^2x. Hence its structure will be more transparent if we define it by means of *commutation relations* imposed on the generators x, dx, d^2x. The only relation in (9, 10), which is not a commutation relation between generators, is the second relation in (9) containing an arbitrary element f of the algebra $\mathcal{A}$. The reason why it contains an arbitrary element f is that in contrast to the commutation relation $dx\,x = \xi(x)\,dx$ the right-hand side of this relation is not a homomorphism of the algebra $\mathcal{A}$ because of the non-homogeneous term $[\partial,\xi]_q(f)$. Obviously imposing the condition

$$[\partial,\xi]_q(f) = 0, \tag{19}$$

we can replace the second relation in (9) by the commutation relation

$$d^2x\,x = \xi(x)\,d^2x, \tag{20}$$

which has exactly the same form as the first one involving the first order differential.

Actually the choice for a homomorphism ξ is not very wide. Indeed every homomorphism ξ of the algebra $\mathcal{A}$ is determined by an element $h_\xi \in \mathcal{A}$ such that $\xi(x) = h_\xi$. From (4) it follows that if the derivative ∂ determined by a homomorphism ξ should satisfy $\partial(x^m) \sim x^{m-1}$ then $\xi(x) = \alpha_\xi\,x$ where α_ξ is a complex number. The condition (19) can be solved with regard to a homomorphism ξ, and the following proposition describes a structure which is induced on one-dimensional space in this case.

Proposition 2 *The condition (19) is satisfied if and only if* $\xi(x) = q\,x$. *This solution leads to the q-differential calculus on anyonic line with derivative*

$$\partial(f) = \sum_{k\geq 1} \alpha_k \, \frac{x^{k-1}}{[k-1]_q!}, \quad f = \sum_{k\geq 0} \alpha_k \, \frac{x^k}{[k]_q!} \tag{21}$$

This means that one can consistently add the relation $x^3 = 0$ *to the relations (9, 10).*

Proof. Using the formula (8) one can find

$$\begin{aligned} \partial(\xi(f)) &= \alpha_\xi \sum_m \alpha_m \sum_{k=0}^{m-1} \xi^k(h)\, h^{m-k-1}, \\ q\,\xi(\partial(f)) &= q \sum_m \alpha_m \sum_{k=0}^{m-1} \xi^k(h)\, h^{m-k-1}. \end{aligned}$$

Thus

$$[\partial, \xi]_q(f) = (\alpha_\xi - q) \sum_m \alpha_m \sum_{k=0}^{m-1} \xi^k(h)\, h^{m-k-1}. \tag{22}$$

From the above formula it immediately follows that $\xi(x) = q\,x$ or $\xi(f(x)) = f(qx)$. Putting $\xi(x) = q\,x$ in (8), one obtains (21). It was explained in [5] that q-differential calculus determined by the derivative (21) is correctly defined at cube root of unity on one-dimensional space generated by a variable x only in the case when $x^3 = 0$. It is easy to show that the relation $x^3 = 0$ can be consistently added to the relations (9, 10).

Now the algebra of differential forms $\Omega_\xi(\mathcal{A})$ on anyonic line can be defined as an unital associative algebra generated by three generators x, dx, d^2x satisfying the following commutation relations:

$$\begin{aligned} x^3 &= 0, && \\ dx\,x &= q\,x\,dx, && d^2x\,x = q\,x\,d^2x \qquad (23) \\ (dx)^3 &= 0, && d^2x\,dx = q^2\,dx\,d^2x. \qquad (24) \end{aligned}$$

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