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ON THE CONSISTENCY OF THE DETERMINISTIC LOCAL VOLATILITY FUNCTION MODEL ('IMPLIED TREE')

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ABSTRACT
We show that the frequent claim that the implied tree prices exotics consistently with the market is untrue if the local volatilities are subject to change and the market is arbitrage-free. In the process, we analyse – in the most general context – the impact of stochastic variables on the P&L of a hedged portfolio, and we conclude that no model can *a priori* be expected to price all exotics in line with the vanilla options market. Calibration of an assumed underlying process from vanilla options alone must not be overly restrictive, yet still unique, and relevant to all exotic options of interest. For the implied tree we show that the calibration to real-world prices allows us to only price vanilla options themselves correctly. This is usually attributed to the incompleteness of the market under traditional stochastic (local) volatility models. We show that some 'weakly' stochastic volatility models without quadratic variation of the volatilities avoid the incompleteness problems, but they introduce arbitrage. More generally, we find that any stochastic tradable either has quadratic variation – and therefore a Γ-like P&L on instruments with non-linear exposure to that asset – or it introduces arbitrage opportunities.

Keywords: Implied trees; Stochastic local volatility; Exotic options; Vega hedging; Skew models; Market incompleteness;

1. Introduction

One of the customary ways in dealing with the volatility smile and term-structure as observed, for example, in the implied Black-Scholes volatilities in the options markets, is to translate the different Black-Scholes volatilities into a 'local' volatility ('implied tree') [1, 2, 3]. This assumes the stock-price follows the process

$$\frac{dS}{S} = \mu(t)dt + \sigma(S(t), t)dW,$$

where σ is a deterministic function of the stock-price and time, calibrated such that the model matches the observed vanilla options prices, and W is a standard Brownian motion (SBM) and the only random source of the model. While the earliest incarnations of this idea led to very erratic behaviour of the local volatility surface,

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much recent work has focused on how to compute a surface that is both reasonably smooth and fast to find [4, 5, 6, 7]. Both the names DVF (deterministic volatility function) and LVF (local volatility function) are customary for this approach. We will use the abbreviation DLVF, for obvious reasons, and to distinguish it from SLVF (stochastic local volatility function, or stochastic implied tree models), which we shall also discuss in this paper. We shall use LVF when we refer to both classes of models.

DLVF models are the preferred skew models of a number of institutions, given that alternative approaches to create a skew effect encounter limitations as to how well they can fit market data [8]. The commonly perceived downside of DLVF models are practical limitations in measuring the volatility exposure of an exotic options portfolio. There are, however, no theoretical grounds on which to affirm that volatility risk can be hedged at all in this framework: A truly deterministic volatility function must be assumed. The two most pressing theoretical reservations one may harbour towards local volatility models, and the problems arising from these concerns, are therefore the following.

- When the stock-price moves, but the LVF is assumed to be constant, this imposes a rule on the Black-Scholes-implied volatilities which is neither of the two popular rules nick-named 'sticky strike' and 'sticky delta'. The 'sticky-implied-tree' rule leads to different stock-deltas than either of the other two rules [9]. Therefore, before putting trust in local volatility models, we must either justify the emergence of this rule, or show that the validity or otherwise of the 'sticky-implied-tree' assumption is immaterial to the P&L of a hedged strategy, which can only be the case if we can hedge against stock and volatility movements separately.
- In general, it is difficult to estimate how the local volatility surface moves when a limited number of Black-Scholes-implied volatilities change in the market. Note, however, that measuring this exposure is one of the primary motivations for using a skew model at all. One of the key arguments used in defence of DLVF models is that volatility risk is already priced into the market, and that the DLVF model is only an *effective theory* which makes use of the fact that volatility risk is already integrated out in the right way by the market-implied local DLVF. This is true for a well-diversified portfolio of vanilla options, but the implied volatility does not compensate sufficiently for *option-specific* vega risk of vanilla or exotic options. A Δ-hedged portfolio short a log-contract has a much more predictable P&L in the real world than a Δ-hedged portfolio short just one at-the-money vanilla option. This is because in the first case the unknown P&L component is only a function of the realised asset price variance (assuming a diffusive asset price process), whereas in the second case the P&L is also very much a function of the average Γ experienced over the life of the option. So, although implied volatilities often compensate the option seller for the vega-risk taken, they do so insufficiently for someone who does not diversify his volatility risk across various strikes. If we want to sell an

exotic option, and hope that it is priced in line with the market, we must also show that we either can diversify our vega exposure in a similar way, and that the vega-risk which can not be diversified away is priced in line with the vega-risk premium that is already integrated out in the observable Black-Scholes implied volatilities. Obviously, the DLVF model does not give us a volatility risk premium above the premium implied in the vanilla prices. This is what should be meant when it is claimed that the implied tree values exotics 'in line with the market'. The concern that needs to be addressed is thus whether the expectations that are already integrated out in the 'vanilla-implieds' can be re-used as 'exotic-implieds'. Note that the only way we can be certain that the effective integration of volatility risk can be applied to exotics is by showing that volatility risk of an exotic option can be hedged with a dynamic trading strategy of vanilla options. Unless one can show that volatility exposure can be hedged against any of the points on the local volatility surface (or any of the variables which parameterise this surface), we may not be entitled to use risk-neutral valuation techniques in a world where implied volatilities are subject to change. This would make the DLVF models unviable.

We will demonstrate in this article that these two concerns can be redressed in an arbitrage-free market only if the future spot-volatilities are indeed deterministically identical to their DLVF values. That they are not is clear from empirical work and from simple intuition [10, 11].

The structure of the paper is as follows: First, we give a general discussion of what conditions we might expect in order for the local volatility model to be viable under risk-neutral valuation with an only mildly time-varying volatility function. We observe that these requirements destroy the no-arbitrage condition, and that we thus need to consider the full SLVF model known also as the stochastic implied tree. This paper can be described as a practical guide to (a) the problem of incompleteness and (b) the HJM-like no-arbitrage condition of stochastic implied trees as described in [12]. The above-mentioned early sections of the paper introduce the theoretical problems. Although the way of presenting these problems may be new, the incompleteness problem has been discussed in much greater detail and mathematical rigor in the literature on stochastic volatility models.

After this practitioner's guide to the theoretical problems, we present a discrete framework in which we are able to construct, if it exists, the replicating trading strategy of any exotic option, involving stock and Arrow-Debreu option positions. We demonstrate that this portfolio always exists in a two-step trinomial world. Next, we analyse a three-step trinomial world and find that the stochastic process required to make the hedge work violates the no-arbitrage condition. This problem persists in the continuum limit.

We then present the conclusion that in a DLVF model risk-neutral valuation of exotics (without additional arbitrary parameters like for example option-dependent volatility risk premia, which can not be calibrated to fit the vanilla options market) is inconsistent with the assumptions of no arbitrage and unknown future volatility.

2. Stochastic Processes with and without Quadratic Variation

Some of the issues discussed here benefit from a brief tour through familiar territory. The Black-Scholes world is based on the assumption that the stock price S follows the process

$$dS = rSdt + \sigma SdW,$$

where the dW are the increments of an SBM. The Black-Scholes PDE

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf,$$

in a slightly different notation, is called the *continuous trading equation*

$$\Theta\, dt + rS\Delta\, dt + \frac{1}{2}\sigma^2 S^2 \Gamma dt = fr\, dt.$$

The left-hand side of the continuous trading equation is equal to $df - \Delta dS + r\Delta S dt$, the P&L of a Δ-hedged portfolio, which is risk-less in the Black-Scholes world, because the volatility σ is a known constant. For the rest of the discussion, it will be easier to think of the continuous trading equation in terms of a futures-hedge, as the funding term then drops out, i.e. we think of the continuous trading equation as

$$df - \frac{\partial f}{\partial F} dF = rf dt,$$

or, after expanding the total differential for $df(F,t)$ to first order in dt,

$$\Theta dt + \frac{1}{2}\sigma^2 F^2 \Gamma_F dt = rf dt.$$

The continuous trading equation thus tells us that if all the other assumptions of the Black-Scholes world hold, the instantaneous P&L of a long, Δ-hedged, option position is a trade-off between the loss of Θ on the option and the gain on Γ, which is proportional to the instantaneous variance rate σ^2 of the stock-price. The fact that we have a Γ-related P&L component is a consequence of the fact that the stock-price has non-vanishing quadratic variation $dS, dF = \mathscr{O}(\sqrt{dt})$. Now imagine that S is stochastic, but of vanishing quadratic variation, for example $dS \equiv \left(r + W - \frac{t^2}{2}\right) Sdt$ or $dF \equiv \left(W - \frac{t^2}{2}\right) Fdt$, then the left-hand side of the continuous trading equation is $df - \Delta_F dF$ or

$$\Theta dt = rf dt,$$

so the Γ-related P&L disappears, even though the asset price process is stochastic. The asset price process has a fundamental problem, however: Even though initially it has the desirable property of $\hat{E}\left[F_t|F_0\right] = F_0$ [1], it is in general not the case that $\hat{E}\left[F_t|F_u\right] = F_u$, $u > 0$, so the futures price is not a martingale: because $\frac{\partial S}{\partial t}$ persists

[1] To prove this, simply use the fact that the random variable $X_t = \int_0^t W(u)du$ with $W(0) = 0$ is normally distributed with zero mean and variance $\frac{t^3}{3}$.

for some time, observing the 'momentum' of the stock price would provide arbitrage opportunities between stock-returns and funding costs. In this model, options are priced at their discounted intrinsic value, which is consistent with the observation that holding Γ is of no value. In reality, people pay a price for the privilege of being long Γ and the Θ is rather smaller than the cost of funding on the option premium and often negative, which thus indicates that the market believes that the stock price does have higher than linear variations. This means that the option market 'knows' that the stock price is either diffusive or exhibits jumps, or both. In turn, this also means that the arbitrage opportunity related to the persistence of the stock-momentum disappears. To state this formally:

Proposition 1 *For the future F of an asset S (in a world of deterministic funding costs) which follows the Ito process*

$$dF = \left(\xi B(t) - \frac{\xi^2 t^2}{2} \right) F \, dt + \sigma(F,t) F \, dW$$

where $B(t)$ and $W(t)$ are SBMs, $F, \xi, \sigma \geq 0$, and the entire information on the asset price is it's history, the following statements are equivalent

1. *$B(t)$ is observable*
2. *$\sigma(F,t) \equiv 0$ and $\xi \neq 0$.*
3. *$\xi \neq 0$, and the market satisfies, at any given time $B(t) \gtrless \frac{\xi t^2}{2} \iff dF \gtrless 0$.*
4. *The future price is predictable to order dt, and, except when $B(t) = \frac{\xi t^2}{2}$, allows arbitrage.*
5. *The price process $F(t)$ is stochastic and differentiable almost everywhere. It thus has no quadratic or higher variation.*

Also, if any of the conditions is true, there can be no risk-neutral measure for F.

Proof. $1 \iff 2$ is trivial, as no two Gaussian variables can be simultaneously inferred from one futures price. $2 \iff 3$ follows from the fact that, if $\sigma \neq 0$, almost all realisations of dW would dominate over the dt-term in dF, so the statements in item 3 could not be made with any certainty, and vice versa. The first part of $2 \iff 4$ is trivial, since the leading order in dF is dt with coefficients which are (by equivalence with item 1) observable, if conditions 2 are satisfied. This, however, would still be true if ξ were zero. However, in this case $dF \equiv 0$, and the market would not allow arbitrage. Yet if $\xi \neq 0$, then we know the sign of dF, and arbitrage exists, in that a zero-price position in F can be acquired that is instantaneously profitable. Thus also $4 \Longrightarrow 2$. $5 \Longrightarrow 2$ can be shown in the following way: if $\sigma \neq 0$ anywhere, the price process would have quadratic variation, as there are terms of order $\sqrt{dt}$. If $\sigma \equiv 0$ and $\xi \equiv 0$, however, the process is deterministic. Thus a stochastic price process without quadratic variation must have $\sigma \equiv 0$ and $\xi \neq 0$. $2 \Longrightarrow 5$ is then just the trivial reversal of this argument.

The non-existence of the risk-neutral measure follows directly from the existence of arbitrage opportunities, according to Harrison and Pliska [13] □.

Note also that the observation, originally made in the HJM framework for interest rates, that the no-arbitrage condition determines the drift of a stochastic, tradable variable is related to this proposition. This observations was extended to stochastic local volatilities by Derman and Kani [12, 14]. For both interest rates and stochastic volatilities it is also true that in the absence of quadratic or higher variations (i.e. zero volatility in a jump-less diffusion model) the no-arbitrage drift is zero. Proposition 1 only applies to a specific type of process, but makes no assumptions as to the nature of the tradable asset.

Let us therefore assume the somewhat more general jump-less, stochastic futures price process

$$dS(t) = \left(r(t) + I(t) - \frac{\partial}{\partial t} \ln \hat{E}_0 \left[e^{\int_0^t I(u)du} \right] \right) S(t)dt + \sigma(W,s)S(t)dW(s), \quad (2.1)$$

where $I(t)$ is an Ito process, and $\hat{E}_0$ is the expectation operator evaluated at time zero. Henceforth, we call a process of the type Eq. (2.1) a *generalised Ito process*, which differs from the usual Ito process because the drift is an Ito process itself, but with a deterministic drift correction $-\frac{\partial}{\partial t} \ln \hat{E}_0 \left[e^{\int_0^t I(u)du} \right]$. Funding rates are deterministic, such that the futures price $F(t)$ follows the same process, but without the interest rate drift component. This then has the solution

$$F(t) = F_0 \frac{\exp\left[\int_0^t I(s)ds + \int_0^t \sigma(F(s),s)dW - \frac{1}{2}\int_0^t \sigma^2(F(s),s)ds \right]}{\hat{E}\left[\exp \int_0^t I(s)ds \right]},$$

which has similar properties to the process of Proposition 1, but is more general. This leads to a natural generalisation of Proposition 1:

Theorem 1 *For a future – in a world of deterministic funding costs – on an asset, which follows the process Eq. (2.1), the following statements are equivalent*

1. $\sigma(F,t) \equiv 0$ *and* $I(t)$ *is stochastic.*
2. $I(t)$ *is stochastic, and, at any given time* $I(t) \gtrless \frac{\partial}{\partial t} \ln \hat{E}_0 \left[\exp \int_0^t I(s)ds \right] \iff dF \gtrless 0$
3. *The future price is predictable to order* dt*, and, except when* $\frac{\partial F}{\partial t_-} = 0$*, allows arbitrage.*
4. *The price process* $F(t)$ *is stochastic without quadratic variation.*

Furthermore, if these conditions are true, there is no risk-neutral measure for F*.*

Proof. As opposed to Proposition 1, here we have to leave out the observability of the realised value of $I(t)$, as this would require knowledge of $\frac{\partial}{\partial t} \ln \hat{E}_0 \left[\exp \int^t I(s)ds \right]$, which can not be taken for granted. $1 \iff 2$ is analogous to Proposition 1, and items 3 and 4 follow immediately $\Box$.

The important lesson to learn from Theorem 1 is that we do not need to know the details of the generalised Ito process that makes F stochastic in order to conclude that in the absence of quadratic variation there are arbitrage opportunities which we can spot simply by observing $\frac{\partial F}{\partial t}$, and the only way to exclude arbitrage at all time is to make $F(t)$ deterministic and $\hat{E}_t [dF_u] = 0 \ \ \forall u > t$, which of course means that $F(t)$ is simply a constant in time.

From this we introduce the important

Lemma 1 *If, in a world of deterministic funding costs, a tradable asset follows a process with no quadratic or higher variation, it is either deterministic or allows arbitrage almost always, or both.*

Proof. The fact that the process has no higher than quadratic variations and that the futures price is a martingale means that it is a generalised Ito process of the type Eq. (2.1), except that the drift term $I(t)$ in Theorem 1 can be generalised to a jump-diffusion process where the set of times where jumps occur are of measure zero (i.e. the jumps are always countable, like in a Poisson process). Then the quadratic variation of the process $F(t)$ is zero, and the price-process is almost always predictable at least to order dt. This means it is differentiable almost everywhere, and continuous everywhere. The differentiability almost everywhere would allow successful arbitrage almost always (with infinitesimal losses in the countable cases where it is not differentiable), if the slope of the process is different from the funding costs on the asset (because the cost of setting up a futures position is zero). Therefore, the process allows no arbitrage only if F is constant almost everywhere. Furthermore, if the generalised Ito process of no quadratic variation does allow arbitrage, it is very easy to spot the arbitrage by observing the momentum of the futures price $\Box$.

It is very important to note that we never implied in any of the proofs of the above theorem and lemma that the future is written on a stock-price: It could have been a local volatility, as long as the funding costs are deterministic. Again, this is a generalisation of the observations of HJM, and Derman and Kani to any stochastic tradable asset under zero volatility [12, 14]. The HJM drifts for interest rates and the local-volatility drifts as calculated by Derman and Kani prove that even for non-zero volatility the drifts must be deterministic, if the no-arbitrage condition is to be satisfied. Lemma 1 is thus also more restricted than the results of [12, 14], in that we only analyse the zero-volatility case, but it is also more general, in that it applies to any tradable asset.

The bit to keep in mind for the next sections is that a stochastic tradable has to have at least quadratic variation in order to preclude arbitrage.

2.1. *Does the quadratic variation matter in discrete hedging?*

Against taking quadratic variation too seriously, one can frequently hear variations of the following objection:

'Quadratic variation has to do with the fractal dimension of the process. It basically says, if I add the moduli of all the stock-price changes in any finite interval, I get infinity. This is patently not the case in reality. Furthermore, I typically re-hedge only after a stock-move of, say, 4%. So I re-hedge maybe once a week on average. Adding all the moduli of stock-returns relevant to my hedging-strategy, they obviously don't diverge. So for practical purposes, the stock-movement relevant to me is proportional to time. In other words: to me the stock-price path has dimensionality 1'.

It is easy to see what is wrong with this argument. Our hedger experiences a typical Γ-related P&L component of $52 \cdot (0.04)^2\Gamma$ on an annualised basis. The crux is that, if his pain-barrier was a stock-move of, say, 2%, he would not re-hedge once a week, or even twice a week on average, but four times a week, if the quadratic variation of the stock-price process is the highest non-vanishing variation. So on an annualised basis, his Γ-related P&L component is $208 \cdot (0.02)^2\Gamma$, which is exactly the same as before. The quadratic variation therefore does matter equally for someone who hedges infrequently as for someone who hedges continuously. One can not escape the Γ-related P&L by simply re-hedging more frequently. In other words: the fractal dimension of the stock-price process matters regardless of the granularity of price observations. If our hedger lived in a world where the dimensionality of the stock-price process is linear, dropping his pain-barrier to 2% would incur a Γ-related P&L component of $104 \cdot (0.04)^2\Gamma$, and he could observe, regardless of the granularity of his hedge, that more frequent re-hedging lets him reduce the Γ-P&L. Let us therefore not be fooled: The fractal dimension of the stochastic process is observable. In particular:

Theorem 2 *The fractal dimension on the scale relevant to discrete-time hedging is also observable.*

2.2. *Hedging 'Exotics' in the Fixed-Income World*

Before we tackle the stochastic local volatility world, let us consider an instructive example of the typical problem we discussed in section 2 from the fixed-income markets. Kani *et. al.* observed that trading and hedging against forward interest rates is similar to trading and hedging against local volatilities [15]. This allows us to use examples which are easy to follow because the set of independent forward rates is one-dimensional, while the set of local volatilities is two-dimensional. Here we bring an example of a very simple fixed income world with only two bond-prices, and a simple 'exotic' (i.e. a simple function of these two prices) which can not be replicated with a dynamic trading strategy without taking new independent variables into account, namely variance and covariance rates of the interest rates.

This is a further qualificatnion of the claim by Kani *et. al.*, namely that everything in the fixed-income world can be replicated by dynamically attainable 'forward rate gadgets' which have zero cost and provide a specified degree of exposure to a particular forward rate [15]. The central conclusion we work towards here is the inapplicability of an effective DLVF theory to an SLVF world. Consider thus the frequent claim that vanilla options make local-volatility gadgets attainable, which complete the market even in the presence of exotic options. This claim and the above-mentioned claim of Kani *et. al.* are are easily mis-interpreted for the same reason, namely that quadratic variations are not negligible in a stochastic, arbitrage-free market, and that they can not be fully hedged against [2] Note, however, that this observation can already be made in the derivation of the Black-Scholes model: Perfect replication of a vanilla option in the Black-Scholes world is only possible when we also know the volatility of the stock. For multi-factor models, we need to know the variance-covariance rate matrix in order to achieve perfect replication. The DLVF models make no reference to this matrix, and there are usually not enough degrees of freedom to calibrate to such a matrix if it were known.

Consider a world with only two bonds B_1 and B_2, maturing at times t_1 and t_2 from now. The corresponding interest rates are defined through $B_1 = \exp(-r_1 t_1)$ and $B_2 = \exp(-r_2 t_2)$. The forward rate is the interest rate for the term between t_1 and t_2 which can be locked in at present, i.e. $f = \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1}$, therefore $\frac{B_1}{B_2} = \exp\left(f \cdot (t_2 - t_1)\right)$. The 'interest rate gadget' is the portfolio $\Lambda = B_1 - \left[\frac{B_1}{B_2}\right] B_2$, where the square brackets indicate a position size ('hedge ratio'), which means a value that stays fixed during instantaneous changes of the market observables r_1 and r_2. Now imagine we have some other instrument, whose exposure to the forward rate f we want to hedge. We observe that

$$\delta\Lambda = B_1 \left[(t_2 - t_1)\delta f - t_1 (t_2 - t_1)\delta f \delta r_1 - \frac{1}{2}(t_2 - t_1)^2 (\delta f)^2 \right] .$$

Kani *et. al.* mention that the bond B_1 can be synthesised by holding one gadget and $\frac{B_1}{B_2}$ units of B_2, which gives exact replication to all orders [15]. Unfortunately, this is not a valid example of hedging, because all we have in the 'replicating' portfolio is exactly one bond B_1. To really keep the correspondence with DLVF models, we need to introduce a new instrument which has a pay-off as a function of f, but which is not already in this market. Consider for example a security which pays exactly the forward rate as, say, a US dollar amount. To hedge this, we need a strategy whose P&L is and stays linear in the forward rate. However, since the hedge ratio itself is a function of the forward rate, we get the second derivatives in $\delta\Lambda$. If we hedge the forward-rate exposure of some portfolio with the gadget Λ, we see that we introduce a negative P&L component proportional to the variance rate of f, and a correlation term between the forward rate and r_1. We do not have enough degrees of freedom to eliminate these new terms, i.e. to synthesise a product which is strictly linear in the forward rate. The corresponding effect on, say, for example, interest rate futures prices is frequently referred to in the fixed income markets as a 'convexity correction'. The size of the convexity correction

[2] We would like to stress that ref. [12] indicates that Kani *et. al.* are not vicitims of this mis-interpretation. What matters here is only that ref. [15] could easily mislead the reader into believing that forward-rate gadgets are perfect hedging instruments, as the paper does not refer to the necessity of making convexity corrections. The point we make is that misreading the paper in this way is analogous to ignoring the quadratic variation of local volatility in DLVF models.

is increasing with the variance rate $(df)^2/dt$ and the covariance rate $(df)(dr_1)/dt$. Note that if we know both these rates, perfect replication is still possible, in the same sense as it is possible in the Black-Scholes world for vanilla options when we know the volatility of the underlying.

3. Stochastic Local Volatilities

Define $\vec{x} = (F_S, \vec{\alpha})$, where $\vec{\alpha}$ is an n-component vector parameterising the local volatility surface, calibrated to market prices, and F_S is the forward price of the stock. Let us now consider a scalar function $f(\vec{x})$ of these $n+1$ stochastic variables, and define the vector-operator

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial F_S}, \frac{\partial}{\partial \alpha_1}, \frac{\partial}{\partial \alpha_2}, \ldots, \frac{\partial}{\partial \alpha_n} \right).$$

Then

$$df = f_t dt + (\vec{\nabla} f) \cdot (d\vec{x}) + \frac{1}{2}(d\vec{x})^T \cdot (\vec{\nabla} \otimes \vec{\nabla} f) \cdot (d\vec{x}) + \ldots$$

If the stochastic process for $\vec{x}$ has no more than quadratic variation, the higher order terms are smaller than order dt. Let us assume that local volatility futures (or local volatility 'gadgets' in the language of ref. [15]) are attainable at zero cost. Then, after hedging the portfolio to first order in the stock price and local volatilities, the continuous trading equation is

$$f_t dt + \frac{1}{2}(d\vec{x})^T \cdot \left(\vec{\nabla} \otimes \vec{\nabla} f \right) \cdot (d\vec{x}) = r f dt\,, \tag{3.1}$$

which makes for a total of one Θ component, the funding on the stock-hedge, and $\frac{(n+1)n}{2}$ terms arising from all the different second derivatives with respect to the different components of $\vec{x}$.[3] We now have to make assumptions on what the asset price process is in the real world. Broadly, we can classify the imaginable price processes into four groups:

- DLVF world: The local volatilities are completely deterministic. The Black-Scholes world is a trivial example. The asset price process of an implied tree is the most general case.
- WSLVF world: The local volatilities are stochastic, continuous everywhere, and differentiable almost everywhere. Let us call this model the 'weakly stochastic local volatility function' model.
- SLVF world: The variables parameterising the volatility function follow stochastic processes with non-vanishing quadratic variation, but vanishing

[3] The assumption that local volatility futures are attainable is probably no more permissible than the assumption in the fixed income world that forward rate gadgets are attainable, that is they are attainable only to first order. In this sense, Eq. (3.1) will not be strictly correct, but will contain other second order terms, arising from the second-order terms of the local volatility gadgets. Qualitatively, the number of P&L terms is predicted correctly, however, as we only need replace $(\nabla \otimes \nabla f)_{ij}$ with $(\nabla \otimes \nabla f)_{ij} + C_{ij}$, where C_{ij} represents the combined second order exposure created by our holding the local volatility gadgets.

higher variations (This means these processes are diffusive). Call this the stochastic local volatility function model.

- Jump-diffusion world: Either the stock-price process or the volatility parameters (as implied by the vanilla options market) or both are not simply diffusive. This would manifest itself in higher than quadratic variations.

Note that the decision on which world a particular market corresponds to can be based on empirical data according to Theorem 2.

Let us consider the suitability of the DLVF model to all of these four worlds.

In the DLVF world the local volatilities are perfect forecasts of future realised instantaneous volatilities. In the absence of arbitrage, the evolution of the local volatility surface is static except for a shift of the time-frame of the observer. In this case, risk-neutral valuation and the DLVF model are obviously applicable. However, the viewpoint that we live in such a DLVF world contradicts empirical evidence [10, 11].

The WSLVF world has the previously described arbitrage opportunities: observe the local volatility 'momentum', and simply go long the gadgets corresponding to growing local vols, and short the others. This provides a risk-less profit almost always.[4] Within this world the implied tree does not work in the way it is customarily used and clearly all vanilla options and all options calculated on the implied tree are priced assuming a sub-optimal hedging strategy. In fact there is no optimal hedging strategy: any given strategy can be beaten by another one using more of the arbitrage opportunities. Clearly, while the world could be WSLVF, we gain nothing by making this assumption, unless we have a model to identify and exploit the arbitrage opportunities. We can make the intriguing observation, however, that everyone gets away with using risk-neutral valuation as long as no market participant spots the arbitrage. If option buyers and sellers all worked with a DLVF model and the associated hedging strategies, the existence of arbitrage is irrelevant for practical purposes. So it is in fact entirely possible that the real world is of the WSLVF type, although, we would clearly prefer to have a stronger cause for the use of risk-neutral valuation in the DLVF model than the reason that it works simply because everybody is using it. Note that the assumption that the real world is WSLVF is easily testable: use a DLVF model, and take local volatility positions according to the 'momentum' of the local volatility parameters. Unless one makes profits almost always with this strategy, the real world is not WSLVF.

Next we have to discuss the SLVF world. A necessary condition for local volatility gadgets to exist is that tradability of all European vanilla options, which is equivalent with the tradability of all Arrow-Debreu options. Since the existence of sufficiently many of these options is a necessary condition to calibrate the LVF surface, we will assume in the remaining part of this paper that sufficiently many Arrow-Debreu options or European vanilla options are available in order to produce gadgets for all the LVF parameters we allow in the model, and that these

[4] A necessary condition for the existence of such arbitrage is that the 'almost never' occurring cases of making a loss produce a bounded loss, which is the case here.

options can be used in any hedging strategy for exotic options. Then we have two possibilities to distinguish:

- SLVF(I) world: Local volatility gadgets exist, but second-order exposure can not be hedged. The gadgets themselves may or may not produce additional second-order exposure.
- SLVF(II) world: In this world, we can hedge against variations of stock price and local volatilities up to second order.

The central issue of this paper is that the SLVF(II) world is theoretically impossible unless there is a very low-parametric representation of the LVF, and that, in any case, volatility of volatility has P&L effects which accrue over time.

We do not consider the jump-diffusion world for two reasons. Firstly, we are not aware of any specific claim that DLVF models could be appropriate to such a world, although the idea that effective theories may also have jump-risk 'effectively' integrated out can not be discarded *a priori*. Secondly, however, we already show that there irreconcilable problems with the application of DLVF models in an SLVF world. Since jump-diffusion models are a super-set of SLVF models, we would have to hedge against even more sources of uncertainty in jump-diffusion worlds. It is thus unnecessary to prove that the problems of the SLVF worlds persist in the jump-diffusion setting.

4. P&L in the SLVF worlds

In an SLVF(I) world, we have all the P&L terms of Eq. (3.1) to consider. This will lead to risk premia on the option proportional to the variance of the to-be-realised 'Γ' P&L, in other words proportional to

$$d\vec{x}^T \cdot \int_0^T (\nabla \otimes \nabla f) \cdot d\vec{x}, \tag{4.1}$$

or simply to all the un-hedged second order terms, because they are of order dt and thus accrue P&L proportional with time.

The question is now whether we can quantify the volatility risk premium appropriate for an exotic option in an SLVF(I) model. Obviously, the terms in Eq. (4.1) are in principle computable, if the market-implied covariances between all local volatilities are known, as the rest is simply path-integration [5]. There is of course no hope to infer these parameters unless the *a priori* parameterisation of the LVF we are willing to consider is extremely low-dimensional: For an n-dimensional parameterisation, we have to infer the n parameters, and all components of the variance-covariance matrix of these parameters plus the stock-price. This is a symmetric $(n+1) \times (n+1)$ matrix. In total, we have thus $\frac{(n+1)(n+2)}{2} + n$ parameters to estimate (in a non-mean-reverting SLVF model). Assuming we have m independent vanilla option prices, we are allowed to use no more than $\frac{\sqrt{41+8m}-7}{2}$ parameters for

[5] We make no statement about the practical difficulties of attempting this computation.

the LVF. We thus need four vanilla option prices for the usual non-mean-reverting stochastic volatility model, and at least eight option prices to accommodate even the simplest skew information, by specifying the SLVF by two volatility parameters. If the SLVF is required to be mean reverting, we require twelve options to calibrate such a model. Now say one of those parameters was designed to capture skew, and the other to capture term structure. If we therefore took a second skew parameter, making the total parameterisation of the LVF three-dimensional, we would require 19 options in the mean-reverting case and 13 otherwise. This analysis underlines the urgent need for 'smoothing' or 'regularising' of the LVF, which has been attempted by many authors recently [4, 5, 6, 7]. However, smoothing alone does not suffice to estimate volatility risk premia.

At this point in the discussion, one might want to ask the question of whether the volatility risk premium really matters. After all, we never worry about it in the Black-Scholes world, even though the Black-Scholes assumption of deterministic volatility is clearly wrong in practice.

4.1. *Volatility Risk Premia and Exotic Options*

It is in the context of Black-Scholes type models, that we find that there is skew: The BS volatilities implied in the market differ for different strikes and maturities. This is ascribed to the BS assumptions being wrong in one way or another. The LVF approach is to say that volatility simply is not flat, the stochastic volatility and jump-diffusion approach is to say that there is risk-aversion and risk-premia are associated with un-hedgeable risks, or in other words with the incompleteness of the market.

When using a BS-implied volatility surface, we do not need to worry what causes the skew. If there are risk-premia, they will already be in the option price. However, there is a qualifying statement to be made: the P&L of a Δ-hedged long option position until expiry is

$$\int_0^T \left(\Theta(t) + \frac{1}{2}\Gamma_F(t)F^2(t)\hat{\sigma}^2(t) - r(t)f(t) \right) dt \tag{4.2}$$

where $\hat{\sigma}(t)$ is the actual realised volatility at time t, and Γ_F is the futures-Γ. All the integrands are stochastic, and we thus do not expect the option to be priced at the expectation of all these parameters. Instead, we expect that the option is priced at the expectation – under some risk-neutral or, if necessary, risk-adjusted measure – of the entire integral Eq. (4.2). This price can be expressed by means of a Black-Scholes implied volatility. Thus, the option can be priced at some implied σ which can, in Eq. (4.2), replace the stochastic $\hat{\sigma}(t)$. The Black-Scholes implied volatility allows us to use the Black-Scholes option pricing theory, which is an *effective theory*, where specific expectations (or integrals) of some of the underlying stochastic variables are extracted from the current prices of the traded assets. The effective dynamics which results is based on some of the sources of uncertainty being 'effectively' integrated out of the full stochastic theory. It can be shown that DLVFs are risk-adjusted expectations of future instantaneous volatilities [12]. In this way, DLVF models are *effective theories of volatility* in the same way as static forward interest rates define an effective theory for interest rates. This is an important point which we will return to later on.

Beyond the volatility risk in Eq. (4.2) of the P&L of a hedged options position, there is an equally important risk of spending an unusually long amount of time in regions of high Γ. However, we should expect that no-one is compensated for this option-specific risk, simply because this can be diversified away by selling a portfolio of vanilla options providing a reasonably constant Γ profile across spot prices and times. This portfolio would still have vega exposure, but no option-specific risk.[6] We thus formulate the following observations.

1 *The implied Black-Scholes volatilities compensate the option seller for volatility risk, but not for option-specific risk.*

If we use implied Black-Scholes volatilities to price exotic options, we ignore that the exotic may have specific risk that can not be diversified away by trading listed options. In particular that risk, because it is a specific risk attached to an un-quoted asset, can not be implied from the market. Furthermore, the same argument holds for any model calibrated only on vanilla options. This leads to the key criterion for successful exotic model valuation:

2 *An option for which there is no known market price must either be hedgeable with quoted securities, or otherwise will incur instrument-specific risk premia.*

Obviously, this condition is satisfied by any sensible model whenever reality stays within the constraints of the model-assumptions. The problems with the DLVF models, as we argue here, is that reality is too far removed from the fundamental DLVF assumption of deterministic (local) volatility parameters. This is underlined by empirical research on market data [11]. A trivial aside of the above is

3 *An exotic option model has to either have a plausible way of incorporating option-specific risk-premia or permit a proof that a replicating trading strategy exists.*

We will show below that the DLVF model applied to a world of non-deterministic volatility parameters does neither if the only allowed hedging and calibration instruments are European vanilla options. It does not matter whether we use a low or high-parametric SLVF model: In low-parametric SLVF models we may create option-specific risk that can not be hedged by other vanilla options, simply because the parameter-family of LVFs do not capture the full volatility exposure of all conceivable exotics. In a high-parametric SLVF, we will normally not have enough vanilla options to calibrate to, and worse: even if we have a continuum of vanilla options for all strikes and expiries, we can not calibrate a continuum of SLVF parameters to that. It is therefore relevant to initially construct the space of all options we are willing to consider as hedging instruments in our model, and then decide which SLVF world this puts us in. Obviously, in an SLVF(II) world, there always

[6] In fact there is some option-specific risk, because when the implied vols change for some options, the Γ-profile of the portfolio changes. This is also a vega risk, because it is caused by the change of implied volatilities.

is a hedging strategy for everything, but it would have to be shown that we are in such a world.

The next step is thus to provide a mechanism which proves the existence of a hedge against stock and volatility risk in an LVF framework. We choose a trinomial world. More generally, we simply try to answer the question of which of the SLVF worlds must be the real one, if the real asset price process is SLVF, and if the set of all exotics we wish to consider are all exotics that can be defined in this discrete world. It therefore leads naturally to an answer to the question whether all well-defined exotics can be hedged in the continuum limit, and whether this continuum limit is an SLVF(I) or SLVF(II) world if we only consider vanilla options in the hedge. What is important to remember, however, is that we will perform the model calibration to the vanilla European options market in the DLVF sense, i.e. we do not calibrate the local volatilities' variance and covariance rates. This is because we want to test the viability of the DLVF model in a world where the asset price process is SLVF, and where only vanilla options, the underlying, and funding instruments are considered as hedging instruments.

5. Hedging an Exotic Option in an SLVF world using a DLVF model

5.1. *Geometry of the tree*

A discrete local volatility tree with pre-determined state space needs to be at least trinomial in order for the discrete process to recombine [7] Since we assume that interest rates can be hedged separately, we work in a zero-interest-rate tree (the appropriate change of numeraire is thus implicitly assumed), which we can specify with the following state-space and transition probabilities

$$\begin{aligned} S_n^i &= S_0 e^{iu} \quad , \qquad -n \le i \le n \\ p\left(S_n^i \to S_{n+1}^{i-1}\right) &= \tfrac{\alpha_n^i}{1+e^{-u}} \\ p\left(S_n^i \to S_{n+1}^{i}\right) &= 1-\alpha_n^i \\ p\left(S_n^i \to S_{n+1}^{i+1}\right) &= \tfrac{\alpha_n^i}{1+e^{u}} \end{aligned} \tag{5.1}$$

where S_0 is the initial value of the stock price. At any step, the stock price change by a factor $e^{\pm u}$ or stay the same. S_n^i is the stock price at the n^{th} non-trivial time-slice after i net up-moves. The choice of notation for the transition probabilities is an arbitrary parameterisation such that the sum of the probabilities to leave a given node is one, and the stock price is a martingale, and the α's are strictly monotonically increasing functions of the local volatilities. The choice of the 'global'

[7] Implied binomial trees are possible, but the state space (i.e. the up and down steps and the time steps) depends on synthetic options which are very short-dated and forward starting a long time in the future. We prefer not to import the complications of the market-calibration into the construction of the state-space, so that volatility changes leave the state-space unchanged. Also, the trinomial approach is much simpler, as the Eqs. (5.1) leave exactly one free local parameter: the local volatility.

tree-parameter u puts an upper bound on the possible local volatilities in this world, because, in order to preserve positivity of the transition probabilities, we require $0 \leq \alpha \leq 1$. This does not put any fundamental restrictions on the conclusions we will draw from analysing this discrete model.[8] We make no assumptions about the process underlying the evolution of the local volatility parameters α_n^i. For example, restrictions on the drift of local volatilities have to be posed by the condition that, if local-volatility gadgets are attainable, the local volatilities and the stock price have to be jointly martingale in order to avoid arbitrage [12]. However, we do not need to place any restrictions on the evolution of the volatilities, except that they are bounded in the interval $[0, u]$.[9]

5.2. *Paths and vanilla options on the tree*

The set of all paths $\mathscr{P}$ has 3^n elements for a tree with n time-slices. Define a path as the n-element sequence of the space-slices the path visits

$$\pi = (\pi_1, \pi_2, \ldots, \pi_n) \quad , \quad (\pi_0 = 0, (\pi_i - \pi_{i-1}) \in \{-1, 0, 1\}) \ .$$

We will also use the alternative definition, the n element sequence of steps (1=up, -1=down)

$$\tilde{\pi} = \{\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_n\} \quad , \quad \tilde{\pi}_i \in \{-1, 0, 1\} \ .$$

The realised stock-path $\Pi(m)$ up to time m gives a filtration on the set of paths $\mathscr{P}$. We define $\mathscr{P}[\Pi(m)] \subset \mathscr{P}$ as the set of the paths whose first m steps match the realised path $\Pi(m)$, and the final stock-value of the realised path as $S_0 e^{\Pi_m(m)u}$. We also define the set of directed paths $\mathscr{D}[m, i, \tau, j] = \{\pi \in \mathscr{P} \ , \ \pi_m = i \ , \ \pi_\tau = j\}$, which means $\mathscr{D}[m, i, \tau, j]$ is the set of all paths which at time m give a stock price of $S_0 e^{iu}$ and at time τ a stock price of $S_0 e^{ju}$.

Let us define the *future cone of a node* as the part of the tree which can be reached from that particular node. The question is whether it is possible to hedge any path-dependent derivative when, at any time-step $m \in \{0, 1, \ldots, n-1\}$, the local volatilities α_k^i in the future cone of $(m, \Pi(m))$ are not known with certainty, and we only use European vanilla options as hedge instruments?

The restriction to use only European options is equivalent to using only Arrow-Debreu options. On the lattice-world, this is trivially true, whereas in the continuous world, some limiting procedure is necessary. The path-probabilities, as measured

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[8] From $\hat{E}\left[\frac{S_{t+1}}{S_t}\right] = 1$ and $\hat{E}\left[\left(\frac{S_{t+1}}{S_t} - 1\right)^2\right] = e^{\sigma^2 \Delta t} - 1$ we get $\alpha_\tau^i = e^u \frac{e^{(\sigma_\tau^i)^2 \Delta t} - 1}{(e^u - 1)^2}$, and the constraint $\alpha_\tau^i \leq 1$ means that all local volatilities have to satisfy $\left(\sigma_\tau^i\right)^2 \leq \frac{1}{\Delta t} \ln(2 \cosh u - 1)$.

[9] A necessary condition for this is that the volatilities of local volatilities are square-integrable over time and that the modulus of the drift are integrable over time, although these two conditions are not sufficient.

from time m, are

$$p\quad(\pi, m, \Pi(m)) = \mathbf{I}_{\pi\in\mathscr{P}[\Pi(m)]} \times$$
$$\prod_{j=m}^{n-1}\left(\mathbf{I}_{\tilde{\pi}_{j+1}=1}\frac{\alpha_j^{\pi_j}(m)}{1+e^{-u}} + \mathbf{I}_{\tilde{\pi}_{j+1}=0}\left(1-\alpha_j^{\pi_j}(m)\right) + \mathbf{I}_{\tilde{\pi}_{j+1}=-1}\frac{\alpha_j^{\pi_j}(m)}{1+e^{u}}\right)$$

where $\mathbf{I}_x$ is the indicator function which is equal to 1 if x is true and zero otherwise.

The Arrow Debreu price $A_\tau^i(m)$ is the price at time m of the derivative paying one unit if the stock-price at time τ is S_0e^{iu}, i.e. if the future realised path satisfies $\Pi(\tau) \in \mathscr{P}[\Pi(m)] \cap \mathscr{D}[0,0,\tau,j]$. Since $p(\pi, m, \Pi(m))$ is zero whenever $\pi \notin \mathscr{P}[\Pi(m)]$,

$$A_\tau^i(m, \Pi(m)) = \sum_{\pi\in\mathscr{D}[0,0,\tau,j]} p(\pi, m, \Pi(m)) \;,\;\; \tau > m.$$

Note that the continuum limit of a discrete Arrow-Debreu price is an ill-defined concept. Instead, a given continuous-world definition of an Arrow-Debreu price needs to be matched to a discretisation-dependent linear combination of discrete-world Arrow-Debreu prices. This will then converge properly to the discrete-world Arrow-Debreu price. This is normal procedure for path-integration problems, and is implicitly done for any tree-model.

5.3. *Path-dependent options*

The most general path dependent option f has a final pay-out of $X = X(\Pi(n))$. In a similar fashion as for the Arrow-Debreu options, we need to ensure that this definition provides a correct continuum limit for the path integral. Thus, for the option to approximate a continuum-world exotic, $X = X(\Pi(n))$ is a discretisation-dependent function. The question is whether such a path-dependent option can always be hedged with a strategy involving only the stock and Arrow-Debreu options, or whether such a hedge becomes valid at least in the continuum limit. Under risk-neutral valuation, the value of the path-dependent option at time m is

$$f(\Pi(m)) = \sum_{\pi\in\mathscr{P}} p(\pi, m, \Pi(m))\, X(\pi)\,.$$

To determine the hedge portfolio, we have to know how many independent securities there are available in this market. There are two constraints on the Arrow Debreu prices

$$\sum_{i=-\tau}^{\tau} A_\tau^i(m, \Pi(m)) = 1$$

$$\sum_{i=-\tau}^{\tau} A_\tau^i(m, \Pi(m))\, e^{iu} = e^{\Pi_m(m)u}\,, \tag{5.2}$$

valid for all $\tau, m\,;\; \tau > m$. The first condition derives from the fact that the stock-price at time τ will with certainty be on one of the nodes of that time-slice, and the second simply says that a portfolio of Arrow Debreu prices which pays exactly the stock-price at any node at time τ is equivalent to a portfolio holding only the stock. Henceforth, for notational simplicity, we shall simply write $\Pi(m)$ instead of $(m, \Pi(m))$, as the time can be inferred from the cardinality of the representation of $\Pi(m)$. At time m, the most general portfolio Λ short f can thus be represented as

$$\Lambda(\Pi(m)) = -f(\Pi(m)) + \omega(\Pi(m))S(\Pi(m)) + \sum_{t=m+1}^{n} \left(\sum_{i=\Pi_m(m)-(\tau-m)+2}^{\Pi_m(m)+(\tau-m)} w_\tau^i(\Pi(m)) A_\tau^i(\Pi(m)) \right) . \tag{5.3}$$

Note that the somewhat strange limits in the inner sum are there in order to avoid specifying weights for worthless and redundant Arrow Debreu options. Specifying the entire time-indexed functionals $\omega(\Pi(m))$ and $w_\tau^i(\Pi(m))$ for all m and $\Pi(m)$ then defines a trading strategy. If there exists a hedged trading strategy, it must be completely determined by the specification of these functional. Note that we assumed we are in a zero-interest-rate environment, so we have disregarded the bond/deposit/loan in our portfolio, and the constraint that the portfolio should be self-financing does not apply in the usual sense. Instead, all we require from a hedged strategy is

$$\Lambda(\Pi(m)) = \Lambda'(\Pi(m+1)) \ \ \forall \Pi(m+1) \in \mathscr{P}\left[\Pi(m)\right] ,$$

where Λ' is the portfolio one time-step ahead, but before re-hedging. $\Lambda'(\Pi(m+1))$ thus has the same portfolio weights as $\Lambda(\Pi(m))$. As time moves on one step, the following relevant market parameters will have changed:

- the stock price
- the local volatilities in the future cone of $\Pi_m(m)$.

Given that we assume that we know the volatility $\alpha_m^{\Pi_m(m)}$ with certainty (and given that, after Δ and Γ-hedging in the trinomial world we are in fact not exposed to it at all), we can work out how many constraints and free variables there are to work out the hedge at every step.

5.4. *Constructing the hedge; tractor options*

Any path-dependent option (and therefore *any* option) can be written as a linear combination of what we shall call tractor options. A tractor options $\mathscr{T}_\pi$ pays out one unit if and only if the stock-price up to time n has followed the path π. In the continuous world, we could call the payoff of any contingent claim (exotic or otherwise) a functional on the space of continuous functions $S(t)$. Each option valuation can be expressed as an infinite-dimensional integral over the set of continuous functions on the time to maturity $C([0,T])$, with the corresponding pay-off functional as the integral kernel, which leads to the path-integral representation of risk-neutral option valuation. It is possible that the analysis which follows can be consistently formulated in this continuous framework, but the path-integral can also be represented as a limiting case of path-integrals over functions on discrete image spaces as long as the mapping from the continuous-world derivative to the corresponding discrete-world linear combination of tractor options is done in a way that demonstrably converges to the continuous-world pay-off.

We are going this route of replacing the path-integral by a sum of paths in a discrete world, and then taking the continuum limit. The question of whether all exotic options can be hedged can now be re-phrased as the question of whether all tractor options can be hedged. Note that the value of a tractor option at time m is simply

$$\mathscr{T}_\pi\left(\Pi(m)\right) = p\left(\pi, \Pi(m)\right) \,,$$

so we see that we have in fact been using tractor options already in the discussion so far, and that – in the language of path-integration – the tractor option prices form the probability measure on the function space for the functions $S(t)$. So we see that $\mathscr{P}$ is the set of all states of the world, and the set of all tractors generates the largest possible σ-algebra on that space. $\mathscr{T} : \mathscr{P} \to [0,1]$ is our probability measure on that space, and $\left(\mathscr{P}, \sigma\left(\{\mathscr{T}_{\pi_1}\}, \{\mathscr{T}_{\pi_2}\}, \ldots, \{\mathscr{T}_{\pi_{3^n}}\}\right), \mathscr{T}\right)$ is the relevant probability space for valuing options in a risk-neutral valuation framework. It can also be seen that our above definition of an Arrow-Debreu price is itself a linear combination of tractor prices

$$A^i_\tau(\Pi(m)) = \sum_{\pi \in \mathscr{D}[0,0,\tau,j]} \mathscr{T}_\pi(\Pi(m)) \,, \ \tau > m \,,$$

which means that the Arrow Debreu prices are the probabilities associated with a particular partition of the state space. Note that this equation holds independently of the local volatilities. We may ask whether the reverse is true, i.e. whether we can write the tractor as a linear combination of Arrow-Debreu options. The answer is strictly no, as can be easily seen by the fact that there are many more tractors than Arrow-Debreu prices, and there are probably no more constraints on the tractor prices than we had for the Arrow-Debreu prices in Eq. (5.2). So there is no general static hedging strategy possible for all tractors. In set-theoretical notation, the σ-algebra generated by the sets $\mathscr{D}[0,0,\tau,j] \cap \mathscr{P}\left[\Pi(m)\right]$ (relevant to observing which Arrow-Debreu option pays off) is much smaller than the σ-algebra generated by the sets $\{\pi_i\}\,,\ \pi_i \in \mathscr{P}\left[\Pi(m)\right]$ (relevant to observing which tractor option pays off). Let us again consider the portfolio Eq. (5.3). How many free parameters are there to adjust the hedge with? It is easy to convince oneself that there will be $(n-m)^2$ independent Arrow-Debreu weights [10] and one stock-weight. The number of constraints depends on our hedging ambitions. We could attempt to hedge against the volatilities to any order, which we will demonstrate to be impossible for any tree of more than two steps. However, this is only necessary in a jump-diffusion

[10]Since $\sum_{\tau=m+1}^{n}\left(\sum_{i=-(\tau-m)+2}^{\tau-m} 1\right) = (n-m)^2$.

world. If we want to be hedged against volatility to second order, as we should be in an SLVF world, the constraints are

$$\Lambda'\left(\Pi(m)\otimes\{1\}\right)=\Lambda'\left(\Pi(m)\otimes\{0\}\right)=\Lambda'\left(\Pi(m)\otimes\{-1\}\right)$$

$$\frac{\partial\Lambda'\left(\Pi(m)\otimes\{1\}\right)}{\partial\alpha_i^j}=\frac{\partial\Lambda'\left(\Pi(m)\otimes\{-1\}\right)}{\partial\alpha_i^j}$$

$$\frac{\partial\Lambda'\left(\Pi(m)\otimes\{0\}\right)}{\partial\alpha_i^j}=0$$

$$\frac{\partial^2\Lambda'\left(\Pi(m)\otimes\{0\}\right)}{\partial\alpha_i^j\partial\alpha_k^l}=0$$

$$\forall i=m+1,\ldots,n-1\,,\;j=\Pi_m(m)-(i-m),\ldots,\Pi_m(m)+(i-m)$$

$$\forall k=m+1,\ldots,n-1\,,\;j=\Pi_m(m)-(k-m),\ldots,\Pi_m(m)+(k-m) \quad (5.4)$$

The first equation enforces Δ and Γ hedging, the second equation enforces cross-terms between volatility and stock, the third equation enforces 'local-vol-parameter' vega hedging, and the fourth enforces all the second derivatives with respect to the volatilities' parameterisation. Note that in the second line in Eq. (5.4) we do not require the first derivatives to be zero, as we do in the third line, as this would also enforce a hedge against the option price derivative $\frac{\partial^3 f}{\partial S^2\partial\alpha_i^j}$, which is more than we need. Because the derivative $\frac{\partial}{\partial S}$ has no unique discrete representation on the trinomial tree, the derivative $\frac{\partial^2 f}{\partial S\partial\alpha_i^j}$ also has no unique representation. However, we need not dwell too much on this technicality, as we will see that even our potentially too undemanding set of Eqs. (5.4) can not be satisfied in a three-step world and beyond.

If we want to be hedged against local volatility movements only to first order, the constraints are

$$\Lambda'\left(\Pi(m)\otimes\{1\}\right)=\Lambda'\left(\Pi(m)\otimes\{0\}\right)=\Lambda'\left(\Pi(m)\otimes\{-1\}\right)$$

$$\frac{\partial\Lambda'\left(\Pi(m)\otimes\{0\}\right)}{\partial\alpha_i^j}=0$$

$$\forall i=m+1,\ldots,n-1\,,\;j=\Pi_m(m)-(i-m),\ldots,\Pi_m(m)+(i-m) \quad (5.5)$$

We will show that the constraints Eqs. (5.5) can always be satisfied. The algebra of the problem is very involved, and from the four-step or even three-step world upwards the equations are far too many to analyse all possible tractor hedges without the help of an algebraic calculations program. An example Mathematica$^{\text{TM}}$ notebook can be requested from the author.

5.5. ***Example: A two-step world***

As a first example, we consider a two-step tree where we are short K tractor options $f_{\{1,-1\}}$. Later, we shall also consider the most general exotic option in this two-step trinomial world. The zero-time hedged portfolio is

$$\Lambda(\emptyset) = -f(\emptyset) + \omega(\emptyset)S_0 + w_1^1(\emptyset)A_1^1(\emptyset)$$
$$+w_2^2(\emptyset)A_2^2(\emptyset) + w_2^1(\emptyset)A_2^1(\emptyset) + w_2^0(\emptyset)A_2^0(\emptyset)$$

After one step, before re-hedging, we have

$$\Lambda(\Pi(1)) = -f(\Pi(1)) + \omega(\emptyset)S_1^{\Pi_1(1)} + w_1^1(\emptyset)A_1^1(\Pi(1))$$
$$+ w_2^2(\emptyset)A_2^2(\Pi(1)) + w_2^1(\emptyset)A_2^1(\Pi(1)) + w_2^0(\emptyset)A_2^0(\Pi(1))$$

In full

$$\begin{aligned}
\Lambda(\{1\}) &= -K\frac{\alpha_1^1(1)}{1+e^{-u}} + \omega(\emptyset)S_0e^u + w_1^1(\emptyset) \\
&\quad +w_2^2(\emptyset)\frac{\alpha_1^1(1)}{1+e^u} + w_2^1(\emptyset)(1-\alpha_1^1(1)) + w_2^0(\emptyset)\frac{\alpha_1^1(1)}{1+e^{-u}} \\
\Lambda(\{0\}) &= \omega(\emptyset)S_0 + w_2^1(\emptyset)\frac{\alpha_1^0(1)}{1+e^u} + w_2^0(\emptyset)(1-\alpha_1^0(1)) \\
\Lambda(\{-1\}) &= \omega(\emptyset)S_0e^{-u} + w_2^0(\emptyset)\frac{\alpha_1^{-1}(1)}{1+e^u}
\end{aligned}$$

If we attempt to hedge to second order, we solve in all the Eqs. (5.4). Each of the portfolios depends only on one of the future local volatilities, so to first order all the first derivatives with respect to the local volatilities have to be zero, which leads to

$$w_2^1(\emptyset) = w_2^0(\emptyset) = 0 \quad , \quad w_2^2(\emptyset) = Ke^u$$

After that, the condition $\Lambda'(\{0\}) = \Lambda'(\{-1\})$ leads to $\omega(\emptyset) = 0$, and $\Lambda'(\{0\}) = \Lambda'(\{1\})$ to $w_1^1 = 0$. So the only hedge we put on initially is e^uK units of A_2^2. We also see that this makes the portfolio worthless initially. If the first step of the stock-price path is flat or down, the portfolio is patently worth-less. If it is up, the portfolio has also value zero, regardless of the actual value of the remaining local volatility in the future cone, α_1^1. The hedge portfolio for this step is then trivially calculated by $\Lambda'(\{1,1\}) = \Lambda'(\{1,0\}) = \Lambda'(\{1,-1\})$. The solution is $\omega(\{1\}) = -\frac{K}{S_0(e^u-1)}$, $w_2^2(\{1\}) = e^uK$. So the entire hedging strategy was found easily. Note that the hedge constraints were all linear in the local volatilities, because the constraints are formulated at the last time-slice. Any exotic option will have collapsed to an Arrow Debreu option by that time. Thus, the σ-algebra generated by the sets $\mathscr{D}[0,0,1,j] \cap \mathscr{P}[\Pi(m)]$ and the σ-algebra generated by the paths still relevant after the first time-step (i.e. when the hedging conditions need to be matched) are identical. Thus we expect the two-step world to provide a hedging strategy for all tractors. To demonstrate this, we can work out the whole hedged trading strategy. Let us write the general two-step exotic as a linear combination

$$f = \sum_{i,j=-1}^{1} K_{\{i,j\}} \mathscr{T}_{\{i,j\}}.$$

Then the part of the hedging strategy relevant to the first time-step is

$$\begin{aligned}
\omega(\emptyset) &= \frac{K_{\{0,0\}} - K_{\{-1,1\}} + e^u K_{\{-1,0\}}}{(1-e^{-u})S_0} + \frac{e^{2u} K_{\{-1,-1\}}}{(1-e^u)S_0} \\
w_1^1(\emptyset) &= K_{\{1,0\}} - K_{\{0,1\}} - e^u K_{\{0,-1\}} + e^u K_{\{-1,0\}} \\
w_2^0(\emptyset) &= K_{\{-1,1\}} - (1+e^u)K_{\{-1,0\}} + e^u K_{\{-1,-1\}} \\
w_2^1(\emptyset) &= K_{\{0,1\}} - (1+e^u)K_{\{0,0\}} + e^u K_{\{0,-1\}} + (1+e^u)w_2^0 \\
w_2^2(\emptyset) &= K_{\{1,1\}} - (1+e^u)K_{\{1,0\}} + e^u K_{\{1,-1\}} + (1+e^u)w_2^1 - e^u w_2^0
\end{aligned}$$

It is easy to check that with these parameters, the value of the portfolio is indeed independent of all the future volatilities. In all the cases the final hedge is trivial, and we will not demonstrate how to compute it. The reason that the hedge is trivial on the last time-step is that all the tractor options will have collapsed to Arrow-Debreu options. For this reason the problem of finding an exactly replicating portfolio in the two-step world is solvable. Note that a three-step world, however, collapses to a two-step world after the first step, and one third of all the tractor options become worthless at that time. The remaining ones are still two-step tractors. We will demonstrate by the example of a particular tractor option that the general three-step exotic can not be replicated to second order in the stock-price and local volatilities with a dynamic trading strategy of vanilla products.

5.6. *Hedging Three-Step Tractors*

Imagine we want to hedge K times the tractor $\mathscr{T}_{\{0,0,0\}}$ in a three-step world, i.e. the exotic which pays off K units only if the stock never moves at all. For convenience, we will generally omit the arguments of w, and α. Then the portfolios after one step are

$$\begin{aligned}
\Lambda'(\{1\}) &= w_1^1 + \frac{\alpha_1^1}{1+e^{-u}} w_2^0 + (1-\alpha_1^1)w_2^1 + \frac{\alpha_1^1}{1+e^u} w_2^2 + \frac{\alpha_1^1\alpha_2^0}{(1+e^{-u})^2} w_3^{-1} \\
&\quad + \frac{\alpha_1^1(1-\alpha_2^0) + (1-\alpha_1^1)\alpha_2^1}{1+e^{-u}} w_3^0 \\
&\quad + \left(\frac{\alpha_1^1(\alpha_2^2+\alpha_2^0)}{(1+e^u)(1+e^{-u})} + (1-\alpha_1^1)(1-\alpha_2^1) \right) w_3^1 \\
&\quad + \frac{\alpha_1^1(1-\alpha_2^2) + (1-\alpha_1^1)\alpha_2^1}{1+e^u} w_3^2 + \frac{\alpha_1^1\alpha_2^2}{(1+e^u)^2} w_3^3 + e^u \omega S_0
\end{aligned}$$

$$\begin{aligned}
\Lambda'(\{0\}) &= (1-\alpha_1^0)w_2^0 + \frac{\alpha_1^0}{1+e^u}w_2^1 + \frac{\alpha_1^0(1-\alpha_2^{-1}) + (1-\alpha_1^0)\alpha_2^0}{1+e^{-u}}w_3^{-1} \\
&\quad + \left(\frac{\alpha_1^0(\alpha_2^1+\alpha_2^{-1})}{(1+e^u)(1+e^{-u})} + (1-\alpha_1^0)(1-\alpha_2^0)\right)w_3^0 \\
&\quad + \frac{\alpha_1^0(1-\alpha_2^1) + (1-\alpha_1^0)\alpha_2^0}{1+e^u}w_3^1 + \frac{\alpha_1^0\alpha_2^1}{(1+e^u)^2}w_3^2 \\
&\quad + \omega S_0 - K(1-\alpha_1^0)(1-\alpha_2^0)
\end{aligned}$$

$$\begin{aligned}
\Lambda'(\{-1\}) &= \frac{\alpha_1^{-1}}{1+e^u}w_2^0 + \left(\frac{\alpha_1^{-1}(\alpha_2^0+\alpha_2^{-2})}{(1+e^u)(1+e^{-u})} + (1-\alpha_1^{-1})(1-\alpha_2^{-1})\right)w_3^{-1} \\
&\quad + \frac{\alpha_1^{-1}(1-\alpha_2^0) + (1-\alpha_1^{-1})\alpha_2^{-1}}{1+e^u}w_3^0 + \frac{\alpha_1^{-1}\alpha_2^0}{(1+e^u)^2}w_3^1 + e^{-u}\omega S_0
\end{aligned}$$

Let us try to find hedge ratios so as to satisfy the Eqs. (5.4). We will not illustrate all the tedious algebra involved, but we present one possible way to proceed, giving only some intermediate steps in the solution. First consider $\frac{\partial\Lambda'(\{0\})}{\partial\alpha_2^1} = 0$, which leads to $w_3^2 = -e^u w_3^0 + (1+e^u)w_3^1$. Next, $\frac{\partial\Lambda'(\{0\})}{\partial\alpha_2^0} = 0$, gives $w_3^1 = -(1+e^u)K - e^u w_3^{-1} + (1+e^u)w_3^0$. $\frac{\partial\Lambda'(\{0\})}{\partial\alpha_2^{-1}} = 0$ leads to $w_3^0 = (1+e^u)w_3^{-1}$. Next, $\frac{\partial\Lambda'(\{1\})}{\partial\alpha_2^{-2}} = \frac{\partial\Lambda'(\{-1\})}{\partial\alpha_2^{-2}}$ yields $w_3^{-1} = 0$. Demanding that $\frac{\partial\Lambda'(\{1\})}{\partial\alpha_2^2} = \frac{\partial\Lambda'(\{-1\})}{\partial\alpha_2^2}$ then gives $w_3^3 = -(1+e^u+e^{2u})(1+e^u)K$. The simplified solutions so far are thus $w_3^0 = w_3^{-1} = 0$, $w_3^1 = -(1+e^u)K$, $w_3^2 = -(1+e^u)^2K$, and $w_3^3 = -(1+e^u+e^{2u})(1+e^u)$. This settles the hedge ratios for the longest-living Arrow-Debreu prices. Next, we satisfy $\frac{\partial\Lambda'(\{1\})}{\partial\alpha_1^1} = \frac{\partial\Lambda'(\{-1\})}{\partial\alpha_1^1}$ by setting $w_2^2 = -e^u w_2^0 + (1+e^u)w_2^1 - e^u\alpha_2^0 K$. The fact that the hedge ratio is dependent on a local volatility is not really a problem. We need to be careful, however, not to treat any α's which appear in a hedge ratio as variable when calculating other hedge-ratios because the implicit assumption is always that, while the stock price or the volatility are allowed to change instantaneously, the hedge ratios must stay constant until after the information on the new levels of stock price and volatilities has become known. With this in mind, the condition $\frac{\partial\Lambda'(\{1\})}{\partial\alpha_1^{-1}} = \frac{\partial\Lambda'(\{-1\})}{\partial\alpha_1^{-1}}$ leads to $w_2^0 = K\alpha_2^0$. Now, without demanding to fulfil the remaining Eqs. (5.4), we are already lead to an inevitable violation of one part of Eqs. (5.4), namely that $\frac{\partial\Lambda'(\{1\})}{\partial\alpha_2^0} - \frac{\partial\Lambda'(\{-1\})}{\partial\alpha_2^0} = \frac{K}{1+e^u}(\alpha_1^{-1} - e^u\alpha_1^1)$. There is no further hedge ratio that we can adjust to remedy this problem ![11]

Now we can of course do the same analysis for all the 27 tractor options in this small trinomial world, but we do this with Mathematica™. It turns out that in general, all but one of the second-order conditions of the Eqs. (5.4) can be satisfied and hedging to first order is possible.

[11] In fact, we can just stubbornly proceed to solve all the other conditions in Eqs. (5.4), namely $\frac{\partial\Lambda'(\{1\})}{\partial\alpha_1^0} = 0$ and $\Lambda'(\{1\}) = \Lambda'(\{0\}) = \Lambda'(\{-1\})$, which give the hedge ratios $w_2^1 = (1+e^u)\alpha_2^0 K$, $w_2^2 = (1+e^u)^2\alpha_2^0 K$, $w_1^1 = 0$, and $\omega = \frac{1-\alpha_2^0}{1-e^{-u}}K$, but this will not change the nature of the problem.

5.7. *Analysis of Larger Trinomial Worlds*

When analysing larger trinomial worlds, we quickly run into constraints in terms of processing power. Our Mathematica$^{\mathrm{TM}}$ implementation frequently runs out of memory and needs copious manual intervention when analysing a five-step trinomial world. Nevertheless, some valuable observations can be made even with our calculations limited to a five-step and smaller trinomial worlds.

The number of violated constraints in Eqs. (5.4) increases very quickly in going from a three-step to a four-step and five-step world. Obviously, the constraints arising in an n-step world are a subset of the constraints in any larger trinomial worlds, so in general we can not hope to find an exact second order hedge in any larger trinomial world. The question is, however, whether these terms matter, or whether their number becomes less and less significant when compared to the relevant terms.

Let us define the number of un-satisfiable constraints in Eqs. (5.4) for a particular tractor $\mathscr{T}_\pi$ as ν_π. Then we can for example measure the number

$$N_n = \sum_{\pi \in \mathscr{P}_n} \nu_\pi,$$

where we have given the set $\mathscr{P}$ a subscript n to indicate how many steps the trinomial world has. We find

$$\{N_1, N_2, \ldots\} = \{0, 0, 21, 210, 1344, \ldots\}$$

Note that on a 300 MHz/192 MB personal computer, the last number took about 12 hours to compute in Mathematica$^{\mathrm{TM}}$. The estimated time for the next element in the series is 160 hours, needing constant human intervention. So we have to make our conclusion from this limited sequence. In fact this sequence already indicates what kind of severity of mishedge we might expect in the continuum limit: In an n-step tree, there are $n^2 - 1$ local volatilities in the future cone of the origin, which means there are n^2 stochastic parameters to consider. This gives $\frac{n^2(n^2+1)}{2}$ P&L terms in the continuous trading equation arising from the quadratic variations of the stochastic variables alone. Assuming that the total vega-risk of vanilla options should not be affected by the discretisation, the relative importance of each particular quadratic term that can not be hedged is thus weighed down by a factor $1/(n^2(n^2+1))$ on average, when we compare different trinomial worlds. There are 3^n tractors in each n-step trinomial world, so the relative importance of each tractor, when defining an exotic option, will be weighed down by a factor 3^{-n}. As a rule of thumb, the relevance of all unsatisfied hedge constraints stays constant if, for large i, $N_i \propto 3^i i^2(i^2+1)$. Starting with $N_3 = 21$, the corresponding estimate of N_4 would be $3N_3 \frac{4^2(4^2+1)}{3^2(3^2+1)} = 190.4$, which is somewhat below the actual N_4, while the estimate of N_5 would be $9N_3 \frac{5^2(5^2+1)}{3^2(3^2+1)} = 1362$, which slightly larger than the actual N_5. With the limited sequence we were able to compute, it thus seems very plausible that the relative importance of the un-hedgeable terms does not disappear in the continuum limit.

Lastly, we note that it is of course possible to construct 'exotics', i.e. linear combination of tractors, which do not exhibit any quadratic or linear volatility exposure after an optimal hedge is put in place. A trivial example are the European vanilla options themselves; another, less trivial, example are the tractors $\mathscr{T}_{\{1,1,1,\ldots,1,1,i\}}$ $i \in \{-1,0,1\}$ and $\mathscr{T}_{\{-1,-1,-1,\ldots,-1,-1,i\}}$ $i \in \{-1,0,1\}$, and linear combinations thereof. Also, arbitrary tractors may often have more complicated linear combinations which are insensitive to local volatilities to second order. In the set of all exotic options, however, these will form a set of measure zero: The linear vector space spanned by the set of all tractor options is the space of all contingent claims on the underlying asset. The space spanned by all independent linear combinations which have no volatility exposure (to second order) after hedging with vanilla options is a subspace of that set. If any tractor option at all has un-hedgeable local-vega risk, this subspace is a true subspace, and thus forms a set of measure zero in the space of all contingent claims. Thus we have shown that – out of all conceivable contingent claims – almost all are unhedgeable with a trading strategy using vanilla options only. The severity of the problem is not mitigated in the continuum limit.

6. Discussion and Conclusions

We have demonstrated that, in an arbitrage-free market, pricing exotic options with a DLVF model (implied tree) is inconsistent with the assumption of unknown future volatilities, inasmuch as the LVF implied in the European vanilla options market does not fully compensate for irreplicable volatility risks exhibited by exotics if vanilla options are the only instruments used for hedging volatility exposure. Intuitively, we can interpret this result in the following manner: Local volatility gadgets exist, but they do not rid us of second-order local volatility exposure, which generates steady additional (positive or negative) P&L over time. This P&L is not priced in .[12]

By contrast, a market where the volatility process has only linear variation does not give us the second-order P&L terms, but is not arbitrage-free. A market where the volatility has no variation at all gives no vega-P&L problems at all, and is arbitrage free. It is under the assumption of this market that the DLVF model is constructed, which is theoretically self-consistent, but not consistent with reality [10, 11].

We stress that the lack of a second-order vega-hedge is not model-specific: even a stochastic volatility model would create that, as long as the volatilities which we try to imply are not substantially fewer than the number of vanilla option prices one calibrates the model to. However, in stochastic volatility models and jump-diffusion models the onus of pricing in volatility risk is not on the implied volatilities alone. Instead, the price of such risks can be calibrated to the market and particular model parameters exist to absorb this calibration explicitly. Low-parametric SLVF models

[12] Furthermore, the implied tree generally gives no description of how the best possible hedge can be attained even within these limitations. This, however, is a purely practical issue.

do therefore not require the volatility risk to be *effectively* integrated out and – in some mysterious way – absorbed into the DLVF calibration.

We have only considered LVF models. It should be noted, however, that what we pointed out here is a general concern which needs to be addressed for any asset-price model one is willing to consider: The model must price exotics consistently with the market parameters that one is calibrating the model to. As we have seen here, it is not always easy to convince oneself whether or not this demand is actually met by a proposed model. All we have discovered in this paper is that it is not possible to satisfy this condition when using an LVF model calibrated to vanilla option prices if (a) we only use the asset and vanilla options as hedge instruments, (b) volatilities are not completely deterministic, (c) the market is arbitrage-free, and (d) the number of local volatilities is of the order of the number of vanilla options available for model calibration. As this is a central conclusion, we sum it up as another

Theorem 3 *Vanilla options do not complete the market in the most general SLVF model.*

However, it is possible that no single model price process satisfies this demand if we use one and the same calibration recipe regardless of what contingent claim we want to price. Thus, we must find a proof for a different model that it does price exotics in line with the vanilla options market. Possible resolutions of this dilemma might be

- A low-parametric LVF surface calibration which includes volatility risk in what must essentially be an SLVF model.
- A different calibration using quoted exotic options as well as vanilla European options. In this case we must also allow the use of these exotics as hedge instruments.
- A jump-diffusion model, or a combination of jump-diffusion with SLVF or with DLVF as in ref. [16]
- A model which makes no self-consistent assumptions about the underlying process at all, but proves in other ways that it prices (maybe only specific) exotics 'in line with' the market. This amounts to finding a model-independent hedging strategy .[13]

There is certainly no guarantee that any of the first three resolutions would work for all possible exotics, and the exit route proposed in the last item is likely to involve substantially different derivations for each different class of exotic options. Also, there is no guarantee that a model-independent hedging-solution exists for any particular exotic option.

Faced with the task of pricing a particular exotic option while not having a bullet-proof hedging model, it must be of paramount concern to know the un-priced

[13] Since one makes no assumptions about the underlying process, this model can not be general enough to allow us to price all conceivable exotics. As an example of the concept just note that it is trivial to price European options with arbitrary payoffs from vanilla European options in a model-independent way.

risks, whatever the model. The DLVF model gives us no information whatsoever on this. However, ideas such as the 'uncertain volatility models' of Avellaneda, Levy, and Parás, and of Lyons have emerged in an attempt to avoid the perils of trusting a single underlying-process to price all exotics correctly [17, 18, 19]. The main premise of their approach is that the exotic pricing should be performed as model-independently as possible, by using optimum amounts of standard traded options to statically hedge out as much of the payoff as possible; thereafter the residual is delta-hedged in order to obtain the tightest possible worst/best price spread for the option within the assumption of a 'certainty interval' or confidence interval for future volatilities [17, 18, 19, 20].

In summary, we should never trust a model-process of the underlying to price a particular exotic correctly, unless we can prove this to be the case .[14] The way in which the DLVF model is often trusted to even price *all conceivable* exotic options correctly – while we have shown that the set of correctly priced options is infinitely smaller (of measure zero in the space of all exotics) – should serve a warning. The effective theory works for almost none of the conceivable derivatives.

As a by-product of the proof, we have also found that any tradable asset, if it is stochastic, must have at least quadratic variation – and will therefore produce a Γ-like P&L on instruments with non-linear exposure to that asset – if the market is arbitrage-free. This result complements the observations of the HJM model and of Derman and Kani that the quadratic variation of their particular diffusive models determines the drift completely if the condition of no arbitrage is imposed [14, 12]. Even in the context of jump-diffusion models as for example in [16], the drift is required to be deterministic for the same reason. Since the drift cannot be stochastic in the absence of arbitrage, we must either have at least quadratic variation or full determinism.

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[14] The best way to prove that the model works is by analysing the hedging strategy that the model would suggest. While we saw in this paper that it is difficult to extract a hedging strategy from an DLVF model, other models may lend themselves more naturally to an interpretation in terms of the proposed hedging method.

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