

Abel ODEs: Equivalence and Integrable Classes

E.S. Cheb-Terrab^{a,b,c} and A.D. Roche^d

^a*Department of Mathematics
University of British Columbia, Vancouver, Canada.*

^b*CECM, Department of Mathematics and Statistics
Simon Fraser University, Vancouver, Canada.*

^c*Department of Theoretical Physics
State University of Rio de Janeiro, Brazil.*

^d*Symbolic Computation Group
Computer Science Department, Faculty of Mathematics
University of Waterloo, Ontario, Canada.*

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Abstract

A classification, according to invariant theory, of non-constant invariant Abel ODEs known as solvable and found in the literature is presented. A set of new integrable classes depending on one or no parameters, derived from the analysis of the works by Abel, Liouville and Appell [2, 3, 4], is also shown. Computer algebra routines were developed to solve ODEs members of these classes by solving their related equivalence problem. The resulting library permits a systematic solving of Abel type ODEs in the Maple symbolic computing environment.

PROGRAM SUMMARY

Title of the software package: Extension to the Maple ODEtools package

Catalogue number: (supplied by Elsevier)

Software obtainable from: CPC Program Library, Queen's University of Belfast, N. Ireland (see application form in this issue)

Licensing provisions: none

Operating systems under which the program has been tested: UNIX, Macintosh, Windows (95/98/NT).

Programming language used: Maple V Release 5

Memory required to execute with typical data: 16 Megabytes.

Keywords: Abel type first order ordinary differential equations (ODEs), equivalence problem, integrable cases, symbolic computation.

Nature of mathematical problem

Analytical solving of Abel type first order ODEs having *non-constant* invariant.

Methods of solution

Solving the equivalence problem between a given ODE and representatives of a set of non-constant invariant Abel ODE classes for which solutions are available.

Restrictions concerning the complexity of the problem

The computational routines presented work when the input ODE belongs to one of the Abel classes considered in this work. This set of Abel classes can be extended, but there are classes - depending intrinsically on many parameters - for which the solution of the equivalence problem, as presented here, may lead to large and therefore untractable expressions. When the invariants of a given Abel ODE depend on analytic functions, the success of the routines depends on Maple's ability to normalize these invariants and recognize zeros (this is well implemented in Maple, but it may nevertheless not work as expected in some cases). Also, when the solution for the class parameter depends on other algebraic symbols entering the ODE being solved, the routines can determine this dependency only when it has rational form.

Typical running time

The methods being presented here have been implemented in the framework of the *ODEtools* Maple package. On the average, over Kamke's [1] first order Abel examples (see sec. 6), the ODE-solver of *ODEtools* is now spending ≈ 6 sec. per ODE when *successful*, and ≈ 11 sec. when *unsuccessful*. The timings in this paper were obtained using Maple R5 on a Pentium II 400 - 128 Mb. of RAM - running Windows98.

Unusual features of the program

These computational routines are able - in principle - to integrate the infinitely many members of all the *non-constant* invariant Abel ODE classes considered in this work. Concretely, when a given Abel ODE belongs to one of these classes, the routines can determine this fact, by solving the related equivalence problem, and then use that information to return a closed form solution without requiring further participation from the user. The ODE families that are covered include, as particular cases, *all* the Abel solvable cases presented in Kamke's and Murphy's books, as well as the Abel ODEs member of other classes not previously presented in the literature to the best of our knowledge. After incorporating the new routines, the ODE solver of the *ODEtools* package succeeds in solving 97 % of Kamke's first order examples.

LONG WRITE-UP

1 Introduction

From some point of view, after Riccati type ODEs, the simplest first order ordinary differential equations (ODEs) are those having as right hand side (RHS) a third degree polynomial in the dependent variable, also called *Abel type* ODEs¹

$$y' = f_3 y^3 + f_2 y^2 + f_1 y + f_0 \quad (1)$$

where $y \equiv y(x)$, and f_0, f_1, f_2 and f_3 are analytic functions of x . As opposed to Riccati ODEs, for which integration strategies can be built around their equivalence to second order linear ODEs, Abel ODEs admit just a few available integration strategies, most of them based on the pioneering works by Abel, Liouville and Appell around 100 years ago [2, 3, 4]. In those works it was shown that Abel ODEs can be organized into equivalence classes. Two Abel ODEs are defined to be equivalent if one can be obtained from the other through the transformation

$$\{x = F(t), \quad y(x) = P(t)u(t) + Q(t)\} \quad (2)$$

where t and $u(t)$ are respectively the new independent and dependent variables, and F, P and Q are arbitrary functions of t satisfying $F' \neq 0$ and $P \neq 0$.

Integration strategies were then discussed in [3, 4], around objects *invariant* under Eq.(2)² (herein called the *invariants*) which can be built with the coefficients $\{f_3, f_2, f_1, f_0\}$ and their derivatives. To each class there corresponds a different set of values of these invariants, and actually any one of them (we shall pick one and call it the *invariant*) is enough to characterize a class. A simple integrable case happens when the invariant *is constant*³; the solution to the ODE then follows straightforwardly in terms of quadratures, as explained in textbooks [1, 5]. On the contrary, when the invariant is not constant, just a few integrable cases are known and the formulation of solving strategies based on the equivalence between two such Abel ODEs, one of which is integrable, appears to be only partially explored in the literature, and not explored in general in computer algebra systems.

Having this in mind, this paper concerns Abel ODEs with non-constant invariant and presents:

1. A discussion and classification of the integrable Abel ODEs found both in Kamke's book and in the works from the late 19th and early 20th century by Abel, Appell, Liouville and other sources;
2. A set of new integrable Abel ODE classes - some depending on arbitrary parameters - derived from those aforementioned works;
3. An explicit method of verifying or denying the equivalence between two given Abel ODEs, one of which we know how to solve since it represents one of the above mentioned classes; and in the positive case, a way to determine the function parameters F, P and Q of the transformation Eq.(2) which maps one into the other;

¹For convenience, in this work, by "Abel ODEs" we mean Abel ODEs of first kind, since Abel ODEs of second kind can always be transformed into first kind by a simple change of variables.

²The invariants change in form for $F(t) \neq t$, but keep their value. See Eq.(5).

³There exists one invariant such that if it is constant then the other invariants are as well.

4. A computational scheme to resolve the equivalence problem in the case of *parameterized* classes, including the determination of the value for the class parameter when the equivalence exists;
5. A set of computer algebra (Maple) routines implementing the algorithms presented in items (3) and (4) above, to systematically solve - in principle - any Abel ODE belonging to one of the classes, parameterized or not, presented here and for which a closed form solution is known (items (1) and (2) above).

Item (1) is interesting since the Abel ODEs shown in textbooks in general, including Kamke's book, are displayed without further classification, and in fact many of them belong to the same class. This classification in terms of invariant theory is also necessary in a computational scheme for solving Abel ODEs as the one being presented, and we have not found it in other references. The integrable classes mentioned in (2) are new to the best of our knowledge, although directly or indirectly derived from previous works. The formulation of the equivalence problem mentioned in (3) is the one given by Liouville in [3], is systematic and does not involve solving any auxiliary differential equations⁴. Concerning item (4), the idea can be viewed as a way of avoiding the untractable expressions which one would encounter when making direct use of Liouville's strategy with *parameterized* classes. The strategy presented is applicable when there exists a solution for some numerical values of the parameter, or when this parameter is a rational function of other symbols entering the input ODE. Regarding item (5), the implementation presented here is, as far as we know, unique in computer algebra systems in its ability to solve non-constant invariant, parameterized or not, Abel ODE classes.

The paper is organized as follows. In sec. 2, the basic definitions and the classic formulation of the equivalence problem in terms of invariants is reviewed and shown to apply straightforwardly to the case of a non-parameterized class. In sec. 3 it is shown how these ideas can be complemented by taking advantage of computers to tackle the equivalence problem in the case of a parameterized class. Section 4 presents a classification of the integrable classes we have found in the literature with some additional comments as to their derivation. In sec. 5 new integrable Abel classes are presented. In sec. 6, a test-suite for the routines presented is discussed and statistics are shown describing the performance with this test-suite as well as with Kamke's first order examples. Finally, the conclusions contain some general remarks about this work and its possible extensions.

Additionally, we present in the Appendix a table listing the distinct Abel ODE classes that we have found, representative ODEs from each class, and their respective solutions.

2 Classical Theory for Abel ODEs

In general, Abel type ODEs can be studied using two related concepts: *invariants* and ODE *equivalence classes*. We define two Abel ODEs to be equivalent⁵ if one can be obtained from the other using a transformation of the form Eq.(2). The equivalence class containing a given ODE is then the set of all the ODEs equivalent to the given one. We note that although the infinitely many members of a class can be mapped into each other by using Eq.(2), there are also infinitely many disjoint Abel classes (Eq.(2) is not sufficient to map *any* Abel ODE into a given one).

To each class one can associate an infinite sequence of absolute invariants [3, 4]. To see this, consider two Abel ODEs, the first Eq.(1), the second obtained from Eq.(1) through the transformation

⁴An approach somewhat similar to this one by Liouville is discussed in [6].

⁵For a more formal definition of *class* see [7]

Eq.(2)

$$u' = \tilde{f}_3 u^3 + \tilde{f}_2 u^2 + \tilde{f}_1 u + \tilde{f}_0 \quad (3)$$

where the coefficients $\{\tilde{f}_0, \tilde{f}_1, \tilde{f}_2, \tilde{f}_3\}$, are related to the those of Eq.(1) by

$$\begin{aligned} \tilde{f}_0 &= \frac{F' (f_0(F) + f_1(F) Q + f_2(F) Q^2 + f_3(F) Q^3) - Q'}{P} \\ \tilde{f}_1 &= \frac{P'}{P} - F' (f_1(F) + 2 f_2(F) Q + 3 f_3(F) Q^2) \\ \tilde{f}_2 &= P F' (f_2(F) + 3 f_3(F) Q) \\ \tilde{f}_3 &= P^2 F' f_3(F) \end{aligned} \quad (4)$$

Following [4], we call an *absolute invariant* of Eq.(1) a function $I(f, x)$ of the coefficients $\{f_0, f_1, f_2, f_3\}$ and their derivatives with respect to x such that, for all $\{F, P, Q\}$ in Eq.(2),

$$I(\tilde{f}, t)|_{\tilde{f}=\tilde{f}(f,t)} = I(f, x)|_{x=F(t)} \quad (5)$$

where $\tilde{f} = \tilde{f}(f, t)$ represents the coefficients $\{\tilde{f}_0, \tilde{f}_1, \tilde{f}_2, \tilde{f}_3\}$ and their derivatives with respect to t , expressed using Eq.(4).

Similarly, we call a *relative invariant* a function, say s , of the coefficients of Eq.(1) and their derivatives such that when changing variables using Eq.(2), the resulting expression is equal to the original one up to a factor, say φ_s , dependent uniquely on the functions F, P , and Q in Eq.(2) and independent of the coefficients themselves [7]:

$$s(\tilde{f})|_{\tilde{f}=\tilde{f}(f,t)} = \varphi_s(F, P, Q) s(f)|_{x=F(t)} \quad (6)$$

Liouville showed that in the case of Abel equations there is a relative invariant of weight 3

$$s_3 \equiv f_0 f_3^2 + \frac{1}{3} \left(\frac{2 f_2^3}{9} - f_1 f_2 f_3 + f_3 f_2' - f_2 f_3' \right) \quad (7)$$

which can be used to recursively generate an infinite sequence of relative invariants s_{2m+1} of weights $2m + 1$ respectively⁶, through the formula

$$s_{2m+1} \equiv f_3 s'_{2m-1} - (2m - 1) s_{2m-1} \left(f_3' + f_1 f_3 - \frac{f_2^2}{3} \right) \quad (8)$$

As is clear from this definition, the product of two relative invariants respectively of weights n and m is a relative invariant of weight $n + m$, and by dividing any two relative invariants of equal weight one can generate an infinite sequence of absolute invariants

$$I_1 = \frac{s_3^3}{s_3^3}, I_2 = \frac{s_7 s_3}{s_5^2}, I_3 = \frac{s_9}{s_3^3}, \text{ etc...} \quad (9)$$

⁶In the case of s_3 , $\varphi_{s_3} = (F'P)^3$; the weight n refers to the degree of φ_{s_n} with respect to $(F'P)$.

In [4], Appell showed that this sequence can also be obtained from two basic absolute invariants - say J_1, J_0 , by expressing J_1 as a function of J_0 and then differentiating the result with respect to J_0 . As a consequence, if I_1 is constant then all the other ones will be too. This fact allows one to identify the constant character of the invariants in Eq.(9) by looking at just the first one. We note also that there are infinitely many *different* classes having I_1 constant, related to the infinitely many possible constant values I_1 can have.

2.1 Integration strategy

A description of a method of integration when the invariants are constant⁷ is found in the works by Abel [2], Liouville [3] and Appell [4]. In such a case, *all* members of the class can be systematically mapped into a *separable* first order ODE (representative of the class), by appropriately choosing F, P and Q in Eq.(2); see for instance [1] and [5].

A quite different situation happens when I_1 is not constant. In such a case, relatively few integrable Abel ODEs are known, and the integration methods used to solve each of them depend in an essential way on non-invariant properties of the coefficients f . Those methods are then useless for solving the other infinitely many members of the same classes, unless one can solve the related equivalence problems; i.e., determining - when they exist -the values of F, P and Q in Eq.(2) linking two Abel ODE which belong to the same class.

2.2 Identifying an ODE as member of a given Abel ODE class

Consider two Abel ODEs; the first one given by Eq.(1), and a second one being of the same form, but with coefficients $\tilde{f}_0, \tilde{f}_1, \tilde{f}_2$ and \tilde{f}_3 . The problem now is to determine whether the second Abel ODE can be obtained from Eq.(1) by changing variables using Eq.(2). This problem can be formulated by equating the coefficients between the *transformed* equation, obtained by applying the transformation Eq.(2) to Eq.(1), and the second Abel ODE, resulting in Eq.(4), which can be seen as an ODE system for $\{F, P, Q\}$. To solve this system, following Liouville [3], we first note that the *absolute* invariants corresponding to the two Abel ODEs don't depend on P or Q (see previous section). Hence the function F entering Eq.(2) can be obtained by just running an elimination process using two of these absolute invariants, for instance $I_1 = s_5^3/s_3^5$ and $I_2 = s_5 s_7/s_3^4$:

$$0 = \frac{\tilde{s}_5^3}{\tilde{s}_3^5} - \frac{s_5^3}{s_3^5} \Big|_{x=F(t)} \quad 0 = \frac{\tilde{s}_5 \tilde{s}_7}{\tilde{s}_3^4} - \frac{s_5 s_7}{s_3^4} \Big|_{x=F(t)} \quad (10)$$

As discussed in [3, 4], the *existence* of a common solution $F(t)$ to both equations above (such that $F' \neq 0$) is the necessary and sufficient condition for the existence of a transformation Eq.(2) relating the two Abel ODEs. Once F is known, the system Eq.(4) becomes trivial in that P and Q can be re-expressed in terms of F by performing fairly simple calculations. In the case of interest of this work - non-constant invariant⁸ - the resulting expressions are:

$$P(t) = \frac{F' \tilde{f}_3^2 s_3}{f_3^2 \tilde{s}_3} \Big|_{x=F(t)} \quad Q(t) = \frac{F' \tilde{f}_2 \tilde{f}_3 s_3 - f_2 f_3 \tilde{s}_3}{3 f_3^2 \tilde{s}_3} \Big|_{x=F(t)} \quad (11)$$

where $\{f_i, \tilde{f}_i\}$ with $i : 0 \rightarrow 3$ are the coefficients of the two Abel equations, s_3 is the relative invariant Eq.(7) expressed in terms of f_i and $\tilde{s}_3 = s_3|_{f_i=\tilde{f}_i}$.

⁷In [6] it is also shown that in the constant invariant case the problem can also be formulated in terms of the symmetries of these ODEs.

⁸When the invariant is non-constant, $s_3 \neq 0$.

Concerning the explicit solution $F(t)$ for Eq.(10), we note that our interest in solving the equivalence problem is in that it leads directly to the solution of other members of an Abel class, when the solution to a representative of the class is known. In turn, all the solvable classes we are aware of have a representative with rational coefficients (see the Appendix), and hence also rational invariants⁹. Hence, assuming that one of the two Abel ODEs has rational coefficients and that Eq.(10) was obtained by applying Eq.(2) to it, the system Eq.(10) will always be polynomial in $F(t)$. In such a case, when a common solution $F(t)$ to both equations exists, the resultant between these polynomials will be zero [3]; i.e.: there will be a common factor, depending on F and t and representing the common solution, which can be obtained by calculating the greatest common divisor (GCD) between the two equations in Eq.(10). Conversely, if that GCD does not depend on F , a transformation Eq.(2) linking the input equation to Eq.(1) does not exist. That the dependence on F of this GCD is a necessary condition for the existence of the desired transformation Eq.(2) is a consequence of the validity of Eq.(5) and hence the system (10). A proof of its sufficiency was given by Appell in [4].

The whole process just described to determine the equivalence between two given Abel ODEs, one of which is rational in x , can be summarized as follows:

1. Calculate two absolute invariants, set up the system Eq.(10), and calculate the GCD between the two equations;
2. When this GCD does not depend on F , the ODEs don't belong to the same class; otherwise determine an explicit expression for $F(t)$ from the result of the GCD calculation;
3. Plug this value for F into the formulas Eq.(11) to determine the values of $P(t)$ and $Q(t)$, arriving in this way at the transformation Eq.(2) mapping one Abel ODE into the other.

Example:

Consider the two non-constant invariant Abel ODEs

$$y' = -\frac{1}{2(x+4)} (xy^3 + y^2) \quad (12)$$

$$y' = \frac{(f'x - f)}{2(f + 3x)} ((x - f)y^3 + y^2) - \frac{y}{x} \quad (13)$$

where in the above $f \equiv f(x)$ is an analytic (arbitrary) function. As in the typical situation one of these ODEs has invariants rational in x and we know its solution; i.e. for Eq.(12) we have

$$C_1 + \frac{\sqrt{y^2x - 4y - 1}}{y} + 2 \arctan \left(\frac{1 + 2y}{\sqrt{y^2x - 4y - 1}} \right) = 0 \quad (14)$$

where C_1 is an arbitrary constant. We would then like to determine whether there are functions $\{F, P, Q\}$ so that Eq.(12) transforms under Eq.(2) into Eq.(13), and if so, determine $\{F, P, Q\}$ and use them together with Eq.(14) to build the answer to Eq.(13). For this purpose, we start (step (1)) by computing the relative invariants s_3, s_5, s_7 , leading to Eq.(10)

⁹On the other hand, there is no reason to expect that the second of the two Abel ODEs being tested for equivalence also has rational invariants. If however the invariants of *both* Abel ODEs are rational in t and the coefficients are *numbers*, then it is also possible to determine $F(t)$ by performing a rational function decomposition as mentioned in [6].

$$\begin{aligned}
0 &= \frac{(87ft+9f^2+184t^2)^3}{t(9f+31t)^5} - \frac{(280+105F+9F^2)^3}{(40+9F)^5} \\
0 &= \frac{(81f^3+1431f^2t+7185ft^2+10903t^3)(9f+31t)}{(87ft+9f^2+184t^2)^2} - \frac{(1674F^2+10290F+81F^3+19600)(40+9F)}{(280+105F+9F^2)^2}
\end{aligned} \tag{15}$$

where in the above $F \equiv F(t)$ is the function we are looking for and f is taken at $x = t$. We recall that when Eq.(13) has non-constant invariant the denominators in above are not zero since $s_3 \neq 0$. Calculating the GCD between the numerators of the expressions above (step (2)) and equating this GCD to zero, we obtain¹⁰

$$27(t + tF - f) = 0 \tag{16}$$

from where the common solution $F(t)$ to both equations is given by

$$F(t) = \frac{f(t)}{t} - 1 \tag{17}$$

Substituting this value of F into Eq.(11), a transformation of the form Eq.(2) mapping Eq.(12) into Eq.(13) is finally given by

$$\left\{ x = \frac{f(t)}{t} - 1, \quad y(x) = tu(t) \right\} \tag{18}$$

from where by changing variables in Eq.(14) using the transformation above and renaming the variables ($t \rightarrow x$, $u \rightarrow y$), the solution to Eq.(13) is obtained

$$C_1 + \frac{\sqrt{\left(\frac{f}{x} - 1\right) x^2 y^2 - 4xy - 1}}{xy} + 2 \arctan \left(\frac{(1 + 2xy)}{\sqrt{\left(\frac{f}{x} - 1\right) x^2 y^2 - 4xy - 1}} \right) = 0 \tag{19}$$

3 Parameterized Abel ODE classes

We formulate here the equivalence problem in the case of *parameterized* classes. By ‘‘parameterized class’’ we mean an (Abel) ODE class depending on symbolic parameters which *cannot be removed* by changing variables using Eq.(2). The interest in *parameterized* solvable classes is clear: to each set of values of the parameters corresponds - roughly speaking - a different Abel class¹¹. Hence, a formulation of the equivalence problem for parameterized classes enables one to solve all the members of infinitely many classes at once.

In order to simplify the discussion, we consider the problem of an Abel ODE class depending on just one parameter¹², say \mathcal{C} . Also, we distinguish between two different types of problems: one is when the equivalence problem has a solution for a specific *numerical* value of \mathcal{C} ; the other happens when to have a solution it is required that \mathcal{C} assumes *symbolic* values, for instance in terms of other *symbols* entering the input ODE. We discuss first the *numerical* case, and in the next subsection we show how the *symbolic* case can be mapped into many *numerical* problems - when the parameter depends on other symbols in a *rational* manner - by using rational interpolation methods.

¹⁰We note that when $f(t)$ is an algebraic mapping involving varied analytic functions, Maple’s procedures to simplify and put them in normal form may fail and consequently the GCD computation may not be successful.

¹¹There may be particular different sets of values for which the resulting ODEs will nevertheless belong to the same class.

¹²The integrable classes presented in the literature depend at most on one parameter (see sec. 4).

3.1 Solution for some numerical value of \mathcal{C}

To facilitate the exposition we present the discussion around a concrete example. Consider the equivalence problem between a given Abel ODE, for instance,

$$y' = 8 \frac{(1 - x^4 - x^8) y^3}{x^7} + 4 \frac{y^2}{x^4} + \frac{y}{x} \quad (20)$$

and the one presented in Abel's memoirs [2]

$$y' = \frac{(\mathcal{C} x^4 + x^2 + 1) y^3}{x^3} + y^2 \quad (21)$$

If this equivalence exists, then it exists for a *specific* value of the parameter \mathcal{C} since there is no solution for arbitrary \mathcal{C} (the existence of such a solution would mean the class does not really depend on any parameter). Hence, the common solution $F(t)$ to the system Eq.(10) will not show up until the correct value of \mathcal{C} is determined¹³, invalidating the itemized algorithm of the previous section.

A natural alternative to this problem would be to take one more absolute invariant, for instance, $s_3 s_7 / s_5^2$, so that our system Eq.(10) becomes

$$0 = \frac{\tilde{s}_5^3}{\tilde{s}_3^5} - \frac{s_5^3}{s_3^5} \Big|_{x=F(t)} \quad 0 = \frac{\tilde{s}_3 \tilde{s}_7}{\tilde{s}_5^2} - \frac{s_3 s_7}{s_5^2} \Big|_{x=F(t)} \quad 0 = \frac{\tilde{s}_5 \tilde{s}_7}{\tilde{s}_3^4} - \frac{s_5 s_7}{s_3^4} \Big|_{x=F(t)} \quad (22)$$

and search for a solution to this problem such that $F' \neq 0$, $\mathcal{C}' = 0$. For that purpose, eliminate \mathcal{C} from the first and second expressions above by taking the resultant with respect to \mathcal{C} , obtaining - say - R_1 . In the same way, eliminate \mathcal{C} from the first and the third expressions of Eq.(22) obtaining R_2 . Hence, when a solution exists, the resultant between R_1 and R_2 with respect to F will vanish. In other words, the algorithm of the previous section will work if instead of performing the calculations over the expressions Eq.(10) we perform them over R_1 and R_2 . The GCD between R_1 and R_2 will then return the factor depending on both F and t , whose solution is the function $F(t)$ we are interested in.

This method, simple and correct in theory, unfortunately does not work in practice because the expressions tend to grow in size so much that the computation of the first of these three resultants may not be possible, even with a simple example such as the one shown above. The problem resides in the fact that multivariate GCDs and resultants are quite expensive operations for the current symbolic computation environments.

An alternative to this problem consists of reducing it to a sequence of bivariate GCD and resultant calculations, for which the available algorithms are relatively fast. The idea can be summarized as follows.

1. From the previous considerations, when a solution to the equivalence problem exists, the resultant between any two of the expressions in Eq.(22) will not vanish for any value of t , since we haven't introduced the correct (unknown at this point) value of \mathcal{C} . Hence, if we insert in Eq.(22) a numerical value¹⁴ for t and calculate the GCD between any two of the resulting expressions, this GCD *cannot contain any factor depending on F* . This gives us a first "existence condition" test for the solution before proceeding further;

¹³There may be more than one solution \mathcal{C} .

¹⁴We note there may exist "invalid evaluation points"; roughly speaking to avoid this problem this evaluation point must not cancel any of the coefficients of the variables remaining in the system - see [8].

2. When Eq.(22) evaluated at $t = \text{number}$ passed the test of the previous step, take two of the resulting three expressions and calculate their resultant with respect to F , obtaining, say, \tilde{R}_1 . Then take a *different* pair and calculate their resultant with respect to F again, obtaining, say, \tilde{R}_2 . Neither of these resultants will vanish since the GCD calculations of the previous step showed no factor depending on F . Also, the calculation of \tilde{R}_1 and \tilde{R}_2 is now quite simpler since the expressions do not involve t ;
3. Then if a solution to the problem exists, the GCD between \tilde{R}_1 and \tilde{R}_2 will yield a factor depending on C ; equating it to zero and solving it for C will give the *common solution C for \tilde{R}_1 and \tilde{R}_2* . More precisely, what we will get in this way is a set of *candidates* (including among them the correct value) for C ; not all of them will necessarily lead to a solution $F(t)$ to the original problem;
4. We now plug these candidates for C into Eq.(22), one at a time, receiving a system of three expressions involving again only two unknowns, now F and t . If there is a common solution $F(t)$ to these expressions, the resultant with respect to F between any two of them will vanish. Hence, the GCD between those two expressions will contain a factor depending both on F and t ; equating this factor to zero and solving for F leads to the solution $F(t)$.

Returning to our example of determining the equivalence between Eq.(20) and Eq.(21), the itemized procedure just outlined runs as follows.

According to step (1), $t = 0$ is tried first, but it is found to be an invalid evaluation point. The next value of t to try, $t = 1$ turns out to be valid, so Eq.(22) was evaluated at $t = 1$; the GCDs between any two of the three resulting expressions do not depend on F , so this first test for the “existence” of a solution passed.

Continuing with step (2), the calculation of \tilde{R}_1 and \tilde{R}_2 is performed without problems concerning the size of the expressions.

The GCD of step (3) results in the three factors: $36C - 5$, $C + 1$ and $9C - 2$; equating them to zero and solving them for C we arrive at three candidates for C .

In step (4), plugging each of these candidates one at a time into Eq.(22) and taking the GCD between two of the three resulting expressions we note that $C = 5/36$ does not lead to any factor depending on both F and t , but $C = -1$ leads to such factor: $F^2t^4 - 1$. So that for $C = -1$ the problem admits two solutions: $F = \pm 1/t^2$.

Finally, by introducing $F = 1/t^2$ into the formulas for P and Q Eq.(11) we arrive at the transformation of the form Eq.(2) mapping Eq.(21) into Eq.(20)

$$\left\{ x = \frac{1}{t^2}, y = -2 \frac{u(t)}{t} \right\} \quad (23)$$

and hence by applying the same change of variables to the answer of Eq.(21) *and* substituting $C = -1$ we obtain the answer to Eq.(20).

A remark however is in order: if, in step (3) of the algorithm just described, the numeric candidates for C involve fractional powers of rational numbers, the Maple system may then enter not efficient expensive computations, exhausting the system resources before determining the expression $F(t)$ solving the problem in step (4). The root of this limitation seems to be in the absence in Maple of built-in normalization for such “numeric radicals”¹⁵.

¹⁵A built-in normalization of radicals is implemented in the computer algebra system “Mathematica”. For typical problems we tried where Maple exhausted the system resources trying to determine the solution $F(t)$, we exported

3.2 Solution when the parameter \mathcal{C} is some *rational* function of other symbols

When \mathcal{C} assumes *symbolic* values, for instance it depends on other *symbols* - say $\{\alpha\}$ - entering the input ODE, if this dependency is rational it is possible to map the determination of $\mathcal{C}(\alpha)$ into a sequence of problems having for solution a *numerical* value of \mathcal{C} . In turn each of these *numerical* problems can be tackled using the algorithm of the previous subsection. The idea consists of attributing numerical values to the symbols $\{\alpha\}$ entering the invariants in order to determine $\mathcal{C}(\alpha)$ by means of a rational interpolation. To simplify the presentation we first discuss the case when $\{\alpha\}$ consists of a *single* parameter, and then show how to extend the algorithm to the case in which $\{\alpha\}$ consists of *many* parameters by commenting on a concrete example. So when α consists of just one parameter this rational interpolation scheme is summarized as follows:

1. Take the system Eq.(22) and attribute a numerical value to α (check for possible wrong evaluation points), so the resulting system depends on just x , \mathcal{C} and F ;
2. Enter step (1) in the algorithm of the previous subsection and run all the steps:
 - (a) If there is no solution for \mathcal{C} and F such that $\mathcal{C}' = 0$ and $F' \neq 0$ then quit the process - the input ODE does not belong to this class, or the solution involves a non rational dependency of \mathcal{C} on $\{\alpha\}$.
 - (b) If however a solution for \mathcal{C} and F was found, record the values for \mathcal{C} and α - they represent a *point* of a curve $\mathcal{C}(\alpha)$;
3. Using the points recorded so far, interpolate \mathcal{C} as a function of α and test if this interpolated value already solves the problem for an *arbitrary* α ¹⁶:
 - (a) If so, the problem has been solved;
 - (b) Otherwise change the evaluation point of α and re-enter step (2) of this enumeration;

We note that the rational interpolation of $\mathcal{C}(\alpha)$ requires the knowledge a priori of the polynomial degrees in α of both numerator and denominator. That information is not available in advance, but we know that these two degrees sum to $n - 1$, where n is the number of points being interpolated. So, when performing the test in step (3) and before going to step (3b) we actually test all possible different interpolations, starting with the maximum possible degree for the numerator and finishing with the maximum possible degree for the denominator.

Concerning the extension of this algorithm for the case when $\{\alpha\}$ involves more than one parameter, for instance $\{a, b\}$, this extension is easy and better illustrated with an example. Consider the equivalence problem between the Abel ODEs

$$y' = \frac{C(2x^2C - 2)y^3 - 3Cy^2 + Cxy}{1 - x^2C} \quad \text{and} \quad y' = \frac{a(2x^2a - 2b)y^3 - 3ay^2b + abxy}{b(b - x^2a)} \quad (24)$$

which solution is just the identity $x \rightarrow x, y \rightarrow y$, but only exists when $\mathcal{C} = \frac{a}{b}$ (we choose the identity without loss of generality and so that the solving process is easy to follow). We want to determine $\mathcal{C} = \frac{a}{b}$ (rational function of *two* parameters) using the algorithm just described.

the mathematical expressions involving radicals to Mathematica and noticed that $F(t)$ was determined in this other computer algebra system in reasonable time.

¹⁶This is done by restoring the symbol “ α ” in the system obtained in step (1) and checking if the system is satisfied.

We start with step (1), take the system Eq.(22) corresponding to Eq.(24) and attribute numerical values for the first parameter, a , checking for possible wrong evaluation points - we end up evaluating this system at $a = 1$. However, the system still depends on the second parameter, b , so we attribute numerical values to b too - check for possible wrong evaluation points - and hence end up evaluating the system altogether at $a = 1, b = 2$. The resulting system now only depends on x, F and \mathcal{C} .

So we enter step (2) (actually the whole algorithm of sec. 3.1) with this evaluated system of the previous step and detect that $\mathcal{C} = 1/2$ leads to the solution $F = x$; so a solution $\mathcal{C}' = 0$ and $F' \neq 0$ exists.

Hence, in step (3) we interpolate $\mathcal{C}(b)$ (so far we have just one point, so $\mathcal{C}(b) = 1/2$) and test this value for arbitrary b , verifying that the interpolation is still incomplete: the system Eq.(22) with $a = 1$ and *arbitrary* b is not satisfied by $\mathcal{C} = 1/2, F = x$.

We are then in step (3.3b); so attribute a new numerical value to b , hence evaluate the system Eq.(22) at $a = 1, b = 3$ and re-enter step (2) finding that $\mathcal{C} = 1/3$ leads to $F = x$.

We record this new point (step (2.2b)) and interpolate - now using the two points obtained so far - our first interpolation is of degree 1 in b : $\mathcal{C}(b) = (5 - b)/6$. This interpolation however does not solve the problem for arbitrary b . So we increase by one the degree in b of the denominator in the interpolation, resulting in $\mathcal{C}(b) = 1/b$, and verify that this second interpolation indeed solves the system for arbitrary b - so the interpolation for b at $a = 1$ is complete.

We return then to the interpolation of $\mathcal{C}(a, b)$ with respect to a , record that for $a = 1$ there is a solution $\mathcal{C}(b) = 1/b$, and test this solution for *arbitrary* a (step (3) with respect to a), verifying that the interpolation for a is still incomplete.

Hence we re-enter step (1) with new evaluation points $a = 2, b = 4$, and re-start the process of determining $\mathcal{C}(b)$.

Following the same steps just described, for $a = 2$ we find that $\mathcal{C}(b) = 2/b$ solves the problem for arbitrary b , leading to the second point in the interpolation of $\mathcal{C}(a, b)$ with respect to a .

So we are now in step (3) again, and interpolating \mathcal{C} using the two points obtained so far we find $\mathcal{C}(a, b) = a/b$ - this value of $\mathcal{C}(a, b)$ is verified to satisfy the system Eq.(22) for arbitrary a , and thus the problem has been solved.

The algorithm just described, though expensive in computations, is successful in solving the equivalence problem in reasonable time for typical situations (see sec. 6). In this example, for instance, it took 12 seconds to: setup the invariants and the system Eq.(22), simplify this system to a normal form, run all the items of the algorithm just described to determine $\mathcal{C}(a, b)$ and $F(t)$, then determine $P(t)$ and $Q(t)$ according to Eq.(11), and finally return a solution to the second of the ODEs in Eq.(24).

3.2.1 Remark on the existence of multiple solutions for the class parameter $\mathcal{C}(\alpha)$

The interpolation algorithm just described is valid provided there is only one solution curve $\mathcal{C}(\alpha)$; otherwise we may end up trying to interpolate $\mathcal{C}(\alpha)$ using points which belong to different solution curves, leading nowhere. In turn, the existence of many curves $\mathcal{C}(\alpha)$ solving a given problem is related to the existence of symmetries in the invariants (of the ODE representative of the class we want to match) entering Eq.(22). Concretely, if the mapping $\{\mathcal{C} \rightarrow \kappa(\mathcal{C}), x \rightarrow \phi(x, \mathcal{C})\}$ is a symmetry of these invariants, then if $\{\mathcal{C}, F(x)\}$ leads to a solution for the equivalence problem, consequently $\{\kappa(\mathcal{C}), \phi(F(x), \mathcal{C})\}$ will also lead to a (different) solution. A concrete example of this situation is discussed in sec. 6.1.

Concerning detecting this situation, we note that if the mapping $\{\mathcal{C} \rightarrow \kappa(\mathcal{C}), x \rightarrow \phi(x, \mathcal{C})\}$ is a symmetry of the invariants, then the *inverse* mapping is also a symmetry, and since these invariants

are *rational* in both x and \mathcal{C} , the form of such a symmetry mapping is

$$\kappa(\mathcal{C}) = \frac{a\mathcal{C} + b}{c\mathcal{C} + d}, \quad \phi(x, \mathcal{C}) = \frac{f(\mathcal{C})x + g(\mathcal{C})}{h(\mathcal{C})x + j(\mathcal{C})} \quad (25)$$

that is, a fractional linear mapping, where $\{a, b, c, d\}$ and $\{f(\mathcal{C}), g(\mathcal{C}), h(\mathcal{C}), j(\mathcal{C})\}$ are in principle constants and functions to be determined. Regarding the use of interpolation methods, the problem happens when $\kappa(\mathcal{C}) \neq \mathcal{C}$. A first manner of detecting this problem then consists of setting up the system of equations and inequations

$$I(x, \mathcal{C}) = I(x, \mathcal{C}) \Big|_{\substack{x=\phi(x, \mathcal{C}) \\ \mathcal{C}=\kappa(\mathcal{C})}}, \quad \phi = \frac{F}{G}, \quad F_{xx} = 0, \quad G_{xx} = 0, \quad \kappa(\mathcal{C}) \neq \mathcal{C} \quad (26)$$

where $I(x, \mathcal{C}) = s_5^3/s_3^5$ is the first invariant and F , G and κ are the unknowns to be determined, and seeing if this system is consistent. This check for consistency can be performed by simplifying this system with respect to its integrability conditions - for this purpose we used the *diffalg* [13] and *RIF* [14] Maple packages. When the system has no solution, this fact is detected by these packages; otherwise the related symmetry of the invariants is obtained from the output of these packages directly.

4 Integrable Abel ODE classes found in the literature

This section is devoted to a compilation of integrable Abel ODE classes found in the literature. The compilation is not intended to be complete, but it nevertheless covers various of the usual references; mainly Kamke's and Murphy's books [1, 5], and the original works by Abel, Liouville and others on these subjects [2, 3, 4, 10].

One of the noticeable things in these references is that the presentation of integrable cases lacks a classification in terms of their invariants. Consequently, many of these ODEs can actually be obtained from one another by means of Eq.(2), that is, they belong to the same class. Since part of this work consisted in writing computer routines addressing the equivalence problem, we performed this classification, and therefore present here a more compact collection of integrable Abel ODE *classes*, as opposed to just integrable ODEs. Classes not depending on parameters are labelled by numbers (e.g., Class 1), while those depending on parameters are labelled with letters (e.g., Class A). The solutions to the representatives of these classes are presented altogether in a table in the Appendix.

While revising the related literature we also noticed that various of the cases presented in books or papers are in fact particular cases of the integrable classes presented by Abel, Liouville and Appell in [2, 3, 4]. In turn the methods they used to obtain new integrable classes seem to be forgotten or not mentioned elsewhere. So, it appeared reasonable to start by reviewing and analyzing selected parts of those works in this section, and then show in the next section how, starting from these ideas, additional integrable classes can be obtained.

The first large presentation of integrable cases is due to Abel himself in [2]. His idea was to consider integrating factors of the form

$$\mu = e^{r(x,y)} \quad (27)$$

for "Abel" equations written in terms of two arbitrary functions p and q as:

$$\Phi \equiv yy' + p(x) + q'(x)y = 0 \quad (28)$$

The first non-trivial case discussed in [2] was found by taking $r(x, y)$ as quadratic in y :

$$\mu = e^{(\alpha + \beta y + \gamma y^2)}$$

where α , β and γ are arbitrary functions of x . Abel formulated this problem by applying Euler's operator to the total derivative $\mu\Phi$, obtaining a system easily solvable for α , β , γ and p . The resulting Abel family has non-constant invariant and is shown in Abel's memoirs as depending on one arbitrary function $q(x)$ and two arbitrary constants C_i :

$$yy' - \frac{q'}{2C_1q + C_2} + q'y \quad (29)$$

(for the corresponding integrating factor see [2]). Now, for the purpose of building computer routines addressing the equivalence problem, it is crucial to determine whether or not a given class depends on parameters since, as explained in sec. 3, in such a case the formulation of that problem is much more difficult. In the case of Eq.(29), the two parameters C_i and the function $q(x)$, can be removed by first converting the ODE to first kind using $y(x) = 1/v(x)$, and then employing a transformation of the form Eq.(2): $\{x = F(t), v(x) = u(t)\sqrt{-2C_1}\}$, with F implicitly defined by $2C_1q(F) - t\sqrt{-2C_1} + C_2 = 0$, arriving at a representative of the class simpler than Eq.(29),

$$y' = \frac{y^3}{x} + y^2 \quad (30)$$

and showing that this class does not depend on parameters. It is then easy to verify that Eq.(30) is a particular case of a parameterized class¹⁷ derived from Appell's work [4].

The next integrable case shown by Abel is obtained by considering for Eq.(28) an integrating factor of the form $\mu = \exp(1/(\alpha + \beta y))$. Proceeding as in the previous case, Abel arrived at another integrable ODE class with non-constant invariant, which however (see [3]) is a particular member of the parameterized class Eq.(33) shown by Abel in the same paper.

Constant Invariant case

Abel then considered an integrating factor of the form $\mu = (\alpha + \beta y)^n$. This ansatz does not lead to a non-constant invariant family. However, this is the first presentation we have found of a method for the *constant* invariant case. Liouville, and others after him, rediscovered this method, presented in Kamke as due to M. Chini [9], and in Murphy's book as a change of variables mapping an Abel ODE into a separable one. A recent discussion of the symmetries of this *constant* invariant problem is found in [6].

Class "A" depending on one arbitrary parameter

The next ansatz considered by Abel was

$$\mu = (A + y)^a (B + y)^b y \quad (31)$$

where $A(x)$ and $B(x)$ are arbitrary functions and a and b are arbitrary constants. By taking $b = -a$ Abel showed that a tractable integrable case results:

$$yy' + \frac{q'}{4q} \left(\left(q + 2\frac{C_1}{q} \right)^2 - \frac{q^2}{a^2} \right) + q'y = 0 \quad (32)$$

The arbitrary function $q(x)$ can be removed together with the constant C_1 by rewriting this ODE in first kind format, and then appropriately choosing $\{F, P, Q\}$ in Eq.(2); so that a simpler representative of this class depending on only one parameter "a", is given by¹⁸

¹⁷Eq.(30) is obtained from Eq.(58) taking $C = 0$ and changing variables $\{x = it, y = iu(t)\}$.

¹⁸A representative of the same class of Eq.(33) is shown in [3] as $y' = \frac{4}{9x^3} \left((x^2 + 1)^2 - cx^4 \right) y^3 + \frac{4y^2}{3}$

$$y' = \left(\alpha x + \frac{1}{x} + \frac{1}{x^3} \right) y^3 + y^2 \quad (33)$$

Class 1

In [10], Halphen noted a connection between doubly-periodic elliptic functions and the Abel type ODE

$$y' = \frac{3y(1+y) - 4x}{x(8y-1)} \quad (34)$$

which transforms into itself under infinitely many rational changes of variables, from where he was able to determine both a parametric and an algebraic solution for it (see the Appendix).

Class 2

In a paper by Liouville [3] mostly dedicated to Abel equations, he discussed the integrable cases known at that time (1903), and presented some new ones. Liouville reviewed Abel's work and considered for Eq.(28) an integrating factor of the form Eq.(27) with $r(x, y)$ cubic in y , arriving at the integrable family $y' = 6axy^2 + 3ay^3$, depending on a parameter a . This parameter however can be removed by changing variables as in $\{y = -u(t)/\sqrt[3]{3a}, x = t/\sqrt[3]{3a}\}$ arriving at the integrable class free of parameters represented by

$$y' = -2y^2x + y^3 \quad (35)$$

Class "B" depending on one arbitrary parameter

As a generalization of Eq.(35), in [3] Liouville also presented the parameterized family

$$y' + \left(3mx^2 + 4m^2x + n \right) y^3 + 3xy^2 = 0 \quad (36)$$

written in terms of two parameters m and n and which can be mapped into a Riccati ODE solvable in terms of special functions. Eq.(35) is a member of the class represented by Eq.(36) after setting $m = 0$. However, when $m = 0$, n can be removed from Eq.(36) by changing variables $\{x = t\sqrt[3]{n}, y(x) = tu(t)/n^{2/3}\}$, leading to a class without parameters - actually represented by Eq.(35). In turn, when $m \neq 0$, m and n can be "merged" by changing variables $\{y = u(t)/m^2, x = mt\}$ and introducing a new parameter $a = n/m^3$, resulting in

$$y' = - \left(3x^2 + 4x + a \right) y^3 - 3xy^2 \quad (37)$$

In summary, Eq.(36) is not a full 2-parameter class, but instead two classes represented by Eqs.(35) and (37), respectively depending on zero and one parameters. A simpler representative for this class and its solution are found in the Appendix.

Class 3

Still in [3] Liouville pointed out that by interchanging the role between the dependent and independent variables in Eq.(35) one arrives at a different Abel integrable class. After rewriting this resulting ODE in first kind format and performing a change of variables of the form Eq.(2), a simpler representative of this integrable class is given by

$$y' = \frac{y^3}{4x^2} - y^2 \quad (38)$$

4.1 Integrable Abel ODE classes shown in Kamke and some others books

One of the most well known collection of (69) Abel ODEs is the one shown in Kamke's book. This collection however makes no distinction between constant or non-constant invariant cases, presents ODEs of the same class as different, and does not discuss what would be the representative for each class depending on the least number of parameters. A first classification for these Abel ODEs is then given by¹⁹:

Classification	ODE numbers as in Kamke's book
4 are too general	50, 219, 250, 269
40 constant invariant	38, 41, 46, 49, 51, 188, 204, 213, 214, 215, 216, 218, 221, 222, 223, 224, 225, 226, 227, 228, 229, 231, 236, 238, 239, 243, 244, 245, 246, 247, 248, 249, 251, 252, 254, 255, 260, 261, 262, 264
24 non-constant invariant	36, 37, 40, 42, 43, 45, 47, 48, 111, 145, 146, 147, 151, 169, 185, 203, 205, 206, 234, 235, 237, 253, 257, 265
10 shown without solution	40, 47, 48, 203, 205, 206, 234, 237, 253, 265

Table 1. First classification for the 69 Abel ODEs shown in Kamke's book.

As mentioned, all constant invariant ODEs can systematically be transformed into separable ODEs (see for instance Murphy's book), so that the interesting subset is the one comprising 24 ODEs having non-constant invariants. We note also that 10 of these 24 ODEs are shown in the book without a solution, and in fact we were unable to solve any of 203, 205, 206, 234, 253 or 265, so that the number of integrable cases for us is 18.

From these 18 ODEs (and hence from the 69 Abel type Kamke's examples), only four - those numbered: 47, 185, 235 and 237- would really lead to additional integrable classes with respect to those presented in the works by Abel, Liouville and Appell. We note however that the examples 47, 185 and 237 are all members of Class "C" (see Eq.(49)), which can be derived from the work by Abel [2] - even when it was not presented in the original work. So that the number of additional integrable classes presented in Kamke reduces to one, represented by the example 235. The classification and details are as follows.

Class 4

$$(xy + a)y' + by = 0 \quad (39)$$

This ODE (K 1.235) is presented in Kamke in terms of two arbitrary parameters $\{a, b\}$; then, a change of variables which transforms it into a linear ODE is shown. A simpler representative of this class - not depending on parameters - can be obtained by rewriting this equation in first kind format via $\{x = t, y = \frac{1}{tu(t)} - \frac{a}{t}\}$ and then changing variables $\{x = \frac{a}{tb}, y = \frac{tu(t)}{a}\}$, leading to

$$y' = y^3 - \frac{(x+1)}{x}y^2 \quad (40)$$

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For the ODE

$$y' - a(x^n - x)y^3 - y^2 = 0 \quad (41)$$

¹⁹In this classification, by "too general" we mean: these ODEs cannot be solved without restricting the example to a concrete particular case. We excluded ODEs - like those numbered 230 and 232 - which are already of Bernoulli type. We note also that the ODEs shown in Kamke without solution can all be transformed into an Emden type second order ODE shown in Kamke as 6.74, for which only a general discussion is presented. In turn, a detailed discussion on the integrable cases of Emden type ODEs is found in [11].

presented in Kamke as K 1.47, there is no solution shown in the book, but instead a suggestion of transforming the ODE into a second order one. We followed that suggestion and then ran a symmetry analysis, noticing that the resulting ODE will have two point symmetries if either $\{a = -\frac{2n+2}{9+6n+n^2}\}$ or $\{n = 2, a = \frac{6}{25}\}$, leading to two integrable classes not shown in the book. In the former case, from Eq.(41), we arrive at

$$y' + \frac{(2n+2)(x^n - x)y^3}{9+n^2+6n} - y^2 = 0 \quad (42)$$

However, this ODE can be transformed into Eq.(49) by changing variables $\{x = t^{\frac{2}{1-n}}, y = -u(t)\frac{n+3}{2}t^{\frac{n+1}{n-1}}\}$ followed by $n = \frac{a+2}{a-2}$, so that it belongs to Class C. In the same line, taking $\{n = 2, a = \frac{6}{25}\}$ in Eq.(41), and changing variables $\{x = \frac{t^2-1}{t^2}, y = 5/2u(t)t^3\}$ one arrives at Eq.(49) with $a = 6$, so that this second branch of Eq.(41) is also a member of Class C.

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$$x(y+a)y' + by + cx = 0 \quad (43)$$

This ODE (K 1.237) depending on three arbitrary parameters $\{a, b, c\}$, is presented in the book without a solution. We note however that changing $\{x \rightarrow y, y \rightarrow x\}$ leads to an ODE also of Abel type and in second kind format. Converting the latter to first kind format via $\{x = t, y = \frac{1}{cu(t)} - \frac{bt}{c}\}$, replacing $y \rightarrow y'$ and running a symmetry analysis, the resulting second order ODE has two symmetries when $a = -2b$, leading to an integrable case. Introducing $a = -2b$ into Eq.(43), rewriting it in first kind format via $\{x = t, y = -\frac{1}{tu(t)} + 2b\}$ and changing variables $\{x = -\frac{b^2(t+4)}{2c}, y = \frac{2cu(t)}{b^3(t+4)}\}$ leads to a simpler representative of the class not depending on any parameters:

$$y' = \frac{-xy^3 + 2y^2}{2(x+4)} \quad (44)$$

However, by changing variables $\{x = 4(1-t^2)/t^2, y = -u(t)t/2\}$ one arrives at Eq.(49) again, this time with $a = -1/2$, so that Eq.(44) is also member of Class C.

A classification for all these 18 non-constant invariant Kamke examples is then as follows²⁰

Class 2	Class 3	Class 4	Class A	Class B	Class C	Class D
36, 40	145, 147	235	257	42, 43	45, 47, 48, 151, 185, 237	37, 111, 146, 169

Table 2. Classification for the 18 non-constant invariant solvable Abel ODEs in Kamke's book.

where classes C and D are defined in sec. 5. In summary, all but one of Kamke's 58 solvable examples (18 non-constant invariant + 40 constant invariant) are particular cases of the integrable classes presented by Abel, Liouville and Appell in [2, 3, 4], or can be derived from there (those belonging to Classes C and D).

Another collection of Abel ODEs is found in the book by Murphy [5]. After selecting those examples not having a constant invariant and for which a solution is shown in the book, we arrived at a set of nine ODEs, numbered in the book as: 78, 79, 80, 86, 275, 304, 345, 383 and 593. None of these ODEs represent an additional integrable class; their distribution among the classes discussed in this work is as follows

²⁰Equations K.1.47, K.1.48 and K.1.237 belong to Class C for infinitely many - however particular - values of one of the two parameters (see Eq.(42)); we don't know their solution for other values.

Class 2	Class 3	Class B	Class C	Class D
78, 80	275	86	304, 383, 593	79, 345

Table 3. Classification for the non-constant invariant solvable Abel ODEs in Murphy's book.

A wider collection of Abel ODEs than the one shown in Kamke's book is found in the book by Polyanin and Zaitsev [11]. This book is rather new (1995) and covers a vast number of integrable ODE problems which we have not found in other books, hence making the examples attractive. On the other hand the Abel ODEs shown there are classified not according to their invariants but according to their form, and the origin of their solutions is not given. Apart from a main section consisting of four tables (82 Abel ODEs - all derived from four basic ones), the book contains other sections illustrating mappings between Abel and higher order ODEs. The quantity of examples is large and the computational routines we prepared for the equivalence problem are not yet covering in full the case in which the parameters of the class may assume *symbolic* values. As a result we still don't have a way to solve the equivalence problem for the whole set of integrable classes presented in [11]. Our analysis of these Abel ODEs of [11] is then still incomplete; consequently we restricted the presentation here to just a sample, constituted by the ODEs of the first of these four tables. These are 20 ODEs obtained from

$$yy' - y = sx + Ax^m \quad (45)$$

by giving particular values to the parameters m and s (A is kept arbitrary). These ODEs appear in section 1.3.1 of [11] under the numbers: 1, 2, 10, 16, 19, 22, 23, 26, 27, 30, 32, 33, 45, 46, 47, 48, 53, 54, 55 and 56. We were not able to classify those numbered 27, 20, 48, 55 and 56. The distribution of the remaining ODEs, in the classes discussed in this work, is as follows:

Constant invariant	Class 1	Class 2	Class 3	Class C	Class D
1, 2, 26	23	32	33	10, 19, 22, 45, 46, 47, 53, 54	16

Table 4. Classification for 15 of the 20 Abel ODE examples of Table 1.1 of [11].

5 New integrable Abel ODE classes derived from previous works

Class "C" depending on one arbitrary parameter

The form of the integrating factor studied by Abel actually leads to other integrable cases not mentioned in the original work [2]. One of them is obtained by taking $b = a$ in Eq.(31), resulting in the ODE family²¹

$$yy' - q'y - \frac{q'n^2 \left(-\frac{q}{n} + C_1^2 \left(\frac{q}{n} \right)^{2n-1} \right)}{(n+1)^2} = 0 \quad (46)$$

where $n \neq -1$. The function $q(x)$ and the parameter C_1 can be removed as done with Eq.(32), leading to

$$y' = n \left(x - x^{2n-1} \right) y^3 - (n+1) y^2 \quad (47)$$

which is turned exact by means of the integrating factor

$$\mu = \frac{\left(1 + \left((x^2 - x^{2n}) y - 2x \right) y \right)^{-\frac{n+1}{2n}}}{y^{\frac{2n-1}{n}}} \quad (48)$$

²¹ n in Eq.(46) is related to a in Eq.(31) by $n = -1/(2a + 1)$

A simpler representative of this class is obtained by changing variables $\{y = u(t)t^{\frac{n}{n-1}}, x = t^{\frac{1}{1-n}}\}$, then introducing a new parameter by means of $n = \frac{\alpha}{\alpha-2}$, arriving at

$$y' = \frac{\alpha(1-x^2)y^3}{2x} + (\alpha-1)y^2 - \frac{\alpha y}{2x} \quad (49)$$

Taking into account Eq.(48), an implicit solution for this class is given by

$$C_1 + \frac{\alpha}{x} \left(1 - \frac{(1-xy)^2}{y^2}\right)^{1/\alpha} - 2 \int \frac{1-xy}{y} \left(1-z^2\right)^{\frac{1-\alpha}{\alpha}} dz = 0 \quad (50)$$

Class “D” depending on one arbitrary parameter

In [4], Appell showed a series of integrable cases derived from the solutions to

$$u' = A(u) + B(u)t \quad (51)$$

By changing variables $\{t = \frac{1}{y} - \frac{A(x)}{B(x)}, u = x\}$, this ODE is transformed into the Abel ODE

$$y' = -\frac{y^3}{B} - \left(\frac{A}{B}\right)' y^2 \quad (52)$$

where A and B are now functions of x . Any particular $\{A, B\}$ leading to a solvable case in Eq.(51) will then also lead to an integrable Abel ODE Eq.(52). Among the choices for $\{A, B\}$ considered in [4] - such that Eq.(51) results linear, homogeneous, or of Riccati type - only this mapping into Riccati type leads to something new. This case is obtained by taking

$$A = ax^2 + bx + c, \quad B = \alpha x^2 + \beta x + \gamma \quad (53)$$

The related Abel ODE family, depending on six parameters $\{a, b, c, \alpha, \beta, \gamma\}$, is given by

$$y' = -\frac{y^3}{\alpha x^2 + \beta x + \gamma} - y^2 \frac{d}{dx} \left(\frac{ax^2 + bx + c}{\alpha x^2 + \beta x + \gamma} \right) \quad (54)$$

and its solution could be expressed in terms of the solution to the Riccati ODE

$$y' = (a + \alpha x)y^2 + (b + \beta x)y + c + \gamma x \quad (55)$$

However, we were not able to solve this Riccati ODE for arbitrary values of the six parameters involved and in [4] there is no indication of how that could be done. The alternative we then investigated is to consider the second order ODE obtained by replacing $y = y'$ in Eq.(55). That ODE has *two* point symmetries if and only if $\alpha = 0$. With these symmetries we were able to solve that second order ODE, and hence Eq.(55) when $\alpha = 0$. Concerning the related Abel family Eq.(54) - now depending on five parameters - an appropriate change of variables of the form Eq.(2)

$$\left\{ x = \frac{t\sqrt{\beta}}{a} - \frac{\gamma}{\beta}, \quad y = \sqrt{\beta}u(t) \right\} \quad (56)$$

followed by the introduction of a new parameter C by means of

$$C = -\frac{(\beta^2 c + \alpha \gamma^2) a - \alpha \beta \gamma b}{\beta^2} \quad (57)$$

transforms Eq.(54) into a simpler representative for the class

$$y' = -\frac{y^3}{x} - \frac{(C + x^2)y^2}{x^2} \quad (58)$$

also showing that this class depends not on five but on one parameter²². It appeared of value to us also to determine the number of parameters on which Eq.(54) depends in the general case, that is *before* taking $\alpha = 0$. For that purpose we searched for the appropriate changes of variables of the form Eq.(2) which would remove as many as possible of these parameters, requiring that both the change and its inverse are finite. We then considered the branches which become infinite for some particular values of the parameters $\{a, b, c, \alpha, \beta, \gamma\}$ entering the transformations found. The results are summarized as follows. If all these parameters are different from zero, introducing new parameters $\{A, B, C, G\}$ by means of

$$\begin{aligned} \alpha &= \frac{\beta^2 + 4A^4}{4\gamma} \\ b &= \frac{8\beta\gamma^2 a A^2 B + C}{2\gamma A^2 B (\beta^2 + 4A^4)} \\ c &= \frac{A^2 B C + 16\gamma^2 a A^6 B + \beta C + 4\beta^2 \gamma^2 a A^2 B}{A^2 B (\beta^2 + 4A^4)^2} \\ \gamma &= \frac{C}{2A^3 B G (\beta^2 + 4A^4)} \end{aligned} \quad (59)$$

followed by changing variables $\{x = \frac{C(2tA^2 - \beta)}{(\beta^2 + 4A^4)^2 A^3 B G}, y = u(t)A\}$ in the six-parameter Eq.(54), one arrives at a 2-parameter representative for the same class

$$y' = -\frac{y^3}{x^2 + 1} + \frac{G(Bx + x^2 - 1)y^2}{(x^2 + 1)^2} \quad (60)$$

Now the case $\alpha = 0$ was already shown to lead to Eq.(58), and all the other possible branches obtained from Eq.(54) by taking some of the other parameters equal to zero lead either to constant invariant families, or to members of the classes already discussed in this work²³

Three new classes not depending on parameters

While analyzing the works [2, 3, 4] and Kamke's examples, a large number of symbolic experiments were performed, sometimes leading to intermediate results which with a bit more work appeared to be new integrable classes by themselves. This happened three times, resulting in classes 5, 6 and 7, for which representatives and solutions are given as follows:

Class 5

$$y' = -\frac{(2x + 3)(x + 1)y^3}{2x^5} + \frac{(5x + 8)y^2}{2x^3} \quad (61)$$

²²We note that in this process we have made two implicit assumptions: $a \neq 0$ and $\beta \neq 0$. To assure that the cases in Eq.(54) are covered by Eq.(58) we then also considered $a = 0$ and $\beta = 0$ separately, arriving at ODEs respectively members of the classes represented by Eq.(58) and Eq.(35).

²³There is a special case, when $b = 4 \frac{\gamma\beta a}{\beta^2 + 4A^4}$, where the resulting Abel ODE can only be obtained from Eq.(60) by taking appropriate limits.

Solution:

$$C_1 + \frac{\sqrt{A}}{\sqrt[4]{4 \frac{(x+1)^2}{x^2 A} + 1}} + \int^{2 \frac{x+1}{x\sqrt{A}}} (z^2 + 1)^{-5/4} dz = 0 \quad (62)$$

where $A = \frac{4}{y} - \frac{10}{x} - \frac{6}{x^2} - 4$.

Class 6

$$y' = -\frac{y^3}{x^2(x-1)^2} + \frac{(1-x-x^2)y^2}{x^2(x-1)^2} \quad (63)$$

Solution:

$$C_1 - \text{Ei} \left(1, \frac{y+x^2-x}{xy(x-1)} \right) + \frac{(x-1)y e^{\frac{x-y-x^2}{xy(x-1)}}}{x-1+y} = 0 \quad (64)$$

where $\text{Ei}(n, x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt$ is the exponential integral.

Class 7

$$y' = \frac{(4x^4 + 5x^2 + 1)y^3}{2x^3} + y^2 + \frac{(1-4x^2)y}{2x(x^2+1)} \quad (65)$$

Solution:

$$C_1 + 2 \frac{x+A}{\sqrt[4]{A^2+1}(Ax-1)} + \int^A (z^2+1)^{-5/4} dz = 0 \quad (66)$$

where $A = \frac{x-2yx^4-3yx^2-y}{x(x+yx^2+y)}$

6 Computer algebra routines, tests and performance

The two itemized algorithms described in sections 2.2 and 3 for solving the equivalence problem between two given Abel ODEs were implemented in Maple R5, in the framework of the ODEtools package [12]. The implementation consists of various routines, mainly accomplishing the following:

1. determine whether a given Abel ODE belongs to one of the solvable classes described in the previous sections; in doing that, determine also the function $F(t)$ entering Eq.(2) and the value of the parameters in the case of a parameterized class;
2. use that information to determine the functions $P(t)$ and $Q(t)$ entering Eq.(2) and return a solution to the given ODE by means of changing variables in the solution available for the representative of the class (see the Appendix).

6.1 Representatives with simpler invariants for Classes A, C and D

While preparing the computational routines being presented, we noticed that: if on the one hand the solutions to the representatives for the parameterized classes A, C and D (Eqs.(33, 49, 58)) can be expressed in a relatively simple manner (see the Appendix), on the other hand, for each of these representatives, the form of the three invariants entering Eq.(22) is much simpler if an appropriate redefinition of the class parameter followed by a change of variables of the form Eq.(2) is performed. In turn, the complexity of these invariants is a relevant issue for a computer algebra implementation since simpler invariants lead to simpler GCD and resultant computations.

In the case of Class A (Eq.(33)), redefining the parameter as $\alpha = 5/36 - \kappa/3$ - where κ is the new parameter - and changing variables $\{x = \frac{\sqrt{12t-6}}{2t-1}, y = \frac{(2t-1)u(t)}{\sqrt{12t-6}}\}$ lead to the class representative

$$y' = \frac{((3\kappa - 2x - x^2)y^3 - 9y^2 - 9y)}{9(2x - 1)} \quad (67)$$

and the first invariant s_5^3/s_3^5 for it is

$$\frac{(9\kappa^2 + (30x^2 - 6x + 3)\kappa - 6x^3 + 5x^4 + 7x^2)^3}{9(\kappa + x^2)^5} \quad (68)$$

as opposed to

$$729 \frac{(27\alpha^2 x^8 - (72x^6 + 270x^4 + 15x^8)\alpha + 54x^4 + 36x^2 + 135 + 15x^6 + 2x^8)^3}{x^4(2x^4 - 9\alpha x^4 + 9x^2 + 27)^5} \quad (69)$$

which is the same first invariant but calculated for Eq.(33). Actually a measure of the Maple computational *length* of the three invariants $[s_5^3/s_3^5, s_3 s_7/s_5^2, s_5 s_7/s_3^4]$ entering Eq.(22) shows the values [99, 215, 223] when calculated on Eq.(67) above and [146, 354, 356] when calculated on Eq.(33). Also, the degrees in x of both numerator and denominator of Eq.(68) are lower than the corresponding degrees of Eq.(69).

A similar situation happens with Class C (Eq.(49)), where redefining the parameter as $\alpha = \frac{\kappa-3}{\kappa}$ and changing variables $\left\{x = \frac{\sqrt{(\kappa+1)(3-\kappa)(3t+4)t}}{(\kappa+1)t}, y = \frac{t^2 u(t) \kappa ((3t+4)\kappa + 3t+4)}{\sqrt{(\kappa+1)(3-\kappa)(3t+4)t}}\right\}$ lead to a class representative a bit more complicated than Eq.(49):

$$y' = 4((\kappa - 2)x + \kappa - 3)\kappa x y^3 + 6y^2 - \frac{(6\kappa x + 5\kappa + 3)y}{(3x + 4)\kappa x} \quad (70)$$

for which, however, the computational *length* of the three invariants $[s_5^3/s_3^5, s_3 s_7/s_5^2, s_5 s_7/s_3^4]$ is [110, 273, 305] as opposed to [173, 382, 412] when calculated for Eq.(49). The degrees with respect to x of the numerators and denominators of the invariants of Eq.(70) are also lower than those of the invariants of Eq.(49).

The same reduction in the complexity of the invariants is achieved for class D (Eq.(58)) by redefining the parameter via $C = \frac{9}{8}\sqrt{\kappa}$ and changing variables $\{x = \frac{3\sqrt{2(1-t^2\kappa)}\sqrt{\kappa}}{4(1-t\sqrt{\kappa})}, y = \sqrt{2(1-t^2\kappa)}\sqrt{\kappa}u(t)\}$, leading to

$$y' = \frac{(2x^2\kappa - 2)\kappa y^3 - 3y^2\kappa + \kappa xy}{1 - \kappa x^2} \quad (71)$$

for which the first invariant, s_5^3/s_3^5 , is given by²⁴

$$\frac{((2x^4 + 15x^3)\kappa^2 - (4x^2 + 15x - 9)\kappa + 2)^3}{4\kappa^3(\kappa x^3 - x + 1)^5} \quad (72)$$

as opposed to

$$729 \frac{(C + x^2)^3(2C^4 + 8x^2C^3 + (12x^4 + 15x^2)C^2 + 8x^6C + 2x^8 - 15x^6 + 9x^4)^3}{(2C^3 + 6C^2x^2 + (9x^2 + 6x^4)C + 2x^6 - 9x^4)^5} \quad (73)$$

²⁴The case $\kappa = 0$ is treated separately.

which is the same first invariant but calculated for the class representative Eq.(58). In Eq.(72), not only the degrees with respect to x but also those with respect to the class parameter κ are lower than the corresponding degrees in Eq.(73).

Moreover, for class D this change in the representative of the class also fixes a problem: the representative Eq.(58) is itself invariant under $C \rightarrow -C$ followed by the change of variables

$$\left\{ x = \frac{iC}{t}, y = iu(t) \right\} \quad (74)$$

where i is the imaginary unit, and hence Eq.(73) is invariant under $\{C \rightarrow -C, x \rightarrow iC/x\}$. Consequently, if $\{C, F(t), P(t), Q(t)\}$ is a solution to the equivalence problem between a given ODE and Eq.(58), then $\{-C, \frac{iC}{F(t)}, iP(t), Q(t)\}$ is also a solution; this fact invalidates the use of the interpolation method described in sec. 3.2 with Eq.(58) since that method can only be used when the interpolated solution $C(\alpha)$ is unique.

6.2 Installation

The programs being presented have been written as one more step in the development of the *ODEtools* Maple package [12] and hence are integrated to it and not distributed separately. To install the new Abel routines then what is necessary is to install *ODEtools* to run in the Maple environment by putting the related libraries (two files - **maple.ind** and **maple.lib**) in any directory - say *DE_libraries_directory* - then opening Maple, and adding that directory to Maple's *libname* variable via

```
> libname := DE_libraries_directory, libname:
```

where ‘>’ is the Maple prompt. This instruction automatically updates Maple's **dsolve** subroutines to make use of the new routines for solving Abel ODEs - no further steps are required. In this way, the new Abel ODE routines are automatically used by Maple's **dsolve** when the input ODE is of Abel type, as well as to solve higher order ODEs when they can be reduced to first order Abel ODEs members of the classes discussed in sec. 4 and 5.

Apart from this integration with **dsolve**, it is also possible to try *just* the routines being presented by giving to **dsolve** the extra argument [**Abel**]. For example (Kamke's ODE 37)

```
> ode[37] := diff(y(x),x)-y(x)^3-a*exp(x)*y(x)^2 = 0;
```

$$ode_{37} := y' - y^3 - ae^x y^2 = 0$$

```
> dsolve(ode[37], [Abel]);
```

$$C_1 + \frac{e^{-1/2(ae^x+y^{-1})^2}}{ae^x} + \frac{\sqrt{2\pi}}{2} \operatorname{erf}\left(\frac{\sqrt{2}}{2}(ae^x+y^{-1})\right) = 0 \quad (75)$$

These implicit answers can also be tested in the Maple worksheet, interactively, using the standard Maple **odetest** command.

Due to the intrinsic complexity of Abel equations, the solution for most of the solvable classes is expressed in implicit form and in terms of elliptic integrals and special or hypergeometric functions. Then, to save the time Maple spends in trying to “integrate” these integrals or to “invert” these algebraic expressions it is frequently convenient to call **dsolve** with the optional extra arguments

'useInt, implicit', meaning: use Maple's inert `Int` and return the solution directly in implicit form²⁵. For example, for Kamke's 185

```
> ode[185] := x^7*diff(y(x),x)+2*(x^2+1)*y(x)^3+5*x^3*y(x)^2 = 0
```

$$ode_{185} := x^7 y' + 2(x^2 + 1)y^3 + 5x^3 y^2 = 0$$

the solution obtained using these extra arguments is:

```
> dsolve(ode[185], [Abel], useInt, implicit);
```

$$C_1 + \frac{x}{\sqrt[4]{A^2 + 1}} + 1/2 \int^A (z^2 + 1)^{-5/4} dz = 0 \quad (76)$$

where $A = x^{-1} + \frac{x^2}{y}$. The explicit computation of the integral above leads to a complicated expression with hypergeometric functions somehow obscuring the structure of the solution.

Concerning Maple's difficulty in solving the problem when the class parameter \mathcal{C} involves radicals (see end of sec. 3.1), we implemented an environment variable controlling how *hard* the routines will work, in order to avoid exhausting system resources unless a hard trial is specifically requested. This environment variable is `_Env_odsolve_Abel_try_hard`, it can be assigned a positive integer from 1 to 5, and by default it is assigned to 4; the meaning of the possible values is as follows:

- if `_Env_odsolve_Abel_try_hard = 1` then the algorithm for parameterized Abel classes is disabled, and for non-parameterized classes only a restricted equivalence using $\{x = t, y = P(t)u + Q\}$, that is: with $F(t) = t$ in Eq.(2), is tried;
- if `_Env_odsolve_Abel_try_hard = 2` then for non-parameterized classes a full equivalence using Eq.(2) is tried;
- if `_Env_odsolve_Abel_try_hard = 3` then the algorithm for parameterized Abel classes is enabled but only for *numerical* solutions for the class parameter \mathcal{C} ;
- if `_Env_odsolve_Abel_try_hard = 4` then for parameterized Abel classes both a *numerical* or a rational interpolation solution for the class parameter \mathcal{C} are tried (symbolic variables in the coefficients are allowed but solutions with radicals are not computed) ;
- if `_Env_odsolve_Abel_try_hard = 5` then the algorithm for parameterized Abel classes will also compute solutions involving radicals for the class parameter \mathcal{C} .

Finally, these routines for Abel ODEs were programmed to provide extensive run-time information on the computations being performed through the Maple standard **userinfo & infolevel** scheme. For example, Kamke's ODE 43 belongs to class B (see Table 2.) and both the way how this is determined and the explicit values found for F , P , Q , and the parameter entering Eq.(37) can be seen by setting the **infolevel** as follows²⁶:

```
> infolevel[dsolve] := 4;
> ode[43] :=
```

²⁵The option `implicit` is standard in Maple's `dsolve` and the implementation of the option `useInt` comes with the routines for Abel ODEs presented here.

²⁶We kept few lines to illustrate the *userinfo* feature, and represented the missing ones by '...'.

$$ode_{43} := y' + (3ax^2 + 4a^2x + b)y^3 + 3y^2x = 0$$

```

> dsolve(ode[43], [Abel], implicit);
      ....
The relative invariant s3 is: -3*a*x^2+b-2*x^3
The first absolute invariant s5^3/s3^5 is: 108*(12*a^2*x^3-8*a*x*b + ...
The second absolute invariant s3*s7/s5^2 is: 1/3*(3*a*x^2-b+2*x^3) * ...
The third absolute invariant s5*s7/s3^4 is: 9*(372*a^3*x^4+450*a^2*x^5 + ...
      ....
      ....
->      =====
->      ...checking Abel class B (by Liouville)
Trying a = 0
Trying a = 1
Trying b = 0
Trying b = 1
-> Step 1: checking for a disqualifying factor on F after evaluating x
Trying x = 0
*** No disqualifying factor on F was found ***
-> Step 2: calculating resultants to eliminate F and get candidates for C
*** Candidates for C are [1, 4, 1/4], ***
-> Step 3: looking for a solution F depending on x
-----
C = 1/4 leads to the solutions [{F = -1-3/2*x}]
Interpolated candidate for the class parameter C is: C = 1/4
General testing of the candidate C = 1/4 with arbitrary b
Interpolation is still incomplete; trying next value of b
-----
Trying b = 2
      ....
      ....
General testing of the candidate C = -3/4*b+1 with arbitrary b
-----
C = -3/4*b+1 leads to the solutions [{F = -1-3/2*x}]
General test of C = -3/4*b+1 passed OK;
interpolation for b in this level is complete.
      ....
      ....
General testing of the candidate C = (-3/4*b+a^3)/a^3 with arbitrary a
-----
C = (-3/4*b+a^3)/a^3 leads to the solutions [{F = -1/2*(3*x+2*a)/a}]
General test of C = (-3/4*b+a^3)/a^3 passed OK;
interpolation for b in this level is complete.
-----
Value of the Class parameter solving the problem is:
      C = 1/4*(-3*b+4*a^3)/a^3
Inverse of the transformation solving the problem is:

```

$$\{t = -1/2*(3*x+2*a)/a, u(t) = -2/3*a^2*y(x)\}$$

Solution:

$$C_1 + \frac{\left(\left(\frac{2a+3x}{2a} - A_1\right) K(A_1, -\sqrt{A_2}) - \sqrt{A_2} K(A_1 + 1, -\sqrt{A_2})\right)}{\left(\left(\frac{2a+3x}{2a} - A_1\right) I(A_1, -\sqrt{A_2}) + \sqrt{A_2} I(A_1 + 1, -\sqrt{A_2})\right)} = 0 \quad (77)$$

where $A_1 = \frac{1}{2}\sqrt{\frac{-3b+4a^3}{a^3}}$, $A_2 = \frac{3b-4a^3}{4a^3} + \frac{(2a+3x)^2}{4a^2} - \frac{3}{2a^2y}$, and I and K are respectively the modified Bessel functions of first and second kind.

6.3 Tests and Performance

The idea was to test these computational routines to confirm the correctness of the returned solutions as well as to obtain the classification presented in the previous sections for solvable Abel ODEs. The first testing arena was the 69 Abel examples found in Kamke plus the 9 solvable examples with non-constant invariant from Murphy's book, plus the first 20 Abel ODE examples from [11] mentioned at the end of sec. 4 - totaling 98 Abel ODE examples. The routines passed these tests - the solutions obtained were confirmed to be correct using other symbolic computation tools interactively - and the resulting classification is that shown in Tables 1, 2, 3 and 4 of sec. 4.

The aforementioned test however involves only 40 "non-constant invariant and solvable" Abel ODE examples, and does not fully test the new routines. We have then set up a more thorough test, which can be taken as a test-suite for other computer algebra implementations of methods for solving Abel ODEs. The ideas behind this additional test-suite are summarized as follows:

- Take the representatives of the seven Abel class not depending on parameters, discussed in sec. 4 and 5, and generate with each four more Abel ODEs of the same class by applying the different types of transformations:

1. The general case with F , P and Q arbitrary:

$$tr_1 := \{x = F(t), y(x) = P(t)u(t) + Q(t)\} \quad (78)$$

2. Rational transformation with symbolic coefficients a and b :

$$tr_2 := \left\{x = \frac{a}{t} + bt, y(x) = \frac{u(t)}{t} + 1\right\} \quad (79)$$

3. Non-rational transformation involving an elementary function and symbols:

$$tr_3 := \left\{x = e^t + 1 + \frac{t}{a}, y(x) = u(t) + 1\right\} \quad (80)$$

4. Non-rational transformation involving abstract powers and symbols:

$$tr_4 := \left\{x = \frac{a}{t^n} + \frac{t}{b}, y(x) = u(t) + 1\right\} \quad (81)$$

For example, the first of these four Abel ODEs generated from Eq.(34) is given by:

$$y' = \frac{F' P^2 (-3 + F) y^3}{8 F} + \frac{((3 F - 9) Q - 10) P F' y^2}{8 F} \tag{82}$$

$$+ \left(\frac{((3 F - 9) Q^2 - 20 Q - 3) F'}{8 F} - \frac{P'}{P} \right) y + \frac{((F - 3) Q^3 - 10 Q^2 - 3 Q) F'}{8 F P} - \frac{Q'}{P}$$

and the other three are obtained from this one by replacing F , P and Q by the values implied by Eqs. (79), (80) and (81). The scheme just outlined generates 28 more solvable Abel ODE examples with non-constant invariant (4 per class), and suffices for testing the solving of classes without parameters.

Concerning classes with parameters:

- Take the representatives of the four Abel classes depending on parameters discussed in sec. 4 and 6.1, generate with each one four more Abel ODEs of the same class by applying the transformations Eq.(79) and Eq.(81), preceded by replacing the single parameter \mathcal{C} entering each class representative by $\mathcal{C} = 2$ and $\mathcal{C} = \alpha/\beta$.

This increases by $4 \times 2 \times 2 = 16$ more ODE examples, and the solving of these examples tests both the scheme for numeric values of the parameter \mathcal{C} as well as the case where \mathcal{C} is a rational function of other symbols entering the given ODE.

The time spent by the routines being presented in solving all these $28 + 16 = 44$ additional Abel ODEs with non-constant invariant, is summarized in Tables 5, 6 and 7 as follows:

Class	transf. (i)	transf. (ii)	transf. (iii)	transf. (iv)
1; Eq.(34)	0.255 sec.	0.879 sec.	0.687 sec.	1.256 sec.
2; Eq.(35)	1.530 sec.	1.900 sec.	1.998 sec.	2.620 sec.
3; Eq.(38)	0.277 sec.	6.760 sec.	5.623 sec.	11.635 sec.
4; Eq.(40)	0.456 sec.	7.126 sec.	3.941 sec.	15.511 sec.
5; Eq.(61)	0.610 sec.	2.505 sec.	3.082 sec.	7.877 sec.
6; Eq.(63)	1.379 sec.	120.952 sec.	45.565 sec.	282.103 sec.
7; Eq.(65)	0.896 sec.	15.441 sec.	24.586 sec.	183.557 sec.

Table 5. *Timings for 28 Abel ODEs with non-constant invariant - 7 classes free of parameters.*

Class parameter $\mathcal{C} = 2$			
Class	transf. (ii)	transf. (iv)	
A; Eq.(67)	25.546 sec.	41.304 sec.	
B; Eq.(37)	29.396 sec.	54.171 sec.	
C; Eq.(70)	18.648 sec.	41.425 sec.	
D; Eq.(71)	36.641 sec.	75.962 sec.	

Table 6. *Timings for 8 Abel ODEs with non-constant invariant - "numeric" class parameter.*

Class parameter $\mathcal{C} = \frac{\alpha}{\beta}$			
Class	transf. (ii)	transf. (iv)	
A; Eq.(67)	67.076 sec.	156.425 sec.	
B; Eq.(37)	174.751 sec.	491.210 sec.	
C; Eq.(70)	73.940 sec.	174.761 sec.	
D; Eq.(71)	88.365 sec.	177.314 sec.	

Table 7. *Timings for 8 Abel ODEs with non-constant invariant - "symbolic" class parameter.*

Concerning a comparison of performances between the new routines and those available in other computer algebra systems (CAS), this appeared to us not justified in this case: roughly speaking

none of these CAS return solutions for Abel ODEs with non-constant invariant. More precisely, from Mathematica 3.0, Macsyma 2.7, Maple 5.1 and Mupad 4.0, all of them failed²⁷ in solving any of the 18 Kamke's examples shown in Table 2. (and hence in solving any of the 44 ODEs of Tables 5, 6 and 7), except for Kamke's example 235 - it is an inverse linear ODE - and anyway none of them solved it after transforming it to first kind format.

6.4 Performance with the 1st order ODE examples from Kamke

Although the main purpose of this paper is to present a computational scheme for finding solutions to Abel ODEs, it is interesting to see how **odsolve** - the ODE-solver of the ODEtools Maple package [12] - performs with the addition of these new routines. The performance with all of Kamke's 553 solvable examples²⁸ after incorporating the computational routines presented in this paper is: 97% are solved. This performance is summarized as follows

Degree in y'	ODEs	Solved	Average time	
			<i>solved</i>	<i>fail</i>
1	350	337	3.2 sec.	12.9 sec.
2	145	140	8.8 sec.	61.1 sec.
3	27	26	7.2 sec.	17 sec.
higher	31	30	13.4 sec.	25.2 sec.
Total:	553	533	\approx 6 sec.	\approx 20 sec.

Table 8. Kamke's first order ODEs, solved by **odsolve**: 97%

The number and classification of Kamke's 1st order ODEs still not solved by **odsolve** is now:

Class	Kamke's numbering
rational	452
Riccati	25
NONE	80, 81, 83, 87, 121, 128, 340, 367, 395, 460, 506, 510, 543, 572

Table 9. Kamke's 1st order solvable ODEs for which **odsolve** fails: 3%

where the Abel ODEs numbered in Kamke's book as 47, 48, 205, 206, 237 and 265 not presented in the tables above are known to be solvable only for specific values of their parameters - and for these values **odsolve** succeeds - and the ODEs 234 and 253 were not included since their solutions are not shown in the book or known to us.

7 Conclusions

In this paper, a first classification, according to invariant theory, of solvable non-constant invariant Abel ODEs found in the literature, was presented. Also, a set of new solvable classes, depending on one or no parameters, derived from the analysis of the works by Abel, Liouville and Appell [2, 3, 4], was shown. Computer algebra routines were then developed, in the framework of the Maple ODEtools package, to solve - in principle - any member of these classes by solving its related equivalence problem. The result is a concrete new tool for solving Abel type ODEs fully integrated with Maple's ODE-solver **dsolve**.

The classification shown has had the intention of giving a first step towards organizing in a single reference the integrable cases widely scattered throughout the literature. The derivation of

²⁷For Kamke's ODE 257, Macsyma (2.7) returns a wrong answer in terms of y' .

²⁸We classified as *unsolvable* in general Kamke's examples 50, 55, 56, 74, 79, 82, 202, 219, 250, 269, 331, 370, 461, 503 and 576.

new solvable parameterized classes from the works by Abel and Appell in the 19th century (Classes “C” and “D”) also showed that valuable information can still be obtained from these old papers. In fact, from Tables 1, 2, 3 and 4 in sec. 4, the larger number of integrable cases found in textbooks are particular members of this Class “C” (Eq.(49)) - an integrable class derived from a case somehow disregarded in Abel’s Memoires [2].

As for the computer routines, the implementation presented here for solving the equivalence problem for *parameterized* classes proved to be a valuable tool in most of the Abel ODE examples we were able to collect. The Abel ODE routines here presented were recently integrated to the Maple computer algebra system and are already part of the upcoming Maple release 6.

On the other hand, we note the intrinsic limitation of this Abel ODE problem: most of the solutions can only be obtained in implicit form and in terms of quadratures; in turn, these integrals are usually elliptic integrals so that they cannot be expressed using elementary functions. Also the classification of integrable cases presented here is incomplete in that it is missing - at least - a more thorough analysis of the integrable cases presented in [11]. We are presently working on this topic and expect to succeed in obtaining reportable results in the near future.

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Appendix A²⁹

Class	Representative equation and solution
1	$y' = \frac{3y^2-3y-x}{x(8y-9)}, \quad C_1 + \frac{x^3(4x^2+(8y^2-36y+27)x+4y^4-4y^3)}{(x^2+2x(y^2-3y)+y^4)^3} = 0$
2	$y' = -2y^2x + y^3, \quad C_1 + \frac{x\text{Ai}\left(x^2-\frac{1}{y}\right)+\text{Ai}\left(1,x^2-\frac{1}{y}\right)}{x\text{Bi}\left(x^2-\frac{1}{y}\right)+\text{Bi}\left(1,x^2-\frac{1}{y}\right)} = 0$
3	$y' = \frac{y^3}{4x^2} - y^2, \quad C_1 + \frac{\left(x-\frac{1}{y}\right)\text{Ai}\left(\left(x-\frac{1}{y}\right)^2-\frac{1}{2x}\right)+\text{Ai}\left(1,\left(x-\frac{1}{y}\right)^2-\frac{1}{2x}\right)}{\left(x-\frac{1}{y}\right)\text{Bi}\left(\left(x-\frac{1}{y}\right)^2-\frac{1}{2x}\right)+\text{Bi}\left(1,\left(x-\frac{1}{y}\right)^2-\frac{1}{2x}\right)} = 0$
4	$y' = y^3 - \frac{x+1}{x}y^2, \quad C_1 + \frac{1}{x}e^{\frac{1}{y}-x} - \text{Ei}\left(1, x - \frac{1}{y}\right) = 0$
5	$y' = -\frac{(2x+3)(x+1)y^3}{2x^5} + \frac{(5x+8)y^2}{2x^3},$ $C_1 + \frac{\sqrt{A}}{\sqrt[4]{4\frac{(x+1)^2}{x^2A}+1}} + \int^2 \frac{x+1}{x\sqrt{A}}(z^2+1)^{-5/4} dz = 0, \quad A = \frac{4}{y} - \frac{10}{x} - \frac{6}{x^2} - 4$
6	$y' = -\frac{y^3}{x^2(x-1)^2} + \frac{(1-x-x^2)y^2}{x^2(x-1)^2}, \quad C_1 - \text{Ei}\left(1, \frac{y+x^2-x}{xy(x-1)}\right) + \frac{(x-1)ye^{\frac{x-y-x^2}{xy(x-1)}}}{x-1+y} = 0$
7	$y' = \frac{(4x^4+5x^2+1)y^3}{2x^3} + y^2 + \frac{(1-4x^2)y}{2x(x^2+1)}$ $C_1 + 2\frac{x+A}{\sqrt[4]{A^2+1(Ax-1)}} + \int^A (z^2+1)^{-5/4} dz = 0 \quad A = \frac{x-2yx^4-3yx^2-y}{x(x+yx^2+y)}$
A	$y' = \left(\alpha x + \frac{1}{x} + \frac{1}{x^3}\right)y^3 + y^2,$ $C_1 + \frac{x^3}{y+x} \exp\left(\int \frac{-yx^2}{y+x} \frac{2dz}{z^2-z-\alpha z^3}\right) - \int \frac{-yx^2}{y+x} \exp\left(\int \frac{2dz}{z^2-z-\alpha z^3}\right) dz = 0$
B	$y' = 2(x^2 - \alpha^2)y^3 + 2(x+1)y^2,$ $C_1 + \frac{(\alpha+x)\text{K}\left(\alpha, -\sqrt{x^2+\frac{1}{y}-\alpha^2}\right) + \sqrt{x^2+\frac{1}{y}-\alpha^2}\text{K}\left(1+\alpha, -\sqrt{x^2+\frac{1}{y}-\alpha^2}\right)}{(\alpha+x)\text{I}\left(\alpha, -\sqrt{x^2+\frac{1}{y}-\alpha^2}\right) - \sqrt{x^2+\frac{1}{y}-\alpha^2}\text{I}\left(1+\alpha, -\sqrt{x^2+\frac{1}{y}-\alpha^2}\right)} = 0$
C	$y' = \frac{\alpha(1-x^2)y^3}{2x} + (\alpha-1)y^2 - \frac{\alpha y}{2x}, \quad C_1 + \frac{\alpha}{x} \left(1 - \left(\frac{1}{y} - x\right)^2\right)^{1/\alpha} - 2 \int \frac{1-xy}{y} (1-z^2)^{\frac{1-\alpha}{\alpha}} dz = 0$
D	$y' = -\frac{y^3}{x} - \frac{(\alpha+x^2)y^2}{x^2},$ $C_1 + \frac{(\alpha+1)\text{M}\left(-\frac{\alpha}{2}-\frac{3}{4}, \frac{1}{4}, \frac{1}{2}\left(x-\frac{\alpha}{x}-\frac{1}{y}\right)^2\right) + \left(\frac{x}{y}-x^2\right)\text{M}\left(-\frac{\alpha}{2}+\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\left(x-\frac{\alpha}{x}-\frac{1}{y}\right)^2\right)}{(\alpha^2+\alpha)\text{W}\left(-\frac{\alpha}{2}-\frac{3}{4}, \frac{1}{4}, \frac{1}{2}\left(x-\frac{\alpha}{x}-\frac{1}{y}\right)^2\right) + 2\left(\frac{x}{y}-x^2\right)\text{W}\left(-\frac{\alpha}{2}+\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\left(x-\frac{\alpha}{x}-\frac{1}{y}\right)^2\right)} = 0$

Representative ODEs and their solutions for the Abel ODE classes presented in this work.

²⁹The solution shown for the representative of class D is not valid when α is an integer, or when 2α is a positive integer. In those cases, the solution of the associated Riccati equation Eq.(55) takes many different forms depending on the value of α , which we found inconvenient to present here. $\text{Ei}(n, x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt$ is the exponential integral, $\text{Ai}(x)$ and $\text{Bi}(x)$ are the Airy wave functions, $\text{K}(x)$ and $\text{I}(x)$ are the modified Bessel functions of the first and second kinds, respectively, and $\text{M}(x)$ and $\text{W}(x)$ are the Whittaker functions.