A note on V-binomials' recurrence for Lucas sequence V_n companion to U_n sequence

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Summary
Following [2] (2009) we deliver V-binomials' recurrence formula for Lucas sequence V_n companion to U_n sequence [1] (1878). This formula is not present neither in [1] (1878) nor in [2] (2009), nor in [3] (1915), nor in [4] (1936), nor in [5] (1949) and neither in all other quoted here as "'Lucas (p,q)-people"' references [1-29]- far more not complete. Meanwhile V-binomials' recurrence formula for Lucas sequence V_n easily follows from the original Theorem 17 in [2] absent in quoted papers except for [2] of course. Our formula may and should be confronted with [3] (1915) Fontené recurrence i.e. (6) or (7) identities in [7] (1969) which, as we indicate, also stem easily from the Theorem 17 in [2].

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1 Preliminaries

Notation a,b $a \neq b$ in [1] (1878) is used for the roots of the equation $x^2 = Px - Q$ or $(a,b) \equiv (u,v)$ in [2] (2009) for the roots of the equation $x^2 = \ell x - 1$. The identification $(a,b) \equiv (p,q)$ i.e. p,q are used in "'Lucas (p,q)-people"' publications recently and in recent past (look into not complete list of references [2 - 29] and p,q-references therein).
Lucas (p,q)-people would then use U-identifications:

$$n_{p,q} = \sum_{j=0}^{n-1} p^{n-j-1} q^j = U_n = \frac{p^n - q^n}{p-q},\ 0_{p,q} = U_0 = 0,\ 1_{p,q} = U_1 = 1,$$

where p,q denote now the roots of the equation $x^2 = sx + t$ hence $p + q = s$ and $pq = -t$ and the empty sum convention was used for $0_{p,q} = 0$. Usually one assumes $p \neq q$. In general also $s \neq t$ - though according to the context [10] (1989) $s = t$ may happen to be the case of interest.

The Lucas U-binomial coefficients $\binom{n}{k}_U \equiv \binom{n}{k}_{p,q}$ are then defined as follows

Definition 1 *Let U be as in [1] i.e $U_n = n_{p,q}$ then U-binomial coefficients for any $n, k \in \mathbb{N} \cup \{0\}$ are defined as follows*

$$(1) \qquad \binom{n}{k}_U = \binom{n}{k}_{p,q} = \frac{n_{p,q}!}{k_{p,q}! \cdot (n-k)_{p,q}!} = \frac{n_{p,q}^{\underline{k}}}{k_{p,q}!}$$

where $n_{p,q}! = n_{p,q} \cdot (n-1)_{p,q} \cdot \ldots \cdot 1_{p,q}$ *and* $n_{p,q}^{\underline{k}} = n_{p,q} \cdot (n-1)_{p,q} \cdot \ldots \cdot (n-k+1)_{p,q}$.

Definition 2 *Let V be as in [1] i.e $V_n = p^n + q^n$, hence $V_0 = 2$ and $V_n = p + q = s$. Then V-binomial coefficients for any $n, k \in \mathbb{N} \cup \{0\}$ are defined as follows*

$$(2) \qquad \binom{n}{k}_V = \frac{V_n!}{V_k! \cdot V_(n-k)!} = \frac{V_n^{\underline{k}}}{V_k!}$$

where $V_n! = V_n \cdot V_{n-1} \cdot \ldots \cdot V_1$ *and* $V_n^{\underline{k}} = V_n \cdot V_{n-1} \cdot \ldots \cdot V_{n-k+1}$.

One easily generalizes L-binomial to L-multinomial coefficients [29] .

Definition 3 *Let L be any natural numbers' valued sequence i.e. $L_n \in \mathbb{N}$ and $s \in \mathbb{N}$. L-**multinomial coefficient** is then identified with the symbol*

$$(3) \qquad \binom{n}{k_1, k_2, \ldots, k_s}_L = \frac{L_n!}{L_{k_1}! \cdot \ldots \cdot L_{k_s}!}$$

where $k_i \in \mathbb{N}$ and $\sum_{i=1}^{s} k_i = n$ for $i = 1, 2, \ldots, s$. Otherwise it is equal to zero.

Naturally for any natural n, k and $k_1 + \ldots + k_m = n - k$ the following holds

$$(4) \qquad \binom{n}{k}_L \cdot \binom{n-k}{k_1, k_2, \ldots, k_m}_L = \binom{n}{k, k_1, k_2, \ldots, k_m}_L$$

2 V-binomial coefficients' recurrence

The authors of [2] prove (Th. 17) the following nontrivial recurrence for the general case of $\binom{r+s}{r,s}_{L[p,q]}$ L-binomial arrays in multinomial notation. Let $s, r > 0$. Then

$$(5) \binom{r+s}{r,s}_{L[p,q]} = g_1(r,s) \cdot \binom{r+s-1}{r-1,s}_{L[p,q]} + g_2(r,s) \cdot \binom{r+s-1}{r,s-1}_{L[p,q]}$$

where $\binom{r}{r,0}_L = \binom{s}{0,s}_L = 1$.

$$(6) \qquad L[p,q]_{r+s} = g_1(r,s)\cdot L[p,q]_r + g_2(r,s)\cdot L[p,q]_s.$$

Taking into account the U-addition formula i.e. the first of two trigonometric-like L-addition formulas (42) from [1] [see also [20], [22]] ($L[p,q] = L = U, V$) i.e.

$$(7) \qquad 2U_{r+s} = U_r V_s + U_s V_r, \qquad 2V_{r+s} = V_r V_s + U_s U_r$$

one readily recognizes that the U-binomial recurrence from the Corollary 18 in [2] is identical with the U-binomial recurrence (58) [1]. However there is not companion V-binomial recurrence neither in [1] (1878) nor in [2] (2009).

This V-binomial recurrence is given right now in the form of (5) adapted to $L[p,q] = V[p,q] = V$ - Lucas sequence case.

$$(8) \quad \binom{r+s}{r,s}_{V[p,q]} = h_1(r,s)\binom{r+s-1}{r-1,s}_{V[p,q]} + h_2(r,s)\binom{r+s-1}{r,s-1}_{V[p,q]},$$

where $p \neq q$ and $\binom{r}{r,0}_L = \binom{s}{0,s}_L = 1$.

$$(9) \qquad V_{r+s} = h_1(r,s)V_r + h_2(r,s)V_s.$$

and where $(p \neq q)$

$$(10) \qquad h_1\cdot(p^r q^s - q^r p^s) = p^{r+s}q^s - q^{r+s}p^s,$$

$$(11) \qquad h_2\cdot(q^r p^s - p^r q^s) = p^{r+s}q^r - q^{r+s}p^r.$$

The recurrent relations (13) and (14) in [28] for $n_{p,q}$-binomial coefficients are special cases of this paper formula (5) i.e. of Th. 17 in [2] with straightforward identifications of g_1, g_2 in (13) and in (14) in [28] as well as this paper recurrence (6) for $L = U[p,q]_n = n_{p,q}$ sequence.

$$(12) \qquad g_1 = p^r, \quad g_2 = q^s,$$

or

$$(13) \qquad g_1 = q^r, \quad g_2 = p^s,$$

whle

$$(14) \qquad (s+r)_{p,q} = p^s r_{p,q} + q^r s_{p,q} = (r+s)_{q,p} = q^r s_{p,q} + p^s r_{p,q}.$$

Now let A is any natural numbers' or even complex numbers' valued sequence. One readily sees that also (1915) Fontené recurrence for Fontené-Ward generalized A-binomial coefficients i.e. equivalent identities (6) , (7)

in [7] **are special cases of** this paper formula **(5)** i.e. of Th. 17 in [2] with straightforward identifications of g_1, g_2 in this paper formula (5) identities while this paper recurrence (6) becomes trivial identity.
Namely, the identities (6) and (7) from [7] (1969) read correspondingly:

$$\binom{r+s}{r,s}_A = 1 \cdot \binom{r+s-1}{r-1,s}_A + \frac{A_{r+s}-A_r}{A_s}\binom{r+s-1}{r,s-1}_A, \tag{15}$$

$$\binom{r+s}{r,s}_A = \frac{A_{r+s}-A_s}{A_r} \cdot \binom{r+s-1}{r-1,s}_A + 1 \cdot \binom{r+s-1}{r,s-1}_A, \tag{16}$$

where $p \neq q$ and $\binom{r}{r,0}_L = \binom{s}{0,s}_L = 1$. And finally we have tautology identity

$$A_{s+r} \equiv \frac{A_{r+s}-A_s}{A_r} \cdot A_r + 1 \cdot A_s. \tag{17}$$

As for **combinatorial interpretations** of L-binomial or L-multinomial coefficients we leave that subject apart from this note because this note is to be deliberately short. Nevertheless we direct the reader to some papers and references therein; these are herethe following: [30] (2010),[2] (2009), [29] (2009), [10] (1989),[11] (1991),[13] (1992), [14] (1993), [15] (1994), [16] (1994), [26] (2004),[26] (2004), [29] (2009) and to this end see [8].

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