Non-divergence harmonic maps

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Abstract

We describe work on solutions of certain non-divergence type and therefore non-variational elliptic and parabolic systems on manifolds. These systems include Hermitian and affine harmonics which should become useful tools for studying Hermitian and affine manifolds, resp. A key point is that in addition to the standard condition of nonpositive image curvature that is well known and understood in the theory of ordinary harmonic maps (which arise from a variational problem), here we also need in addition a global topological condition to guarantee the existence of solutions.

Introduction

In this paper, we shall describe concepts and tools from geometric analysis that we have developed for particular classes of manifolds, namely Hermitian and affine ones. Hermitian manifolds are complex manifolds that are also equipped with an Hermitian metric. Similarly, an affine manifold can be equipped with a Riemannian metric as an auxiliary structure. Here, a manifold is said to be flat or affine if it admits an atlas whose coordinate changes are affine transformations.

Basic tools of Riemannian geometry are the geodesics and their higher dimensional generalizations, the harmonic maps. They are the critical points of an energy integral that involves the metric. Therefore, they are backed by a variational structure. This depends on the Levi-Cività connection underlying the Riemannian metric. A Hermitian manifold, however, naturally possesses a different connection, the complex one that respects the complex structure. This connection is different from the Levi-Cività connection unless the manifold is Kähler. Similarly, an affine manifold carries a flat affine connection that has nothing to do with the Levi-Cività connection of the auxiliary Riemannian metric. In particular, that Riemannian metric need not be flat.

Thus, harmonic maps are not naturally defined on such manifolds, and the main point of this paper is to discuss suitable substitutes. Thus, Hermitian harmonic maps, as introduced and studied in [JY], are defined through the complex connection, and affine harmonic maps, as introduced and studied in [JŞ], are determined by the affine connection, and the resulting equations do not satisfy a variational principle. This is already the case for affine geodesics, as there is in general no length or energy functional that they could locally minimize. Also, the Euler-Lagrange equations of variational problems necessarily have a special, divergence-type structure which in general the affine harmonic map equations do not possess. The absence of a variational structure makes the analysis more difficult. Therefore, we need an additional global non-triviality condition to guarantee the existence of an affine harmonic map in a given homotopy class. As in the case of ordinary harmonic maps, nonpositive curvature of the target manifold is also required.

In this paper, we overview the results of [JŞ], and its connections with the previous work of [JY]. For all geometric concepts and notations not explained here, as well as for a recent treatment and survey of the theory of harmonic maps, we refer to [J3] as our standard reference. In particular, we shall use the heat equation method as introduced in the seminal paper [MR] and applied in many subsequent papers in geometric analysis (see [J4] for a more detailed history). However, as we do not have a variational structure at our disposal, we cannot utilize the arguments of those papers and have to proceed rather differently.

1 Coordinate transformations and invariant differential operators

A Riemannian metric γ on a manifold M is locally, that is, w.r.t. local coordinates x^α, of the form

$$\gamma = \gamma_{\alpha\beta} dx^\alpha \otimes dx^\beta, \tag{1}$$

and under coordinate transformations $x = x(y)$, it transforms as

$$\gamma_{\alpha\beta} dx^\alpha \otimes dx^\beta = \gamma_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^\delta} \frac{\partial x^\beta}{\partial y^\eta} dy^\delta dy^\eta =: h_{\delta\eta} dy^\delta dy^\eta. \tag{2}$$

Therefore, the coefficients of the inverse metric tensor transform according to

$$\gamma^{\alpha\beta} \frac{\partial y^\delta}{\partial x^\alpha} \frac{\partial y^\eta}{\partial x^\beta} = h^{\delta\eta}. \tag{3}$$

Now, the second derivative of a function ϕ,

$$\frac{\partial^2 \phi}{\partial x^\alpha \partial x^\beta} \tag{4}$$

transforms into

$$\frac{\partial^2 \phi}{\partial x^\alpha \partial x^\beta} \frac{\partial x^\alpha}{\partial y^\delta} \frac{\partial x^\beta}{\partial y^\eta} + \frac{\partial \phi}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial y^\delta \partial y^\eta}, \tag{5}$$

that is, there is an additional term with *second* derivatives of the coordinate transformation. Therefore, in general,

$$\gamma^{\alpha\beta} \frac{\partial^2 \phi}{\partial x^\alpha \partial x^\beta} \tag{6}$$

is not invariant – unless that second derivative $\frac{\partial^2 x^\alpha}{\partial y^\delta \partial y^\eta}$ vanishes for all indices. This is the case when the coordinate transformations are *affine linear*. In other words, on an affine manifold, the metric trace of the second derivatives of a function ϕ is coordinate invariant.

Analogously, on a complex manifold, we may consider a Hermitian metric

$$\gamma = \gamma_{\alpha\bar{\beta}} dz^\alpha \otimes dz^{\bar{\beta}}, \tag{7}$$

and the Hermitian trace

$$\gamma^{\alpha\bar{\beta}} \frac{\partial^2 \phi}{\partial z^\alpha \partial z^{\bar{\beta}}} \tag{8}$$

is invariant under *holomorphic* coordinate transformations. More generally, the same applies for a map $f : M \to N$ from M into some Riemannian manifold N in place of a function ϕ. Thus, denoting the Christoffel symbols of N in local coordinates by Γ^i_{jk}, we have the invariant operator on an affine manifold M

$$\gamma^{\alpha\beta}\left(\frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta} + \Gamma^i_{jk}\frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta}\right), \quad i = 1, \dots, n. \tag{9}$$

Similarly, on a complex manifold, we obtain the operator

$$\gamma^{\alpha\bar{\beta}}\left(\frac{\partial^2 f^i}{\partial z^\alpha \partial z^{\bar{\beta}}} + \Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\bar{\beta}}}\right), \quad i = 1, \dots, n. \tag{10}$$

We then call a solution ϕ, resp., f of

$$\gamma^{\alpha\beta}\frac{\partial^2 \phi}{\partial x^\alpha \partial x^\beta} = 0; \qquad \gamma^{\alpha\beta}\left(\frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta} + \Gamma^i_{jk}\frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta}\right) = 0, \quad i = 1, \dots, n \tag{11}$$

an affine harmonic function, resp., map. Analogously, a solution of

$$\gamma^{\alpha\bar{\beta}}\frac{\partial^2 \phi}{\partial z^\alpha \partial z^{\bar{\beta}}} = 0; \qquad \gamma^{\alpha\bar{\beta}}\left(\frac{\partial^2 f^i}{\partial z^\alpha \partial z^{\bar{\beta}}} + \Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\bar{\beta}}}\right) = 0, \quad i = 1, \dots, n \tag{12}$$

on a complex manifold is called Hermitian harmonic.

We note that the equations (systems) (11), (12) are not in divergence form, in contrast to the equation (system) for ordinary harmonic functions (maps) on a Riemannian manifold. This makes the existence and regularity theory more difficult.

In fact, ordinary harmonic functions (maps) on a Riemannian manifold M satisfy

$$\frac{1}{\sqrt{\det\gamma}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{\det\gamma}\,\gamma^{\alpha\beta}\frac{\partial \phi}{\partial x^\beta}\right) = 0 \tag{13}$$

$$\frac{1}{\sqrt{\det\gamma}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{\det\gamma}\,\gamma^{\alpha\beta}\frac{\partial f^i}{\partial x^\beta}\right) + \gamma^{\alpha\beta}\Gamma^i_{jk}\frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta} = 0, \quad i = 1, \dots, n, \tag{14}$$

that is, some derivatives of the metric need to compensate second derivatives of the function under coordinate changes in order to make the diferential equation invariant.

2 Harmonic maps without variational or divergence structure

A geometric structure usually induces a particular type of connection that preserves that structure. In Riemannian geometry, the Levi-Cività connection is the unique torsion free connection that preserves the Riemannian metric. For a complex structure we get a canonical complex connection. Similarly, for an affine structure, we have the affine flat connection. As a result, different structures on the same manifold induce different connections. To investigate such structures, we need appropriate tools from geometric analysis. In Riemannian geometry, geodesics and their higher dimensional analogues, harmonic maps, are such tools. Commonly, geodesics and harmonic maps are defined in terms of a variational principle, as critical points of the energy integral. This, however, is special for the Levi-Civitá connection in Riemannian geometry and does not generalize to Hermitian or affine geometry. Thus, as described above, we rather define such objects directly in terms of the relevant connection. We then obtain an elliptic system that can be written in local coordinates, but whose solutions are invariant under coordinate changes in the appropriate category (differentiable, complex, affine). These solutions then yield suitable classes of functions (when the target is $\mathbb{R}$) or maps (when the target is a Riemannian manifold). For instance, maps defined in this way, between Hermitian and Riemannian manifolds are called Hermitian harmonic and harmonic maps from affine flat to Riemannian manifolds are called affine harmonic. The notion of Hermitian harmonic maps was first introduced and investigated by Jost and Yau [JY] and that of affine harmonic maps was first introduced and investigated by Jost and Şimşir [JŞ]. In either case, a solution of the elliptic system was obtained, under suitable conditions, from the associated parabolic system.

Parabolic and elliptic systems with a nonlinearity as in the harmonic map problem and without a variational or divergence structure have been investigated by von Wahl [W1]. However, he was mainly interested in boundary value problems on Euclidean domains and not in the case of closed manifolds. Therefore, in order to treat the central problem of analyzing when the solution of the parabolic system converges to that of elliptic one, a more global approach had to be developed by Jost and Yau for the Hermitian harmonic maps in [JY]. Extensions of existence and uniques results for the Dirichlet problem in the work of Jost and Yau to noncompact but complete domain manifolds were first considered by Lei Ni [N]. Subsequently, Grunau and Kühnel [GK] developed a more flexible method. Throughout this work, harmonic map systems without a variational structure in which the underlying equations is of non-divergence form will be called non-divergence harmonic maps.

2.1 Hermitian harmonic maps

Let M be a compact complex manifold with a Hermitian metric $(\gamma_{\alpha\bar{\beta}})_{\alpha,\beta=1,\ldots,m}$ in local coordinates $z=(z^1,\ldots,z^m)$, and N be a compact Riemannian manifold with $(g_{ij})_{i,j=1,\dots,n}$ in local coordinates $(f^1,\dots,f^n)$. Hermitian harmonic maps $f: M \longrightarrow N$ are defined as the solutions of the semi linear elliptic system

$$\gamma^{\alpha\bar{\beta}}(\frac{\partial^2 f^i}{\partial z^\alpha \partial z^{\bar{\beta}}} + \Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\bar{\beta}}}) = 0, \quad i = 1,\dots,n \tag{15}$$

This is the system first studied by Jost and Yau [JY]. As discussed above, when M is not Kähler, the system (15) is not in divergence form. The method of Jost and Yau consists in studying the associated parabolic equation,

$$f \quad : \quad M \times [0,\infty) \longrightarrow N \tag{16}$$

$$\frac{\partial f^i}{\partial t} = \gamma^{\alpha\bar{\beta}}(\frac{\partial^2 f^i}{\partial z^\alpha \partial z^{\bar{\beta}}} + \Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial x^{\bar{\beta}}}) \tag{17}$$

$$f(z,0) = g(z) \tag{18}$$

where $g: M \longrightarrow N$ is a continuous map. They show that a solution exists for $0 \le t < \infty$, under the assumption that N has non positive sectional curvature and converges to a solution of (15) under the geometric assumption of the following theorem:

Theorem 2.1 (Jost-Yau) *Let M be a compact Hermitian manifold. Let N be a compact Riemannian manifold of negative sectional curvature. Let $g: M \longrightarrow N$ be continuous, and suppose that g is not homotopic to a map g_0 for which there is a nontrivial parallel section of $g_0^{-1}TN$; for instance, assume that g is not homotopic to a map onto a closed geodesic of N. Then there is a Hermitian harmonic map $f: M \longrightarrow N$ homotopic to g.*

In fact, an example in [JY] shows that without this global geometric assumption, a solution of the parabolic system need not converge as $t \to \infty$, but may rather circle around N forever. This is in contrast to the case of ordinary harmonic maps where the variational structure forces a decay of the energy integral along a solution of the parabolic flow which in turn implies that the solution has to settle down asymptotically to a solution of the elliptic system.
As remarked above, when M is Kähler, then something special happens: The Hermitian harmonic map f is simply an ordinary harmonic map.

2.2 Affine harmonic maps

As described above, on an affine manifold M with metric tensor $\gamma_{\alpha\beta}$, we can define an affinely invariant differential operator, $L := \gamma^{\alpha\beta}\frac{\partial^2}{\partial x^\alpha \partial x^\beta}$. A function $f: M \longrightarrow \mathbb{R}$ that satisfies $Lf = 0$ is called affine harmonic. More generally, a map $f: M \longrightarrow N$ where N is a Riemannian manifold with metric g_{ij} and Christoffel symbols Γ^i_{jk} is called affine harmonic if it satisfies

$$\gamma^{\alpha\bar{\beta}}(\frac{\partial^2 f^i}{\partial x^\alpha \partial x^{\bar{\beta}}} + \Gamma^i_{jk}\frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^{\bar{\beta}}}) = 0, \quad i = 1,\dots,n \tag{19}$$

in local coordinates on N. In invariant notation (19) can be written as

$$\gamma^{\alpha\beta} D_\alpha D_\beta f = 0 \tag{20}$$

where D is the connection on the bundle $T^*M \otimes f^{-1}TN$ induced by the flat connection on M and the Levi-Civitá connection on N. Jost and Şimşir obtained the following general existence result for affine harmonic maps, [JŞ].

Theorem 2.2 (Jost - Şimşir) *Let M be a compact affine manifold, N a compact Riemannian manifold of nonpositive sectional curvature. Let $g : M \to N$ be continuous, and suppose g is not homotopic to a map $g_0 : M \to N$ for which there is a nontrivial parallel section of $g_0^{-1}TN$. Then g is homotopic to an affine harmonic map $f : M \to N$.*

In fact, this result is stronger than the one stated in [JŞ]; the latter was formulated only for the special case of *Kähler* affine manifolds in the sense of [CY]. However, in the next section, we shall describe the analytic scheme for showing existence in such a way that it applies to any compact affine manifold M.

Again, one may construct examples to show that the global topological condition is needed in general, see [JŞ]. Using the argument of Al'ber [A], one can also show that the affine harmonic map is unique in its homotopy class under the assumptions of the above theorem. In fact, here, we also need the global condition.

3 Analytic aspects of the existence scheme

Consider the system (19), (20)

$$\frac{\partial f}{\partial t} = \gamma^{\alpha\beta} D_\alpha D_\beta f \tag{21}$$

Linearizing and using standard results about linear parabolic system, which follow from the implicit function theorem , it follows that (21) has a solution for a short time interval $[0, \tau)$, and the interval of existence is open. Dealing with the global situation needs the following steps which are harder.

1. Showing the closedness of the existence interval, for which one needs the nonpositive sectional curvature of the target manifold.

2. Showing that the solution of (21) converges to a non-divergence harmonic map as t approaches ∞, i.e., show that as t approaches ∞, $\frac{\partial f}{\partial t}$ converges to 0.

In order to handle the first step one should show the local boundedness of the energy density function $\eta(f) = \frac{1}{2}\langle df, df\rangle_{T^*M\otimes f^{-1}TN}$ where df stands for the first derivatives of f w.r.t. the spatial variables x. For a detailed treatment of the procedure one may see [JY] and [JŞ]. Closedness of the existence interval and thus the global existence follows from the regularity theory for parabolic equations. In the following, we shall discuss the affine case; the complex case is analogous. Thus, x will now stand for affine coordinates.

One first shows

$$\sup_{x\in M} g_{ij}\frac{\partial f^i}{\partial t}\frac{\partial f^j}{\partial t} \tag{22}$$

is nonincreasing in t. Next, $\eta(f)$ satisfies a linear differential inequality, and we therefore obtain

$$\eta(f(x,t)) \leq c \sup_{t_0\leq\tau\leq t}\int_M \eta(f(.,\tau), \tag{23}$$

for any $t_0 > 0$, see e.g. [J1], Section 3.3. Here and in the sequel, c stands for some constant that can be controlled by the geometry of the manifolds involved, but which we do not make explicit here.
Next, using Jacobi field estimates [J1] and the procedure in [JŞ] one controls the norm of df with respect to the spatial variable x.

$$|df(x,t)| \leq c\left(\int_M \tilde{d}^2(f(.,\tau),f^0)\right)^{1/2} + c \tag{24}$$

where $\tilde{d}(f(.,\tau),f^0)$is the homotopy distance between the initial map $f^0 = f(.,0)$ and the map $f(.,t)$ at time t. It is defined as the length of the shortest geodesic from $f(x,t)$ to $f^0(x)$ in the homotopy class of curves determined by the homotopy between them. Further computation leads to

$$|df(x,t)| \leq c(1+t). \tag{25}$$

Then, (22) and (25) yield C^1-bounds for the solution of (21). In order to get $C^{2,\alpha}$ bounds, one may apply the regularity theory for solutions of linear parabolic equations by the standard bootstrapping argument.

For the second step of the proof one needs to show the convergence of the solution $f(x,t)$ of (21) to a non-divergence harmonic map at ∞. In this case, one needs to require a topological non-triviality condition as expressed in the Theorems 2.1, 2.2 and also once more the nonpositive sectional curvature of the target manifold.

We first choose a point $x_0 \in M$ where $\tilde{d}^2(f(y,t),f^0(y))$ attains its minimum and apply the maximum principle on both the ball $B(x_0,R)$ of radius R and on its complement, to get

$$\int_M \eta(f(.,t)) \leq c\sup_{y\in M}\tilde{d}(f(y,\tau),f^0(y)) + c. \tag{26}$$

Then (23) gives the pointwise estimate

$$|df(x,t)| \leq c(\sup_{y\in M}\tilde{d}(f(y,\tau),f^0(y)))^{1/2} + c. \tag{27}$$

Therefore, for any $x_1, x_2 \in M$, denoting the lift to universal covers by $\tilde{f}$

$$d(\tilde{f}(x_1,t),\tilde{f}(x_2,t)) \leq c(\sup_{y\in M}\tilde{d}(f(y,\tau),f^0(y)))^{1/2} + c. \tag{28}$$

The essential point of the proof then is to exclude that for some sequence $t_n \to \infty$ for all $y \in M$,

$$\tilde{d}(f(y,t_n), f^0(y)) \longrightarrow \infty. \tag{29}$$

For the details, we refer to [JŞ]. For a family of solutions $f(x,t,s) := f(x, t+s)$ depending on a parameter s, using (21)

$$\left(\gamma^{\delta\epsilon}\frac{\partial^2}{\partial x^\delta \partial x^\epsilon} - \frac{\partial}{\partial t}\right)\left(g_{ij}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}\right) = 2\gamma^{\delta\epsilon}\left(g_{ij}\frac{\partial^2 f^i}{\partial x^\delta \partial s}\frac{\partial^2 f^j}{\partial x^\epsilon \partial s} - \frac{1}{2}R_{ijkl}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial x^\delta}\frac{\partial f^k}{\partial s}\frac{\partial f^l}{\partial x^\epsilon}\right) \tag{30}$$

one can conclude that, as t tends to ∞, $\frac{\partial f(x,t)}{\partial t}$ converges to a parallel section $v(x)$ along f_∞ which, however, is excluded in the assumptions of Theorem 2.2. Hence,

$$\frac{\partial f(x,t)}{\partial t} \to 0 \text{ for } t \to \infty. \tag{31}$$

This, together with the smooth convergence of $f(.,t_n)$ to f_∞, shows that f_∞ solves the elliptic system, i.e., it is affine harmonic.
In fact, (30) is also the key for the uniqueness of an affine harmonic map in its homotopy class.

4 Some possible future developments

1. The theory of non-divergence harmonic maps can be investigated in a more general setting.
2. Dirichlet and Neumann boundary value problems for affine harmonic maps can be studied. In this case, the eternal circling of the solution is prevented by the Dirichlet boundary values. Hence, here we do not need a global topological condition. Of course, one now needs to prove boundary regularity, but this problem can be solved by the methods of Jost and Yau [JY], or that of von Wahl [W2].
3. The method of Grunau and Kühnel [GK] should be extended to show the existence of affine harmonic maps from a complete affine to a complete Riemannian manifold.
4. Most importantly, the results of Theorem 2.2 should be applied to obtain rigidity results in affine differential geometry.

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