THE NUMBER OF GOLDBACH REPRESENTATIONS OF AN INTEGER

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1. Introduction

Let Λ be the von Mangoldt function and

$$R(n) = \sum_{h_1 + h_2 = n} \Lambda(h_1)\Lambda(h_2)$$

be the counting function for the Goldbach numbers. This paper is devoted to study the behaviour of the average order of magnitude of $R(n)$ for $n \in [1, N]$, where N is a large integer. We have the following

Theorem 1. *Let $N \geq 2$ and assume the Riemann Hypothesis (RH) holds. Then*

$$\sum_{n=1}^{N} R(n) = \frac{N^2}{2} - 2 \sum_{\rho} \frac{N^{\rho+1}}{\rho(\rho+1)} + \mathcal{O}\big(N \log^3 N\big),$$

where $\rho = 1/2 + i\gamma$ runs over the non-trivial zeros of the Riemann zeta function $\zeta(s)$.

The first result of this kind was proved in 1991 by Fujii who subsequently improved it (see [4]-[5]-[6]) until reaching the error term $\mathcal{O}\big((N \log N)^{4/3}\big)$. Then Granville [8]-[9] gave an alternative proof of the same result and, finally, Bhowmik and Schlage-Puchta [2] were able to reach the error term $\mathcal{O}\big(N \log^5 N\big)$; in [2] they also proved that the error term is $\Omega(N \log \log N)$.

Our result improves the upper bound in Bhowmik and Schlage-Puchta [2] by a factor $\log^2 N$. In fact, this seems to be the limit of the method in the current state of the circle-method technology: see the remark after the proof.

If one admits the presence of some suitable weight in our average, this loss can be avoided. For example, using the Fejér weight we could work with $L(N; \alpha) = \sum_{n=-N}^{N} (N - |n|)e(n\alpha) = |T(N; \alpha)|^2$ instead of $T(N; \alpha)$ in (23). The key property is that, for $1/N < |\alpha| \leq 1/2$, the function $L(N; \alpha)$ decays as α^{-2} instead of $|\alpha|^{-1}$ and so the dissection argument in (26) is now more efficient and does not cause any loss of logs. Such a phenomenon is well-known from the literature about the existence of Goldbach numbers in short intervals, see, *e.g.*, Languasco and Perelli [12].

In fact we will obtain Theorem 1 as a consequence of a weighted result. Letting $\psi(x) = \sum_{m \leq x} \Lambda(m)$, we have

Theorem 2. *Let $2 \leq y \leq N$ and assume the Riemann Hypothesis (RH) holds. Then*

$$\max_{y \in [2,N]} \left| \sum_{n=1}^{y} \Big[R(n) - (2\psi(n) - n) \Big] e^{-n/N} \right| \ll N \log^3 N. \tag{1}$$

The key reason why we are able to derive Theorem 1 from Theorem 2 via partial summation is that the exponential weight in (1) just varies in the range $[e^{-1/N}, e^{-1}]$ and so it does not change the order of magnitude of the functions involved.

We will use the original Hardy and Littlewood [10] circle method setting, *i.e.*, the weighted exponential sum

$$\widetilde{S}(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n/N} e(n\alpha), \tag{2}$$

where $e(x) = \exp(2\pi i x)$, since it lets us avoid the use of Gallagher's Lemma (Lemma 1 of [7]) and hence, in this conditional case, it gives slightly sharper results, see Lemma 1 below. Such a function was also used by Linnik [13, 14]. The new ingredient in this paper is Lemma 5 below in which we unconditionally detect the existence of the term $-2 \sum_{\rho} N^{\rho+1}/(\rho(\rho+1))$ by solving an arithmetic problem connected with the original one (see eq. (11) below). In the previously mentioned papers this is obtained applying the explicit formula for $\psi(n)$ twice.

The ideas that lead to Theorem 1 and 2 work also for the sum of $k \geq 3$ primes, *i.e.*, for the function

$$R_k(n) = \sum_{h_1+\ldots+h_k=n} \Lambda(h_1) \cdots \Lambda(h_k).$$

We can prove the following

Theorem 3. *Let $k \geq 3$ be an integer, $N \geq 2$ and assume the Riemann Hypothesis (RH) holds. Then*

$$\sum_{n=1}^{N} R_k(n) = \frac{N^k}{k!} - k \sum_{\rho} \frac{N^{\rho+k-1}}{\rho(\rho+1)\cdots(\rho+k-1)} + \mathcal{O}_k\left(N^{k-1} \log^k N\right),$$

where $\rho = 1/2 + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$.

The proof of Theorem 3 is completely similar to the one of Theorems 1 and 2. We just remark that the main differences are in the use of the explicit formula for

$$\psi_j(t) := \frac{1}{j!} \sum_{n \leq t} (t-n)^j \Lambda(n)$$

where j is a non-negative integer, and of the following version of Lemma 1 of [11]:

Lemma. *Assume the Riemann Hypothesis (RH) holds. Let $N \geq 2$, $z = 1/N - 2\pi i\alpha$ and $\alpha \in [-1/2, 1/2]$. Then*

$$\left|\widetilde{S}(\alpha) - \frac{1}{z}\right| \ll N^{1/2}\left(1 + (N|\alpha|)^{1/2}\right) \log N.$$

Another connected problem we can address with this technique is a short-interval version of Theorem 1. We can prove the following

Theorem 4. *Let $2 \leq H \leq N$ and assume the Riemann Hypothesis (RH) holds. Then*

$$\sum_{n=N}^{N+H} R(n) = HN + \frac{H^2}{2} - 2 \sum_{\rho} \frac{(N+H)^{\rho+1} - N^{\rho+1}}{\rho(\rho+1)} + \mathcal{O}\left(N \log^2 N \log H\right),$$

where $\rho = 1/2 + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$.

Also in this case we do not give a proof of Theorem 4; we just remark that the main difference is in the use of the exponential sum $\sum_{n=N}^{N+H} e(n\alpha)$ instead of $\sum_{n=1}^{N} e(n\alpha)$.

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2. Setting of the circle method

For brevity, throughout the paper we write

$$z = \frac{1}{N} - 2\pi i \alpha \qquad \text{where } N \text{ is a large integer and } \alpha \in [-1/2, 1/2]. \tag{3}$$

The first lemma is a L^2-estimate for the difference $\widetilde{S}(\alpha) - 1/z$.

Lemma 1 (Languasco and Perelli [12]). *Assume RH. Let N be a sufficiently large integer and z be as in (3). For $0 \le \xi \le 1/2$, we have*

$$\int_{-\xi}^{\xi} \left| \widetilde{S}(\alpha) - \frac{1}{z} \right|^2 \mathrm{d}\alpha \ll N\xi \log^2 N.$$

This follows immediately from the proof of Theorem 1 of [12] since the quantity we would like to estimate here is $\widetilde{R}_1 + \widetilde{R}_3 + \widetilde{R}_5$ there.

Lemma 1 is the main reason why we use $\widetilde{S}(\alpha)$ instead of its truncated form $S(\alpha) = \sum_{n=1}^{N} \Lambda(n)e(n\alpha)$ as in Bhowmik and Schlage-Puchta [2]. In fact Lemma 1 lets us avoid the use of Gallagher's Lemma [7] which leads to a loss of a factor $\log^2 N$ in the final estimate (compare Lemma 1 with Lemma 4 of [2]). For a similar phenomenon in a slightly different situation see also Languasco [11].

The next four lemmas do not depend on RH. By the residue theorem one can obtain

Lemma 2 (Eq. (29) of [12]). *Let $N \ge 2$, $1 \le n \le N$ and z be as in (3). We have*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e(-n\alpha)}{z^2} \, \mathrm{d}\alpha = n e^{-n/N} + \mathcal{O}(1)$$

uniformly for every $n \le N$.

Lemma 3. *Let N be a sufficiently large integer and z be as in (3). We have*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \widetilde{S}(\alpha) - \frac{1}{z} \right|^2 \, \mathrm{d}\alpha = \frac{N}{2} \log N + \mathcal{O}\big(N(\log N)^{1/2}\big).$$

Proof. By the Parseval theorem and the Prime Number Theorem we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\widetilde{S}(\alpha)|^2 \, \mathrm{d}\alpha = \sum_{m=1}^{\infty} \Lambda^2(m)e^{-2m/N} = \frac{N}{2} \log N + \mathcal{O}(N).$$

Recalling that the equation at the beginning of page 318 of [12] implies

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}\alpha}{|z|^2} = \frac{N}{\pi} \arctan(\pi N),$$

the Lemma immediately follows using the relation $|a - b|^2 = |a|^2 + |b|^2 - 2\Re(a\overline{b})$ and the Cauchy-Schwarz inequality. $\qquad\square$

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Let

$$V(\alpha) = \sum_{m=1}^{\infty} e^{-m/N} e(m\alpha) = \sum_{m=1}^{\infty} e^{-mz} = \frac{1}{e^z - 1}. \tag{4}$$

Lemma 4. *If z satisfies (3) then $V(\alpha) = z^{-1} + \mathcal{O}(1)$.*

Proof. We recall the that the function $w/(e^w - 1)$ has a power-series expansion with radius of convergence 2π (see for example Apostol [1], page 264). In particular, uniformly for $|w| \leq 4 < 2\pi$ we have $w/(e^w - 1) = 1 + \mathcal{O}(|w|)$. Since z satisfies (3) we have $|z| \leq 4$ and the result follows. $\square$

Let now

$$T(y; \alpha) = \sum_{n=1}^{y} e(n\alpha) \ll \min\left(y; \frac{1}{\|\alpha\|}\right). \tag{5}$$

Lemma 5. *Let N be a large integer, $2 \leq y \leq N$ and z be as in (3). We have*

$$\int_{-1/2}^{1/2} T(y; -\alpha) \frac{(\widetilde{S}(\alpha) - 1/z)}{z} \, d\alpha = \sum_{n=1}^{y} e^{-n/N}(\psi(n) - n) + \mathcal{O}\big((yN \log N)^{1/2}\big). \tag{6}$$

We remark that Lemma 5 is unconditional and hence it implies, using also Lemma 6, that the ability of detecting the term depending on the zeros of the Riemann ζ-function in Theorem 1 does not depend on RH.

Proof. Writing $\widetilde{R}(\alpha) = \widetilde{S}(\alpha) - 1/z$, by Lemma 4 we have

$$\int_{-1/2}^{1/2} T(y; -\alpha) \frac{\widetilde{R}(\alpha)}{z} \, d\alpha = \int_{-1/2}^{1/2} T(y; -\alpha) \widetilde{R}(\alpha) V(\alpha) \, d\alpha + \mathcal{O}\left(\int_{-1/2}^{1/2} |T(y; -\alpha)| \, |\widetilde{R}(\alpha)| \, d\alpha\right)$$

$$= \int_{-1/2}^{1/2} T(y; -\alpha) \widetilde{R}(\alpha) V(\alpha) \, d\alpha + \mathcal{O}\big((yN \log N)^{1/2}\big), \tag{7}$$

since, by the Parseval theorem and Lemma 3, the error term above is

$$\ll \left(\int_{-1/2}^{1/2} |T(y; -\alpha)|^2 \, d\alpha\right)^{1/2} \left(\int_{-1/2}^{1/2} |\widetilde{R}(\alpha)|^2 \, d\alpha\right)^{1/2} \ll (yN \log N)^{1/2}.$$

Again by Lemma 4, we have

$$\widetilde{R}(\alpha) = \widetilde{S}(\alpha) - \frac{1}{z} = \widetilde{S}(\alpha) - V(\alpha) + \mathcal{O}(1)$$

and hence (7) implies

$$\int_{-1/2}^{1/2} T(y; -\alpha) \frac{\widetilde{R}(\alpha)}{z} \, d\alpha = \int_{-1/2}^{1/2} T(y; -\alpha) \big(\widetilde{S}(\alpha) - V(\alpha)\big) V(\alpha) \, d\alpha$$

$$+ \mathcal{O}\left(\int_{-1/2}^{1/2} |T(y; -\alpha)| \, |V(\alpha)| \, d\alpha\right) + \mathcal{O}\big((yN \log N)^{1/2}\big). \tag{8}$$

The Cauchy-Schwarz inequality and the Parseval theorem imply that

$$\int_{-1/2}^{1/2} |T(y; -\alpha)| \, |V(\alpha)| \, d\alpha \leq \left(\int_{-1/2}^{1/2} |T(y; -\alpha)|^2 \, d\alpha\right)^{1/2} \left(\int_{-1/2}^{1/2} |V(\alpha)|^2 \, d\alpha\right)^{1/2}$$

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$$\ll \left(y \sum_{m=1}^{\infty} e^{-2m/N}\right)^{1/2} \ll (yN)^{1/2}. \tag{9}$$

By (8)-(9), we have

$$\int_{-1/2}^{1/2} T(y; -\alpha) \frac{\widetilde{R}(\alpha)}{z} \, d\alpha = \int_{-1/2}^{1/2} T(y; -\alpha) \big(\widetilde{S}(\alpha) - V(\alpha)\big) V(\alpha) \, d\alpha + \mathcal{O}\big((yN \log N)^{1/2}\big). \tag{10}$$

Now, by (2) and (4), we can write

$$\widetilde{S}(\alpha) - V(\alpha) = \sum_{m=1}^{\infty} (\Lambda(m) - 1) e^{-m/N} e(m\alpha)$$

so that

$$\int_{-1/2}^{1/2} T(y; -\alpha) \big(\widetilde{S}(\alpha) - V(\alpha)\big) V(\alpha) \, d\alpha$$

$$= \sum_{n=1}^{y} \sum_{m_1=1}^{\infty} (\Lambda(m_1) - 1) e^{-m_1/N} \sum_{m_2=1}^{\infty} e^{-m_2/N} \int_{-1/2}^{1/2} e((m_1 + m_2 - n)\alpha) \, d\alpha$$

$$= \sum_{n=1}^{y} \sum_{m_1=1}^{\infty} (\Lambda(m_1) - 1) e^{-m_1/N} \sum_{m_2=1}^{\infty} e^{-m_2/N} \begin{cases} 1 & \text{if } m_1 + m_2 = n \\ 0 & \text{otherwise} \end{cases}$$

$$= \sum_{n=1}^{y} e^{-n/N} \sum_{m_1=1}^{n-1} (\Lambda(m_1) - 1) = \sum_{n=1}^{y} e^{-n/N} (\psi(n-1) - (n-1)), \tag{11}$$

since the condition $m_1 + m_2 = n$ implies that both variables are $< n$. Now $\psi(n) = \psi(n - 1) + \Lambda(n)$, so that

$$\sum_{n=1}^{y} e^{-n/N} (\psi(n-1) - (n-1)) = \sum_{n=1}^{y} e^{-n/N} (\psi(n) - n) + \mathcal{O}(y).$$

By (10)-(11) and the previous equation, we have

$$\int_{-1/2}^{1/2} T(y; -\alpha) \frac{\widetilde{R}(\alpha)}{z} \, d\alpha = \sum_{n=1}^{y} e^{-n/N} (\psi(n) - n) + \mathcal{O}\big(y + (yN \log N)^{1/2}\big)$$

$$= \sum_{n=1}^{y} e^{-n/N} (\psi(n) - n) + \mathcal{O}\big((yN \log N)^{1/2}\big)$$

since $y \leq N$, and hence (6) is proved. $\qquad \square$

Lemma 6. *Let $M > 1$ be an integer. We have that*

$$\sum_{n=1}^{M} (\psi(n) - n) = -\sum_{\rho} \frac{M^{\rho+1}}{\rho(\rho+1)} + \mathcal{O}(M).$$

Proof. We recall the definition of $\psi_0(t)$ as $\psi(t) - \Lambda(t)/2$ if t is an integer and as $\psi(t)$ otherwise. Hence

$$\sum_{n=1}^{M} \psi(n) = \sum_{n=1}^{M} \psi_0(n) + \frac{1}{2} \sum_{n=1}^{M} \Lambda(n) = \sum_{n=1}^{M} \psi_0(n) + \mathcal{O}(M)$$

by the Prime Number Theorem. Using the fact that $\psi_0(n) = \psi_0(t)$ for every $t \in (n, n+1)$, we also get

$$\sum_{n=1}^{M} \psi_0(n) = \sum_{n=1}^{M} \int_{n}^{n+1} \psi_0(t) \, \mathrm{d}t = \int_{0}^{M} \psi_0(t) \, \mathrm{d}t + \mathcal{O}(M).$$

Remarking that

$$\sum_{n=1}^{M} n = \int_{0}^{M} t \, \mathrm{d}t + \mathcal{O}(M),$$

we can write

$$\sum_{n=1}^{M} (\psi(n) - n) = \int_{0}^{M} (\psi_0(t) - t) \, \mathrm{d}t + \mathcal{O}(M) = \int_{2}^{M} (\psi_0(t) - t) \, \mathrm{d}t + \mathcal{O}(M), \qquad (12)$$

since the integral on $(0, 2]$ gives a contribution $\mathcal{O}(1)$. For $t \geq 2$ we will use the explicit formula (see eq. (9)-(10) of §17 of Davenport [3])

$$\psi_0(t) = t - \sum_{|\gamma| \leq Z} \frac{t^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2} \log\left(1 - \frac{1}{t^2}\right) + R_\psi(t, Z), \qquad (13)$$

where

$$R_\psi(t, Z) \ll \frac{t}{Z} \log^2(tZ) + (\log t) \min\left(1; \frac{t}{Z\|t\|}\right). \qquad (14)$$

The term $-\frac{\zeta'}{\zeta}(0) - \frac{1}{2} \log\left(1 - \frac{1}{t^2}\right)$ gives a contribution $\mathcal{O}(M)$ to the integral over $[2, M]$ in (12). We need now a L^1 estimate of the error term defined in (14). Let

$$\mathcal{E}(M, Z) := \int_{2}^{M} |R_\psi(t, Z)| \mathrm{d}t. \qquad (15)$$

The first term in (14) gives a total contribution to $\mathcal{E}(M, Z)$ which is

$$\ll \frac{M^2}{Z} \log^2(MZ). \qquad (16)$$

The second term in (14) gives a total contribution to $\mathcal{E}(M, Z)$ which is

$$\ll \log M \sum_{n=2}^{M} \left[\int_{n}^{n+1/2} \min\left(1; \frac{t}{Z(t-n)}\right) \mathrm{d}t + \int_{n+1/2}^{n+1} \min\left(1; \frac{t}{Z(n+1-t)}\right) \mathrm{d}t \right]$$

$$\ll \log M \sum_{n=2}^{M} \left(\int_{n}^{n+1/Z} \mathrm{d}t + \int_{n+1/Z}^{n+1/2} \frac{t \, \mathrm{d}t}{Z(t-n)} + \int_{n+1-1/Z}^{n+1} \mathrm{d}t + \int_{n+1/2}^{n+1-1/Z} \frac{t \, \mathrm{d}t}{Z(n+1-t)} \right)$$

$$\ll \log M \sum_{n=2}^{M} \left(\frac{3}{Z} + \frac{2n+1}{Z} \log\left(\frac{Z}{2}\right) \right) \ll \frac{M^2}{Z} \log M \log Z. \qquad (17)$$

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Combining (15)-(17), for $Z = M \log^2 M$ we have that

$$\mathcal{E}(M, Z) \ll M. \tag{18}$$

Inserting now (13) and (18) into (12) we obtain

$$\sum_{n=1}^{M} (\psi(n) - n) = - \sum_{|\gamma| \le Z} \frac{M^{\rho+1}}{\rho(\rho+1)} + \mathcal{O}(M). \tag{19}$$

The lemma follows from (19), by remarking that

$$\sum_{|\gamma| > Z} \frac{M^{\rho+1}}{\rho(\rho+1)} \ll M^2 \int_{Z}^{+\infty} \frac{\log t}{t^2} \, \mathrm{d}t \ll \frac{M^2 \log Z}{Z} \ll \frac{M}{\log M}$$

since $Z = M \log^2 M$. $\qquad\square$

3. Proof of Theorem 1

We will get Theorem 1 as a consequence of Theorem 2. By partial summation we have

$$\sum_{n=1}^{N} \Big[R(n) - (2\psi(n) - n) \Big] = \sum_{n=1}^{N} e^{n/N} \Big\{ \Big[R(n) - (2\psi(n) - n) \Big] e^{-n/N} \Big\}$$

$$= e \sum_{n=1}^{N} \Big[R(n) - (2\psi(n) - n) \Big] e^{-n/N}$$

$$- \frac{1}{N} \int_{0}^{N} \Big\{ \sum_{n=1}^{y} \Big[R(n) - (2\psi(n) - n) \Big] e^{-n/N} \Big\} e^{y/N} \, \mathrm{d}y + \mathcal{O}(1). \tag{20}$$

Inserting (1) in (20) we get

$$\sum_{n=1}^{N} \Big[R(n) - (2\psi(n) - n) \Big] \ll N \log^3 N$$

and hence

$$\sum_{n=1}^{N} R(n) = \sum_{n=1}^{N} n + 2 \sum_{n=1}^{N} (\psi(n) - n) + \mathcal{O}\big(N \log^3 N\big). \tag{21}$$

Theorem 1 now follows inserting Lemma 6 and the identity $\sum_{n=1}^{N} n = N^2/2 + \mathcal{O}(N)$ in (21).

4. Proof of Theorem 2

Let $2 \le y \le N$. We first recall the definition of the singular series of the Goldbach problem: $\mathfrak{S}(k) = 0$ for k odd and

$$\mathfrak{S}(k) = 2 \prod_{p > 2} \Big(1 - \frac{1}{(p-1)^2} \Big) \prod_{\substack{p \mid k \\ p > 2}} \frac{p-1}{p-2}$$

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for k even. Hence, using the well known estimate $R(n) \ll n\mathfrak{S}(n) \ll n \log \log n$, we remark that

$$\sum_{n=1}^{y} \Big[R(n) - (2\psi(n) - n) \Big] e^{-n/N} \ll \sum_{n=1}^{y} n \log \log n \ll y^2 \log \log y. \qquad (22)$$

So it is clear that (1) holds for every $y \in [2, N^{1/2}]$.

Assume now that $y \in [N^{1/2}, N]$ and let $\alpha \in [-1/2, 1/2]$. Writing $\widetilde{R}(\alpha) = \widetilde{S}(\alpha) - 1/z$, recalling (5) we have

$$\sum_{n=1}^{y} e^{-n/N} R(n) = \sum_{n=1}^{y} \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{S}(\alpha)^2 e(-n\alpha) \, \mathrm{d}\alpha = \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{S}(\alpha)^2 T(y; -\alpha) \, \mathrm{d}\alpha$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{T(y; -\alpha)}{z^2} \, \mathrm{d}\alpha + 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{T(y; -\alpha) \widetilde{R}(\alpha)}{z} \, \mathrm{d}\alpha + \int_{-\frac{1}{2}}^{\frac{1}{2}} T(y; -\alpha) \widetilde{R}(\alpha)^2 \, \mathrm{d}\alpha$$

$$= I_1(y) + I_2(y) + I_3(y), \qquad (23)$$

say.

Evaluation of $I_1(y)$. By Lemma 2 we obtain

$$I_1(y) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{T(y; -\alpha)}{z^2} \, \mathrm{d}\alpha = \sum_{n=1}^{y} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e(-n\alpha)}{z^2} \, \mathrm{d}\alpha = \sum_{n=1}^{y} \Big(n e^{-n/N} + \mathcal{O}(1) \Big)$$

$$= \sum_{n=1}^{y} n e^{-n/N} + \mathcal{O}(y). \qquad (24)$$

Estimation of $I_2(y)$. By (6) of Lemma 5 we obtain

$$I_2(y) = 2 \sum_{n=1}^{y} e^{-n/N} (\psi(n) - n) + \mathcal{O}\big((yN \log N)^{1/2} \big). \qquad (25)$$

Estimation of $I_3(y)$. Using (5) and Lemma 1 we have that

$$I_3(y) \ll \int_{-\frac{1}{2}}^{\frac{1}{2}} |T(y; -\alpha)| |\widetilde{R}(\alpha)|^2 \, \mathrm{d}\alpha \ll y \int_{-\frac{1}{y}}^{\frac{1}{y}} |\widetilde{R}(\alpha)|^2 \, \mathrm{d}\alpha + \int_{\frac{1}{y}}^{\frac{1}{2}} \frac{|\widetilde{R}(\alpha)|^2}{\alpha} \, \mathrm{d}\alpha + \int_{-\frac{1}{2}}^{-\frac{1}{y}} \frac{|\widetilde{R}(\alpha)|^2}{|\alpha|} \, \mathrm{d}\alpha$$

$$\ll N \log^2 N + \sum_{k=1}^{\mathcal{O}(\log y)} \frac{y}{2^k} \int_{\frac{2^k}{y}}^{\frac{2^{k+1}}{y}} |\widetilde{R}(\alpha)|^2 \, \mathrm{d}\alpha \ll N \log^2 N + \sum_{k=1}^{\mathcal{O}(\log y)} \frac{y}{2^k} N \frac{2^{k+1}}{y} \log^2 N$$

$$\ll N \log^2 N \log y. \qquad (26)$$

End of the proof. Inserting (24) and (25)-(26) into (23) we immediately have

$$\sum_{n=1}^{y} e^{-n/N} R(n) = \sum_{n=1}^{y} n e^{-n/N} + 2 \sum_{n=1}^{y} e^{-n/N} (\psi(n) - n) + \mathcal{O}\big(N \log^2 N \log y \big).$$

Hence

$$\sum_{n=1}^{y} e^{-n/N} \Big[R(n) - (2\psi(n) - n) \Big] \ll N \log^2 N \log y$$

and the maximum of the right hand side is attained at $y = N$. Thus we can write

$$\max_{y \in [N^{1/2}, N]} \left| \sum_{n=1}^{y} \left[R(n) - (2\psi(n) - n) \right] e^{-n/N} \right| \ll N \log^3 N. \tag{27}$$

Combining (22) and (27) we get that Theorem 2 is proved.

Remark. Let

$$f(\alpha) = f_N(\alpha) = \begin{cases} \frac{1}{2} N^{1/2} \log N & \text{if } \|\alpha\| \le (\log N)^{-1}, \\ 0 & \text{if } \|\alpha\| > (\log N)^{-1}. \end{cases}$$

Then f satisfies both Lemma 1 and Lemma 3 in the sense that

$$\int_{-\xi}^{\xi} |f(\alpha)|^2 \, d\alpha \ll N\xi(\log N)^2$$

for all $\xi \in [0, 1/2]$, and

$$\int_{-1/2}^{1/2} |f(\alpha)|^2 \, d\alpha = \frac{1}{2} N \log N,$$

but

$$\int_{1/y}^{1/2} \frac{|f(\alpha)|^2}{\alpha} \, d\alpha = \frac{1}{4} N (\log N)^2 \int_{1/y}^{1/\log N} \frac{d\alpha}{\alpha} = \frac{1}{4} N (\log N)^2 \log(y/\log N) \asymp N (\log N)^3$$

for $y = N^{1/2}$ and sufficiently large N. This means that the crucial bound for $I_3(y)$ in (26) is essentially optimal in the present state of knowledge, and that it can not be improved without deeper information on $\widetilde{S}(\alpha) - z^{-1}$, such as the stronger analogue of Lemma 1 that follows from a suitable form of Montgomery's Pair-Correlation Conjecture.

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