On the secondary Steenrod algebra

Christian Nassau

Abstract. We introduce a new model for the secondary Steenrod algebra at the prime 2 which is both smaller and more accessible than the original construction of H.-J. Baues.

We also explain how BP can be used to define a variant of the secondary Steenrod algebra at odd primes.

Contents

1. Introduction

Let A be the Steenrod algebra. In [Bau06], H.-J. Baues has constructed an exact sequence $B_\bullet$

$$A \rightarrowtail B_1 \xrightarrow{\partial} B_0 \twoheadrightarrow A \tag{1.1}$$

which captures the algebraic structure of secondary cohomology operations in ordinary mod p cohomology. This sequence is called the *secondary Steenrod algebra* and its knowledge allows, among other things, to give a purely

Date: November 11, 2010.
1991 *Mathematics Subject Classification.* 55S20.
Key words and phrases. secondary Steenrod algebra.

algebraic description of the d_2-differential in the classical Adams spectral sequence (see [BJ06]).

Unfortunately, the construction of $B_\bullet$ is not very explicit and apparently not many topologists have become familiar with it. The aim of the present note is to show that there is a smaller and much more accessible model which captures the same information. In fact our model is so simple that we can describe it in this introduction:

Fix $p = 2$ and let D_0 be the Hopf algebra that represents power series

$$f(x) = \sum_{k\geq 0} \xi_k x^{2^k} + \sum_{0\leq k<l} 2\xi_{k,l} x^{2^k+2^l}$$

under composition modulo 4. There is a natural map $\pi : D_0 \twoheadrightarrow A$ and a decomposition

$$D_0 = \mathbb{Z}/4\{\mathrm{Sq}(R)\} \oplus \sum_{-1\leq k<l} Y_{k,l} A \tag{1.2}$$

where $\mathrm{Sq}(R), Y_{k,l} \in D_0$ are dual to ξ^R resp. $\xi_{k+1,l+1}$ with respect to the natural basis $\{\, \xi^R,\, 2\xi^R\xi_{k,l} \,\}$ of $D_{0*} = \mathbb{Z}/4[\xi_n, 2\xi_{k,l}]$.

Here are some computations that can help to become familiar with D_0: $\mathrm{Sq}^1\mathrm{Sq}^1 = 2\mathrm{Sq}^2 + Y_{-1,0}$, $\mathrm{Sq}^1 Y_{-1,0} = Y_{-1,0}\mathrm{Sq}^1 + 2\mathrm{Sq}(0,1)$. Let $Q_k = \mathrm{Sq}(\Delta_{k+1})$ for the exponent sequence Δ_k with $\xi^{\Delta_k} = \xi_k$ and $P^s_t = \mathrm{Sq}(2^s\Delta_t)$. Then $Q_0Q_k = \mathrm{Sq}(\Delta_1 + \Delta_{k+1}) + Y_{-1,k}$ and $[Q_0, Q_k] = Y_{-1,k}$ if $k > 0$. One also finds

$$P^s_t P^s_t = \begin{cases} 2P^{s+1}_t & (s+1<t), \\ 2P^{s+1}_t + Y_{t-2,2s}\mathrm{Sq}\left((2^s-1)\Delta_t\right) & (s+1=t). \end{cases}$$

So for example $\mathrm{Sq}(0,2)\cdot\mathrm{Sq}(0,2) = 2\mathrm{Sq}(0,4) + Y_{0,2}\mathrm{Sq}(0,1)$. More computations can be found in Figure 1.

For products involving $Y_{k,l}$ there is the simple formula

$$aY_{k,l} = \sum_{i,j\geq 0} Y_{k+i,l+j} \urcorner(\xi_i^{2^{k+1}}\xi_j^{2^{l+1}}, a) \tag{1.3}$$

if we interpret the $Y_{k,l}$ with $k \geq l$ as

$$Y_{k,l} = \begin{cases} Y_{l,k} & (l<k), \\ 2\mathrm{Sq}(\Delta_{k+2}) & (l=k). \end{cases} \tag{1.4}$$

Here we have written $\urcorner(p, a)$ for the contraction of $a \in A$ by $p \in A_*$ defined via $\langle \urcorner(p,a), q\rangle = \langle a, pq\rangle$ for $q \in A_*$. Let $\kappa(a) = \urcorner(\xi_1, a)$.

We now define our model $D_\bullet$ for the secondary Steenrod algebra to be the sequence

$$A \rightarrowtail \underbrace{(A + \mu_0 A + \textstyle\sum_{-1\le k,\,0\le l} U_{k,l}A)/\!\sim}_{=:D_1} \xrightarrow{\partial} D_0 \xrightarrow{\pi} A.$$

D_1 is an A-bimodule via $a\mu_0 = \mu_0 a + \kappa(a)$ and

$$aU_{k,l} = \sum_{i,j\ge 0} U_{k+i,l+j} \urcorner(\xi_i^{2^{k+1}}\xi_j^{2^{l+1}}, a). \tag{1.5}$$

The relations defining D_1 are

$$U_{k,l} = \begin{cases} U_{l,k} + \mathrm{Sq}(\Delta_{k+1} + \Delta_{l+1}) & (l < k), \\ \mu_0 \mathrm{Sq}(\Delta_{k+2}) + \mathrm{Sq}(2\Delta_{k+1}) & (l = k). \end{cases} \tag{1.6}$$

∂ is zero on $A \subset D_1$ and otherwise given by $\partial\mu_0 a = 2a$ and $\partial U_{k,l}a = Y_{k,l}a$.

The following is our main result:

Theorem 1.1. *There is a weak equivalence $B_\bullet \to D_\bullet$ of crossed algebras that is the identity on π_0 and π_1.*

Recall that a crossed algebra [Bau06, 5.1.6] is an exact sequence of the form $B_\bullet$ with B_0 an algebra, B_1 a B_0-bimodule and a bilinear differential $\partial : B_1 \to B_0$ with $(\partial b)b' = b(\partial b')$ for $b, b' \in B_1$. The homotopy groups $\pi_0(B_\bullet) := \operatorname{coker}\partial$ and $\pi_1(B_\bullet) := \ker\partial$ will mostly be A in our examples.

This theorem makes it easy to compute threefold Massey products in the Steenrod algebra. Think of $D_\bullet$ as the splice of the two short exact sequences

$$\begin{tikzcd} A \arrow[r, tail] & D_1 \arrow[r, two heads, "\partial"'] & R_D \arrow[l, bend right, "u"'] \end{tikzcd}, \qquad \begin{tikzcd} R_D \arrow[r, tail] & D_0 \arrow[r, two heads] & A \arrow[l, bend right, "\sigma"'] \end{tikzcd}$$

and pick sections σ and u as indicated. A simple choice, for example, would be $\sigma(\sum c_i \mathrm{Sq}(R_i)) = \sum \widehat{c_i}\mathrm{Sq}(R_i)$ with $\widehat{(-)} : \mathbb{Z}/2 \to \mathbb{Z}/4$ given by $\widehat{0} = 0$ and $\widehat{1} = 1$. For u one can take the map

$$2\mathrm{Sq}(R) \mapsto \mu_0\mathrm{Sq}(R), \qquad Y_{k,l}\mathrm{Sq}(R) \mapsto U_{k,l}\mathrm{Sq}(R) \qquad (\text{for } k < l) \tag{1.7}$$

which is right-linear. For $a, b \in A$ one then has $\sigma(ab) = \sigma(a)\sigma(b) + \partial\tau(a,b)$ with $\tau(a,b) = u\,(\sigma(ab) - \sigma(a)\sigma(b)) \in D_1$. Associativity of the multiplication in A dictates that

$$\langle a, b, c\rangle := \tau(ab, c) - \tau(a,b)\sigma(c) - \tau(a, bc) + \sigma(a)\tau(b,c)$$

is a ∂-cycle, hence in A. $\langle a, b, c\rangle$ is the Massey product in question. It is only defined up to an indeterminacy coming from the choices of σ and u.

As an example, consider the case $a = b = c = \mathrm{Sq}(0,2)$. With σ and u chosen as above one has $\sigma(a)\sigma(b) = 2\mathrm{Sq}(0,4) + Y_{0,2}\mathrm{Sq}(0,1)$, so $\tau(a,b) =$

$\mu_0\mathrm{Sq}(0,4) + U_{0,2}\mathrm{Sq}(0,1)$. One finds

$$\begin{aligned}\langle a,b,c\rangle &= \mathrm{Sq}(0,2)\tau(b,c) - \tau(a,b)\mathrm{Sq}(0,2)\\ &= \mu_0 \underbrace{[\mathrm{Sq}(0,2),\mathrm{Sq}(0,4)]}_{=\mathrm{Sq}(0,1,0,1)} + U_{0,2} \underbrace{[\mathrm{Sq}(0,2),\mathrm{Sq}(0,1)]}_{=0} + U_{2,2}\mathrm{Sq}(0,1)\\ &= \mu_0\mathrm{Sq}(0,1,0,1) + (\mu_0\mathrm{Sq}(0,0,0,1) + \mathrm{Sq}(0,0,2))\,\mathrm{Sq}(0,1)\\ &= \mathrm{Sq}(0,1,2)\end{aligned}$$

which recovers a result of Kristensen and Madsen [KM69]. A straightforward computation, whose details we leave to the interested reader, now generalizes this to

Corollary 1.2. *Let $t \geq 1$. Then $\langle P_t^s, P_t^s, P_t^s\rangle$ is zero for $s < t-1$ and $\langle P_t^{t-1}, P_t^{t-1}, P_t^{t-1}\rangle \ni \mathrm{Sq}\left((2^{t-1}-1)\Delta_t + 2^t\Delta_{t+1}\right)$.*

The plan of the paper is as follows. In the first section we will review the definition and structure of $D_\bullet$ and sketch proofs for the claims in this introduction. In section 3 we will construct an intermediate sequence $E_\bullet$ with a weak equivalence $E_\bullet \to D_\bullet$. We then construct a comparison map $B_\bullet \to E_\bullet$ in section 4, thereby proving the main Theorem. Finally, the appendix sketches the relation of the odd-primary secondary Steenrod algebra with the algebra of BP operations.

Before we proceed, however, I want to thank Mamuka Jibladze for many stimulating emails on the subject. The first such email arrived in May 2004 and this is when my interest in the secondary Steenrod algebra began. Without his guidance it would have been a lot more difficult to wrap my head around Baues's wonderful construction. I also thank Hans-Joachim Baues for very constructive comments on an earlier draft of this paper.

2. The construction of $D_\bullet$

2.1. Definition. As in the introduction, we let

$$D_{0*} = \mathbb{Z}/4[\xi_k, 2\xi_{k,l}\,|\,0 \leq k < l,\, \xi_0 = 1].$$

This is turned into a Hopf algebra with coproduct

$$\begin{aligned}\Delta(\xi_n) &= \sum_{i+j=n} \xi_i^{2^j} \otimes \xi_j + 2\sum_{0\leq k<l} \xi_{n-1-k}^{2^k}\xi_{n-1-l}^{2^l} \otimes \xi_{k,l}\\ \Delta(\xi_{n,m}) &= \xi_{n,m}\otimes 1 + \sum_{k\geq 0} \xi_{n-k}^{2^k}\xi_{m-k}^{2^k} \otimes \xi_{k+1}\\ &\quad + \sum_{0\leq k<l}\left(\xi_{n-k}^{2^k}\xi_{m-l}^{2^l} + \xi_{m-k}^{2^k}\xi_{n-l}^{2^l}\right)\otimes \xi_{k,l}.\end{aligned}$$

We list some basic properties of its dual in the following

Lemma 2.1. *Let $D_0 = \mathrm{Hom}(D_{0*}, \mathbb{Z}/4)$ be the dual algebra and let $\mathrm{Sq}(R)$, $Y_{k,l}(R) \in D_0$ be defined by*

$$\langle \mathrm{Sq}(R), \xi^S \rangle = \delta_{R,S}, \qquad \langle \mathrm{Sq}(R), 2\xi_{m,n}\xi^S \rangle = 0,$$
$$\langle Y_{k,l}(R), \xi^S \rangle = 0, \qquad \langle Y_{k,l}(R), 2\xi_{m,n}\xi^S \rangle = 2\delta_{k+1,m}\delta_{l+1,n}\delta_{R,S}.$$

Write $Y_{k,l}$ for $Y_{k,l}(0)$. The following is true:

1. *There is a multiplicative map $\pi : D_0 \to A$ with $\mathrm{Sq}(R) \mapsto \mathrm{Sq}(R)$.*
2. *One has $Y_{k,l}(R) = Y_{k,l}\mathrm{Sq}(R)$.*
3. *The kernel $R_D = \ker \pi$ is $2D_0 + \sum_{-1 \le k < l} Y_{k,l}A$ and satisfies $R_D^2 = 0$.*
4. *The commutation rule (1.3) holds with $Y_{k,l}$ as in (1.4) for $k \ge l$.*

Proof. The verification is straightforward. □

We will encounter the following A-bimodules more than once.

Lemma 2.2. *There are A-bimodules U, V with*

$$V = \sum_{-1 \le k} V_k A, \quad U = \sum_{-1 \le k,l} U_{k,l}A$$

and relations

$$aV_k = \sum_{i \ge 0} V_{k+i} \urcorner(\xi_i^{2^{k+1}}, a), \quad aU_{k,l} = \sum_{i,j \ge 0} U_{k+i,l+j} \urcorner(\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a).$$

Furthermore, let $R_{k,l} = U_{k,l} + U_{l,k}$ and $R_{k,k} = U_{k,k}$ for $-1 \le k < l$ and

$$K = \sum_{-1 \le k < l} R_{k,l}A + \sum_{-1 \le k} R_{k,k}A.$$

Then

$$aR_{k,l} = \sum_{-1 \le n < m} R_{n,m} \urcorner(\xi_{n-k}^{2^{k+1}} \xi_{m-l}^{2^{l+1}} + \xi_{m-k}^{2^{k+1}} \xi_{n-l}^{2^{l+1}}, a), \tag{2.1}$$

$$aR_{k,k} = \sum_{0 \le i} R_{k+i,k+i} \urcorner(\xi_i^{2^{k+2}}, a) + \sum_{0 \le i < j} R_{k+i,l+j} \urcorner(\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a) \tag{2.2}$$

and K is a bimodule, too. All of U, V and K are free A-modules from both left and right with basis the $U_{k,l}$, V_k, resp. $R_{k,l}$ and $R_{k,k}$. The same is true for the sub-bimodules

$$V' = \sum_{0 \le k} V_k A, \qquad U' = \sum_{-1 \le k,\, 0 \le l} U_{k,l}A, \qquad K' = \sum_{0 \le k < l} R_{k,l}A + \sum_{0 \le k} R_{k,k}A$$

where the generators V_{-1}, $U_{*,-1}$ and $R_{-1,*}$ have been left out.

Proof. This is also straightforward. □

We will need the following computation in A.

Lemma 2.3. *Let $a \in A$ and $k \geq 0$, $l \geq 1$. Then*

$$aQ_k = \sum_{i \geq 0} Q_{k+i} \urcorner \left(\xi_i^{2^{k+1}}, a \right), \tag{2.3}$$

$$\begin{aligned} aP_l^1 = \sum_{i \geq 0} P_{l+i}^1 \urcorner \left(\xi_i^{2^{l+1}}, a \right) + \kappa(a) Q_{l+1} \\ + \sum_{l \leq i < j} Q_i Q_j \urcorner \left(\xi_{l-i}^{2^l} \xi_{l-j}^{2^l}, a \right). \end{aligned} \tag{2.4}$$

Proof. Recall that A_* is canonically an A-bimodule with

$$\Delta(p) = \sum_R \mathrm{Sq}(R) p \otimes \xi^R = \sum_R \xi^R \otimes p \mathrm{Sq}(R).$$

One has $\langle a\mathrm{Sq}(R), p \rangle = \langle a, \mathrm{Sq}(R)p \rangle$ and $\langle \mathrm{Sq}(R)a, p \rangle = \langle a, p\mathrm{Sq}(R) \rangle$. Upon dualization (2.3) therefore becomes the identity

$$Q_k p = \sum_{i \geq 0} (pQ_{k+i}) \cdot \xi_i^{2^{k+1}}.$$

Here both sides are derivations in p, so it only remains to check equality on the ξ_n which is easily done.

The second claim can be proved similarly, but with messier details. We leave this to the skeptical reader. □

The following Lemma is the key to the definition of D_1. Recall that $A + \mu_0 A$ carries the bimodule structure $a\mu_0 = \mu_0 a + \kappa(a)$.

Lemma 2.4. *There is a bilinear map $\lambda : K' \to A + \mu_0 A$ with*

$$\begin{aligned} R_{k,l} &\mapsto \mathrm{Sq}(\Delta_{k+1} + \Delta_{l+1}), \\ R_{k,k} &\mapsto \mathrm{Sq}(2\Delta_{k+1}) + \mu_0 \mathrm{Sq}(\Delta_{k+2}). \end{aligned}$$

Proof. We need to show that λ respects the relations (2.1) and (2.2).

By (2.3) one has

$$aQ_k Q_l = \sum_{i,j \geq 0} Q_{k+i} Q_{l+j} \urcorner (\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a).$$

Using $Q_k Q_l = Q_l Q_k$ and $Q_k^2 = 0$ this immediately implies compatibility with (2.1).

For (2.2) note $a\lambda(R_{k,k}) = aP_{k+1}^1 + \kappa(a) Q_{k+1} + \mu_0 a Q_{k+1}$. The claim is therefore equivalent to

$$\begin{aligned} aP_{k+1}^1 + \kappa(a) Q_{k+1} &= \sum_{0 \leq i} P_{k+i+1}^1 \urcorner (\xi_i^{2^{k+2}}, a) + \sum_{0 \leq i < j} Q_{k+i} Q_{l+j} \urcorner (\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a), \\ aQ_{k+1} &= \sum_{0 \leq i} Q_{k+i+1} \urcorner (\xi_i^{2^{k+2}}, a). \end{aligned}$$

These are again just variants of (2.3) and (2.4). □

Now let $D_1 = (A + \mu_0 A + U')/\operatorname{graph}(\lambda)$. This is easily seen to agree with the definition in the introduction.

Lemma 2.5. *Let $\partial U_{k,l} = Y_{k,l}$ and $\partial\mu_0 = 2$. This defines an exact sequence*

$$A \rightarrowtail D_1 \xrightarrow{\partial} D_0 \xrightarrow{\pi} A.$$

Proof. Lemma 2.4 shows that D_1 is indeed a bimodule. That ∂ is well-defined and bilinear follows from the relations (1.3). Finally, D_1 can be written as the direct sum

$$D_1 = A + \mu_0 A + \sum_{-1 \le k < l} U_{k,l} A.$$

From this the exactness of the sequence is obvious. □

2.2. Represented Functors. Some of the previous constructions can be given meaningful descriptions when we look at their associated functors. Unfortunately, we have not been able to find a good explication for the map λ, so we eventually have to resort to pure algebra in our construction of $D_\bullet$.

Let $\mathrm{Alg}^{\mathrm{c}}_{\mathbb{Z}/4}$ be the category of commutative algebras over $\mathbb{Z}/4$.

Lemma 2.6. *There is a natural isomorphism $\operatorname{Hom}_{\mathrm{Alg}^{\mathrm{c}}_{\mathbb{Z}/4}}(D_{0*}, -) \xrightarrow{\cong} G(-)$ where $G(R) \subset R[[x]]$ is the group*

$$\Big\{ f(x) = \sum_{k \ge 0} t_k x^{2^k} + \sum_{0 \le k < l} t_{k,l} x^{2^k + 2^l} \,\Big|\, t_0 = 1,\ J^2 = 0 \text{ for } J = (2, t_{k,l}) \subset R \Big\}.$$

Proof. A $\phi : D_{0*} \to R$ maps to the f with $t_k = \phi(\xi_k)$ and $t_{k,l} = \phi(2\xi_{k,l})$. □

The bimodules U and V can be understood by looking at the functors

$$V_!(R) = G(R) \times \Big\{ v(x) = \sum_{k \ge 0} v_k x^{2^k} \,\Big|\, v(x)^2 = 2v(x) = 0 \Big\},$$

$$U_!(R) = G(R) \times \Big\{ f_2(x,y) = \sum_{k,l \ge 0} u_{k,l} x^{2^k} y^{2^l} \,\Big|\, f_2(x,y)^2 = 2f_2(x,y) = 0 \Big\}.$$

The group operation is given by $(f_1, v) \circ (g_1, w) = (f_1 g_1, v g_1 + w)$ resp. $(f_1, f_2) \circ (g_1, g_2) = (f_1 g_1, f_2(g_1 \times g_1) + g_2)$.

$V_!$ and $U_!$ are represented by algebras $D_{0*}[v_k]/J^2$ and $D_{0*}[u_{k,l}]/J^2$ where J is the ideal $(2, v_k)$ resp. $(2, u_{k,l})$. V and U can then be recovered as the duals of the degree 1 part of these algebras.

We can use this to at least partially explain the map from U to D_0.

Lemma 2.7. *The map $\phi : U \to D_0$ with $U_{k,l} \mapsto Y_{k,l}$ and $U_{k,k} \mapsto 2Q_{k+1}$ is associated to the natural transformation*

$$U(R) \ni f = (f_1, f_2) \mapsto f^{\mathrm{eff}} \in G(R)$$

with $f^{\mathrm{eff}}(x) = f_1(x) + f_2(x,x)$.

Proof. We have an isomorphism $D_{0*}[u_{k,l}]/J^2 = D_{0*}[2w_{k,l}]$ and will use the $w_{k,l}$ in our computation for the sake of clarity. Recall that $\langle Q_k a, p\rangle = \langle a, (\partial p)/(\partial\xi_{k+1})\rangle$ for $a \in A$, $p \in A_*$. Therefore the dual $\phi_* : D_{0*} \to U_*$ is given by

$$p \mapsto 2\sum_{k\geq 0}(\partial p)/(\partial\xi_{k+1})w_{k,k} + \sum_{0\leq k<l} 2(\partial p)/(\partial\xi_{k,l})\,(w_{k,l}+w_{l,k})\,.$$

The map $\widehat{\phi_*} : D_{0*} \to D_{0*}[2w_{k,l}]$ with $p \mapsto p + \phi_*(p)$ is multiplicative since ϕ_* is a derivation. It therefore does correspond to a natural transformation $U_!(R) \to G(R)$. To see that this transformation is $f \mapsto f^{\mathrm{eff}}$ one just has to check that $\widehat{\phi_*}(\xi_{n+1}) = \xi_{n+1} + 2w_{n,n}$ and $\widehat{\phi_*}(2\xi_{k,l}) = 2\xi_{k,l} + 2w_{k,l} + 2w_{l,k}$. □

The bilinearity of ϕ expresses the fact, that $f \mapsto f^{\mathrm{eff}}$ is multiplicative. This is also easy to see computationally.

Lemma 2.8. *One has* $(fg)^{\mathrm{eff}} = f^{\mathrm{eff}} \circ g^{\mathrm{eff}}$.

Proof. We have

$$\begin{aligned}(fg)^{\mathrm{eff}}(x) &= f_1(g_1(x)) + f_2(g_1(x), g_1(x)) + g_2(x,x),\\ f^{\mathrm{eff}}(g^{\mathrm{eff}}(x)) &= f_1(g_1(x)+g_2(x,x)) + f_2(g_1(x)+g_2(x,x), g_1(x)+g_2(x,x)).\end{aligned}$$

Since $g_2^k = 0$ for $k \geq 2$ we have

$$\begin{aligned}f_1(g_1(x)+g_2(x,x)) &= f_1(g_1(x)) + g_2(x,x),\\ f_2(g_1(x)+g_2(x,x), g_1(x)+g_2(x,x)) &= f_2(g_1(x), g_1(x))\end{aligned}$$

which implies $(fg)^{\mathrm{eff}}(x) = f^{\mathrm{eff}}(g^{\mathrm{eff}}(x))$. □

3. The construction of $E_\bullet$

We now prepare ourselves for the comparison between our $D_\bullet$ and the $B_\bullet$ of Baues. It turns out that an intermediate $E_\bullet$ is required. The reason is that $D_\bullet$, although sufficient for the computational applications of the theory, does not capture all of the structure of $B_\bullet$. The latter carries a comultiplication which turns it into a *secondary Hopf algebra* and the associated invariants L and S are crucial for the comparison. We will therefore now pass to a slightly larger $E_\bullet$ where this extra structure can be expressed.

3.1. Definition. Let $X = \sum_{-1\leq k,l} X_{k,l}A$ be a copy of U with $U_{k,l}$ renamed $X_{k,l}$ and let $X' \subset X$ be the subspace without $X_{-1,-1}A$. Let $\widehat{E_k} = D_k + X' + \mu_0 X'$ for $k = 0, 1$. We will write $e = e_D + e_X$ for the decomposition of $e \in \widehat{E_k}$ into the D_k and $X + \mu_0 X$ components. Let $\rho : E_\bullet \to D_\bullet$ denote the projection $e \mapsto e_D$. We extend ∂ to $\widehat{E_\bullet}$ via $\partial e = \partial e_D + e_X$. This defines an exact sequence

$$A \rightarrowtail \widehat{E_1} \xrightarrow{\partial} \widehat{E_0} \xrightarrow{\pi} A. \tag{3.1}$$

We need to define a multiplication on $\widehat{E_0}$. Note that there is an isomorphism $U \cong V \otimes_A V$ where $U_{k,l} \leftrightarrow V_k \otimes V_l$. We can therefore write $X_{k,l} = X_k X_l$ where the X_k are generators of a copy V_X of V. Let $\psi : A \to V_X'$ be given by $\psi(a) = \sum_{k\geq 0} X_k \urcorner(\xi_{k+1}, a)$. ψ is a derivation because one has $\psi(a) = X_{-1}a - aX_{-1}$. Recall that $\kappa : A \to A$ is also a derivation.

Lemma 3.1. *Let* $* : D_0 \otimes D_0 \to D_0 + X + \mu_0 X$ *be given by*

$$a * b = ab + \psi(a)\psi(b)\mu_0 + X_{-1}\psi(a)\kappa(b) \tag{3.2}$$

and extend this to all of $\widehat{E_0}$ *via* $d*m = \pi(d)m$, $m*d = m\pi(d)$ *and* $mm' = 0$ *for* $d \in D_0$ *and* $m, m' \in X + \mu_0 X$. *Then* $*$ *is associative.*

Proof. The only questionable case is when all three factors are in D_0. But this is a straightforward computation:

$$\begin{aligned}
&(a*b)*c = \\
&= abc + \psi(ab)\psi(c)\mu_0 + X_{-1}\psi(ab)\kappa(c) + \psi(a)\psi(b)\mu_0 c + X_{-1}\psi(a)\kappa(b)c \\
&= abc + \psi(a)b\psi(c)\mu_0 + a\psi(b)\psi(c)\mu_0 + X_{-1}\psi(a)b\kappa(c) + X_{-1}a\psi(b)\kappa(c) \\
&\qquad + \psi(a)\psi(b)c\mu_0 + \psi(a)\psi(b)\kappa(c) + X_{-1}\psi(a)\kappa(b)c, \\
&a*(b*c) = \\
&= abc + \psi(a)\psi(bc)\mu_0 + X_{-1}\psi(a)\kappa(bc) + a\psi(b)\psi(c)\mu_0 + aX_{-1}\psi(b)\kappa(c) \\
&= abc + \psi(a)b\psi(c)\mu_0 + \psi(a)\psi(b)c\mu_0 + X_{-1}\psi(a)\kappa(b)c + X_{-1}\psi(a)b\kappa(c) \\
&\qquad + a\psi(b)\psi(c)\mu_0 + X_{-1}a\psi(b)\kappa(c) + \psi(a)\psi(b)\kappa(c).
\end{aligned}$$

□

Figure 1 illustrates the multiplication in E_0 with the computation of the first few Adem relations.

We will define $E_0 \subset \widehat{E_0}$ by a condition on the coefficients of $Y_{-1,*}$, $X_{-1,*}$ and $X_{*,-1}$. To formulate that condition we need to define two more maps.

Lemma 3.2. *Let* $\theta_D : D_0 \to V$ *be the map that extracts the* $Y_{-1,k}$. *In other words, let*

$$\theta_D(\mathrm{Sq}(R)) = 0, \quad \theta_D(Y_{-1,n}a) = V_n a, \quad \theta_D(Y_{k,l}a) = 0 \quad \textit{for } k \neq -1.$$

Then $\widehat{\theta_D} : D_0 \to V + \mu_0 V$ *with* $\widehat{\theta_D}(d) = \theta_D(d) + \psi(d)\mu_0$ *is a derivation.*

Proof. We sketch a quick computational proof here. A better argument will be given later from the functorial point of view.

We already know that ψ is a derivation, so we just need to show $\theta_D(de) = d\theta_D(e) + \theta_D(d)e + \psi(d)\kappa(e)$. Since θ_D sees only the $\xi_{0,n}$ we can compute $\theta_D(de)$ from the coproduct formula

$$\Delta\xi_{0,n} = \xi_{0,n} \otimes 1 + \sum_{k\geq 0} \xi_{n-k}^{2^k} \otimes \xi_{0,k} + \xi_{n-1} \otimes \xi_1$$

and these summands translate to $\theta_D(d)e$, $d\theta_D(e)$ and $\psi(d)\kappa(e)$. □

Similarly, let $\theta_E : \widehat{E_0} \to V$ extract the $X_{-1,k}$:

$$\theta_E(X_{-1,k}a) = V_k a, \quad \theta_E(X_{l,-1}a) = 0,$$
$$\theta_E\left(D_0 + \mu_0 X + \sum\nolimits_{k,l\geq 0} X_{k,l}A\right) = 0.$$

Lemma 3.3. *One has $\theta_E(d * e) = \theta_E(d)e + d\theta_E(e) + \psi(d_D)\kappa(e_D)$ for $d, e \in \widehat{E_0}$.*

Proof. This is a straightforward computation. See also the discussion in Remark 3.9 below. □

Lemma 3.4. *Define*

$$\widetilde{E_0} = D_0 + \sum_{k,l\geq 0} X_{k,l}A + \sum_{k,l\geq 0} \mu_0 X_{k,l}A + \sum_{k\geq 0} X_{-1,k}A \subset \widehat{E_0}$$

and let $E_0 \subset \widetilde{E_0}$ be the subset where $\theta_D \circ \rho$ and θ_E coincide. Then E_0 is closed under the multiplication $*$.

Proof. It's clear that $\widetilde{E_0}$ is multiplicatively closed since $*$ cannot generate any $X_{k,-1}$ if this is not already part of one factor.

That E_0 is also multiplicatively closed follows from the identical formulas for $\theta_D(de)$ and $\theta_E(de)$. □

Corollary 3.5. *Let $E_1 = \partial^{-1}(E_0) \subset \widehat{E_1}$. Then*

$$A \rightarrowtail E_1 \xrightarrow{\partial} E_0 \xrightarrow{\pi} A. \tag{3.3}$$

is a crossed algebra $E_\bullet$ with a canonical projection $\rho : E_\bullet \twoheadrightarrow D_\bullet$.

Proof. Clear. □

3.2. Represented Functors.

Lemma 3.6. *For $f(x) \in G(R)$ let $\tau_f(x)$ and $\theta_f(x)$ be defined by the decomposition*

$$f(x) = x + \tau_f(x^2) + x\theta_f(x^2) \tag{3.4}$$

and write $\overline{f}(x) = f(x) - x$. Then

$$\overline{fg}(x) = \overline{f}(g(x)) + \overline{g}(x), \tag{3.5}$$
$$\theta_{fg}(x) = \theta_f(g(x)) + \theta_g(x) + \xi_1^f \overline{g}(x), \tag{3.6}$$

where $\xi_1^f = \tau_f'(0)$ is the coefficient of x^2 in $f(x)$.

Proof. This is a straightforward computation. □

$[n,m]$	Definition	D_0	$X+\mu_0 X$
$[1,1]$	$1\cdot 1$	$2\,\mathrm{Sq}(2) + Y_{-1,0}$	$X_{-1,0} + \mu_0 X_{0,0}$
$[1,2]$	$1\cdot 2+3$	$Y_{-1,0}\mathrm{Sq}(1)$	$X_{-1,0}\mathrm{Sq}(1) + \mu_0 X_{0,0}\mathrm{Sq}(1) + X_{0,0}$
$[2,2]$	$2\cdot 2+3\cdot 1$	$2\,\mathrm{Sq}(1,1) + 2\,\mathrm{Sq}(4) + Y_{-1,0}\mathrm{Sq}(2)$	$X_{-1,0}\mathrm{Sq}(2) + X_{0,0}\mathrm{Sq}(1) + \mu_0 X_{0,0}\mathrm{Sq}(2) + \mu_0 X_{0,1}$
$[1,3]$	$1\cdot 3$	$Y_{-1,0}\mathrm{Sq}(2)$	$X_{-1,0}\mathrm{Sq}(2) + \mu_0 X_{0,0}\mathrm{Sq}(2) + X_{0,0}\mathrm{Sq}(1)$
$[3,2]$	$3\cdot 2$	$2\,\mathrm{Sq}(2,1) + 2\,\mathrm{Sq}(5) + Y_{-1,0}(\mathrm{Sq}(0,1)+\mathrm{Sq}(3))$	$X_{-1,0}(\mathrm{Sq}(0,1) + \mathrm{Sq}(3)) + X_{0,0}\mathrm{Sq}(2) + X_{0,1} + \mu_0 X_{0,0}(\mathrm{Sq}(0,1) + \mathrm{Sq}(3)) + \mu_0 X_{0,1}\mathrm{Sq}(1)$
$[2,3]$	$2\cdot3+4\cdot1+5$	$2\,\mathrm{Sq}(2,1)$	$X_{0,1} + \mu_0 X_{0,1}\mathrm{Sq}(1)$
$[1,4]$	$1\cdot 4+5$	$2\,\mathrm{Sq}(5) + Y_{-1,0}\mathrm{Sq}(3)$	$X_{-1,0}\mathrm{Sq}(3) + X_{0,0}\mathrm{Sq}(2) + \mu_0 X_{0,0}\mathrm{Sq}(3)$
$[3,3]$	$3\cdot 3+5\cdot 1$	$2\,\mathrm{Sq}(6) + Y_{-1,0}(\mathrm{Sq}(1,1)+\mathrm{Sq}(4))$	$X_{-1,0}(\mathrm{Sq}(1,1) + \mathrm{Sq}(4)) + X_{0,0}(\mathrm{Sq}(0,1) + \mathrm{Sq}(3)) + \mu_0 X_{0,0}(\mathrm{Sq}(1,1)+\mathrm{Sq}(4))$
$[2,4]$	$2\cdot4+5\cdot1+6$	$2\,\mathrm{Sq}(3,1) + 2\,\mathrm{Sq}(6) + Y_{-1,0}\mathrm{Sq}(4)$	$X_{-1,0}\mathrm{Sq}(4) + X_{0,0}\mathrm{Sq}(3) + X_{0,1}\mathrm{Sq}(1) + \mu_0 X_{0,0}\mathrm{Sq}(4) + \mu_0 X_{0,1}\mathrm{Sq}(2)$
$[1,5]$	$1\cdot 5$	$2\,\mathrm{Sq}(6) + Y_{-1,0}\mathrm{Sq}(4)$	$X_{-1,0}\mathrm{Sq}(4) + X_{0,0}\mathrm{Sq}(3) + \mu_0 X_{0,0}\mathrm{Sq}(4)$
$[4,3]$	$4\cdot 3+5\cdot 2$	$2\,\mathrm{Sq}(1,2) + 2\,\mathrm{Sq}(4,1) + Y_{-1,0}(\mathrm{Sq}(2,1) + \mathrm{Sq}(5))$	$X_{-1,0}(\mathrm{Sq}(2,1) + \mathrm{Sq}(5)) + X_{0,0}(\mathrm{Sq}(1,1) + \mathrm{Sq}(4)) + \mu_0 X_{0,0}(\mathrm{Sq}(2,1) + \mathrm{Sq}(5)) + \mu_0 X_{0,1}\mathrm{Sq}(0,1)$
$[3,4]$	$3\cdot 4+7$	$Y_{-1,0}\mathrm{Sq}(2,1)$	$X_{-1,0}\mathrm{Sq}(2,1) + X_{0,1}\mathrm{Sq}(2) + \mu_0 X_{0,0}\mathrm{Sq}(2,1) + \mu_0 X_{0,1}\mathrm{Sq}(3) + X_{0,0}\mathrm{Sq}(1,1)$
$[2,5]$	$2\cdot 5+6\cdot 1$	$2\,\mathrm{Sq}(4,1)$	$X_{0,1}\mathrm{Sq}(2) + \mu_0 X_{0,1}\mathrm{Sq}(3)$
$[1,6]$	$1\cdot 6+7$	$Y_{-1,0}\mathrm{Sq}(5)$	$X_{-1,0}\mathrm{Sq}(5) + \mu_0 X_{0,0}\mathrm{Sq}(5) + X_{0,0}\mathrm{Sq}(4)$

FIGURE 1. List of Adem relations in E_0.

Recall that V represents the functor

$$V_!(R) \cong G(R) \times \Big\{ v(x) = \sum_{k\geq 1} v_k x^{2^k} \mid v(x)^2 = 0,\, 2v(x) = 0 \Big\}.$$

This extends to $M = V + \mu_0 V$ as

$$M_!(R) \cong G(R) \times \Big\{ v(x) = v_0(x) + \mu_0 v_1(x) \mid v_0,\, v_1 \text{ as in } V_!(R) \Big\}$$

where

$$(f, v_0 + \mu_0 v_1) \circ (g, w_0 + \mu_0 w_1) = (fg, v_0 g + w_0 + \xi_1^f w_1 + \mu_0(v_1 g + w_1)).$$

We can use this to give an explanation of ψ and θ_D.

Lemma 3.7. *Let $\widehat{\theta_D}$ be the derivation $D_0 \to V + \mu_0 V = M$ from Lemma 3.2 and let $\widetilde{\theta_D} : \mathrm{Sym}_{D_{0*}}(M_*) \to D_{0*}$ be the multiplicative extension with $\widetilde{\theta_D}|_{M_*} = \widehat{\theta_{D*}}$. Then $\widetilde{\theta_D}$ represents the transformation $G(R) \to M_!(R)$ with $f \mapsto (f, \theta_f(x) + \mu_0 \overline{f}(x))$.*

Proof. For an $f(x)$ of the form $\sum_{k\geq 0} x^{2^k} + \sum_{0\leq k<l} 2\xi_{k,l} x^{2^k+2^l}$ one has

$$\begin{aligned} \tau_f(x) &= \sum_{k\geq 1} \xi_k x^{2^{k-1}} + \sum_{1\leq k<l} 2\xi_{k,l} x^{2^{k-1}+2^{l-1}}, \\ \theta_f(x) &= \sum_{k\geq 0} 2\xi_{0,k} x^{2^k}. \end{aligned}$$

The map $f \mapsto (f, \theta_f(x) + \mu_0 \overline{f}(x))$ therefore corresponds to the $M_* \to D_{0*}$ with $v_k \mapsto 2\xi_{0,k}$ and $\mu_0^* v_k \mapsto \xi_k$. But this is just $\widehat{\theta_{D*}}$. □

The multiplicative properties of ψ and θ_D that we established in Lemma 3.2 are therefore just a reformulation of (3.5) and (3.6).

We can now translate the definition of E_0 into the functorial context.

Lemma 3.8. *The ring $\widehat{E_0}$ represents pairs $(f_1(x), f_2(x,y))$ with $f_1(x) \in G(R)$ and $f_2(x,y) = f_2^{(0)}(x,y) + \mu_0 f_2^{(1)}(x,y)$ with $(f_1, f_2^{(j)}) \in U_!(R)$. The multiplication $*$ corresponds to the composition*

$$\begin{aligned} (f \circ g)_2 (x,y) = {} & f_2 (g_1(x), g_1(y)) + \xi_1^f \cdot g_2^{(1)}(x,y) + g_2(x,y) \\ & + \mu_0^f \overline{g}(x) \cdot \overline{f}(g(y)) + \xi_1^f x \cdot \overline{g}(y). \end{aligned}$$

The subset of those (f_1, f_2) with

$$f_2(x,y) = x \cdot \theta_{f_1}(y^2) + f_2^{(0)}(x^2, y^2) + \mu_0 f_2^{(1)}(x^2, y^2)$$

is closed under $*$ and represented by E_0.

Proof. Again this is straightforward. □

Remark 3.9. Rephrasing the previous discussion one could say that in E_0 we are studying certain pairs $f = (f_1, f_2)$ under the transformation rule

$$(fg)_1 = f_1 g_1, \quad (fg)_2(x, y) = (fg)_2^{\text{basic}}(x, y) + \text{correction terms}$$

where

$$(fg)_2^{\text{basic}}(x, y) = f_2 \left(g_1(x), g_1(y)\right) + \xi_1^f \cdot g_2^{(1)}(x, y) + g_2(x, y).$$

Here the correction terms are specifically crafted to preserve the conditions

$$\begin{aligned} f_2(x, y) &\equiv 0 \mod y^2, \\ f_2(x, y) &\equiv x\theta_{f_1}(y^2) \mod x^2 \end{aligned}$$

that define E_0. To us this suggests that the basic object of study should be the composition $(fg)_2^{\text{basic}}$ and the subspace E_0, both of which have a reasonably elementary definition. The precise structure of the correction terms might then count as an artifact of the retraction from $\widehat{E_0}$ to E_0.

4. The Hopf structure on $E_\bullet$

The secondary Steenrod algebra comes equipped with a diagonal $B_\bullet \to B_\bullet \hat{\otimes} B_\bullet$ that extends the usual coproducts on A and B_0. This extra structure is essential for the characterization of $B_\bullet$ in the Uniqueness Theorem [Bau06, 15.3.13]. In this section we are going to exhibit a similar structure on $E_\bullet$, which is a key step in our proof that $B_\bullet \sim E_\bullet$.

4.1. E_0 as Hopf algebra.

Lemma 4.1. *There is a unique multiplicative* $\Delta_0 : E_0 \to E_0 \otimes E_0$ *with*

$$\Delta_0 \left(\mathrm{Sq}(R)\right) = \sum_{E+F=R} \mathrm{Sq}(E) \otimes \mathrm{Sq}(F)$$

and $\Delta_0(Z) = Z \otimes 1 + 1 \otimes Z$ *for* $Z \in \{Y_{k,l}, X_{k,l}, \mu_0 X_{k,l}\}$.

Proof. The uniqueness is clear. To show existence, we begin with the dual of the multiplication map $D_{0*} \otimes D_{0*} \to D_{0*}$. This defines a $\Delta_0 : D_0 \to D_0 \otimes D_0$ with $\Delta_0(Y_{k,l}) = Y_{k,l} \otimes 1 + 1 \otimes Y_{k,l}$. We extend this to all of E_0 via $\Delta_0(Z \cdot \mathrm{Sq}(R)) = (Z \otimes 1 + 1 \otimes Z) \cdot \Delta(\mathrm{Sq}(R))$ for $Z \in \{X_{k,l}, \mu_0 X_{k,l}\}$. We have to show that this map is multiplicative.

This is a straightforward computation, and we will work out only one representative case. Let $a \in A$ and $\Delta a = \sum a' \otimes a''$. Then

$$\begin{aligned}
\Delta_0\left(aX_{k,l}\right) &= \Delta_0\Big(\sum\nolimits_{i,j\geq 0} X_{k+i,l+j} \urcorner\left(\xi_i^{2^{k+1}}\xi_j^{2^{l+1}}, a\right)\Big) \\
&= \sum_{i,j\geq 0}\left(X_{k+i,l+j}\otimes 1 + 1\otimes X_{k+i,l+j}\right)\Delta_0\left(\urcorner\left(\xi_i^{2^{k+1}}\xi_j^{2^{l+1}}, a\right)\right) \\
&= \sum_{a',a''}\sum_{i,j\geq 0}\Big\{\left(X_{k+i,l+j}\urcorner\left(\xi_i^{2^{k+1}}\xi_j^{2^{l+1}}, a'\right)\right)\otimes a'' \\
&\qquad\qquad + a'\otimes\left(X_{k+i,l+j}\urcorner\left(\xi_i^{2^{k+1}}\xi_j^{2^{l+1}}, a''\right)\right)\Big\} \\
&= \sum_{a',a''}\left(a'X_{k,l}\otimes a'' + a'\otimes a''X_{k,l}\right)
\end{aligned}$$

where we have used $\Delta\urcorner(p,a) = \sum \urcorner(p,a')\otimes a'' = \sum a'\otimes\urcorner(p,a'')$. This shows $\Delta_0(aX_{k,l}) = \Delta_0(a)\Delta_0(X_{k,l})$. We leave the remaining cases to the reader. □

There is also a canonical augmentation $\epsilon : E_0 \to \mathbb{Z}/4$ which is dual to the inclusion $\mathbb{Z}/4 \subset D_{0*} \subset E_{0*}$. The following corollary is then obvious.

Corollary 4.2. *E_0 is a Hopf algebra over $\mathbb{Z}/4$ with augmentation ϵ and coproduct Δ_0. The projection $E_0 \to A$ is a map of Hopf algebras.*

4.2. The folding product. We next want to define a secondary diagonal $\Delta_1 : E_1 \to (E \mathbin{\hat{\otimes}} E)_1$. This requires a short discussion of the folding product $(E \mathbin{\hat{\otimes}} E)_\bullet$ that figures on the right hand side. The necessary algebraic background is developped in [Bau06, Ch. 12] and [Bau06, Introduction (B5-B6)].

Let p for the moment be an arbitrary prime and $\mathbb{G} = \mathbb{Z}/p^2$. We consider exact sequences of $\mathbb{G}$-modules of the form

$$M_\bullet = \left(A^{\otimes m} \xrightarrow{\iota} M_1 \xrightarrow{\partial} M_0 \xrightarrow{\pi} A^{\otimes m}\right)$$

Under certain assumptions (e.g., if both factors are $[p]$-algebras in the sense of [Bau06, 12.1.2]) one can define the folding product

$$(M \mathbin{\hat{\otimes}} N)_\bullet = \left(A^{\otimes(m+n)} \xrightarrow{\iota_\sharp} (M \mathbin{\hat{\otimes}} N)_1 \xrightarrow{\partial_\sharp} \underbrace{(M \mathbin{\hat{\otimes}} N)_0}_{=M_0\otimes N_0} \xrightarrow{\pi\otimes\pi} A^{\otimes(m+n)}\right)$$

of two such sequences. Here $(M \mathbin{\hat{\otimes}} N)_1$ is a quotient of $M_1\otimes N_0 \oplus N_0\otimes M_1$, so we can represent its elements as tensors $m \mathbin{\hat{\otimes}} n$ where either $m \in M_1$, $n \in N_0$ or $m \in M_0$, $n \in N_1$. Let $R_M = \ker(M_0 \to A)$ and $R_N = \ker(N_0 \to A)$ be the relation modules. Then $(M \mathbin{\hat{\otimes}} N)_1$ fits into the short exact sequence

$$A^{\otimes(m+n)} \xrightarrow{\iota_\sharp} (M \mathbin{\hat{\otimes}} N)_1 \xrightarrow{\partial} R_M\otimes N_0 + M_0\otimes R_N = R_{M \hat{\otimes} N}$$

with $\partial(m \mathbin{\hat{\otimes}} n) = (\partial m)\otimes n + (-1)^{|m|} m\otimes(\partial n)$.

Unfortunately, $D_\bullet$ and $E_\bullet$ are not $[p]$-algebras in the sense of [Bau06, 12.1.2], because D_0 and E_0 fail to be $\mathbb{G}$-free. It is easy to see, however, that in both cases ∂ restricts to an isomorphism $\mu_0 M_0 \longrightarrow pM_0$, so the reduction $\tilde{M}_\bullet$ with $\tilde{M}_1 = M_1/\mu_0 M_0$ and $\tilde{M}_0 = M_0/pM_0$ is again an exact sequence. A careful reading of Baues's theory shows that this suffices for the construction of the folding product.

Assume now that we have a right-linear splitting $u : R_M \hookrightarrow M_1$ of ∂. For $B_\bullet$ such a splitting has been established in [Bau06, 16.1.3-16.1.5]. For $D_\bullet$ we take the map $R_D \to D_1$

$$2\mathrm{Sq}(R) \mapsto \mu_0 \mathrm{Sq}(R), \quad Y_{k,l}a \mapsto U_{k,l}a \quad (\text{for } k < l,\ a \in A).$$

from (1.7) in the introduction. We extend this to $R_E = R_D \oplus W \to E_1 = D_1 \oplus W$ via $u_E = u_D \oplus \mathrm{id}_W$ where $W = X + \mu_0 X$. We then get an induced splitting $u_\sharp$ for $(M \hat{\otimes} M)_\bullet$ with $u_\sharp(r \otimes m) = u(r) \hat{\otimes} m$ and $u_\sharp(m \otimes r) = m \hat{\otimes} u(r)$ for $r \in R_M$, $m \in M_0$.

The splitting u allows us to decompose M_1 as the direct sum $M_1 = \iota(A) \oplus u(R_M)$. However, this decomposition is only valid for the *right* action of M_0 on $M_\bullet$. We also have an action from the left and this is described by the associated *multiplication map*[1] $\mathrm{op} : M_0 \otimes R_M \to A^{\otimes m}$ with

$$m \cdot u(r) = u(m \cdot r) + \iota(\mathrm{op}(m, r)).$$

In our examples, op actually factors through $M_0 \otimes R_M \twoheadrightarrow A \otimes R_M$. For $B_\bullet$ this is proved in [Bau06, 16.3.3]. For $D_\bullet$ and $E_\bullet$ it is obvious as both D_1 and E_1 are A-bimodules to begin with.

We will now compute op and $\mathrm{op}_\sharp$ explicitly for $D_\bullet$ and $E_\bullet$.

Lemma 4.3. *For $d \in D_0$ and $-1 \le k < l$ one has $\mathrm{op}(a, 2d) = \kappa(a)\pi(d)$ and*

$$\mathrm{op}(a, Y_{k,l}) = \sum_{\substack{i,j \ge 0, \\ k+i \ge l+j}} \mathrm{Sq}(\Delta_{k+i+1} + \Delta_{l+j+1}) \urcorner (\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a).$$

Furthermore, $\mathrm{op}(a, x) = 0$ *for all* $x \in X + \mu_0 X$.

Proof. Since $u(2d) = \mu_0 \pi(d)$ one finds $au(2d) = \kappa(a)\pi(d) + u(a \cdot 2d)$ which proves $\mathrm{op}(a, 2d) = \kappa(a)\pi(d)$.

We have $a \cdot u(Y_{k,l}) = \sum_{i,j \ge 0} U_{k+i,l+j} \urcorner (\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a)$. Using the relations (1.6) we can write

$$U_{k+i,l+j} = \begin{cases} u(Y_{k+i,l+j}) & (k+i < l+j), \\ u(2\mathrm{Sq}(\Delta_{k+i+2})) + \mathrm{Sq}(2\Delta_{k+i+1}) & (k+i = l+j), \\ u(Y_{l+j,k+i}) + \mathrm{Sq}(\Delta_{k+i+1} + \Delta_{l+j+1}) & (k+i > l+j). \end{cases}$$

Therefore

$$a \cdot u(Y_{k,l}) = u(aY_{k,l}) + \sum_{\substack{i,j \ge 0, \\ k+i \ge l+j}} \mathrm{Sq}(\Delta_{k+i+1} + \Delta_{l+j+1}) \urcorner (\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a)$$

[1] This map is denoted A in Baues's theory.

as claimed.

Finally, $\operatorname{op}(a,-)$ vanishes on $M = X + \mu_0 X$ because $u|_M = \mathrm{id}$ is left-linear. $\square$

For $\operatorname{op}_\sharp$ there is a similar result.

Lemma 4.4. *Write* $B_{k,l,i,j} = \mathrm{Sq}(\Delta_{k+i+1} + \Delta_{l+j+1})$. *Then*

$$\operatorname{op}_\sharp(a, \Delta(2d)) = \Delta \operatorname{op}(a, 2d), \quad \textit{(for } d \in D_0\textit{)},$$
$$\operatorname{op}_\sharp(a, \Delta(Y_{k,l})) = \sum_{\substack{i,j\geq 0,\\ k+i\geq l+j}} (B_{k,l,i,j} \otimes 1 + 1 \otimes B_{k,l,i,j}) \urcorner(\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a).$$

One has $\operatorname{op}_\sharp(a, \Delta(x)) = 0$ *for* $x \in X + \mu_0 X$.

Proof. The first claim follows from

$$\operatorname{op}_\sharp(a, \Delta(2d)) = \kappa(a)\Delta(2d) = \Delta\left(\kappa(a) \cdot 2d\right) = \Delta \operatorname{op}(a, 2d).$$

For the second we use $\operatorname{op}_\sharp(a, \Delta(Y_{k,l})) = \operatorname{op}_\sharp(a, Y_{k,l} \otimes 1 + 1 \otimes Y_{k,l})$. From Lemma 4.3 we find

$$\begin{aligned} \operatorname{op}_\sharp(a, Y_{k,l} \otimes 1) &= \sum \operatorname{op}(a', Y_{k,l}) \otimes a'' \\ &= \sum B_{k,l,i,j} \urcorner(\cdots, a') \otimes a'' \\ &= \sum (B_{k,l,i,j} \otimes 1) \urcorner(\cdots, a) \end{aligned}$$

where we have temporarily suppressed some details. There is a similar formula for $\operatorname{op}_\sharp(a, 1 \otimes Y_{k,l})$ and together they make up the second claim.

That $\operatorname{op}_\sharp(-, \Delta(X + \mu_0 X))$ vanishes is clear from the vanishing of op on $A \otimes (X + \mu_0 X)$. $\square$

4.3. The secondary coproduct. We can now define the secondary diagonal $\Delta_\bullet : E_\bullet \to (E \hat{\otimes} E)_\bullet$. We still need a few preparations.

Lemma 4.5. *Let* $U'' \subset U$ *be the sub-bimodule on the* $U_{k,l}$ *with* $k, l \geq 0$. *There is a bilinear* $\nabla : U'' \to A \otimes A$ *with* $U_{k,l} \mapsto Q_l \otimes Q_k$.

Proof. One has

$$\begin{aligned} a\,(Q_k \otimes 1) &= \sum (a' Q_k \otimes a'') = \sum_{i\geq 0} Q_{k+i} \urcorner(\xi_i^{2^{k+1}}, a') \otimes a'' \\ &= \sum_{i\geq 0} (Q_{k+i} \otimes 1) \urcorner(\xi_i^{2^{k+1}}, a). \end{aligned}$$

Therefore

$$a\,(Q_k \otimes Q_l) = a\,(Q_k \otimes 1)\,(1 \otimes Q_l) = \sum_{i,j\geq 0} (Q_{k+i} \otimes Q_{l+j}) \urcorner(\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a)$$

which is the same commutation relation as for the $U_{k,l}$. $\square$

Lemma 4.6. *There is a right-linear $\nabla : R_E \to A \otimes A \oplus \mu_0 A \otimes A$ with*

$$\begin{aligned} \nabla X_{k,l} &= Q_l \otimes Q_k, \quad \nabla \mu_0 X_{k,l} = \mu_0 Q_l \otimes Q_k && (0 \leq k, l) \\ \nabla Y_{k,l} &= Q_l \otimes Q_k && (0 \leq k < l) \end{aligned}$$

and $\nabla|_{2D_0} = \nabla|_{Z_*} = 0$ where $Z_k = X_{-1,k} + Y_{-1,k}$. Let $\Phi(a,r) = \nabla(ar) - a(\nabla r)$ be the left linearity defect of ∇. Then

$$\Phi(a,r) = \Delta \operatorname{op}(a,r) + \operatorname{op}_\sharp(a, \Delta r) \tag{4.1}$$

for $a \in A$ and $r \in R_E$.

Proof. R_E is free as a right A-module with basis 2, Z_k (for $0 \leq k$), $Y_{k,l}$ (for $0 \leq k < l$) and $X_{k,l}$, $\mu_0 X_{k,l}$ (for $0 \leq k, l$). Therefore ∇ is well-defined and right-linear.

We have $\Phi(a, X_{k,l}) = 0$ and $\Phi(a, \mu_0 X_{k,l}) = 0$ by Lemma 4.5, $\Phi(a, 2) = 0$ and $\Delta \operatorname{op}(a, 2) + \operatorname{op}_\sharp(a, \Delta 2) = 0$ by Lemma 4.4, so it just remains to prove the formula for $r = Y_{k,l}$ and $r = Z_k$.

Combining Lemmas 4.3 and 4.4 we find

$$\begin{aligned} &\Delta \operatorname{op}(a, Y_{k,l}) + \operatorname{op}_\sharp(a, \Delta Y_{k,l}) \\ &\quad = \sum_{\substack{i,j \geq 0, \\ k+i \geq l+j}} \underbrace{(\Delta B_{k,l,i,j} - B_{k,l,i,j} \otimes 1 + 1 \otimes B_{k,l,i,j})}_{=:C_{k,l,i,j}} \urcorner (\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a) \end{aligned}$$

where

$$C_{k,l,i,j} = \begin{cases} Q_{k+i+1} \otimes Q_{l+j+1} + Q_{l+j+1} \otimes Q_{k+i+1} & (k+i+1 \neq l+j+1), \\ Q_{k+i+1} \otimes Q_{l+j+1} & (k+i+1 = l+j+1). \end{cases}$$

To see that this is $\Phi(a, Y_{k,l})$ note first that $\nabla(aU_{k,l}) - a\nabla(U_{k,l}) = 0$ by Lemma 4.5. We can compute $\Phi(a, Y_{k,l}) = \nabla(aY_{k,l}) - a\nabla(Y_{k,l})$ from this by changing every $\nabla U_{n,m}$ to $\nabla Y_{n,m}$. Since $\nabla U_{k,l} = \nabla Y_{k,l}$ for $k < l$ and

$$\nabla U_{k+i,l+j} = \begin{cases} \nabla Y_{k+i,l+j} + C_{k,l,i,j} & (k+i \geq l+j) \\ \nabla Y_{k+i,l+j} & (k+i < l+j) \end{cases}$$

this introduces exactly the error terms from the $C_{k,l,i,j}$.

The case of Z_k is similar and left to the reader. ∎

Now define $\mathfrak{X}$, $L : R_E \to A \otimes A$ by $\nabla(r) = \mathfrak{X}(r) + \mu_0 L(r)$. Recall that $E_1 = \iota(A) \oplus u(R_E)$ and let $\Delta_1 : E_1 \to (E \hat{\otimes} E)_1$ be given by

$$\Delta_1(\iota(a)) = \iota_\sharp(\Delta(a)), \quad \Delta_1(u(r)) = u_\sharp(\Delta_0(r)) + \iota_\sharp(\mathfrak{X}(r)). \tag{4.2}$$

Lemma 4.7. *With this coproduct $E_\bullet$ becomes a secondary Hopf algebra.*

Proof. First note that Δ_1 is right-linear and fits into a commutative diagram

$$\begin{tikzcd}
A \arrow[r, tail, "\iota"] \arrow[d, "\Delta"] & E_1 \arrow[r, "\partial"] \arrow[d, "\Delta_1"] & E_0 \arrow[r, two heads] \arrow[d, "\Delta_0"] & A \arrow[d, "\Delta"] \\
A \otimes A \arrow[r, tail, "\iota_\sharp"] & (E \hat{\otimes} E)_1 \arrow[r, "\partial"] & E_0 \otimes E_0 \arrow[r, two heads] & A \otimes A
\end{tikzcd}$$

$\Delta_\bullet : E_\bullet \to (E \hat{\otimes} E)_\bullet$ is therefore a map of $[p]$-algebras in the sense of [Bau06, 12.1.2 (4)]. There is also a natural augmentation $\epsilon_\bullet : E_\bullet \to G_\bullet$ where $G_\bullet = (\mathbb{F} \hookrightarrow \mathbb{F} + \mu_0\mathbb{F} \to \mathbb{G} \twoheadrightarrow \mathbb{F})$ is the unit object for the folding product.

It remains to verify the usual identities

$$(\epsilon_\bullet \hat{\otimes} \mathrm{id})\Delta_\bullet = \mathrm{id} = (\mathrm{id} \hat{\otimes} \epsilon_\bullet)\Delta_\bullet, \quad (\Delta_\bullet \hat{\otimes} \mathrm{id})\Delta_\bullet = (\mathrm{id} \hat{\otimes} \Delta_\bullet)\Delta_\bullet.$$

This can be done on the A generators $\mu_0, U_{k,l}, X_{k,l}, \mu_0 X_{k,l} \in E_1$. We have $\Delta_1(\mu_0) = \mu_0 \hat{\otimes} 1 = 1 \hat{\otimes} \mu_0$ and

$$\begin{aligned}
\Delta_1(U_{k,l}) &= U_{k,l} \hat{\otimes} 1 + 1 \hat{\otimes} U_{k,l} + Q_l \hat{\otimes} Q_k, \qquad\qquad (\text{for } k < l) \\
\Delta_1(X_{k,l}) &= X_{k,l} \hat{\otimes} 1 + 1 \hat{\otimes} X_{k,l} + Q_l \hat{\otimes} Q_k, \\
\Delta_1(\mu_0 X_{k,l}) &= \mu_0 X_{k,l} \hat{\otimes} 1 + 1 \hat{\otimes} \mu_0 X_{k,l}.
\end{aligned}$$

Then, for example,

$$\begin{aligned}
(\mathrm{id} \hat{\otimes} \Delta_1)\Delta_1(U_{k,l}) &= (\mathrm{id} \otimes \Delta_1)\big(U_{k,l} \hat{\otimes} 1 + 1 \hat{\otimes} U_{k,l} + Q_l \hat{\otimes} Q_k\big) \\
&= U_{k,l} \hat{\otimes} 1 \hat{\otimes} 1 + 1 \hat{\otimes} U_{k,l} \hat{\otimes} 1 + 1 \hat{\otimes} 1 \hat{\otimes} U_{k,l} \\
&\qquad + 1 \hat{\otimes} Q_l \hat{\otimes} Q_k + Q_l \hat{\otimes} 1 \hat{\otimes} Q_k + Q_l \hat{\otimes} Q_k \hat{\otimes} 1 \\
&= (\Delta_1 \hat{\otimes} \mathrm{id})\Delta_1(U_{k,l}).
\end{aligned}$$

We leave the remaining cases to the reader. □

Our Δ_1 fails to be left-linear or symmetric; as in [Bau06, 14.1] that failure is captured by the *left action operator* L and the *symmetry operator* S as defined in the following Lemma.

Lemma 4.8. *For $e \in E_1$ and $a \in A$ one has*

$$\Delta_1(ae) = a\Delta_1(e) + \iota_\sharp(\kappa(a)L(\partial e)), \quad T\Delta_1(e) = \Delta_1(e) + \iota_\sharp(S(\partial e))$$

with $S(r) = (1+T)\mathfrak{X}(r)$ where $T : A \otimes A \to A \otimes A$ is the twist map.

Proof. That $S(r) = (1+T)\mathfrak{X}(r)$ is obvious from the definition. For the left-linearity defect one computes

$$\begin{aligned}
\Delta_1(a \cdot u(r)) &= \Delta_1(u(ar) + \iota(\mathrm{op}(a,r))) \\
&= u_\sharp(\Delta_0(ar)) + \iota_\sharp(\mathfrak{X}(ar) + \Delta\,\mathrm{op}(a,r)), \\
a \cdot \Delta_1(u(r)) &= a \cdot (u_\sharp(\Delta_0(r)) + \iota_\sharp(\mathfrak{X}(r))) \\
&= u_\sharp(a \cdot \Delta_0(r)) + \iota_\sharp\big(\mathrm{op}_\sharp(a, \Delta_0(r)) + a \cdot \mathfrak{X}(r)\big).
\end{aligned}$$

Therefore $\Delta_1(au(r)) - a\Delta_1(u(r))$ is

$$\iota_\sharp\left(\mathfrak{X}(ar) - a\mathfrak{X}(r) + \Delta\,\mathrm{op}(a,r) - \mathrm{op}_\sharp(a,\Delta_0(r))\right)$$

which by Lemma 4.6 is

$$\iota_\sharp\left(\mathfrak{X}(ar) - a\mathfrak{X}(r) + \nabla(ar) - a\nabla(r)\right) = \iota_\sharp(\kappa(a)L(r)).$$

□

Note that in Baues's book L was originally defined as a certain map $L : A \otimes R \to A \otimes A$. However, it was shown in [BJ04, 12.7] that $L(a \otimes r) = \kappa(a)L(\mathrm{Sq}^1 \otimes r)$, so our $L(r)$ corresponds to $L(\mathrm{Sq}^1 \otimes a)$ in [Bau06].

4.4. Proof of $B_\bullet \sim E_\bullet$. We are now very close to establishing the weak equivalence between $E_\bullet$ and the secondary Steenrod algebra $B_\bullet$. Recall that B_0 is the free associative algebra over $\mathbb{Z}/4$ on the Sq^k with $k > 0$. Let $\mathfrak{c}_0 : B_0 \to E_0$ be the multiplicative map with $B_0 \ni \mathrm{Sq}^n \mapsto \mathrm{Sq}^n \in D_0$. It's easily checked that $\mathfrak{c}_0$ is also comultiplicative.

Let $\mathfrak{c}_0^* E_1$ be defined as the pullback of $E_1 \to E_0$ along $\mathfrak{c}_0$. We then have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & E_1 \arrow[r] & E_0 \arrow[r, two heads] & A \arrow[d, equal] \\
A \arrow[r, tail] & \mathfrak{c}_0^* E_1 \arrow[r] \arrow[u, "\mathfrak{c}_1"] & B_0 \arrow[r, two heads] \arrow[u, "\mathfrak{c}_0"] & A
\end{tikzcd}$$

that defines a new sequence $\mathfrak{c}^* E_\bullet$ together with a weak equivalence to $E_\bullet$. We will prove that $\mathfrak{c}^* E_\bullet \cong B_\bullet$.

Lemma 4.9. *$\mathfrak{c}^* E$ inherits a secondary Hopf algebra structure from $E_\bullet$ such that the map $\mathfrak{c}^* E_\bullet \to E_\bullet$ is a map of secondary Hopf algebras.*

Proof. Indeed, using the splitting $(\mathfrak{c}^* E \,\hat{\otimes}\, \mathfrak{c}^* E)_1 = \iota'_\sharp(A \otimes A) \oplus u'_\sharp(R_{B \otimes B})$ we can transport the definition (4.2) to

$$\Delta_1\left(\iota'(a)\right) = \iota'_\sharp(\Delta(a)), \quad \Delta_1\left(u'(r)\right) = u'_\sharp(\Delta_0(r)) + \iota'_\sharp(\mathfrak{X}(\mathfrak{c}_0(r))).$$

We leave the details to the reader. □

Note that the left action and symmetry operators of $\mathfrak{c}^* E_\bullet$ are given by $L' = L \circ \mathfrak{c}_0$ and $S' = S \circ \mathfrak{c}_0$. The following Lemma therefore shows that these agree with the operators from the secondary Steenrod algebra.

Lemma 4.10. *Decompose $\nabla \mathfrak{c}_0|_{R_B} : R_B \to A \otimes A \,\oplus\, \mu_0 A \otimes A$ as*

$$\nabla(\mathfrak{c}_0(r)) = \mathfrak{X}(r) + \mu_0 L(r) \quad \textit{with } \mathfrak{X},\ L : R_B \to A \otimes A.$$

Then $r \mapsto L(r)$ resp. $r \mapsto (1+T)\mathfrak{X}(r)$ coincide with the left-action resp. symmetry operator of $B_\bullet$.

Proof. For $0 < n < 2m$ let $[n,m] \in R_B$ denote the Adem relation

$$\underbrace{\mathrm{Sq}^n \otimes \mathrm{Sq}^m + \sum_{1\le k\le \frac{n}{2}} \binom{m-k-1}{n-2k} \mathrm{Sq}^{m+n-k} \otimes \mathrm{Sq}^k}_{=\langle n,m\rangle} + \underbrace{\binom{m-1}{n}\mathrm{Sq}^{m+n}}_{=\Lambda_{n,m}}.$$

Together with $2 \in R_B$ the $[n,m]$ generate R_B as a B_0-bimodule. We let $F^1 = \mathbb{Z}/2\{\mathrm{Sq}^n | n \geq 1\}$, so $\langle n,m\rangle \in F^1 \otimes F^1$ and $\Lambda_{n,m} \in F^1$.

According to [BJ04, 12.7] or [Bau06, 14.4.3] the left action map is the unique bilinear $L : R_B \to A \otimes A$ with $L([n,m]) = L_R(\langle n,m\rangle)$ where $L_R : F^1 \otimes F^1 \to A \otimes A$ is given by

$$L_R(\mathrm{Sq}^n \otimes \mathrm{Sq}^m) = \sum_{\substack{n_1+n_2=n\\ m_1+m_2=m\\ m_1,n_2 \text{ odd}}} \mathrm{Sq}^{n_1}\mathrm{Sq}^{m_1} \otimes \mathrm{Sq}^{n_2}\mathrm{Sq}^{m_2}.$$

Lemma 4.6 proves that the L that we extracted from ∇ is also bilinear, so we only have to verify that it gives the right value on the Adem relations. We now compute

$$\begin{aligned} \mathrm{Sq}^n * \mathrm{Sq}^m &= \mathrm{Sq}^n\mathrm{Sq}^m + \psi(\mathrm{Sq}^n)\psi(Sq^m)\mu_0 + X_{-1}\psi(\mathrm{Sq}^n)\kappa(\mathrm{Sq}^m) \\ &= \mathrm{Sq}^n\mathrm{Sq}^m + X_0\mathrm{Sq}^{n-1}X_0\mathrm{Sq}^{m-1}\mu_0 + X_{-1,0}\mathrm{Sq}^{n-1}\mathrm{Sq}^{m-1}. \end{aligned} \tag{4.3}$$

For the μ_0-component we then find

$$\begin{aligned} \nabla(X_0\mathrm{Sq}^{n-1}X_0\mathrm{Sq}^{m-1}) &= \big((1\otimes Q_0)\mathrm{Sq}^{n-1}\big) \cdot \big((Q_0 \otimes 1)\mathrm{Sq}^{m-1}\big) \\ &= \Big(\sum_{\substack{n_1+n_2=n,\\ n_2 \text{ odd}}} \mathrm{Sq}^{n_1} \otimes \mathrm{Sq}^{n_2}\Big) \cdot \Big(\sum_{\substack{m_1+m_2=m,\\ m_1 \text{ odd}}} \mathrm{Sq}^{m_1} \otimes \mathrm{Sq}^{m_2}\Big) \end{aligned}$$

as claimed.

The identification of $S = (1+T)\mathfrak{X}$ with the symmetry operator proceeds similarly. We first evaluate $S([n,m])$. Moving μ_0 to the right gives

$$\nabla(\mathfrak{c}_0(r)) = \mu_0 L(r) + \mathfrak{X}(r) = L(r)\mu_0 + \underbrace{\kappa(L(r)) + \mathfrak{X}(r)}_{=:\tilde{\mathfrak{X}}(r)}.$$

We claim that $\mathrm{Sq}^n\mathrm{Sq}^m \in D_0$ does not have any $Y_{k,l}$-component with $0 \leq k,l$. Indeed, from the coproduct formula in D_0 we find

$$\Delta\xi_{n,m} \equiv \xi_n\xi_m \otimes \xi_1 \mod \xi_{k,l} \otimes 1,\ 1 \otimes \xi_{k,l},\ 1 \otimes \xi_j \text{ with } j \geq 2.$$

From (4.3) we then find

$$\tilde{\mathfrak{X}}(\mathrm{Sq}^n\mathrm{Sq}^m) = \nabla\mathrm{Sq}^n\mathrm{Sq}^m + \nabla X_{-1,0}\mathrm{Sq}^{n-1}\mathrm{Sq}^{m-1} = 0.$$

It follows that $S([n,m]) = (1+T)\kappa\left(L([n,m])\right) = (1+T)L\left(\kappa([n,m])\right)$. We still need to show that this is the expected outcome. Let $\langle n,m\rangle = \sum_i \mathrm{Sq}^{n_i} \otimes \mathrm{Sq}^{m_i}$. Expanding slightly on the computation above, we see that

$$L([n,m]) = \nabla\left(X_{0,0}\mathrm{Sq}^{n_i-1}\mathrm{Sq}^{m_i-1} + X_{0,1}\mathrm{Sq}^{n_i-3}\mathrm{Sq}^{m_i-1}\right).$$

Therefore

$$(1+T)L(\kappa([n,m])) = (1+T)\nabla X_{0,1}\left(\mathrm{Sq}^{n_i-4}\mathrm{Sq}^{m_i-1} + \mathrm{Sq}^{n_i-3}\mathrm{Sq}^{m_i-2}\right)$$

where we have ignored the $X_{0,0}(\cdots)$ because $(1+T)\nabla X_{0,0} = 0$. Since $\Lambda_{n,m} = \sum_i \mathrm{Sq}^{n_i}\mathrm{Sq}^{m_i} \in F^1$ we have

$$\begin{aligned} 0 &= \urcorner(\xi_2, \sum_i \mathrm{Sq}^{n_i}\mathrm{Sq}^{m_i}) = \sum_i \mathrm{Sq}^{n_i-2}\mathrm{Sq}^{m_i-1}, \\ 0 &= \urcorner(\xi_1^2, \urcorner(\xi_2, \sum_i \mathrm{Sq}^{n_i}\mathrm{Sq}^{m_i})) = \sum_i \left(\mathrm{Sq}^{n_i-4}\mathrm{Sq}^{m_i-1} + \mathrm{Sq}^{n_i-2}\mathrm{Sq}^{m_i-3}\right). \end{aligned}$$

We finally arrive at

$$(1+T)L(\kappa([n,m])) = (1+T)\nabla X_{0,1}\left(\mathrm{Sq}^{n_i-2}\mathrm{Sq}^{m_i-3} + \mathrm{Sq}^{n_i-3}\mathrm{Sq}^{m_i-2}\right).$$

In the notation of the remark following [Bau06, 16.2.3] this is just $(1+T)K[n,m]$ where it is also affirmed that this is the correct value for $S([n,m])$.

The proof of the Lemma will be complete, once we have verified that S has the right linearity properties. From Lemma 4.6 we see that the linearity defect of ∇ is symmetrical; therefore $(1+T)\nabla = S + \mu_0(1+T)L$ is actually bilinear. For S this translates into

$$S(ra) = S(r)a, \quad S(ar) = aS(r) + (1+T)\kappa(a)L(r).$$

This agrees with the characterization in [Bau06, 14.5.2]. □

Corollary 4.11. *There is an isomorphism $\mathfrak{c}^*E_\bullet \cong B_\bullet$.*

Proof. Apply the Uniqueness Theorem [Bau06, 15.3.13]. □

This also proves Theorem 1.1 since we have by construction a chain of weak equivalences $\mathfrak{c}^*E_\bullet \xrightarrow{\sim} E_\bullet \xrightarrow{\sim} D_\bullet$.

Remark 4.12. The map $S : R_E \to A \otimes A$ does not factor through the projection $R_E \to R_D$. This can be seen from the computation

$$\begin{aligned} [3,2] &= 2\mathrm{Sq}(2,1) + 2\mathrm{Sq}(5) + (X_{-1,0} + Y_{-1,0})(\mathrm{Sq}(0,1) + \mathrm{Sq}(3)) \\ &\quad + X_{0,0}\mathrm{Sq}(2) + X_{0,1} + \mu_0 X_{0,0}(\mathrm{Sq}(0,1) + \mathrm{Sq}(3)) + \mu_0 X_{0,1}\mathrm{Sq}(1), \\ [2,2]\mathrm{Sq}^1 &= 2\mathrm{Sq}(2,1) + 2\mathrm{Sq}(5) + (X_{-1,0} + Y_{-1,0})(\mathrm{Sq}(0,1) + \mathrm{Sq}(3)) \\ &\quad + \mu_0 X_{0,0}(\mathrm{Sq}(0,1) + \mathrm{Sq}(3)) + \mu_0 X_{0,1}\mathrm{Sq}(1). \end{aligned}$$

One finds that $S([3,2]) = Q_1 \otimes Q_0 + Q_0 \otimes Q_1$ and $S([2,2]\mathrm{Sq}^1) = 0$ even though $[3,2]$ and $[2,2]\mathrm{Sq}^1$ have the same image in D_0. This shows that the secondary diagonal $\Delta_1 : B_1 \to (B \hat{\otimes} B)_1$ has no analogue over $D_\bullet$.

Appendix A. EBP and a model at odd primes

Let p be a prime and let BP denote the Brown-Peterson spectrum at p. In this appendix we show how a model of the secondary Steenrod algebra can be extracted from BP if $p > 2$.

Recall that the homology $H_*\mathrm{BP}$ is the polynomial algebra over $\mathbb{Z}_{(p)}$ on generators $(m_k)_{k=1,2,\ldots}$ and that $\mathrm{BP}_* \subset H_*\mathrm{BP}$ is the subalgebra generated by the Araki generators $(v_k)_{k=1,2,\ldots}$. Let $\mathrm{EBP}_* = E(\mu_k \mid k \geq 0) \otimes \mathrm{BP}_*$ with exterior algebra generators μ_k of degree $|\mu_k| = |v_k| + 1$. EBP_* is a free BP_*-module and defines a Landweber exact homology theory EBP. Obviously, the representing spectrum is just a wedge of copies of BP. As usual, we let $I = (v_k) \subset \mathrm{BP}_*$ be the maximal invariant ideal.

The cooperation Hopf algebroid $\mathrm{EBP}_*\mathrm{EBP}$ is very easy to compute:

Lemma A.1. *One has* $\mathrm{EBP}_*\mathrm{EBP} = E(\mu_k) \otimes_{\mathbb{Z}_{(p)}} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathbb{Z}_{(p)}} E(\tau_k)$ *with*

$$\eta_R(\mu_n) = \sum_{k=0}^{n} \mu_k t_{n-k}^{p^k} + \tau_n \tag{A.1}$$

and

$$\Delta\tau_n = 1 \otimes \tau_n + \sum_{k=0}^{n} \tau_k \otimes t_{n-k}^{p^k} + \sum_{0 \leq a \leq n} \mu_a \left(-\Delta t_{n-a}^{p^a} + \sum_{b+c=n-a} t_b^{p^a} \otimes t_c^{p^{a+b}} \right).$$

The other structure maps are inherited from $\mathrm{BP}_*\mathrm{BP}$.

Proof. We use (A.1) to define the $\tau_k \in \mathrm{EBP}_*\mathrm{EBP} = E(\mu_k) \otimes \mathrm{BP}_*\mathrm{BP} \otimes E(\mu_k)$. $\Delta\tau_n$ can then be computed from $(\eta_R \otimes \mathrm{id})\eta_R(\mu_n) = \Delta\eta_R(\mu_n)$. □

We can put a differential on EBP by setting $\partial\mu_k = v_k$ and this turns $\mathrm{EBP}_*\mathrm{EBP}$ into a differential Hopf algebroid.

Corollary A.2. *For $p > 2$ the homology Hopf algebroid of* $\mathrm{EBP}_*\mathrm{EBP}$ *with respect to ∂ is the dual Steenrod algebra A_*.*

Proof. We have $\partial\tau_n = \eta_R(v_n) - \sum_{k=0}^{n} v_k t_{n-k}^{p^k} \equiv 0 \bmod I^2$, so there are $\tau_n' \equiv \tau_n \bmod I$ with $\partial\tau_n' = 0$. Therefore $H^*(\mathrm{EBP}_*; \partial) = \mathbb{F}_p$ and

$$H^*(\mathrm{EBP}_*\mathrm{EBP}; \partial) = \mathbb{F}_p[t_k | k \geq 1] \otimes E(\tau_n' | n \geq 0) = A_*.$$

Lemma A.1 then shows that the induced coproduct on A_* coincides with the usual one. □

We prefer to work with operations rather than cooperations. Write $E = \mathrm{EBP}_*$, $\Gamma_* = \mathrm{EBP}_*\mathrm{EBP}$ and let $\Gamma = \mathrm{Hom}_E(\Gamma_*, E)$ be the operation algebra $\mathrm{EBP}^*\mathrm{EBP}$ of EBP. Then Γ is a differential algebra and for odd p its homology $H(\Gamma; \partial)$ can be identified with the Steenrod algebra A. We therefore get an exact sequence $P_\bullet$

$$A \rightarrowtail \operatorname{coker}\partial \xrightarrow{\partial} \ker\partial \twoheadrightarrow A. \tag{A.2}$$

by splicing $H(\Gamma;\partial) \hookrightarrow \Gamma/_{\operatorname{im}\partial} \twoheadrightarrow \operatorname{im}\partial$ and $\operatorname{im}\partial \hookrightarrow \ker\partial \twoheadrightarrow H(\Gamma;\partial)$. We claim that for odd p this sequence is a model for the secondary Steenrod algebra.

Theorem A.3. *Let $p > 2$ and let $B_\bullet \to G_\bullet$ be the secondary Steenrod algebra with its canonical augmentation to $G_\bullet = (\mathbb{F}_p \hookrightarrow \mathbb{F}_p\{1,\mu_0\} \to \mathbb{Z}_{(p)} \twoheadrightarrow \mathbb{F}_p)$. Then there is a diagram of crossed algebras*

$$\begin{tikzcd} P_\bullet \arrow[r, two heads] \arrow[d] & (P/J^2)_\bullet \arrow[d] & T_\bullet \arrow[l, tail] \arrow[d] & B_\bullet \arrow[l] \arrow[d] \\ G^P{}_\bullet \arrow[r, two heads] & G^{P/J^2}{}_\bullet & G^T{}_\bullet \arrow[l, tail] & G_\bullet \arrow[l] \end{tikzcd} \tag{A.3}$$

where all horizontal maps are weak equivalences.

Note that $P_\bullet$ itself cannot be the target of a comparison map from $B_\bullet$ as p^2 is zero in B_0 but not in P_0. In the statement we have also singled out an intermediate sequence $T_\bullet$. This sequence is of independent interest because it is quite small and given by explicit formulas.

To construct (A.3) we first establish the diagram of augmentations. Let $J = I \cdot E \subset E$.

Lemma A.4. *Let $ZE = \ker E \xrightarrow{\partial} E$ and $w_k = v_k\mu_0 - p\mu_k = -\partial(\mu_0\mu_k) \in J$. Then there is a commutative diagram*

$$\begin{tikzcd} G^P{}_\bullet \arrow[d, two heads] & \mathbb{F}_p \arrow[r, tail] \arrow[d, equal] & E/\partial E \arrow[r, "\partial"] \arrow[d, two heads] & ZE \arrow[r, two heads] \arrow[d, two heads] & \mathbb{F}_p \arrow[d, equal] \\ G^{P/J^2}{}_\bullet & \mathbb{F}_p \arrow[r, tail] & E/J \arrow[r, "\partial"] & \left(\ker E/J^2 \xrightarrow{\partial} E/J^3\right) \arrow[r, two heads] & \mathbb{F}_p \\ G^T{}_\bullet \arrow[u, tail] & \mathbb{F}_p \arrow[r, tail] \arrow[u, equal] & \mathbb{F}_p\{1,\mu_k,\mu_0\mu_k\} \arrow[r, "\partial"] \arrow[u, tail] & \mathbb{Z}_{(p)}[v_k,w_k]/J^2 \arrow[r, two heads] \arrow[u, tail] & \mathbb{F}_p \arrow[u, equal] \\ G_\bullet \arrow[u, tail] & \mathbb{F}_p \arrow[r, tail] \arrow[u, equal] & \mathbb{F}_p\{1,\mu_0\} \arrow[r, "\partial"] \arrow[u, tail] & \mathbb{Z}/(p^2) \arrow[r, two heads] \arrow[u, tail] & \mathbb{F}_p \arrow[u, equal] \end{tikzcd}$$

with exact rows.

Proof. This is straightforward, except for the exactness of $G^{P/J^2}{}_\bullet$. First note that

$$\mathbb{F}_p \rightarrowtail J/J^2 \xrightarrow{\partial} J^2/J^3 \xrightarrow{\partial} J^3/J^4 \xrightarrow{\partial} \cdots$$

is exact because it can be identified with the super deRham complex $\Omega^n = \mathbb{F}_p\{\mu^\epsilon d\mu_{i_1}\cdots d\mu_{i_n}\}$ with $df = \sum \frac{\partial f}{\partial\mu_k} d\mu_k$ via $v_k = d\mu_k$. Let E_J denote the complex

$$E/J \xrightarrow{\partial} E/J^2 \xrightarrow{\partial} E/J^3 \xrightarrow{\partial} E/J^4 \xrightarrow{\partial} \cdots .$$

Its associated graded with respect to the J-adic filtration is the sum of shifted copies Ω^{k+*} for $k \geq 0$, so one has $H_k(E_J) = \mathbb{F}_p$ for all k. The exactness of $\mathbb{F}_p \hookrightarrow E/J \to \big(\ker \partial : E/J^2 \to E/J^3\big) \twoheadrightarrow \mathbb{F}_p$ is an easy consequence. $\square$

Now let $P(R)Q(\epsilon) \in \Gamma = \mathrm{Hom}_E(\Gamma_*, E)$ denote the dual of $t^R\tau^\epsilon$ with respect to the monomial basis of Γ_*. (One easily verifies that this is indeed the product of $P(R) := P(R)Q(0)$ and $Q(\epsilon) := P(0)Q(\epsilon)$ as suggested by the notation.) We can think of Γ as the set $E\{\{P(R)Q(\epsilon)\}\}$ of infinite sums $\sum a_{R,\epsilon} P(R)Q(\epsilon)$ with coefficients $a_{R,\epsilon} \in E$.

It is important to realize that the $P(R)$ are *not* ∂-cycles: for $p = 2$, for example, one finds that $\partial\tau_n \equiv v_{n-1}^2 t_1 \mod I^3$ which shows that $\partial P^1 \equiv 4Q(0,1) + v_1^2 Q(0,0,1) + \cdots \mod I^3$.

Lemma A.5. *Let $p > 2$. Then $\partial\tau_n \equiv 0$ mod I^3.*

Proof. The claim is equivalent to $\eta(v_n) \equiv \sum_{0\leq k\leq n} v_k t_{n-k}^{p^k} \mod I^3$. We leave this as an exercise. $\square$

The following Lemma defines $(P/J^2)_\bullet$ and its weak equivalence with $P_\bullet$.

Lemma A.6. *Let $Z\Gamma = \ker \partial : \Gamma \to \Gamma$. There is a commutative diagram*

$$\begin{tikzcd}
P_\bullet \arrow[d, two heads] & A \arrow[r, tail] \arrow[d, equal] & \Gamma/\partial\Gamma \arrow[r, "\partial"] \arrow[d, two heads] & Z\Gamma \arrow[r, two heads] \arrow[d, two heads] & A \arrow[d, equal] \\
P/J^2{}_\bullet & A \arrow[r, tail] & \Gamma/J\Gamma \arrow[r, "\partial"] & \left(\ker \Gamma/J^2\Gamma \xrightarrow{\partial} \Gamma/J^3\Gamma\right) \arrow[r, two heads] & A
\end{tikzcd}$$

with exact rows.

Proof. Choose $\tilde{\tau}_k \in \Gamma_*$ with $\tilde{\tau}_k \equiv \tau_k \mod I$ and $\partial\tilde{\tau}_k = 0$. Let $X(R;\epsilon) \in \Gamma$ be dual to $t^R\tilde{\tau}^\epsilon$. Then $\Gamma = \prod_{R,\epsilon} E \cdot X(R;\epsilon)$ and $\partial X(R;\epsilon) = 0$. It follows that the exactness can be checked on the coefficients alone where it was established in Lemma A.4. $\square$

The construction of $T_\bullet$ requires a more explicit understanding of Γ_*/I^2.

Lemma A.7. *For a family (x_k) let $\Phi_{p^n}(x_k) \in \mathbb{F}_p[x_k]$ be defined by $\sum x_k^{p^n} - (\sum x_k)^{p^n} = p\Phi_{p^n}(x_k)$. Then modulo I^2 one has*

$$\Delta t_n \equiv \sum_{n=a+b} t_a \otimes t_b^{p^a} + \sum_{0<k\leq n} v_k \Phi_{p^k}\left(t_a \otimes t_b^{p^a} \,\middle|\, a+b = n-k\right).$$

Let $w_k = -\partial(\mu_0\mu_k) = v_k\mu_0 - p\mu_k$. Then

$$\Delta\tau_n \equiv 1 \otimes \tau_n + \sum_{n=a+b} \tau_a \otimes t_b^{p^a} + \sum_{0<k\leq n} w_k \Phi_{p^k}\left(t_a \otimes t_b^{p^a} \,\middle|\, a+b = n-k\right).$$

Furthermore,

$$\eta_R(v_n) \equiv \sum_{0\le k\le n} v_k t_{n-k}^{p^k},$$
$$\eta_R(w_n) \equiv -p\tau_n + \sum_{1\le k<n} w_k t_{n-k}^{p^k} + \sum_{0\le k\le n} v_k t_{n-k}^{p^k}\tau_0,$$

Proof. The v_k are defined by $pm_n = \sum_{n=a+b} m_a v_b^{p^a}$ and it follows easily that $v_n \equiv pm_n$ modulo $I^2 \cdot H_*(\mathrm{EBP})$. Recall that $\eta_R(m_n) = \sum_{n=a+b} m_a t_b^{p^a}$ and that Δt_n can be computed from $(\eta_R \otimes \mathrm{id})\eta_R(m_n) = \Delta\eta_R(m_n)$. Inductively, this gives

$$\begin{aligned}\Delta t_n &= \sum_{n=a+b} t_a \otimes t_b^{p^a} + \sum_{0<k\le n} m_k \left(-\Delta t_{n-k}^{p^k} + \sum_{n-k=a+b} t_a^{p^k} \otimes t_b^{p^{k+a}}\right)\\ &\equiv \sum_{n=a+b} t_a \otimes t_b^{p^a} + \sum_{0<k\le n} v_k \Phi_{p^k}\left(t_a \otimes t_b^{p^a} \,\middle|\, a+b=n-k\right)\end{aligned}$$

as claimed. The formula for $\Delta\tau_n$ now follows with Lemma A.1. We leave the computation of $\eta_R(v_n)$ and $\eta_R(w_n)$ to the reader. □

Let $S_\bullet = G^T{}_\bullet$ and recall that

$$S_0 = \mathbb{Z}/p^2 + \mathbb{F}_p\{v_k, w_k \,|\, k \ge 1\} \subset E/J^2,$$
$$S_1 = \mathbb{F}_p\{1, \mu_k, \mu_0\mu_k\} \subset E/J.$$

We now define

$$T_0 = S_0\{\{P(R)Q(\epsilon)\}\} \subset \Gamma/J^2\Gamma,$$
$$T_1 = S_1\{\{P(R)Q(\epsilon)\}\} \subset \Gamma/J\Gamma.$$

Lemma A.8. *This defines a crossed algebra $T_\bullet \subset (P/J^2)_\bullet$ as claimed in Theorem A.3.*

Proof. Lemma A.7 shows that $(S_0, S_0[t_k,\tau_k])$ is a sub Hopf algebroid of $(E/J^2, \Gamma_*/J^2)$ with $\Gamma_*/J^2 = E/J^2 \otimes_{S_0} S_0[t_k,\tau_k]$. Therefore

$$T_0 = \mathrm{Hom}_{S_0}(S_0[t_k,\tau_k], S_0) \hookrightarrow \mathrm{Hom}_{E/J^2}(\Gamma_*/J^2, E/J^2) = \Gamma/J^2$$

is the inclusion of a subalgebra. By Lemma A.5, T_0 is actually contained in $(P/J^2)_0 = \ker\partial : \Gamma/J^2 \to \Gamma/J^3$. The remaining details are left to the reader. □

To prove the Theorem it only remains to establish the weak equivalence $B_\bullet \to T_\bullet$. Recall that B_0 is the free $\mathbb{Z}/p^2$-algebra on generators Q_0 and P^k, $k \ge 1$. We can therefore define a multiplicative $\mathfrak{p}_0 : B_0 \to T_0$ via $Q_0 \mapsto Q(1)$ and $P^k \mapsto P(k)$.

Lemma A.9. *There is a weak equivalence $\mathfrak{p} : B_\bullet \to T_\bullet$ that extends $\mathfrak{p}_0$.*

Proof. The multiplication on Γ_* dualizes to a coproduct $\Delta_\Gamma : \Gamma \to \Gamma \mathbin{\widetilde{\otimes}_E} \Gamma$ were $\widetilde{\otimes}_E$ denotes a suitably completed tensor product. This turns Γ into a topological Hopf algebra over E. We define the completed folding product $(P \mathbin{\widehat{\otimes}_E} P)_\bullet$ as the pullback

$$\begin{tikzcd}
A \otimes A \arrow[r, tail] \arrow[d, equal] & (\Gamma \mathbin{\widetilde{\otimes}_E} \Gamma)/\operatorname{im}\partial_\otimes \arrow[r, "\partial_\otimes"] & \ker \partial_\otimes \arrow[r, two heads] & A \otimes A \arrow[d, equal] \\
A \otimes A \arrow[r, tail] & (P \mathbin{\widehat{\otimes}_E} P)_1 \arrow[r, "\partial_\otimes"] \arrow[u, tail] & P_0 \mathbin{\widetilde{\otimes}_E} P_0 \arrow[r, two heads] \arrow[u, tail] & A \otimes A
\end{tikzcd}$$

where $\partial_\otimes = \partial \otimes \mathrm{id} + \mathrm{id} \otimes \partial$ is the differential on $\Gamma \mathbin{\widetilde{\otimes}_E} \Gamma$. Δ_Γ then restricts to a coproduct $\Delta_\bullet : P_\bullet \to (P \mathbin{\widehat{\otimes}_E} P)_\bullet$. Note that Δ_1 is bilinear and symmetric, since this is true for Δ_Γ. By restriction we get a $\Delta_\bullet : T_\bullet \to (T \mathbin{\widehat{\otimes}_S} T)_\bullet$ where the right hand side is given by

$$(T \mathbin{\widehat{\otimes}_S} T)_0 = S_0\{\{P(R_1)Q(\epsilon_1) \otimes P(R_2)Q(\epsilon_2)\}\} \subset (P \mathbin{\widehat{\otimes}_E} P)_1/J^2,$$
$$(T \mathbin{\widehat{\otimes}_S} T)_1 = S_1\{\{P(R_1)Q(\epsilon_1) \otimes P(R_2)Q(\epsilon_2)\}\} \subset (P \mathbin{\widehat{\otimes}_E} P)_1/J.$$

Let $\mathfrak{p}^*T_\bullet$ be the pullback of $T_\bullet$ along $B_0 \to T_0$. It inherits a secondary Hopf algebra structure from $T_\bullet$. This structure has $L = S = 0$ since the same is true for $P_\bullet$. Baues's Uniqueness Theorem thus implies $B_\bullet \cong \mathfrak{p}^*T_\bullet$. $\square$

References

[Bau06] Hans-Joachim Baues. *The algebra of secondary cohomology operations*, volume 247 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2006. MRNUMBER = 2220189 (2008a:55015) Zbl 1091.55001

[BJ04] H.-J. Baues and M. Jibladze. The algebra of secondary cohomology operations and its dual. MPIM2004-111, 2004. To appear in the *Journal of K-theory*.

[BJ06] Hans-Joachim Baues and Mamuka Jibladze. Secondary derived functors and the Adams spectral sequence. *Topology*, 45(2):295–324, 2006. MRNUMBER = 2193337 (2006k:55031), Zbl 1095.18003

[KM69] Leif Kristensen and Ib Madsen. On the structure of the operation algebra for certain cohomology theories. In *Conf. on Algebraic Topology (Univ. of Illinois at Chicago Circle, Chicago, Ill., 1968)*, pages 134–160. Univ. of Illinois at Chicago Circle, Chicago, Ill., 1969. MRNUMBER = 0250301 (40 #3540) Zbl 0212.28102

`nassau@nullhomotopie.de, http://nullhomotopie.de`