GROMOV-WITTEN INVARIANTS AND RATIONAL CURVES ON GRASSMANNIANS

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Abstract. We study the enumerative significance of the s-pointed genus zero Gromov-Witten invariant on a homogeneous space X. For that, we give an interpretation in terms of rational curves on X.

1. Introduction

Since their appearance in the algebraic context, Gromov-Witten invariants have proven to be an indispensable tool for enumerative geometry. The problem of determining the number N_d of rational curves of degree d passing through $3d-1$ points in general position in the complex projective plane $\mathbb{P}^2$ was solved, by means of Gromov-Witten theory, by Kontsevich (see [12]).

Gromov-Witten invariants arose as enumerative invariants of stable maps, which had been previously introduced independently by Ruan and Tian [16] in the symplectic case, and by Kontsevich and Manin [12] in the algebraic case. Let X be a smooth projective variety over $\mathbb{C}$, and let β be a curve class on X. The set of isomorphism classes of pointed maps $(C, p_1, \ldots, p_s, f)$, where C is a projective nonsingular curve and f is a morphism from C to X with $f_*([C]) = \beta$, is denoted as $M_{g,s}(X, \beta)$. Its compactification, the moduli space $\overline{M}_{g,s}(X, \beta)$, parameterizes stable maps. The stability condition is equivalent to finiteness of automorphisms of the map.

The purpose of our work is to study the enumerative significance of genus zero Gromov-Witten invariants with a Grassmannian target. While Gromov-Witten theory in the algebraic context in general requires the sophisticated machinery of virtual fundamental classes [15, 1] and the invariants have no clear enumerative significance in general, in the case of genus zero with target a homogeneous variety $X = G/P$ the moduli stack is smooth of the expected dimension and it makes sense to ask for the enumerative significance of the Gromov-Witten invariants. Given a tuple of classes of subvarieties of X of suitable dimensions, the Gromov-Witten invariant counts isomorphism classes of $(C \cong \mathbb{P}^1, p_1, \ldots, p_s, f)$ such that $f_*[\mathbb{P}^1]$ is equal to a given curve class and f maps p_i into a general translate of the ith subvariety for all i, according to Fulton and Pandharipande [10]. (The p_i are not fixed here; as explained in op. cit., page 93, there are alternative invariants in which the p_i are fixed points in $\mathbb{P}^1$.) The purpose of this article is to rephrase

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this enumerative interpretation in terms of rational curves on X satisfying incidence conditions. For a large class of enumerative conditions, we are able to exclude that due to reparameterizations of the source curve a map f contributes multiply to the Gromov-Witten invariant. Then the Gromov-Witten invariant simply counts rational curves in a given curve class incident to general translates of the given subvarieties. The main Theorem asserts this to be the case when X is a Grassmannian variety and the subvarieties are Schubert varieties, up to a correction factor of the degree of the curve class for each Schubert variety of codimension one.

2. Preliminaries

Let G be a complex simple Lie group of classical type and P a maximal parabolic subgroup. The homogeneous space $X = G/P$ is a Grassmannian variety, a usual Grassmannian of subspaces of some finite-dimensional complex vector space V when G is of type A, or of subspaces isotropic for a given nondegenerate symmetric or skew-symmetric bilinear form on V in the other classical Lie types. Throughout we assume that $\dim X \geq 2$.

The *quantum Schubert calculus* is a set of combinatorial rules that determine the genus zero three-point Gromov-Witten invariants of X. Quantum analogues of the classical Pieri and Giambelli formulas are given for the usual Grassmannians by Bertram [2] and for isotropic Grassmannians by Buch, Kresch, and Tamvakis [13, 14, 7, 5, 4]. In type A, an explicit combinatorial rule for the invariants, i.e., a quantum Littlewood-Richardson rule, is available, due to Coskun [8].

For the sake of completeness and to fix notation, let us recall some definitions. The Schubert varieties on the usual Grassmannian of m-planes in $V \cong \mathbb{C}^n$ are indexed by integer partitions of length at most m and biggest part at most $n - m$. There are analogous descriptions in the other Lie types, based on *k-strict partitions*; the reader is referred to [7] for a detailed description and facts about Schubert varieties in the isotropic Grassmannians. Here k is $n - m$ when the bilinear form on V is skew symmetric and $X = \mathrm{IG}(m, 2n)$ and in the case of a symmetric bilinear form k is $n - m$, respectively $n+1-m$, when X is $\mathrm{OG}(m, 2n+1)$, respectively $\mathrm{OG}(m, 2n+2)$; a partition is k-strict when there are no repeated parts $> k$. We will generally denote by X_λ the Schubert variety in X corresponding to a partition λ. Its codimension is $|\lambda|$, where for $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ the weight is $|\lambda| := \lambda_1 + \cdots + \lambda_\ell$. The kernel and span of a rational map are important notions, introduced and discussed by Buch in [3]. A rational map of degree d to X is a morphism $f : \mathbb{P}^1 \to X$ such that $f_*[\mathbb{P}^1]$ has degree d. Here degree is understood with respect to the projective embedding of X corresponding to a fundamental representation of G with point stabilizer P. In dimension 1, the unique Schubert class has degree 1 and the corresponding Schubert variety is a line under this embedding. Since $\dim X \geq 2$, X contains a projective plane or a nonsingular quadric threefold, and therefore there exist rational curves of every degree $d \geq 1$ on X.

The moduli space of stable genus zero degree d maps $\overline{M}_{0,s}(X,d)$ is smooth (as a stack) of dimension $\dim X + s - 3 + d \deg c_1(X)$. There are evaluation maps $ev_i : \overline{M}_{0,s}(X,d) \to X$. If $\alpha_1, \ldots, \alpha_s \in A^*(X)$ are classes in the Chow (or cohomology) group of X whose codimensions sum to $\dim \overline{M}_{0,s}(X,d)$ then there is the (non-gravitational, genus zero) *Gromov-Witten invariant*

$$I_d(\alpha_1 \cdots \alpha_s) := \int_{\overline{M}_{0,s}(X,d)} ev_1^*\alpha_1 \cup \cdots \cup ev_s^*\alpha_s.$$

If $\Gamma_1, \ldots, \Gamma_s$ are subvarieties of X with α_i (Poincaré dual to) the fundamental class of Γ_i for each i, then for general $(g_1, \ldots, g_s) \in G^s$ the Gromov-Witten invariant is equal to the number of degree d maps $\mathbb{P}^1 \to X$ sending the ith marked point into $g_i\Gamma_i$ for each i. For proofs of these facts, see [10].

3. Main result

We adopt the notation of Section 2 and prove the following result.

Theorem. *If $X = G/P$ where G is a complex simple Lie group of classical type and P is a maximal parabolic subgroup, d and s are positive integers, and $\Gamma_1,\ldots,\Gamma_s$ are Schubert varieties whose codimensions sum to $\dim \overline{M}_{0,s}(X,d)$, then the Gromov-Witten invariant $I_d([\Gamma_1] \cdots [\Gamma_s])$ is divisible by d^r, where $r = \#\{i : \operatorname{codim} \Gamma_i = 1\}$, and the quotient is equal to the number of degree d rational curves on X incident to general translates of the Γ_i.*

We note that if $\Gamma_i = X$ for any i then the Gromov-Witten invariant is zero (fundamental class axiom) and by Lemma 14 in [10] the set of degree d rational curves on X incident to general translates of the Γ_i is empty. If some $r \geq 1$ of the Γ_i have codimension 1, then the Gromov-Witten invariant is equal to d^r times the $(s-r)$-point Gromov-Witten invariant with the divisor classes omitted (divisor axiom). This implies the divisibility assertion. We first prove the enumerative claim assuming $\operatorname{codim} \Gamma_i \geq 2$ for all i, then we treat the case when some of the Γ_i are divisors.

Lemma. *Let Γ be a Schubert variety in X of codimension at least 2 with Schubert cell Γ^0. Fix a point $p_1 \in \mathbb{P}^1$. Then for each $d \geq 1$ there exists a degree d map $f : \mathbb{P}^1 \to X$ such that*

i. *f is an unramified morphism;*
ii. *$f(p_1) \in \Gamma^0$;*
iii. *f maps a nonzero tangent vector at p_1 to a tangent vector at $f(p_1)$ not contained in the tangent space to Γ^0;*
iv. *$f^{-1}(\Gamma) = \{p_1\}$.*

Recall that $\overline{M}_{0,s}(X,d)$ is irreducible [17, 11]. For a point $x \in X$ and for each i we observe that by the group action there is a birational isomorphism between $ev_i^{-1}(x) \times \mathbb{A}^{\dim X}$ and $\overline{M}_{0,s}(X,d)$, and hence $ev_i^{-1}(x)$ is irreducible as well. In the situation of the Lemma we therefore obtain that $ev_i^{-1}(\Gamma)$ is irreducible, by [9, Theorem 4.17].

Proof. It suffices to verify (ii)-(iv) for a point of $\overline{M}_{0,1}(X,d)$, since the combination of these is an open condition in $ev_1^{-1}(\Gamma)$ and (i) is satisfied on a dense open subset of $ev_1^{-1}(\Gamma) \cap M_{0,1}(X,d)$. When $d = 1$ it is clear that (ii)-(iv) may be satisfied. When $d = 2$ and X is an orthogonal Grassmannian, this follows from the description in [7, Lemma 2.1, proof of Thm. 2.3 or Lemma 3.1, proof of Thm. 3.3].

Otherwise, we consider two cases. There is a critical degree d_0, the smallest for which two general points on X are joined by a rational curve of that degree: $d_0 = \min(m, n-m)$ when $X = G(m,n)$; otherwise d_0 is m when $X = IG(m, 2n)$ and m rounded up to the next even integer for the orthogonal Grassmannians (divided by two in the case of the maximal orthogonal Grassmannians). The first case we consider is when $d \leq d_0$. Then we have the following set-up from [6, §2.2, 3.2, 4.2], [7, §1.4, 2.4, 3.4]; specifically:

- a variety Y_d parameterizing pairs (A, B) with $\dim A = m-d$, $\dim B = m+d$, $A \subset B$, and $B \subset A^{\perp}$ when X is an isotropic Grassmannian;
- an incidence correspondence

$$\begin{array}{ccc} T_d & \longrightarrow & X \\ {\scriptstyle \pi}\big\downarrow & & \\ Y_d & & \end{array}$$

  (where T_d consists of triples (A, Σ, B) with $A \subset \Sigma \subset B$ and Σ a point of X);
- "modified" Schubert varieties $Y_\lambda \subset Y_d$ each defined as the image by π of the preimage $T_\lambda \subset T_d$ of X_λ; and
- a result identifying the three-point genus zero Gromov-Witten invariants in degree d with (in some cases up to a certain power of 2) intersection points of modified Schubert varieties in Y_d. (In types B and D when d is odd there is a codimension 3 subvariety of Y_d denoted Z_d° in [7] to which we need to restrict our attention.)

The fiber of π above a general point of Y_λ is a Grassmannian of particular type (e.g., the Lagrangian Grassmannian of a symplectic $2d$-space in type C). Letting λ^+ be the smallest partition containing λ so that according to [6, proof of Cor. 2, 4 or 6] or [7, Lemma 1.3, 2.1 or 3.1] the map $T_{\lambda^+} \to Y_{\lambda^+}$ is generically finite, the inequality $\dim Y_\lambda \geq \dim Y_{\lambda^+} = \dim T_{\lambda^+}$ is enough to guarantee that the intersection of T_λ with the fiber of π above a general point of Y_λ has codimension at least 2 in the fiber and is generically smooth. Choose a general point of the intersection to be $f(p_1)$; then a general rational map of degree d in the fiber of π meets requirements (ii)-(iv), since we have an explicit description of a general rational curve on the fiber of π by [7, Prop. 1.1]. For $d > d_0$, we simply have to take a degree d_0 curve as just described and attach $d - d_0$ copies of a degree 1 tail. If the point of attachment is general with respect to $f(p_1)$, then a general line will for dimension reasons be disjoint from the Schubert variety, and (ii)-(iv) remain valid. □

Proof of the Theorem. In [10, Lemma 14] it is proven that, in the conditions of the Theorem, the intersection $ev_1^{-1}(g_1\Gamma_1) \cap \cdots \cap ev_s^{-1}(g_s\Gamma_s)$ is a finite set of reduced points, each corresponding to an irreducible source curve $\mathbb{P}^1$ with automorphism-free map to the target variety X, for general $(g_1, \ldots, g_s) \in G^s$. The number of these is the Gromov-Witten invariant.

We prove the Theorem first in the case when each of the Γ_i has codimension at least 2. We claim, for general $(g_1, \ldots, g_s) \in G^s$, that each of these finitely many intersection points satisfies (i)-(iv) of the Lemma, with $\Gamma = g_1\Gamma_1$. Given this, we may repeat the argument with the other Γ_i in place of Γ_1 and find for general $(g_1, \ldots, g_s) \in G^s$ that for each of the finitely many intersection points $(C \cong \mathbb{P}^1, p_1, \ldots, p_s, f)$ and each i, the point p_i is the unique one on C having image contained in $g_i\Gamma_i$.

To prove the claim, by homogeneity we may fix $g_1 = e$ and verify the conditions for general $(g_2, \ldots, g_s) \in G^{s-1}$. Let $ev_1^{-1}(\Gamma_1^0)^*$ denote the open subset of $ev_1^{-1}(\Gamma_1^0) \cap M_{0,s}(X, d)$ satisfying (i)-(iv) of the Lemma. Now we apply the Kleiman-Bertini theorem to the diagram

$$\begin{array}{ccc} & & ev_1^{-1}(\Gamma_1^0)^* \\ & & \downarrow \\ \Gamma_2^0 \times \cdots \times \Gamma_s^0 & \longrightarrow & X^{s-1} \end{array}$$

with action of G^{s-1}. Then, for general $(g_2, \ldots, g_s)$ the intersection is a finite set of reduced points, and there is no contribution to the intersection from points of $ev_1^{-1}(\Gamma_1^0)$ not in $ev_1^{-1}(\Gamma_1^0)^*$. The claim is verified.

For the case when some of the Γ_i are divisors, we repeat the above argument but taking $ev_1^{-1}(\Gamma_1^0)^*$ to be defined by only conditions (i)-(iii) of the Lemma. Then for each of the surviving intersection points, the map $f : \mathbb{P}^1 \to X$ has properties (i)-(iii) at every preimage point of Γ. It follows that there are d distinct choices for a marked point on $\mathbb{P}^1$ mapping to Γ_1. Hence there is a d^r-to-one correspondence (with r as in the statement of the Theorem) between intersection points in $\overline{M}_{0,s}(X, d)$ and degree d rational curves on X satisfying the incidence conditions. □

Remark. The three-point genus zero Gromov-Witten invariants are those that arise as structure constants in the small quantum cohomolgy ring; they are those given by the quantum Schubert calculus. For these, according to the Theorem, the Gromov-Witten invariant precisely counts rational curves on X except in one case: when $d = 2$ and one of the Γ_i has codimension 1, then the Gromov-Witten invariant is twice the number of conics satisfying the incidence conditions. This is only possible when X is an orthogonal

Grassmannian, reflecting the presence of q^2 terms in the quantum Pieri formula for multiplication with the codimension 1 Schubert class (where q is the formal parameter of the quantum cohomology ring).

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