THE MAXIMAL RANK OF ELLIPTIC DELSARTE SURFACES

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ABSTRACT. Shioda described in his article [4] a method to compute the Lefschetz number of a Delsarte surface. In one of his examples he uses this method to compute the rank of an elliptic curve over $k(t)$. In this article we find all elliptic curves over $k(t)$ for which his method is applicable. For each of these curves we also compute the Mordell-Weil rank.

1. INTRODUCTION

Shioda described in [4] a method to compute the Lefschetz number of a Delsarte surface. In one of the examples he used his method to compute the rank over $k(t)$ of the elliptic surface given by:

$$Y^2 = X^3 + at^n X + bt^m.$$

Here as in the rest of the article k is a algebraically closed field of characteristic 0. This rank is bounded by 56 and equal to 56 if and only if m is even and $2m \equiv 3n \mod 2^3 \cdot 3^2 \cdot 5 \cdot 7$. Later in [5] Shioda used this method to find a elliptic surface with rank 68 over k. This is the highest rank known for an elliptic surface over $\mathbb{C}$.

In this article we will briefly describe the method Shioda used. This method works for all elliptic curves over $k(t)$ that can be defined by a polynomial of the form:

$$f = \sum_{i=0}^{3} t^{a_{i0}} X^{a_{i1}} Y^{a_{i2}}. \tag{1}$$

The first theorem that we will prove is:

Theorem 1.1. *Let f be a polynomial as over the field $k(t)$. Suppose f defines a curve of genus 1 over $k(t)$. This curve is birational to a curve given by a polynomial g wich also has four terms, and moreover its Newton polygon $\Gamma(g)$ is one of the polygons from figure 1.*

After the proof of this result we will compute the rank of the elliptic curves defined by (1) for all corresponding Newton polygons. This will then lead us to the final theorem of this article:

Theorem 1.2. *Suppose $E/k(t)$ is an elliptic curve defined by (1), then $rank(E(k(t))) \leq 68$.*

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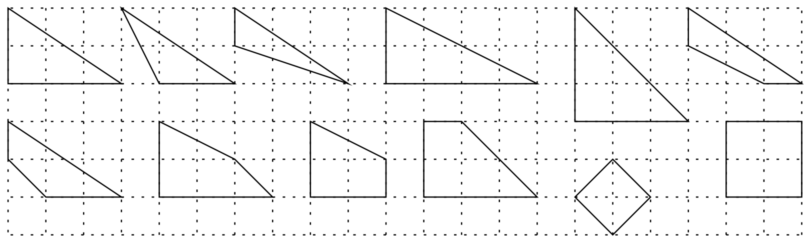


FIGURE 1. All polygons with exactly one interior point and at most 4 corners.

2. Shioda's method/Delsarte Surfaces

Assume that f is irreducible of the form 1, and f defines an elliptic curve E over $k(t)$.

We consider the surface $S \subset \mathbb{P}^3$ over k defined by the homogenized polynomial F corresponding to f:

$$F = t^{a_{00}} X^{a_{01}} Y^{a_{02}} Z^{a_{03}} + t^{a_{10}} Y^{a_{11}} Y^{a_{12}} Z^{a_{13}} + t^{a_{20}} X^{a_{21}} Y^{a_{22}} Z^{a_{23}} + t^{a_{30}} X^{a_{31}} Y^{a_{32}} Z^{a_{33}}.$$

Let $A_f = (a_{ij})$ be the matrix consisting of the powers appearing in the polynomial. We assume A_f to be nonsingular. Then Shioda's method gives us an algorithm to compute the Lefschetz number of the surface. We will give a slightly adapted description of his method, which appears somewhat more convenient to work with. For more details we refer to Shioda's orginal paper [4].

Define L to the subgroup of $(\mathbb{Q}/\mathbb{Z})^4$ generated by $(1,0,0,-1)A_f^{-1}$, $(0,1,0,-1)A_f^{-1}$ and $(0,0,1,-1)A_f^{-1}$. Define:

$$\Lambda = \left\{ (a_i)_i \in L : \begin{array}{l} \forall i: \ a_i \neq 0 \text{ and} \\ \exists t \in \mathbb{Z} \text{ such that } \forall i, \\ ord(ta_i) = ord(a_i) \text{ and } \sum_{i=0}^{3} \{ta_i\} \neq 2 \end{array} \right\}.$$

Here ord is the order in the additive group $\mathbb{Q}/\mathbb{Z}$ and $\{a_i\}$ is the natural bijection between $[0,1) \cap \mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$.

Theorem 2.1 (Shioda)**.** *The Lefschetz number of S is $\lambda = \#\Lambda$.*

Proof: For the proof see [4]. □

3. Genus calculation.

To see which f defines a genus 1 curve, we describe a method which calculates the genus for a given f as in (1). To do this we will first need three definitions:

Definition 3.1. *An* integral polygon *is the convex hull of a finite subset of $\mathbb{Z}^2$.*

Definition 3.2. *Take $f = \sum_{(a,b)\in S} \alpha_{(a,b)} X^a Y^b$ in the ring of Laurent polynomials $k[X^{\pm 1}, Y^{\pm 1}]$, with all $\alpha_{(a,b)} \neq 0$ and S a finite subset of $\mathbb{Z}^2$. Define the* Newton polygon*, $\Gamma(f)$, of f as the convex hull of S.*

Definition 3.3. *Take $f = \sum_{(a,b)\in S} \alpha_{(a,b)} X^a Y^b \in k[X^{\pm 1}, Y^{\pm 1}]$. For every edge, γ, of the Newton polygon define $f_\gamma = \sum_{(a,b)\in S\cap\gamma} \alpha_{(a,b)} X^a Y^b$. We say that f is nondegenerate with respect to its Newton polygon if for every γ we have f_γ, $\frac{\partial f_\gamma}{\partial X}$ and $\frac{\partial f_\gamma}{\partial Y}$ generate the unit ideal in $k[X^{\pm 1}, Y^{\pm 1}]$.*

We now give the following theorem.

Theorem 3.4. *Let $f(X,Y) \in k[X^{\pm 1}, Y^{\pm 1}]$ be absolutely irreducible. Then the curve C defined by f has genus*

$$g \leq \#\{\text{integral points in the interior of } \Gamma(f)\}.$$

Equality holds if f is nondegenerate with respect to its Newton polygon and the singular points of the projective closure of C in $\mathbb{P}^2$ are all among $(0:0:1)$, $(0:1:0)$ and $(1:0:0)$.

Proof: See [1, Theorem 4.2] □

Lemma 3.5. *Let C be the projective closure in $\mathbb{P}^2$ of a curve over $k(t)$ defined by a polynomial f as in (1). Assume that A_f is nonsingular. Then C does not have singular points outside the point $(0:0:1)$, $(0:1:0)$ and $(1:0:0)$ over $\overline{k(t)}$.*

Proof: Since A_f is nonsingular at least one of the minors corresponding to deleting the first column and a row is nonzero. Without loss of generality we can assume that this happens for the first row. Define $X' = s^{b_1}X$, $Y' = s^{b_2}Y$ and $Z' = s^{b_3}Z$, where $s \in \overline{k(t)}$ is some root of t and suitable $b_i \in \mathbb{Z}$. This gives an isomorphism between C and $\tilde{C}$, where $\tilde{C}$ is the curve given by:

$$\tilde{f} := s^n X^{a_{01}} Y^{a_{02}} Z^{m-a_{01}-a_{02}} + \sum_{i=1}^{3} X^{a_{i1}} Y^{a_{i2}} Z^{m-a_{i1}-a_{i2}}.$$

To prove the lemma, it will suffice to prove that $\tilde{C}$ has no singular points outside $(1:0:0), (0:1:0), (0:0:1)$. Assume $(x:y:z)$ is a singular point on $\tilde{C}$. Write $v_0 = s^n x^{a_{i1}} y^{a_{i2}} z^{m-a_{i1}-a_{i2}}$ and $v_i = x^{a_{i1}} y^{a_{i2}} z^{m-a_{i1}-a_{i2}}$ for $i \in \{1,2,3\}$ and $v = (v_0, v_1, v_2, v_3)$. Let B be the matrix defined by:

$$B = \begin{pmatrix} a_{01} & a_{02} & m - a_{01} - a_{02} \\ a_{11} & a_{12} & m - a_{11} - a_{12} \\ a_{21} & a_{22} & m - a_{21} - a_{22} \\ a_{31} & a_{32} & m - a_{31} - a_{32} \end{pmatrix}.$$

Then we have $vB = 0$. Note that the last three rows of B are linearly independent, since we assume that the minor of A obtained by deleting the first row and the first column is nonzero. Now there are two possibilities. The first possibility is that $v_0 = 0$. In that case we find that $v = (0,0,0,0)$. This can only happen when $(x:y:z)$ is one of $(1:0:0)$, $(0:1:0)$ or $(0:0:1)$.

If $v_0 \neq 0$ we can assume $v_0 = 1$. Since B has full rank we now find an unique solution v. Since it is found with linear algebra over $\mathbb{Q}$ we have $v_1, v_2, v_3 \in \mathbb{Q}$. Again using that the last three rows of B are linearly independent this gives that x, y, z are algebraic over $\mathbb{Q}$. This contradicts the fact that $v_0 = 1$. □

Lemma 3.6. *Let f be as in (1). Assume that $\det(A_f) \neq 0$, then f is nondegenerate with respect to its Newton polygon.*

Proof: For any edge γ we find f_γ has either two or three terms. Four terms on one edge is not possible, since then $\det(A_f) = 0$. The case where f_γ has only two terms is simple.

We wil only do the case where f_γ has three terms. Without loss of generality we can assume that $f_\gamma = X^a + Y^b + s^n X^{\lambda a} Y^{(1-\lambda)b}$, where s a root of t. Here n

is nonzero, since otherwise $\det(A_f) = 0$. Define $\eta = X^a/Y^b$. Now we assume that $a \neq 0$, then $\frac{\partial f_\gamma}{\partial X} = 0$ and $f_\gamma - \frac{X}{a}\frac{\partial f_\gamma}{\partial X} = 0$ gives:

$$\eta + \lambda s^n \eta^\lambda = 0$$

$$1 + (1-\lambda) s^n \eta^\lambda = 0$$

Since s is transcendental over k this has no solution. □

Combining the previous two lemma's with the result on the genus gives the following theorem.

Theorem 3.7. *Let C be a curve over the field $k(t)$, defined by an absolutely irreducible polynomial of the form given in (1). Also assume that* $\det(A_f) \neq 0$ *, then the genus of C equals the number of interior lattice points of its Newton polygon.*

4. The forms of the Newton polygon

In this section we will define the concept of equivalent polygons. We will use this to give a result on equivalence of elliptic curves.

Definition 4.1. *We call two polygons A, B* integrally equivalent *if B is the image of A under a linear map given by a matrix in $GL_2(\mathbb{Z})$ possibly composed with a translation.*

Note that being integrally equivalent is an equivalence relation.

If $f \in k[X,Y]$ is irreducible and is not divisible by X or Y, then $\Gamma(f)$ has at least one point on both the x and y-axis. Furthermore $\Gamma(f)$ is contained in the first quadrant. Any integral polygon can be shifted in a unique way such that it satisfies these criteria, i.e. it is contained in the first quadrant and it has a point on each of the axis. We shall consider this to be the *default position* of the polygon.

Proposition 4.2. *Let $f(X,Y) = \sum_{(a,b)\in S} \alpha_{(a,b)} X^a Y^b$ be a bivariate polynomial over a field, defining an irreducible curve C. Assume that all $\alpha_{(a,b)} \neq 0$. Given a polygon A, in default position, integrally equivalent to the polygon $\Gamma(f)$, then there exists an irreducible bivariate polynomial $g(X,Y)$, such that $A = \Gamma(g)$. Moreover the coefficients of f and g will be the same and the curves defined by f and g will be birationally equivalent.*

Proof: Let $M = \begin{pmatrix} k & l \\ m & n \end{pmatrix}$ be the matrix such that $M\Gamma(f)$ is a shift of A. Define $g(U,V) = U^\lambda V^\mu \sum_{(a,b)\in S} \alpha_{(a,b)} U^{ak+bl} V^{am+bn}$. Here λ and μ are so that $\Gamma(g)$ is in default position. By definition g has the same nonzero coefficient as f. The birational equivalence between the curves given by g and f is defined by:

$$\phi : Z(g) \longrightarrow Z(f)$$

$$(U,V) \longrightarrow (U^k V^m, U^l V^n).$$

□

It is a well known result (see [1] and [3]) that up to integral equivalence there are exactly 16 polygons with exactly one interior point. Four of these polygons have more than 4 corners. This gives the final result 1.1.

5. An example

In this section we present one example of a computation of the rank. For the other families of elliptic curve we will provide less details. The methods employed will remain the same however.

We will consider the elliptic curves over $k(t)$ that are defined by a polynomial of the form:

$$f = t^a + (t^b + t^c)X^3 + t^d Y^2 = 0,$$

where a, b, c, d are integers ≥ 0 with $c > b$. We want to find the maximal rank that occurs in this family.

Let E be the curve defined by f and E' the curve defined by:

$$t^{6a} + (t^{6b} + t^{6c})X^3 + t^{6d}Y^2 = 0.$$

Then we have a natural monomorphism $\phi : E(k(t)) \longrightarrow E'(k(t))$, defined by $\phi(x(t), y(t)) = (x(t^6), y(t^6))$. In particular we find the rank of $E(k(t))$ is at most the rank of $E'(k(t))$. So we will restrict ourselves to computing the rank of E'.

Divide the equation of E' by t^{6a} and define $\xi = t^{2(b-a)}X$, $\eta = t^{3(d-a)}Y$ and $n = 6(c - b)$. Then we see that E' is isomorphic to the curve $\tilde{E}$ defined by:

$$\tilde{f} = 1 + (1 + t^n)\xi^3 + \eta^2 = 0.$$

So we only have to determine the maximal rank for curves of this form. Note that for all $m > 0$ there is an injective map:

$$\tilde{E}(k(t)) \longrightarrow \tilde{E}'(k(t)),$$

$$(\xi(t), \eta(t)) \longrightarrow (\xi(t^m), \eta(t^m)).$$

Here $\tilde{E}'$ is the curve given by:

$$1 + (1 + t^{nm})\xi^3 + \eta^2 = 0.$$

From this we see that without loss of generality we can assume that $m|n$.

We will compute the Lefschetz number using the technique from Shioda. To do this we first homogenizing $\tilde{f}$. This gives:

$$\tilde{F} = Z^{n+3} + T^n X^3 + X^3 Z^n + Y^2 Z^{n+1}.$$

Then we compute the matrices A and A^{-1}.

$$A = \begin{pmatrix} 0 & 0 & n+3 & 0 \\ 3 & 0 & 0 & n \\ 3 & 0 & n & 0 \\ 0 & 2 & n+1 & 0 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} -\frac{n}{3(n+3)} & 0 & \frac{1}{3} & 0 \\ -\frac{n+1}{2(n+3)} & 0 & 0 & \frac{1}{2} \\ \frac{1}{n+3} & 0 & 0 & 0 \\ \frac{1}{n+3} & \frac{1}{n} & -\frac{1}{n} & 0 \end{pmatrix}.$$

By definition L is the supgroup of $(\mathbb{Q}/\mathbb{Z})^*$ generated by

$$w_1 = (1, 0, 0, -1)A^{-1} = (-\frac{1}{3}, -\frac{1}{n}, \frac{n+3}{3n}, 0),$$

$$w_2 = (0, 1, 0, -1)A^{-1} = (-\frac{1}{2}, -\frac{1}{n}, \frac{1}{n}, \frac{1}{2}),$$

$$w_3 = (0, 0, 1, -1)A^{-1} = (0, -\frac{1}{n}, \frac{1}{n}, 0).$$

By inspecting these generators we see that L is also generated by:

$$v_1 = w_1 - w_3 = (-\frac{1}{3}, 0, \frac{1}{3}, 0),$$

$$v_2 = w_2 - w_3 = (-\frac{1}{2}, 0, 0, \frac{1}{2}),$$
$$v_3 = w_3 = (0, -\frac{1}{n}, \frac{1}{n}, 0).$$

We see that L consists of elements of the form iv_3, $v_1 + iv_3$, $2v_1 + iv_3$, $v_2 + iv_3$, $v_1 + v_2 + iv_3$ and $2v_1 + v_2 + iv_3$. For each form there are exactly n elements. To compute λ we have to find out which of these elements lie in Λ.

Elements of the form $iv_3, v_1 + iv_3$ and $2v_1 + iv_3$ do not lie in Λ, since they all have zero as their last coordinate.

An element of the form $v_2 + iv_3$ does not lie in Λ. If $i = 0$ this follows from the fact that the second and third coordinate are zero. If $i \neq 0$ then this follows from the the fact that we can compute for all t with $(t, 2n) = 1$:

$$\{\frac{ti}{n}\} + \{-\frac{ti}{n}\} + \{\frac{t}{2}\} + \{\frac{t}{2}\} = 2.$$

We will now determine when $v_1 + v_2 + iv_3 \in \Lambda$. Take $j, m \in \mathbb{Z}_{\geq 0}$ such that $j/m = i/n$ and $(j, m) = 1$ and write $v_1 + v_2 + iv_3 = (\frac{1}{6}, -\frac{j}{m}, \frac{1}{3} + \frac{j}{m}, \frac{1}{2})$. We have $v_1 + v_2 + iv_3 \in \Lambda$ if and only if there exists a t such that $(t, 6m) = 1$ and

$$\{\frac{t}{6}\} + \{-\frac{jt}{m}\} + \{\frac{t}{3} + \frac{jt}{m}\} + \{\frac{t}{2}\} \neq 2.$$

It is easy to compute:

$$\{\frac{t}{6}\} + \{-\frac{jt}{m}\} + \{\frac{t}{3} + \frac{jt}{m}\} + \{\frac{t}{2}\} = \begin{cases} 1 & \text{if } t \equiv 1 \mod 6 \text{ and } \{\frac{tj}{m}\} > \frac{2}{3} \\ 3 & \text{if } t \equiv 5 \mod 6 \text{ and } \{\frac{tj}{m}\} < \frac{1}{3} \\ 2 & \text{elsewhere} \end{cases}$$

By considering a pair $\pm t$, this means that $v_1 + v_2 + iv_3 \in \Lambda$ if and only if $\{\frac{tj}{m}\} < \frac{1}{3}$ for some $t \equiv 5 \mod 6$, with $(t, 6m) = 1$. We now distinguish between the various possibilities :

- The case $m \leq 3$ is easy and leads to $(v_1 + v_2 + iv_3) \notin \Lambda$. This happens precisely when $i \in \{0, n/2, n/3, 2n/3\}$.
- Assume $m > 3$ and $3 \not| m$ or $j \equiv 2 \mod 3$. Then $t \in \mathbb{Z}$ exists with $t \equiv 5 \mod 6$ and $t \equiv j^{-1} \mod m$. For this t we find $\{\frac{tj}{m}\} < \frac{1}{3}$, hence $(v_1 + v_2 + iv_3) \in \Lambda$.
- In the case that $m > 3$, $3|m$, $j \equiv 1 \mod 3$, assume moreover that there exists a $c \equiv 2 \mod 3$, with $(c, m) = 1$ and $\{\frac{c}{m}\} < \frac{1}{3}$. We can find $t \equiv 5 \mod 6$ such that $t \equiv cj^{-1} \mod m$. For that t we have $\{\frac{tj}{m}\} < \frac{1}{3}$. This means $(v_1 + v_2 + iv_3) \in \Lambda$. This happens for all $m > 3$ except when $m \in \{6, 12, 30\}$, as is shown in lemma 5.1 below.
- The final case is $m > 3$, $3|m$, $j \equiv 1 \mod 3$ and there exists no $c \equiv 2 \mod 3$, with $(c, m) = 1$ and $\{\frac{c}{m}\} < \frac{1}{3}$. Assume that $v_1 + v_2 + iv_3 \in \Lambda$. Then $t \equiv 5 \mod 6$ exists, coprime to $6m$ such that $\{\frac{tj}{m}\} < \frac{1}{3}$. Hence $c = jm$ satisfies $c \equiv 2 \mod 3$, $gcd(c, m) = 1$ and $\{\frac{c}{m}\} < \frac{1}{3}$, contrary to our assumption.

  So in this case we find $(v_1 + v_2 + iv_3) \notin \Lambda$. By the following lemma, this final possibility for m and j happens only if $m \in \{6, 12, 30\}$. In other words only if $i \in \{\frac{n}{6}, \frac{n}{12}, \frac{7n}{12}, \frac{n}{30}, \frac{7n}{30}, \frac{13n}{30}, \frac{19n}{30}\}$.

Lemma 5.1. *6, 12 and 30 are the only integers $n > 3$ with the property that there does not exist a prime $p \equiv 2 \mod 3$ such that $3p < n$ and $p \not| n$.*

Proof: If n satisfies this property then it can be written as $n = Kp_1p_2\dots p_t$, with the p_i all primes with $p_i \equiv 2 \mod 3$ and $3p_i < n$. Order the p_i such that $p_i < p_{i+1}$. We construct the number $N = 3p_1\dots p_{t-1} + p_t$ and see that it has a prime $p \equiv 2 \mod 3$ dividing it, with $p \neq p_i$. If $n > 51$ we find:

$$p/n \leq N/n = \frac{3}{Kp_t} + \frac{1}{Kp_1\dots p_{t-1}} \leq \frac{3}{17} + \frac{1}{2\cdot 5\cdot 11} < \frac{1}{3}.$$

This means $3p < n$, but p is not any of the p_i, a contradiction. So if n satisfies the conditions of the lemma we have $n \leq 51$. Checking the lemma for $n \leq 51$ is easy. □

The cases $v_1 + v_2 + iv_3$ and $2v_1 + v_2 + iv_3$ are similar, since $-(v_1 + v_2 + iv_3) = 2v_1 + v_2 + (n-i)v_3$ and the fact that $v \in \Lambda \Leftrightarrow -v \in \Lambda$.

To ensure that all the special values $i \in \{0, \frac{n}{2}, \frac{n}{3}, \frac{2n}{3}, \frac{n}{6}, \frac{n}{12}, \frac{7n}{12}, \frac{n}{30}, \frac{7n}{30}, \frac{13n}{30}, \frac{19n}{30}\}$ encountered in the calculations are actually integers we assume that $60|n$. In that case we find $\lambda = 2n - 22$.

To compute the rank of the curve we still have to compute both the ρ_{triv} and h^2. Both of these we will compute for the curve in short Weierstrass form. Define $\tilde{\eta} = (1+t^n)\eta$ and $\tilde{\xi} = (1+t^n)\xi$ then we get the formula:

$$\tilde{\eta}^2 + \tilde{\xi}^3 + (1+t^n)^2.$$

From here we read off the second Betti number $h^2 = 4n - 2$. We also compute:

$$\Delta = -432(t^n + 1)^4.$$

$$j = 0.$$

From this we see that the elliptic surface has n singular fibres of type IV at the roots of $t^n + 1 = 0$ and no other singular fibres. So we find $\rho_{triv} = (2n + 2)$.

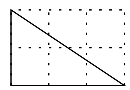
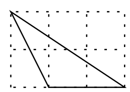
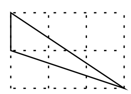
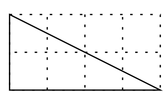
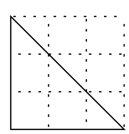
Combining these facts gives:

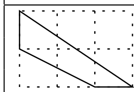
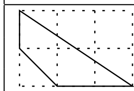
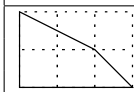
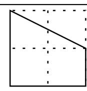
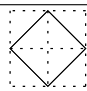
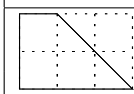
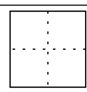
$$r = h^2 - \lambda - \rho_{triv} = 4n - 2 - (2n - 22) - (2n + 2) = 18.$$

This concludes the example we find that the rank of E over $k(t)$ is ≤ 18 and it equals 18 when $60|n$.

6. Results

The following table is a complete list of all integer polygons with exactly one interior point and at most four corners, up to equivalence. For each polygon we give a list of curves over $k(t)$, such that any elliptic curve with Newton polygon equal to the given one can be injected in one of these curves. As a consequence we create 42 families of elliptic curves over $k(t)$. Any other Delsarte elliptic curve over $k(t)$ can be injected into at least one of the 42 families.

Picture	Name	Form with maximal rank	Maximal rank	Occurring for n
	E_n^{1a}	$1+t^n+X^3+Y^2$	68	360
	E_n^{1b}	$1+t^nX+X^3+Y^2$	56	840
	E_n^{1c}	$1+t^nX^2+X^3+Y^2$	9	20
	E_n^{1d}	$1+(1+t^n)X^3+Y^2$	18	60
	E_n^{1e}	$t^n+Y+X^3+Y^2$	68	360
	E_n^{1f}	$1+t^nXY+X^3+Y^2$	9	10
	E_n^{1g}	$1+X^3+(1+t^n)Y^2$	4	6
	E_n^{2a}	$(1+t^n)X+X^3+Y^2$	24	24
	E_n^{2b}	$t^nX+X^2+X^3+Y^2$	3	12
	E_n^{2c}	$X+(1+t^n)X^3+Y^2$	24	24
	E_n^{2d}	$t^nX+XY+X^3+Y^2$	3	12
	E_n^{2e}	$X+X^3+(1+t^n)Y^2$	6	12
	E_n^{3a}	$(1+t^n)Y+X^3+Y^2$	18	60
	E_n^{3b}	$Y+(1+t^n)X^3+Y^2$	18	60
	E_n^{3c}	$Y+X^3+(1+t^n)Y^2$	18	60
	E_n^{3d}	$Y+t^nXY+X^3+Y^2$	1	2
	E_n^{4a}	$1+t^n+X^4+Y^2$	24	24
	E_n^{4b}	$t^n+X+X^4+Y^2$	56	840
	E_n^{4c}	$t^n+X^2+X^4+Y^2$	3	12
	E_n^{4d}	$1+X^3+t^nX^4+Y^2$	56	840
	E_n^{4e}	$1+(t^n+1)X^4+Y^2$	24	24
	E_n^{4f}	$1+t^nY+X^4+Y^2$	24	12
	E_n^{4g}	$t^n+XY+X^4+Y^2$	3	12
	E_n^{4h}	$1+t^nX^2Y+X^4+Y^2$	24	12
	E_n^{4i}	$1+X^4+(1+t^n)Y^2$	6	12
	E_n^{5a}	$1+t^n+X^3+Y^3$	18	60
	E_n^{5b}	$1+t^nX+X^3+Y^3$	68	120
	E_n^{5c}	$1+t^nX^2+X^3+Y^3$	68	120
	E_n^{5d}	$1+(1+t^n)X^3+Y^3$	18	60
	E_n^{5e}	$1+t^nY+X^3+Y^3$	68	120
	E_n^{5f}	$1+t^nXY+X^3+Y^3$	1	2
	E_n^{5g}	$1+t^nX^2Y+X^3+Y^3$	68	120
	E_n^{5h}	$1+t^nY^2+X^3+Y^3$	68	120
	E_n^{5i}	$1+t^nXY^2+X^3+Y^3$	68	120
	E_n^{5j}	$1+X^3+(1+t^n)Y^3$	18	60

Picture	Name	Form with maximal rank	Maximal rank	Occurring for n
	E_n^6	$t^nX^2 + Y + X^3 + Y^2$	9	20
	E_n^7	$t^nX + Y + X^3 + Y^2$	56	840
	E_n^8	$1 + t^nX^2Y + X^3 + Y^2$	56	420
	E_n^9	$t^n + X^2Y + X^2 + Y^2$	3	12
	E_n^{10}	$t^nX + Y + X^2Y + XY^2$	0	1
	E_n^{11}	$t^n + XY^2 + X^3 + Y^2$	18	120
	E_n^{12}	$t^n + X^2 + Y^2 + X^2Y^2$	0	1

7. Equivalence

Before we proceed to compute the rank of the elliptic curves in the above table, we give some criteria explaining why the ranks of certain elliptic curves are the same.

First of all if there is an isogeny between two curves E_1 and E_2 then the ranks of the two curves will be the same.

Secondly if two elliptic curves E_1 and E_2 are defined over a field $k(t)$ and there is a k-linear automorphism $\phi : k(t) \to k(t)$ bijecting the point of E_1 to E_2 then these two curves will have the same rank.

Of course sometimes a combination of these two methods can be used to show that two curves have the same rank. In this section we will use these methods to show why certain families of elliptic curves in our table have the same maximal rank.

Definition 7.1. *As a matter of terminology, we will say that two curves are k-*isogenous *(respectively k-*equivalent*) if they are isogenous (respectively isomorphic) after a k-linear automorphism.*

7.1. **1a.** In this section we explain the relation between the families of curves E_n^{1a}, E_n^{1e}, E_n^{5b}, E_n^{5c}, E_n^{5e}, E_n^{5g}, E_n^{5h} and E_n^{5i}.

Permuting homogeneous coordinates X, Y, Z gives isomorphisms between the families described by E^{5b}, E^{5c}, E^{5e}, E^{5g}, E^{5h} and E^{5i}.

A short Weierstrass form for the curves E_n^{1e} is

$$1 - 4t^n + \xi^3 + \eta^2 = 0.$$

The field automorphism defined by $t \to \sqrt[n]{-4}t$ brings this precisely in the form described by E_n^{1a}. We conclude that the curves E^{1a} and E^{1e} are k-equivalent.

Finally using ideas described in [6, Ch. 8] one finds that the curve E_n^{5b} isomorphic to the curve given by:

$$\eta^2 + \xi^3 + 1 + \frac{4}{27}t^{3n}$$

The field automorphism defined by $t \to \sqrt[3n]{-4/27}t$ brings this precisely to the curve E_{3n}^{1a}. From this it follows that E_{3n}^{1a} and E_n^{5b} are k-equivalent.

We conclude that the curves E_{3n}^{1a}, E_{3n}^{1e}, E_n^{5b}, E_n^{5c}, E_n^{5e}, E_n^{5g}, E_n^{5h} and E_n^{5i} are all k-equivalent, and hence have the same rank.

7.2. **1b.** Here we show that E_{2n}^{1b}, E_{2n}^{4b}, E_{2n}^{4d}, E_{2n}^{7} and E_n^{8} are k-equivalent. From this we can conclude that these families of curves have the same maximal rank.

The short Weierstrass form of E_n^7 is

$$1 + \frac{4}{\sqrt[3]{-4}}t^n\xi + \xi^3 + \eta^2 = 0.$$

The field automorphism defined by $t \to (-4)^{2/(3n)}t$ brings this in the form described by E_n^{1b}.

Using ideas from [6, Ch. 8] we find that the curve E_n^{4b} is isomorphic to the one given by:

$$\eta^2 + \eta + \xi^3 - \frac{2}{\sqrt[3]{2}}\xi = 0.$$

After a field automorphism this gives exactly the curve E_n^7.

The curve of the form E_n^{4b} and E_n^{4d} are isomorphic by the isomorphism $(X, Y) \to (\frac{1}{X}, \frac{Y}{X^2})$.

The curve E_n^8 is isomorphic to the curve given by:

$$\eta^2 + 1 + \xi^3 - \frac{t^{2n}}{4}\xi^4 = 0.$$

There is a field homomorphism bringing this precisely to E_{2n}^{1b}.

7.3. **1c.** We will show that the curves E_{2n}^{1c}, E_n^{1f} and E_{2n}^{6} are k-equivalent.

The curve E_n^{1f} is isomorphic to the curve given by:

$$\eta^2 + 1 + \xi^3 - \frac{1}{4}t^{2n}\xi^2 = 0.$$

There is a field homomorphism bringing this curve precisely to E_{2n}^{1c}. Likewise there is an isomorphism from E_n^6 to the curve given by

$$\eta^2 + 1 + \xi^3 + \sqrt[3]{-4}t^n\xi^2 = 0,$$

and there is a field homomorphism sending this curve to E_n^{1c}.

7.4. **1d.** We will show that the curves E_n^{1d}, E_n^{3a}, E_n^{3b}, E_n^{3c}, E_n^{5a}, E_n^{5d} and E_n^{5j} are isomorphic.

Permuting X, Y, Z gives isomorphisms between the curves E_n^{5a}, E_n^{5d} and E_n^{5j}.

Likewise the curves E_n^{3a}, E_n^{3b} and E_n^{3c} are isomorphic by the morphisms: $(X, Y) \to (X, (1 + t^n)Y)$ from E_n^{3c} to E_n^{3b}, and $(X, Y) \to ((1 + t^n)X, (1 + t^n)Y)$ from E_n^{3b} to E_n^{3a}.

Using ideas from [6, Ch. 8] we find that both the curves E_n^{5a} and E_n^{3b} are isomorphic to E_n^{1d}.

7.5. **2a.** We will show that the curves E_{2n}^{2a}, E_{2n}^{2c}, E_{2n}^{4a}, E_{2n}^{4e}, E_n^{4f} and E_n^{4h} are k-equivalent.

The curves E_n^{2a} and E_n^{2c} are isomorphic and the isomorphism from E_n^{2a} to E_n^{2c} is given by $(X,Y) \to (\frac{1}{X}, \frac{Y}{X^2})$. The curves E_n^{4a} and E_n^{4e} are also isomorphic, with isomorphism given by $(X,Y) \to (\frac{1}{X}, \frac{Y}{X^2})$. The curves E_n^{4f} and E_n^{4h} are likewise isomorphic, with isomorphism given by $(X,Y) \to (\frac{1}{X}, \frac{Y}{X^2})$.

Bringing E_n^{4a} in short Weierstrass form gives E_n^{2a}. So these two forms are also isomorphic.

The curve E_n^{4f} can by taking $(X,Y) \to (X, Y + \frac{1}{2}t^n)$ be brought to the form:

$$\xi^4 + \eta^2 + 1 - \frac{1}{4}t^{2n} = 0.$$

There is a field homomorphism bringing this precisely E_{2n}^{4a}.

7.6. **2b.** We show here that the curves E_n^{2b}, E_n^{2d}, E_n^{4c}, E_n^{4g} and E_n^{9} are k-isogenous and hence have the same rank.

The curve E_n^{2d} is isomorphic to the curve defined by:

$$\eta^2 + \xi^3 + \xi^2 + 16t^n\xi = 0.$$

There is a field homomorphism sending this curve to E_n^{2b}, so E_n^{2d} and E_n^{2b} are k-equivalent.

There is an isogeny from E_{2n}^{4c} to E_n^{2d} given by: $(x,y) \to (x^2, xy)$, so these two curves are isogenous.

There is an isomorphism from E_n^{4g} to the curve given by

$$\eta^2 + \xi^4 + \xi^2 + 16t^n = 0.$$

There is a field homomorphism sending this curve to E_n^{4c}.

Likewise there is is an isomorphism from E_n^{9} to the curve given by

$$\eta^2 - \frac{1}{4}t^n + \xi^2 + \xi^4 = 0.$$

There is a field homomorphism sending this curve to E_n^{4c}.

7.7. **2e.** The curves E_n^{2e} and E_n^{4i} are isomorphic. An isomorphism from E_n^{4i} to E_n^{2e} is given by: $(X,Y) \to (\frac{\sqrt[4]{-1}-X}{\sqrt[4]{-1}+X}, \frac{\sqrt{2}Y}{(X+\sqrt[4]{-1})^2})$.

7.8. **3d.** There is an isogeny from E_n^{5f} to E_n^{3d}, given by $(x,y) \to (xy, y^3)$. Hence the curves E_n^{5f} and E_n^{3d} are isogenous.

7.9. **10.** There is an also isogeny from E_n^{12} to E_n^{10}, given by $(x,y) \to (xy^{-1}, xy)$, hence the curves E_n^{10} and E_n^{12} are isogenous.

8. Calculation of the ranks.

In the previous section we found a number of families that for various reasons have the same maximal rank. In this section we will use Shioda's method to calculate these ranks.

8.1. **1a.** From the previous section we know that the curves E_{3n}^{1e}, E_n^{5b}, E_n^{5c}, E_n^{5e}, E_n^{5g}, E_n^{5h} and E_n^{5i} are all k-isomorphic to a curve E_{3n}^{1a}. In particular that means that these families all have the same maximal rank. We will give the computation of the maximal rank of a curves in the family E_n^{1a}.

The maximal rank of this family was already computed by Shioda [5] in 1992. More details about this curve can be found in the work of Usui [9] and in the article by Chahal, Meijer and Top [2] in 2000. We will repeat their results here out of a sense of completeness. The curve E_n^{1a} is defined by:

$$f = 1 + t^n + X^3 + Y^2 = 0.$$

This is already in short Weierstrass form, with

$$\Delta = -432(1 + t^n)^2, \text{ and } j = 0.$$

The corresponding elliptic surface has a smooth fibre at $t = \infty$ precisely when $6|n$. For the rest of this calculation we will assume $6|n$. In this case the surface has precisely n singular fibres of type II. This gives $\rho_{triv} = 2$. We have that the second Betti number $h^2 = 2n - 2$

Shioda's method can now be used to find λ. Homogenizing f gives:

$$Z^n + T^n + X^3 Z^{n-3} + Y^2 Z^{n-2}.$$

This gives the matrix:

$$A = \begin{pmatrix} 0 & 0 & n & 0 \\ 0 & 0 & 0 & n \\ 3 & 0 & n-3 & 0 \\ 0 & 2 & n-2 & 0 \end{pmatrix}.$$

From here we find the vectors generating L:

$$v_1 = (-\frac{1}{3}, 0, \frac{1}{3}, 0),$$

$$v_2 = (-\frac{1}{2}, 0, 0, \frac{1}{2}),$$

$$v_3 = (\frac{1}{n}, -\frac{1}{n}, 0, 0).$$

It can easily be shown that $iv_3, iv_3 + v_1, iv_3 + 2v_1, iv_3 + v_2 \notin \Lambda$. We will now determine when $iv_3 + v_1 + v_2$ is an element of Λ. Write $t(iv_3 + v_1 + v_2) = (\frac{jt}{m}, -\frac{jt}{m} - \frac{t5}{6}, \frac{t}{3}, \frac{t}{2})$. Here $\frac{j}{m} = \frac{i}{n} - \frac{5}{6}$. Compute:

$$\{\frac{tj}{m}\} + \{-\frac{tj}{m} - \frac{t5}{6}\} + \{\frac{t}{3}\} + \{\frac{t}{2}\} = \begin{cases} 1 & \text{if } t \equiv 1 \mod 6 \text{ and } \frac{tj}{m} < \frac{1}{6} \\ 2 & \text{if } t \equiv 1 \mod 6 \text{ and } \frac{tj}{m} > \frac{1}{6} \\ 2 & \text{if } t \equiv 5 \mod 6 \text{ and } \frac{tj}{m} < \frac{5}{6} \\ 3 & \text{if } t \equiv 5 \mod 6 \text{ and } \frac{tj}{m} > \frac{5}{6} \end{cases}$$

This means that $iv_3 + v_1 + v_2 \notin \Lambda$ precisely when either $m \leq 6$ or $m > 6$ and there does not exist a $j' \equiv 2 \mod 3$ such that $j' \equiv j \mod m$ and $\{\frac{j'}{m}\} < \frac{1}{6}$. The only m for which this happens are $1, 2, 3, 4, 5, 6, 9, 12, 18, 24, 30, 60$. For elements $iv_3 + 2v_1 + v_2 \in L$ we get a similar result. This gives that λ is at most $2n - 72$, with equallity if $360|n$.

We can now compute the maximal rank:

$$r = h^2 - \lambda - \rho_{triv} = 2n - 2 - (2n - 72) - 2 = 68.$$

8.2. **1b.** In the previous section we found that the curves E_{2n}^{1b}, E_{2n}^{4b}, E_{2n}^{4d}, E_{2n}^{7} and E_n^8 are all k-equivalent. This means that the maximal rank of these families of elliptic curves will be the same. We will give the details of the computation of the maximal rank for the family E_n^{1b}. Note that his example has already been treated by Shioda in [4].

A maximal curve will be of the form:

$$f = 1 + t^n X + X^3 + Y^2 = 0.$$

This is in short Weierstrass form so we can easily compute

$$\Delta = -64t^{3n} - 432, \text{ and } j = 1728\frac{2t^{3n}}{2t^{3n}+27}.$$

From this point we will assume $4|n$ so that the corresponding elliptic surface has a smooth fibre at $t = \infty$. The surface has as its only singular fibres $3n$ fibres of type I_1. This gives $\rho_{triv} = 2$. The second Betti number can also be determined as $h^2 = 3n - 2$.

We use Shioda's method to find λ. Homogenizing f gives:

$$Z^{n+1} + T^n X + X^3 Z^{n-2} + Y^2 Z^{n-1}.$$

This gives the matrix:

$$A = \begin{pmatrix} 0 & 0 & n+1 & 0 \\ 1 & 0 & 0 & n \\ 3 & 0 & n-2 & 0 \\ 0 & 2 & n-1 & 0 \end{pmatrix}.$$

From here we can compute the vectors generating L:

$$v_1 = (-\frac{1}{2}, 0, 0, \frac{1}{2}),$$

$$v_2 = (\frac{2}{3n}, -\frac{1}{n}, \frac{1}{3n}, 0).$$

We easily find that $iv_2 \notin \Lambda$. For $iv_2 + v_1$ we write $t(iv_2 + v_1) = (2\frac{jt}{m}, -3\frac{jt}{m} - \frac{3t}{4}, \frac{jt}{m} + \frac{t}{4}, \frac{t}{2})$, where $\frac{j}{m} = \frac{i}{3n} - \frac{1}{4}$. Now compute

$$\{2\frac{jt}{m}\} + \{-3\frac{jt}{m} - \frac{3t}{4}\} + \{\frac{jt}{m} + \frac{t}{4}\} + \{\frac{t}{2}\} = \begin{cases} 1 & \text{if } t \equiv 1 \mod 4 \text{ and } \{\frac{tj}{m}\} < \frac{1}{12} \\ 2 & \text{if } t \equiv 1 \mod 4 \text{ and } \frac{1}{12} < \{\frac{tj}{m}\} < \frac{5}{12} \\ 3 & \text{if } t \equiv 1 \mod 4 \text{ and } \frac{5}{12} < \{\frac{tj}{m}\} < \frac{1}{2} \\ 2 & \text{if } t \equiv 1 \mod 4 \text{ and } \{\frac{tj}{m}\} > \frac{1}{2} \\ 2 & \text{if } t \equiv 3 \mod 4 \text{ and } \{\frac{tj}{m}\} < \frac{1}{2} \\ 1 & \text{if } t \equiv 3 \mod 4 \text{ and } \frac{1}{2} < \{\frac{tj}{m}\} < \frac{7}{12} \\ 2 & \text{if } t \equiv 3 \mod 4 \text{ and } \frac{7}{12} < \{\frac{tj}{m}\} < \frac{11}{12} \\ 3 & \text{if } t \equiv 3 \mod 4 \text{ and } \{\frac{tj}{m}\} > \frac{11}{12} \end{cases}$$

From this we can see that that $iv_2 + v_1 \notin \Lambda$ precisely when either $m \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$ or $j \equiv 3 \mod 4$ and $m \in \{20, 28, 36, 60, 84\}$. This gives that λ is at least $3n - 60$, with equality if $840|n$.

Combining the results gives (for $840|n$) that:

$$r = h^2 - \lambda - \rho_{triv} = 3n - 2 - 2 - (3n - 60) = 56.$$

8.3. **1c.** In the previous section we proved that the curves E_{2n}^{1c}, E_n^{1f} and E_{2n}^{6} are k-equivalent. To find the maximal rank of these families of elliptic curves it suffices to find the maximal rank of the family of curves E_n^{1c}.

The curve E^{1c} is given by:

$$f = 1 + t^n X^2 + X^3 + Y^2 = 0.$$

Bringing this to short Weierstrass gives:

$$\eta^2 + \xi^3 - \frac{t^{2n}}{3}\xi + 1 + \frac{2}{27}t^{3n} = 0.$$

and we find

$$\Delta = -64t^{3n} - 432, \text{ and } j = -\frac{256t^{6n}}{4t^{3n} + 27}.$$

For the rest of this calculation we will assume that $2|n$, so that the only singular fibres of the corresponding elliptic surface are $3n$ fibres of type I_1 and one of type I_{3n}. From here we see that $\rho_{triv} = 1 + 3n$. We can also compute the second Betti number $h^2 = 6n - 2$

We use Shioda's method to find λ. Homogenizing f gives:

$$Z^{n+2} + T^n X^2 + X^3 Z^{n-1} + Y^2 Z^n.$$

From here we determine the matrix:

$$A = \begin{pmatrix} 0 & 0 & n+2 & 0 \\ 2 & 0 & 0 & n \\ 3 & 0 & n-1 & 0 \\ 0 & 2 & n & 0 \end{pmatrix}.$$

This gives the generators of L:

$$v_1 = (-\frac{1}{2}, 0, 0, \frac{1}{2}),$$

$$v_2 = (\frac{1}{3n}, -\frac{1}{n}, \frac{2}{3n}, 0).$$

We trivially find that $iv_2 \notin \Lambda$. For $iv_2 + v_1$ we write $t(iv_2 + v_1) = (\frac{jt}{m}, -3\frac{jt}{m} + \frac{t}{2}, 2\frac{jt}{m}, \frac{t}{2})$. Here $\frac{j}{m} = \frac{i}{3n} - \frac{1}{2}$. We compute

$$\{\frac{jt}{m}\} + \{-3\frac{jt}{m} + \frac{t}{2}\} + \{2\frac{jt}{m}\} + \{\frac{t}{2}\} = \begin{cases} 1 & \text{if } \{\frac{tj}{m}\} < \frac{1}{6} \\ 2 & \text{if } \frac{1}{6} < \{\frac{tj}{m}\} < \frac{5}{6} \\ 3 & \text{if } \frac{5}{6} < \{\frac{tj}{m}\} \end{cases}$$

This means that $iv_2 + v_1 \notin \Lambda$ precisely when $m \in \{1, 2, 3, 4, 5, 6\}$. This gives that λ is at least $3n - 12$, and equality holds if $20|n$.

Combining the results gives, for $20|n$ that the rank is:

$$r = h^2 - \lambda - \rho_{triv} = 6n - 2 - (3n + 1) - (3n - 12) = 9.$$

8.4. **1d.** We already found that the curves E_n^{1d}, E_n^{3a}, E_n^{3b}, E_n^{3c}, E_n^{5a}, E_n^{5d} and E_n^{5j} are k-invariant. In the example we already computed the maximal rank of the curve E_n^{1d} and found $r = 18$.

8.5. **1g.** We will now compute the maximal rank of the family of curves E_n^{1g}. The curve E_n^{1g} is given by:

$$f = 1 + X^3 + (1 + t^n)Y^2 = 0.$$

Bringing this is in short Weierstrass form gives:

$$\eta^2 + \xi^3 + (1 + t^n)^3 = 0.$$

And we can compute:

$$\Delta = -432(t^n + 1)^6, \text{ and } j = 0.$$

For the rest of the calculation we will assume $2|n$. In that case the corresponding elliptic surface has exactly n singular fibres of type I_0^* and no other singular fibres. From this we find $\rho_{triv} = 2 + 4n$. The second Betti number can also be found to be $h^2 = 6n - 2$

We use Shioda's method to find λ. Homogenizing f gives:

$$Z^{n+2} + X^3Z^{n-1} + Y^2Z^n + Y^2T^n.$$

This gives the matrix:

$$A = \begin{pmatrix} 0 & 0 & n+2 & 0 \\ 3 & 0 & n-1 & 0 \\ 0 & 2 & n & 0 \\ 0 & 2 & 0 & n \end{pmatrix},$$

and generators for L:

$$v_1 = (-\frac{1}{3}, \frac{1}{3}, 0, 0),$$

$$v_2 = (-\frac{1}{2}, 0, \frac{1}{2}, 0),$$

$$v_3 = (0, 0, \frac{1}{n}, -\frac{1}{n}).$$

It can easilly be seen that $iv_3, v_1 + iv_3, 2v_1 + iv_3, v_2 + iv_3 \notin \Lambda$. We will now determine when $v_1 + v_2 + iv_3 \in L$ Write $t(v_1 + v_2 + iv_3) = (-\frac{5t}{6}, \frac{t}{3}, \frac{t}{2} + \frac{jt}{m}, -\frac{jt}{m})$, where j and m are minimal such that $j/m = i/n$. Compute

$$\{-\frac{5t}{6}\} + \{\frac{t}{3}\} + \{\frac{t}{2} + \frac{jt}{m}\} + \{-\frac{jt}{m}\} = \begin{cases} 1 & \text{if } t \equiv 1 \mod 6 \text{ and } \{\frac{tj}{m}\} > \frac{1}{2} \\ 3 & \text{if } t \equiv 5 \mod 6 \text{ and } \{\frac{tj}{m}\} < \frac{1}{2} \\ 2 & \text{otherwise} \end{cases}$$

This means that $v_1 + v_2 + iv_3 \notin \Lambda$ precisely when $m \in \{1, 2\}$ or when $m \in \{3, 6\}$ and $j \equiv 1 \mod 3$. For $2v_1 + v_2 + iv_3$ we get a similar result. This gives that λ is at least $2n - 8$, and equality holds if $6|n$.

Combining this gives if $6|n$ the rank:

$$r = h^2 - \lambda - \rho_{triv} = 6n - 2 - (2n - 8) - (4n + 2) = 4.$$

8.6. **2a.** In the previous section we found that the curves E_{2n}^{2a}, E_{2n}^{2c}, E_{2n}^{4a}, E_{2n}^{4e}, E_n^{4f} and E_n^{4h} are all k-equivalent. We will only have to compute the maximal rank of the family of curves E_n^{2a}. The curve E_n^{2a} is defined by:

$$f = (1+t^n)X + X^3 + Y^2 = 0.$$

This is already on short Weierstrass form so we can easily compute.

$$\Delta = -64(t^n+1)^3, \text{ and } j = 1728.$$

From here we will assume $4|n$. In this case the corresponding elliptic surfaces has n singular fibres of type III and no other singular fibres. This gives $\rho_{triv} = 2+n$. The second Betti number will be $h^2 = 3n-2$.

We use Shioda's method to find λ. Homogenizing f gives:

$$XZ^n + XT^n + X^3Z^{n-2} + Y^2Z^{n-1}.$$

This gives the matrix:

$$A = \begin{pmatrix} 1 & 0 & n & 0 \\ 1 & 0 & 0 & n \\ 3 & 0 & n-2 & 0 \\ 0 & 2 & n-1 & 0 \end{pmatrix}.$$

From this we can compute generators for L:

$$v_1 = (-\frac{3}{4}, 0, \frac{1}{4}, \frac{1}{2}),$$

$$v_2 = (\frac{1}{n}, -\frac{1}{n}, 0, 0).$$

It can be easily seen that $iv_2, 2v_1 + iv_2 \notin \Lambda$. We now have to determine when $v_1 + iv_2 \in \Lambda$. Write $t(v_1+iv_2) = (\frac{jt}{m} - \frac{3t}{4}, -\frac{jt}{m}, \frac{t}{4}, \frac{t}{2})$, where j, m are minimal such that $j/m = i/n$. Now compute

$$\{\frac{jt}{m} - \frac{3t}{4}\} + \{-\frac{jt}{m}\} + \{\frac{t}{4}\} + \{\frac{t}{2}\} = \begin{cases} 1 & \text{if } t \equiv 1 \mod 4 \text{ and } \{\frac{tj}{m}\} > \frac{3}{4} \\ 3 & \text{if } t \equiv 3 \mod 4 \text{ and } \{\frac{tj}{m}\} < \frac{1}{4} \\ 2 & \text{otherwise} \end{cases}$$

This means that $v_1 + iv_2 \notin \Lambda$ precisely when $m \in \{1,2,3,4\}$ or when $m \in \{8,12,24\}$ and $j \equiv 1 \mod 4$. For $3v_1 + iv_2$ we get a similar result. This gives that λ is at least $2n-28$, and equality holds if $24|n$.

Combining the we find that if $24|n$ the rank:

$$r = h^2 - \lambda - \rho_{triv} = 3n-2-(2n-28)-(n+2) = 24.$$

8.7. **2b.** In the previous section we proved that the curves E_n^{2b}, E_n^{2d}, E_n^{4c}, E_n^{4g} and E_n^9 are k-isogenous, and as such have the same rank. We will compute the maximal rank of the family of curves E_n^{2b}. The curve E_n^{2b} is given by

$$f = t^nX + X^2 + X^3 + Y^2 = 0.$$

In short Weierstrass form this curve is given by:

$$\eta^2 + \xi^3 + (t^n - \frac{1}{3})\xi + \frac{2}{27} - \frac{t^n}{3} = 0.$$

From this we can compute:

$$\Delta = -64t^{3n} + 16t^{2n}, \text{ and } j = 256\frac{(3t^n-1)^3}{4t^{3n} - t^{2n}}.$$

From here on we will assume that $4|n$. In this case the corresponding elliptic surface has n singular fibres of type I_1, one singular fibre of type I_{2n} and no other singular fibres. This gives $\rho_{triv} = 2n+1$. The second Betti number is given by $h^2 = 3n-2$.

We use Shioda's method to find λ. Homogenizing f gives:

$$XT^n + X^2Z^{n-1} + X^3Z^{n-2} + Y^2Z^{n-1}.$$

This gives the matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & n \\ 2 & 0 & n-1 & 0 \\ 3 & 0 & n-2 & 0 \\ 0 & 2 & n-1 & 0 \end{pmatrix},$$

and generators for L:

$$v_1 = (0, -\frac{1}{2}, 0, \frac{1}{2}),$$
$$v_2 = (-\frac{1}{n}, \frac{2}{n}, -\frac{1}{n}, 0).$$

It is easily seen that $iv_2 \notin \Lambda$. For $v_1 + iv_2$ we write $t(v_1 + iv_2) = (-\frac{jt}{m}, 2\frac{jt}{m} - \frac{t}{2}, -\frac{jt}{m}, \frac{t}{2})$. Here j, m are minimal such that $j/m = i/n$. We can compute:

$$\{-\frac{jt}{m}\} + \{2\frac{jt}{m} - \frac{t}{2}\} + \{-\frac{jt}{m}\} + \{\frac{t}{2}\} = \begin{cases} 3 & \text{if } \{\frac{tj}{m}\} < \frac{1}{4} \\ 2 & \text{if } \frac{1}{4} < \{\frac{tj}{m}\} < \frac{3}{4} \\ 1 & \text{if } \frac{3}{4} < \{\frac{tj}{m}\} \end{cases}$$

This means that $v_1 + iv_2 \notin \Lambda$ precisely when $m \in \{1, 2, 3, 4\}$. This gives that λ is at least $n-6$, and equality holds if $12|n$.

It follows that if $12|n$ the rank is:

$$r = h^2 - \lambda - \rho_{triv} = 3n - 2 - (n-6) - (2n+1) = 3.$$

8.8. **2e.** The curves E_n^{2e} and E_n^{4i} are isomorphic, as such they have the same rank. We will compute the maximal rank of the family E_n^{2e}. The curve E_n^{2e} is defined by:

$$f = X + X^3 + (1+t^n)Y^2 = 0.$$

In short Weierstrass form this gives:

$$\eta^2 + \xi^3 + (t^n+1)^2\xi = 0.$$

For this curve we can compute:

$$\Delta = -64(1+t^n)^6, \text{ and } j = 1728.$$

We will from here on assume that $2|n$. In that case the corresponding elliptic surface has n singular fibres of type I_0^* and no other singular fibres. This gives $\rho_{triv} = 4n+2$. The second Betti number can be determined $h^2 = 6n-2$.

We use Shioda's method to find λ. Homogenizing f gives:

$$XZ^{n+1} + X^3Z^{n-1} + Y^2Z^n + Y^2T^n.$$

This gives the matrix:

$$A = \begin{pmatrix} 1 & 0 & n+1 & 0 \\ 3 & 0 & n-1 & 0 \\ 0 & 2 & n & 0 \\ 0 & 2 & 0 & n \end{pmatrix}.$$

This gives as generators for L:

$$v_1 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0),$$

$$v_2 = (0, 0, \frac{1}{n}, -\frac{1}{n}).$$

It is easy to see that $iv_2, 2v_1 + iv_2 \notin \Lambda$. Write $t(v_1 + iv_2) = (\frac{t}{4}, \frac{t}{4}, \frac{jt}{m} + \frac{t}{2}, -\frac{jt}{m})$, where j, m are minimal such that $j/m = i/n$. We can compute

$$\{\frac{t}{4}\} + \{\frac{t}{4}\} + \{\frac{jt}{m} + \frac{t}{2}\} + \{-\frac{jt}{m}\} = \begin{cases} 1 & \text{if } t \equiv 1 \mod 4 \text{ and } \{\frac{tj}{m}\} > \frac{1}{2} \\ 3 & \text{if } t \equiv 3 \mod 4 \text{ and } \frac{1}{2} > \{\frac{tj}{m}\} \\ 2 & \text{elsewhere} \end{cases}$$

This means that $v_1 + iv_2 \notin \Lambda$ precisely when $m \in \{1, 2\}$ or when $m \in \{4, 12\}$ and $j \equiv 1 \mod 4$.

For $3v_1 + iv_2$ we get a similar result. Now it follows that λ is at least $2n - 10$, and equality holds if $12|n$. We conclude that if $12|n$ the rank of our curve is:

$$r = h^2 - \lambda - \rho_{triv} = 6n - 2 - (2n - 10) - (4n + 2) = 6.$$

8.9. **3d.** The curves E_n^{5f} and E_n^{3d} are isogenous, as such they have the same rank. We will now compute the maximal rank of the family of curves E_n^{3d}. The curve E_n^{3d} is given by:

$$f = Y + t^n XY + X^3 + Y^2 = 0.$$

Bringing this in short Weierstrass form gives:

$$\eta^2 + \xi^3 - (\frac{1}{48}t^{4n} + \frac{1}{2}t^n)\xi - \frac{1}{4} - \frac{1}{24}t^{3n} - \frac{1}{864}t^{6n}.$$

For this form we compute the invariants:

$$\Delta = -(t^{3n} + 27), \text{ and } j = -\frac{(t^{4n} + 24t^n)^3}{t^{3n} + 27}.$$

This has $3n$ singular fibres of type I_1, one singular fibre of type I_{9n}. From this we can compute $\rho_{triv} = 9n + 1$. The second Betti number can also be determined $h^2 = 12n - 2$.

We use Shioda's method to find λ. Homogenizing f gives:

$$YZ^{n+1} + T^n XY + X^3 Z^{n-1} + Y^2 Z^n.$$

This gives the matrix:

$$A = \begin{pmatrix} 0 & 1 & n+1 & 0 \\ 1 & 1 & 0 & n \\ 3 & 0 & n-1 & 0 \\ 0 & 2 & n & 0 \end{pmatrix}.$$

Using this we can find that L is generated by:

$$v_1 = (\frac{1}{3n}, -\frac{1}{n}, \frac{1}{3n}, \frac{1}{3n}).$$

It has to determine whether $iv_1 \in \Lambda$ or not. Write $t(iv_1) = (\frac{jt}{m}, -3\frac{jt}{m}, \frac{jt}{m}, \frac{jt}{m})$. Here j, m are minimal such that $j/m = i/3n$. We can compute

$$\{\frac{jt}{m}\} + \{-3\frac{jt}{m}\} + \{\frac{jt}{m}\} + \{\frac{jt}{m}\} = \begin{cases} 1 & \text{if } \{\frac{tj}{m}\} < \frac{1}{3} \\ 2 & \text{if } \frac{1}{3} < \{\frac{tj}{m}\} < \frac{1}{3} \\ 3 & \text{if } \frac{2}{3} < \{\frac{tj}{m}\} \end{cases}$$

This means that $iv_1 \notin \Lambda$ precisely when $m \in \{1,2,3\}$. This gives that λ is at least $3n-4$, and equality holds if n is even. We conclude that for even n the rank will be:

$$r = h^2 - \lambda - \rho_{triv} = 12n - 2 - (3n-4) - (9n+1) = 1.$$

8.10. **11.** We will determine the maximal rank of curves of the family E_n^{11}. The curve E_n^{11} is given by.

$$f = t^n + XY^2 + X^3 + Y^2.$$

We will for our calculation assume that $6|n$. In this case we find that the short Weierstrass form becomes:

$$\eta^2 + \xi^3 - 3t^{n/3}\xi + 1 + t^n.$$

This has the following invariants:

$$\Delta = -432(1-t^n)^2 \text{ and}$$

$$j = -\frac{110592t^n}{(1-t^n)^2}.$$

The corresponding surface has exactly n singular fibres all of type I_2. It follows that $\rho_{triv} = n+2$. The second Betti number in this case is $h^2 = 2n-2$.

We use Shioda's method to find λ. Homogenizing f gives:

$$T^n + XY^2Z^{n-3} + X^3Z^{n-3} + Y^2Z^{n-2}.$$

This gives the matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & n \\ 1 & 2 & n-3 & 0 \\ 3 & 0 & n-3 & 0 \\ 0 & 2 & n-2 & 0 \end{pmatrix},$$

from which we compute generators for L:

$$v_1 = (0, \frac{1}{2}, \frac{1}{2}, 0),$$

$$v_2 = (-\frac{1}{n}, -\frac{3}{n}, \frac{1}{n}, \frac{3}{n}).$$

It is easily seen that $iv_2, \notin \Lambda$. For $v_1 + iv_2$ we write $t(v_1 + iv_2) = (-\frac{jt}{m} - \frac{t}{6}, -3\frac{jt}{m}, \frac{2t}{3} + \frac{jt}{m}, \frac{t}{2} + 3\frac{jt}{m})$. Here j, m are minimal such that $j/m = i/n - 1/6$. We can compute:

$$\{-\frac{jt}{m} - \frac{t}{6}\} + \{-3\frac{jt}{m}\} + \{\frac{2t}{3} + \frac{jt}{m}\} + \{\frac{t}{2} + 3\frac{jt}{m}\} = \begin{cases} 3 & \text{if } t \equiv 1 \mod 6 \text{ and } \{\frac{tj}{m}\} < \frac{1}{6} \\ 1 & \text{if } t \equiv 1 \mod 6 \text{ and } \frac{1}{2} < \{\frac{tj}{m}\} < \frac{2}{3} \\ 3 & \text{if } t \equiv 5 \mod 6 \text{ and } \frac{1}{3} < \{\frac{tj}{m}\} < \frac{1}{2} \\ 1 & \text{if } t \equiv 5 \mod 6 \text{ and } \frac{5}{6} < \{\frac{tj}{m}\} \\ 2 & \text{otherwise.} \end{cases}$$

This means that $v_1 + iv_2 \notin \Lambda$ precisely when $m \in \{1,2,3,4,6\}$. or when $m \in 12, 24, 60$ and $j \equiv 2 \mod 3$. This gives that λ is at least $n-22$, and equality holds if $120|n$. Combining these results gives if $120|n$:

$$r = h^2 - \lambda - \rho_{triv} = 2n - 2 - (n-22) - (n+2) = 18.$$

8.11. **12.** Finally we will look at the family E_n^{12}. This family consists of elliptic curves in the Edwards form. The curve E_n^{12} is given by:

$$f = t^n + X^2 + Y^2 + X^2Y^2 = 0.$$

Reducing to short Weierstrass form gives:

$$\eta^2 + \xi^3 - (\frac{1}{3} + \frac{14}{3}t^n + \frac{1}{3}t^{2n})\xi + \frac{2}{27} - \frac{22}{9}t^n - \frac{22}{9}t^{2n} + \frac{2}{27}t^{3n}.$$

For this we compute the invariants:

$$\Delta = -256t^n(1-t^n)^4, \text{ and}$$

$$j = -16\frac{(1+14t^n+t^{2n})^3}{t^n(1-t^n)4}.$$

We will for the rest of this calculation assume that n is even. The surface now has n singular fibres of type I_4 and 2 singular fibres of type I_n and no other singular fibres. This gives $\rho_{triv} = 5n$. From here on we will assume that n is even. Under this assumption the second Betti number is $h^2 = 6n - 2$.

We use Shioda's method to find λ. Homogenizing f gives:

$$T^n + X^2Z^{n-2} + Y^2Z^{n-2} + X^2Y^2Z^{n-4}.$$

This gives the matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & n \\ 2 & 0 & n-2 & 0 \\ 0 & 2 & n-2 & 0 \\ 2 & 2 & n-4 & 0 \end{pmatrix}.$$

This gives the following generators for L:

$$v_1 = (0, 0, \frac{1}{2}, \frac{1}{2}),$$

$$v_2 = (0, \frac{1}{2}, 0, \frac{1}{2}),$$

$$v_3 = (-\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, -\frac{1}{n}).$$

It turns out that $iv_3, v_1 + iv_3, v_2 + iv_3 \notin \Lambda$. Write $t(v_1 + v_2 + iv_3) = (-\frac{jt}{m}, \frac{jt}{m} + \frac{t}{2}, \frac{jt}{m} + \frac{t}{2}, -\frac{jt}{m})$, where j, m are minimal such that $j/m = i/n$. Now compute

$$\{-\frac{jt}{m}\} + \{\frac{jt}{m} + \frac{t}{2}\} + \{\frac{jt}{m} + \frac{t}{2}\} + \{-\frac{jt}{m}\} = \begin{cases} 1 & \text{if } \{\frac{tj}{m}\} < \frac{1}{2} \\ 3 & \text{if } \frac{1}{2} < \{\frac{tj}{m}\} \end{cases}$$

This means that $v_1 + v_2 + iv_3 \notin \Lambda$ precisely when $m \in \{1, 2\}$. This gives that λ is at least $n - 2$, and equality holds if $2|n$.

Combining the results gives for even n:

$$r = h^2 - \lambda - \rho_{triv} = 6n - 2 - (n-2) - 5n = 0.$$

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