Convexity of Momentum Maps: A Topological Analysis

Wolfgang Rump and Jenny Santoso

Institute for Algebra and Number Theory, University of Stuttgart
Pfaffenwaldring 57, D-70550 Stuttgart, Germany

Dedicated to Tudor S. Ratiu on the occasion of his 60th birthday and Alan D. Weinstein on the occasion of his "retirement" from UC Berkeley

Abstract. We extend the Local-to-Global-Principle used in the proof of convexity theorems for momentum maps to not necessarily closed maps $f\colon X \to Y$ whose target space Y carries a convexity structure which need not be based on a metric. Using a new factorization of f, convexity of its image is proved without local fiber connectedness, and for almost arbitrary spaces X.

Introduction

Convexity for momentum maps was discovered independently by Atiyah [3] and Guillemin-Sternberg [14] in the case of a Hamiltonian torus action on a compact symplectic[I] manifold X. It was proved that the image of the momentum map μ is a convex polytope, namely, the convex hull of $\mu(X^T)$, where X^T denotes the set of fixed points under the action of the torus T. In this case, μ is open onto its image, and the fibers of μ are compact and connected. Two years later, in 1984, Kirwan [21] (see also [15]) extended this result to the action of a compact connected Lie group G. Here the image of $\mu\colon X \to \mathrm{Lie}(G)^*$ has to be restricted to a closed Weyl chamber in a Cartan subalgebra of $\mathrm{Lie}(G)$, i.e. a fundamental domain of G with respect to its coadjoint action on $\mathrm{Lie}(G)^*$. Equivalently, this amounts to a composition of the momentum map μ with the projection onto the quotient space $Y := \mathrm{Lie}(G)^*/G$ modulo the coadjoint action of G. Up to this time, convexity of μ was proved by means of Morse theory, applied to the components of μ. This works well as long as μ is defined on a compact manifold X.

2000 *Mathematics Subject Classification.* Primary: 52A01, 53C23, 54C10, 53D20. Secondary: 54E18.

Key words and phrases. Convexity space, geodesic manifold, momentum map, Lokal-global-Prinzip.

[I]For introduction to symplectic manifolds, torus actions on symplectic manifolds, and momentum maps resp. see, e.g., [37, 36, 2, 34, 26, 29, 1]

In 1988, Condevaux, Dazord, and Molino [11] reproved these results in an entirely new fashion. They factor out the connected components of the fibers of μ to get a monotone-light factorization $\mu\colon X \to \widetilde{X} \to Y$ (see [24]). If μ is proper, i.e. closed and with quasi-compact fibers, the metric of Y can be lifted to $\widetilde{X}$. Then a shortest path between two points of $\widetilde{X}$ maps to a straight line in Y, which proves the convexity of $\mu(X)$. Based on this method, Hilgert, Neeb, and Plank [17] extended Kirwan's result to non-compact connected manifolds X under the assumption that μ is proper.

After this invention, the proof of convexity now splits into two parts: A geometric part where certain local convexity data have to be verified, and a topological part, a kind of "Lokal-global-Prinzip" [17] which proves global convexity à la Condevaux, Dazord, and Molino.

A further step was taken by Birtea, Ortega, and Ratiu [6, 7] who consider a closed, not necessarily proper map $\mu\colon X \to \widetilde{X} \to Y$, defined on a normal, first countable, arcwise connected Hausdorff space X. The map μ has to be locally open onto its image, locally fiber connected, having local convexity data. Using Vaĭnšteĭn's Lemma, they prove that the light part $\widetilde{X} \to Y$ of μ is proper. This yields global convexity of $\mu(X)$ for two almost disjoint kinds of target spaces Y, either the dual of a Banach space [7] (which implies that the closed unit ball of Y is weak* compact), or a complete locally compact length metric space Y [6]. The second case applies to the cylinder-valued momentum map [30, 31], another invention of Condevaux, Dazord, and Molino [11]: For a symplectic manifold (X, ω), the 2-form ω gives rise to a flat connection on the trivial principal fiber bundle $X \times \mathrm{Lie}(G)^*$ with holonomy group H. The cylinder-valued momentum map $\overline{\mu}$ is obtained from μ by factoring out $\overline{H}$ from the target space Y. The new target space $\overline{\mu}(X) = Y/\overline{H}$ is a cylinder, hence geodesics on it may differ from shortest paths. The convexity theorem then states that $\overline{\mu}(X)$ is *weakly convex*, i.e. any two points of $\overline{\mu}(X)$ are connected by a geodesic arc.

In the present paper, we analyse the topological part of convexity, that is, the passage from local to global convexity. We show that the Lokal-global-Prinzip, as developed thus far, admits a substantial improvement in at least three respects.

Firstly, we replace the monotone-light factorization $f\colon X \to \widetilde{X} \to Y$ that was used for a momentum map $f = \mu$ by a new factorization

$$f\colon X \xrightarrow{q^f} X^f \xrightarrow{f^\#} Y$$

of any continuous map $f\colon X \to Y$ which is locally open onto its image. In a sense, X^f is closer to Y than the leaf space $\widetilde{X}$ since $q^f\colon X \to X^f$ factors through the monotone part $X \to \widetilde{X}$ of f. We show that q^f is an open surjection, while X^f admits a basis of open sets U such that $f^\#$ maps U homeomorphically onto a subspace of Y (Proposition 5). Therefore, $f^\#$ can take the rôle of the light part of f, which means that we can drop the assumption that f (the momentum map) is locally fiber connected.

Secondly, we concentrate on the target space Y instead of X to derive the desired properties of X^f. In this way, the various assumptions on X boil down to a single one, namely, its connectedness as a topological space. Nevertheless, we need no extra assumptions on the target space Y.

Thirdly, we merely assume that the map $f^\#$ is closed, a much weaker condition than the closedness of f. Even the light part of f need not be closed. For example, $f^\#$ is trivial for a local homeomorphism f - a light map which need not be closed, and with fibers of arbitrary size. Using the properties of Y, we prove that the fibers of $f^\#$ are finite (Proposition 10), so that the convexity structure of Y can be lifted along $f^\#$ (Theorem 2).

To make the interaction between convexity and topology more visible, we untie the Lokal-global-Prinzip from its metric context by means of a general concept of convexity, which might be of interest in itself. This also unifies the two above mentioned types of target space considered in [6] and [7]. In the linear case [7], the target space Y may be an arbitrary (not necessarily complete) metrizable locally convex space instead of a dual Banach space. (Metrizability can be weakened by the condition that Y does not contain a locally convex direct sum $\mathbb{R}^{(\aleph_0)}$ as a subspace.) In general, geodesics in our (non-linear) target space Y are one-dimensional continua which need not be metrizable.

In previous versions of the Lokal-global-Prinzip, geodesic arcs or connecting lines between two points of the target space Y are obtained by a metric on Y. Without a concept of length, of course, geodesics are no longer available by shortening of arcs in the spirit of the Hopf-Rinow Theorem. Instead, we obtain geodesics by continued *straightening*, using a local convexity structure. In other words, we deal with a "manifold", that is, a Hausdorff space Y covered by open subspaces U with an additional structure of convexity. The axioms of such a *convexity space* U are very simple: For any pair of points $x, y \in U$, there is a minimal connected subset $C(x, y)$ containing x and y, varying continuously with the end points. In a topological vector space, $C(x, y)$ is just the line segment between x and y, while in a uniquely geodesic space, $C(x, y)$ is the unique shortest path between x and y. With respect to the $C(x, y)$, there is a natural concept of convexity, and for a convexity space U, we just require that the $C(x, y)$ are convex and that U has a basis of convex open sets (see Definition 1).

If convexity is given by a metric, straightening and shortening of arcs leads to the same result, namely, a geodesic of minimal length. For a non-metrizable arc A between two points x and y, there is a substitute for the length of A, namely, the closed convex hull $\overline{C(A)}$ which is diminished by straightening. As a first step, an inscribed "line path" L (in a geodesic sense) satisfies $\overline{C(L)} \subset \overline{C(A)}$, and $\overline{C(L)}$ is the closed convex hull of the finitely many extreme points of L. For a given line path L between x and y, assume that the closed convex hull $\overline{C(L)}$ is compact. Using Zorn's Lemma, we minimize the connected set $\overline{C(L)}$ to a compact convex set C with $x, y \in C$. In contrast to the Hopf-Rinow situation, where the shortening of

L is achieved via the Arzelà-Ascoli Theorem, the straightening method needs the compactness of $\overline{C(L)}$ to show that connectedness carries over to C. By the local convexity structure, it then follows that C contains a line path L_0 between x and y. Thus if $C = L_0$, the line path L_0 must be a geodesic.

So we require two properties to get the straightening process work: First, the closed convex hull of a finite set must be compact; second, a minimal compact connected convex set C containing x and y has to be a geodesic.

To establish a Lokal-global-Prinzip for continuous maps $X \to Y$, possible self-intersec- tions of the arcs to be straightened have to be taken into account. Precisely, this means that closed convex subsets of Y have to be replaced by *étale* maps, i.e. closed locally convex maps $e\colon C \to Y$, such that the connected space C admits a covering by open sets mapped homeomorphically onto convex subsets of Y. We call Y a *geodesic manifold* if the above two properties hold with an adaption to étale maps $e\colon C \to Y$, that is, the second property now states that if C is compact and minimal with respect to $x, y \in C$, then e can be regarded as a geodesic with possible self-intersections. (Such a geodesic is transversal if and only if $e = e^{\#}$.) If the charts U of Y are regular Hausdorff spaces which satisfy a certain finiteness condition (see Definition 2) which holds, for example, if U is either locally compact or first countable, we call Y a *geodesic q-manifold* (the "q" refers to the finiteness condition). Obvious examples of geodesic q-manifolds are complete locally compact length metric spaces, or metrizable locally convex topological linear spaces (Examples 6 and 7). Our main result consists in the following

Lokal-global-Prinzip. *Let $f\colon X \to Y$ be a locally convex continuous map from a connected topological space X to a geodesic q-manifold Y. Assume that $f^{\#}$ is closed. Then any two points of $f(X)$ are connected by a geodesic arc.*

For an inclusion map $f\colon C \hookrightarrow Y$, the conditions on f turn into the assumptions of the Tietze-Nakajima Theorem (see [28]), i.e. the subset C is closed, connected, and locally convex. Thus in case of a locally convex topological vector space Y, the result for $C \hookrightarrow Y$ yields Klee's Convexity Theorem [22], while for a complete Riemannian manifold Y, we get a Theorem of Bangert [4].

1 Convexity spaces

Let X be a Hausdorff space. We endow the power set $\mathfrak{P}(X)$ with a topology as follows. For any open set U of X, define

$$\widetilde{U} := \{C \in \mathfrak{P}(X) \mid C \subset U\}. \tag{1}$$

The collection $\mathfrak{B}$ of sets (1) is closed under finite intersection. We take $\mathfrak{B}$ as a basis of open sets for the topology of $\mathfrak{P}(X)$.

Definition 1. Let X be a Hausdorff space together with a continuous map

$$C\colon X \times X \to \mathfrak{P}(X). \tag{2}$$

We call a subset $A \subset X$ *convex* if $C(x,y) \subset A$ holds for all $x, y \in A$. We say that X is a *convexity space* with respect to a map (2) if the following are satisfied.

(C1) The $C(x,y)$ are convex for all $x, y \in X$.

(C2) The $C(x,y)$ are minimal among the connected sets $C \subset X$ with $x, y \in C$.

(C3) X has a basis of convex open sets.

Note that (C1) implies that $C(y,x) \subset C(x,y)$. Hence C is symmetric:

$$C(x,y) = C(y,x). \tag{3}$$

From (C2) we infer that

$$C(x,x) = \{x\}. \tag{4}$$

Moreover, (C2) implies that every convexity space X is connected. The restriction of the map (2) to a convex subset $A \subset X$ makes A into a convexity space. Hence (C3) implies that X is locally connected.

Lemma 1. *Let X be a convexity space. For $x, y \in X$, the set $C(x,y) \smallsetminus \{y\}$ is connected.*

Proof. Let A be the connected component of x in $C(x,y) \smallsetminus \{y\}$. Since $\{y\}$ is closed, every $z \in C(x,y) \smallsetminus \{y\}$ admits a convex neighbourhood U with $y \notin U$. Hence $C(x,y) \smallsetminus \{y\}$ is locally connected, and thus A is open in $C(x,y)$. Since $C(x,y)$ is connected, it follows that A cannot be closed in $C(x,y)$. Thus $y \in \overline{A}$, which shows that $A \cup \{y\}$ is connected. By (C2), this gives $A \cup \{y\} = C(x,y)$, whence $A = C(x,y) \smallsetminus \{y\}$. □

As a consequence, the $C(x,y)$ can be equipped with a natural ordering.

Proposition 1. *Let X be a convexity space. For $x, y \in X$, the set $C(x,y)$ is linearly ordered by*

$$z \leqslant t \;:\Longleftrightarrow\; z \in C(x,t) \;\Longleftrightarrow\; t \in C(z,y) \tag{5}$$

for $z, t \in C(x,y)$.

Proof. For any $z \in C(x,y)$, the set $C(x,z) \cup C(z,y)$ is connected. Therefore, (C1) and (C2) give

$$C(x,y) = C(x,z) \cup C(z,y). \tag{6}$$

To verify the second equivalence in (5), it suffices to show that

$$z \in C(x,t) \;\Rightarrow\; t \in C(z,y)$$

holds for $z, t \in C(x, y)$. By Eq. (6), it is enough to prove the implication

$$z \in C(x, t) \setminus \{t\} \;\Rightarrow\; t \notin C(x, z). \tag{7}$$

Assume that $z \in C(x,t) \setminus \{t\}$. Then Eq. (4) gives $x \in C(x,t) \setminus \{t\}$. Hence Lemma 1 and (C2) yield $C(x, z) \subset C(x, t) \setminus \{t\}$, which proves (7). Clearly, the relation (5) is reflexive and transitive. By (7), it is a partial order. Furthermore, (5) and (6) imply that it is a linear order. □

Note that the ordering of $C(x, y)$ depends on the pair (x, y) which determines the initial choice $x \leqslant y$. Thus as an ordered set, $C(y, x)$ is dual to $C(x, y)$.

Example 1. Let Ω be a linearly ordered set. A subset I of Ω is said to be an *interval* if $a \leqslant c \leqslant b$ with $a, b \in I$ implies that $c \in I$. The intervals $\{c \in \Omega \mid c < b\}$ and $\{c \in \Omega \mid c > a\}$ with $a, b \in \Omega$ form a sub-basis for the *order topology* of Ω. Note that an open set of Ω is a disjoint union of open intervals. Therefore, Ω is connected if and only if it is a *linear continuum*, i.e. if every partition $\Omega = I \sqcup J$ into non-empty intervals I, J determines a unique element between I and J. With the order topology, a linear continuum Ω is a locally compact convexity space with

$$C(x, y) = \{z \in \Omega \mid x \leqslant z \leqslant y\} \tag{8}$$

in case that $x \leqslant y$. Here the convex sets of Ω are just the connected sets of Ω.

Example 2. More generally, we define a *tree continuum* to be a Hausdorff space X for which every two points $x, y \in X$ are contained in a smallest connected set $C(x, y)$ such that each $C(x, y)$ is a linear continuum, and X carries the finest topology for which the inclusions $C(x, y) \hookrightarrow X$ are continuous. Thus $U \subset X$ is open if and only if every $x \in U$ is an "algebraically inner" point (see [23], §16.2), i.e. if for each $y \in X \setminus \{x\}$, there exists some $z \in C(x, y) \setminus \{x\}$ with $C(x, z) \setminus \{z\} \subset U$. Then X is a convexity space. For example, every one-dimensional CW-complex without cycles is of this type.

Example 3. In the Euclidean plane $\mathbb{R}^2$, consider the solution curves $c \colon \mathbb{R} \to \mathbb{R}^2$ of the differential equation $y' = 3y^{\frac{3}{2}}$ (including the singular solution $c \colon x \mapsto \binom{x}{0}$). With the finest topology making the solution curves continuous, $\mathbb{R}^2$ becomes a tree continuum. Here every point of the singular line is a branching point of order 4.

The following lemma is well-known (see [38], Theorem 26.15).

Lemma 2. *Let X be a connected topological space with an open covering $\mathfrak{U}$. For any pair of points $x, y \in X$, there is a finite sequence $U_1, \ldots, U_n \in \mathfrak{U}$ with $x \in U_1$, $y \in U_n$, and $U_i \cap U_{i+1} \neq \varnothing$ for $i < n$.*

Proposition 2. *Let X be a convexity space. For $x, y \in X$, the subspace $C(x, y)$ is compact and carries the order topology.*

Proof. Let $C(x,y) = \bigcup \mathfrak{U}$ be a covering by convex open sets. By Lemma 2, there is a finite sequence $U_1, \dots, U_n \in \mathfrak{U}$ with $x \in U_1, y \in U_n$, and $U_i \cap U_{i+1} \neq \varnothing$ for $i < n$. Hence $C(x,y) = U_1 \cup \dots \cup U_n$, which shows that $C(x,y)$ is compact.

For $u < v$ in $C(x,y)$, the sets $C(x,u)$ and $C(v,y)$ are compact, hence closed in $C(x,y)$. So the open intervals of $C(x,y)$ are open sets in $C(x,y)$. Conversely, a convex open set in $C(x,y)$ is an interval which must be an open interval since $C(x,y)$ is connected. □

Up to here, we have not used the continuity of the map (2) in Definition 1.

Proposition 3. *Let X be a convexity space. The closure of any convex set $A \subset X$ is convex.*

Proof. Let $A \subset X$ be a convex set, and let $x, y \in \overline{A}$ be given. For any $z \in C(x,y)$, we have to show that $z \in \overline{A}$. Suppose that there is a convex neighbourhood W of z with $W \cap A = \varnothing$. Then $z \neq x, y$. By Proposition 2, there exist $u, v \in W \cap C(x,y)$ with $u < z < v$. Since $C(x,u)$ and $C(v,y)$ are compact, there are disjoint open sets U, V in X with $C(x,u) \subset U$ and $C(v,y) \subset V$ (see, e.g., [20], chap. V, Theorem 8). Hence $C(x,y) \subset U \cup V \cup W$. So there are neighbourhoods $U' \subset U$ of x and $V' \subset V$ of y with $C(x',y') \subset U \cup V \cup W$ for all $x' \in U'$ and $y' \in V'$. Choose $x', y' \in A$. Then $C(x',y') \subset A$, which yields $C(x',y') \subset U \cup V$, where $x' \in U' \subset U$ and $y' \in V' \subset V$, contrary to the connectedness of $C(x',y')$. □

Definition 2. Let X be a convexity space. Define a *star* in X with *center* $x \in X$ and *end set* $E \subset X \setminus \{x\}$ to be a subspace $S(x,E) := \bigcup\{C(x,z) \mid z \in E\}$ with $C(x,z) \cap C(x,z') = \{x\}$ for different $z, z' \in E$ such that $S(x,E)$ carries the finest topology which makes the embeddings $C(x,z) \hookrightarrow S(x,E)$ continuous for all $z \in E$. We call X *star-finite* if every closed star in X has a finite end set.

Thus every star is a tree continuum (Example 2). Recall that a topological space X is said to be a *q-space* [25] if every point of X has a sequence $(U_n)_{n\in\mathbb{N}}$ of neighbourhoods such that every sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in U_n$ admits an accumulation point. For example, every locally compact space, and every first countable space X is a q-space.

Proposition 4. *Let X be a convexity space which is a q-space. Then X is star-finite.*

Proof. Let $S(x,E)$ be a closed star in X, and let $(U_n)_{n\in\mathbb{N}}$ be a sequence of neighbourhoods of x such that every sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in U_n$ has an accumulation point. Suppose that E is infinite. Since $U_n \cap C(x,z) \neq \{x\}$ for all $n \in \mathbb{N}$ and $z \in E$, we find a subset $\{z_n | n \in \mathbb{N}\}$ of E and a sequence $(x_n)_{n\in\mathbb{N}}$ with $x \neq x_n \in C(x,z_n) \cap U_n$.

Thus $(x_n)_{n\in\mathbb{N}}$ has an accumulation point z. Because of the star-topology, z cannot belong to $S(x,E)$, contrary to the assumption that $S(x,E)$ is closed. □

Example 4. A topological vector space X is a convexity space with respect to straight line segments if and only if X is locally convex. Moreover, a locally convex space X is star-finite if and only if X does not contain a locally convex direct sum $\mathbb{R}^{(\aleph_0)}$ as a subspace. In fact, every subspace $\bigoplus_{x\in E}\mathbb{R}x$ of X with $|E|=\aleph_0$ is complete ([33], II.6.2) and gives rise to a closed star $S(0,E)$. Conversely, let $S(x,E)\subset X$ be a closed star with E infinite. Since finite dimensional subspaces of X are star-finite by Proposition 4, we can assume that E is linearly independent and $x=0$. Then the subspace $\bigoplus_{x\in E}\mathbb{R}x$ of X is a locally convex direct sum.

Note that every metrizable locally convex space X is first countable ([33], I, Theorem 6.1), hence star-finite by Proposition 4.

2 Local openness onto the image

For a topological space X, the infinitesimal structure at a point x is given by the set $\mathfrak{D}_x$ of filters on X which converge to x. Let $\mathfrak{F}(X)$ denote the set of all filters on X. We make $\mathfrak{F}(X)$ into a topological space with a basis of open sets

$$\widetilde{U} := \{\alpha\in\mathfrak{F}(X) \mid U\in\alpha\}, \tag{9}$$

where U runs through the class of open sets in X. Every continuous map $f\colon X\to Y$ induces a map $\mathfrak{F}(f)\colon\mathfrak{F}(X)\to\mathfrak{F}(Y)$. For an open set V in Y, we have

$$\mathfrak{F}(f)^{-1}(\widetilde{V}) = \widetilde{f^{-1}(V)}, \tag{10}$$

which shows that $\mathfrak{F}(f)$ is continuous. Consider the subspace

$$\mathfrak{D}(X) := \{(x,\alpha)\in X\times\mathfrak{F}(X) \mid \alpha\in\mathfrak{D}_x\} \tag{11}$$

of $X\times\mathfrak{F}(X)$. Note that for every $x\in X$, the neighbourhood filter $\mathscr{U}(x)$ of x is the coarsest filter in $\mathfrak{D}_x$. Thus, regarding $\mathfrak{D}_x$ as a subset of $\mathfrak{D}(X)$, we get a pair of continuous maps

$$X \xrightarrow{\mathscr{U}} \mathfrak{D}(X) \xrightarrow{\lim} X \tag{12}$$

with $\lim(x,\alpha):=x$ and $\lim\circ\,\mathscr{U}=1_X$. In particular, $\mathfrak{D}_x=\lim^{-1}(x)$.

For a continuous map $f\colon X\to Y$, the local behaviour at $x\in X$ is given by the induced map $\mathfrak{D}_x f\colon\mathfrak{D}_x\to\mathfrak{D}_{f(x)}$. Thus we get an endofunctor $\mathfrak{D}\colon\mathbf{Top}\to\mathbf{Top}$ of the category **Top** of topological spaces with continuous maps as morphisms. The functor $\mathfrak{D}$ is augmented by the natural transformation $\lim\colon\mathfrak{D}\to 1$. On the other hand, the equation $\mathscr{U}\circ f=\mathfrak{D}(f)\circ\mathscr{U}$ holds if and only if f is open.

Definition 3. A continuous map $f\colon X \to Y$ between topological spaces is said to be *locally open onto its image* [5] if every $x \in X$ admits an open neighbourhood U such that the induced map $U \twoheadrightarrow f(U)$ is open onto the subspace $f(U)$ of Y. We call f *filtered* if f is locally open onto its image and $\mathfrak{D}(f) \circ \mathscr{U}$ is injective.

For example, the identity map $1_X\colon X \to X$ is filtered if and only if every point of X is determined by its neighbourhood filter, i.e. if X is a T_0-space. The following structure theorem holds for continuous maps which are locally open onto their image.

Proposition 5. *Let $f\colon X \to Y$ be a continuous map which is locally open onto its image. Up to isomorphism, there is a unique factorization $f = pq$ in* **Top** *into an open surjection q and a filtered map p. If f is filtered, then every point $x \in X$ has an open neighbourhood which is mapped homeomorphically onto a subspace of Y.*

Proof. Consider the following commutative diagram

$$\begin{array}{rcccc}
1\colon & X & \xrightarrow{\ \mathscr{U}\ } & \mathfrak{D}(X) & \xrightarrow{\ \lim\ } & X \\
& \Big\downarrow{\scriptstyle q^f} & & \Big\downarrow{\scriptstyle \mathfrak{D}(f)} & & \Big\downarrow{\scriptstyle f} \\
f^{\#}\colon & X^f & \overset{e}{\hookrightarrow} & \mathfrak{D}(Y) & \xrightarrow{\ \lim\ } & Y,
\end{array}$$

where X^f is the image of $\mathfrak{D}(f) \circ \mathscr{U}$, regarded as a quotient space of X, and $f^{\#} := \lim \circ\, e$. We will prove that $f = f^{\#} \circ q^f$ gives the desired factorization. Let us show first that q^f is open. Thus let U be an open set of X. We have to verify that $(q^f)^{-1}q^f(U)$ is open in X. Since f is locally open onto its image, we can assume that the induced map $U \twoheadrightarrow f(U)$ is open. Let $x \in (q^f)^{-1}q^f(U)$ be given. Then $q^f(x) \in q^f(U)$. So there exists some $y \in U$ with $q^f(x) = q^f(y)$, i.e. $f(x) = f(y)$ and $f(\mathscr{U}(x)) = f(\mathscr{U}(y))$. Hence there is an open neighbourhood $V \in \mathscr{U}(x)$ with $f(V) \subset f(U)$. Again, we can assume that the induced map $V \twoheadrightarrow f(V)$ is open. Furthermore, there is an open neighbourhood $U' \subset U$ of y with $f(U') \subset f(V)$, and $f(U')$ is open in $f(U)$, hence in $f(V)$. Therefore, $V' := V \cap f^{-1}(f(U'))$ is an open neighbourhood of x with $f(V') = f(U')$.

For any $x' \in V'$, there is a point $y' \in U'$ with $f(x') = f(y')$. So the continuity of f implies that $f(\mathscr{U}(x')) = f(\mathscr{U}(y'))$, which gives $q^f(x') = q^f(y')$, and thus $V' \subset (q^f)^{-1}q^f(U') \subset (q^f)^{-1}q^f(U)$. This proves that q^f is open. Consequently, $f^{\#}$ is locally open onto its image.

Since q^f is open, we have a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{\ q^f\ } & X^f \\
\Big\downarrow{\scriptstyle \mathscr{U}} & & \Big\downarrow{\scriptstyle \mathscr{U}} \\
\mathfrak{D}(X) & \xrightarrow{\ \mathfrak{D}(q^f)\ } & \mathfrak{D}(X^f).
\end{array}$$

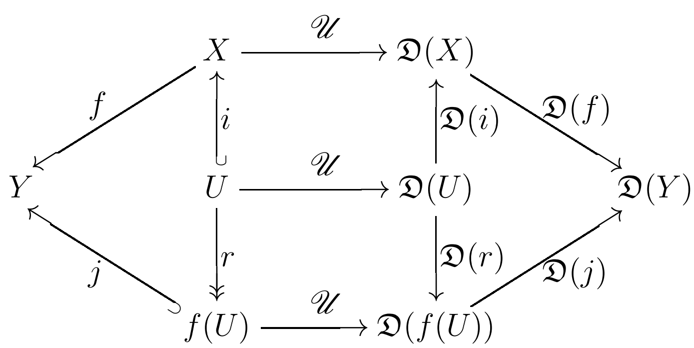
Hence $\mathfrak{D}(f^{\#}) \circ \mathscr{U} \circ q^f = \mathfrak{D}(f^{\#}) \circ \mathfrak{D}(q^f) \circ \mathscr{U} = \mathfrak{D}(f) \circ \mathscr{U} = e \circ q^f$. Therefore, $\mathfrak{D}(f^{\#}) \circ \mathscr{U} = e$, which implies that $f^{\#}$ is filtered.

Now let $f = pq = p'q'$ be two factorizations with p, p' filtered and q, q' open. Then $\mathfrak{D}(p') \circ \mathscr{U} \circ q' = \mathfrak{D}(p') \circ \mathfrak{D}(q') \circ \mathscr{U} = \mathfrak{D}(p) \circ \mathfrak{D}(q) \circ \mathscr{U} = \mathfrak{D}(p) \circ \mathscr{U} \circ q$. Since $\mathfrak{D}(p') \circ \mathscr{U}$ is injective, there exists a map $e \colon E \to E'$ with $q' = eq$. Since q is open, the map e is continuous. So we get a commutative diagram

$$\begin{array}{ccccc} X & \overset{q}{\twoheadrightarrow} & E & \overset{p}{\longrightarrow} & Y \\ \| & & \downarrow{\scriptstyle e} & & \| \\ X & \overset{q'}{\twoheadrightarrow} & E' & \overset{p'}{\longrightarrow} & Y \end{array}$$

in **Top**. By symmetry, we find a continuous map $e' \colon E' \to E$ with $q = e'q'$ and $p' = pe'$. Since q and q' are surjective, e must be a homeomorphism. This proves the uniqueness of the factorization.

Finally, let $f \colon X \to Y$ be filtered. For a given $x \in X$, let U be an open neighbourhood such that the induced map $r \colon U \twoheadrightarrow f(U)$ is open. Since $i \colon U \hookrightarrow X$ is open, we have a commutative diagram



which shows that $\mathfrak{D}(j) \circ \mathscr{U} \circ r = \mathfrak{D}(f) \circ \mathscr{U} \circ i$ is injective. Hence r is injective. □

In the sequel, we keep the notation of Proposition 5 and write

$$f \colon X \overset{q^f}{\twoheadrightarrow} X^f \overset{f^{\#}}{\longrightarrow} Y \tag{13}$$

for the factorization of a map f which is locally open onto its image.

Remarks. 1. Although the factorization (13) is unique up to isomorphism, it does not give rise to a factorization system [12, 10], i.e. a pair $(\mathscr{E}, \mathscr{M})$ of subcategories such that every commutative square

$$\begin{array}{ccc} E_1 & \overset{f_1}{\longrightarrow} & M_1 \\ {\scriptstyle e}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle m} \\ E_0 & \overset{f_0}{\longrightarrow} & M_0 \end{array} \tag{14}$$

with $e \in \mathscr{E}$ and $m \in \mathscr{M}$ admits a unique diagonal d with $f_1 = de$ and $f_0 = md$ (see [16], Proposition 1.4). Apart from the fact that local openness onto the image is not closed under composition (consider the maps $\mathbb{R} \overset{i}{\hookrightarrow} \mathbb{R}^2 \overset{p}{\twoheadrightarrow} \mathbb{R}$ with $i(x) = \binom{x}{x^3-3x}$ and $p\colon \binom{x}{y} \mapsto y$), there cannot be a factorization system since open surjections are not stable under pushout (take, e.g., the pushout of the open surjection $\mathbb{R} \twoheadrightarrow \{0\}$ and the inclusion $\mathbb{R} \hookrightarrow \mathbb{R}^2$).

2. If $f\colon X \to Y$ is locally open onto its image and locally fiber connected [5, 17], the Lemma of Benoist ([5], Lemma 3.7) states that the monotone part π of the monotone-light factorization $f = \widetilde{f} \circ \pi$ is open. Here the local fiber-connectedness of f implies that π is locally open onto its image. Hence $\pi = q^\pi$ is open by Proposition 5. In general, q^f always factors through π, but the two factorizations need not be isomorphic. For example, a local homeomorphism $f\colon X \twoheadrightarrow Y$ is open, but its fibers are discrete.

3 Convexity of maps

In this brief section, we introduce local convexity and extend this concept from subsets to continuous maps (cf. [19] for a notion of convex maps in terms of paths).

Definition 4. Let X be a topological space. We define a *local convexity structure* on X to be an open covering $X = \bigcup \mathfrak{U}$ by convexity spaces $U \in \mathfrak{U}$ (with the induced topology) such that for any $U \in \mathfrak{U}$, every convex open subspace of U belongs to $\mathfrak{U}$ (as a convexity space). We call a subset $C \subset X$ *convex* if $C \cap U$ is convex for all $U \in \mathfrak{U}$. We say that C is *locally convex* if every $z \in C$ admits a neighbourhood $U \in \mathfrak{U}$ such that $C \cap U$ is convex.

The covering $\mathfrak{U}$ will be referred to as the *atlas* of the local convexity structure. In the special case $X \in \mathfrak{U}$, the atlas $\mathfrak{U}$ just consists of the convex open sets of a convexity space X.

In contrast to local convexity, our concept of convexity refers to all sets in $\mathfrak{U}$. So the intersection of convex sets is convex, and every subset $A \subset X$ admits a *convex hull* $C(A)$, that is, a smallest convex set $C \supset A$. The next proposition generalizes Proposition 3.

Proposition 6. *Let X be a topological space with a local convexity structure $\mathfrak{U}$. The closure of any convex set $A \subset X$ is convex.*

Proof. For every $U \in \mathfrak{U}$, we have $\overline{A} \cap U = \overline{A \cap U} \cap U$. This set is convex by Proposition 3. Hence $\overline{A}$ is convex. □

Definition 4 admits a natural extension to continuous maps.

Definition 5. Let $f: X \to Y$ be a continuous map between topological spaces, where Y has a local convexity structure $\mathfrak{V}$. We call f *locally convex* if every $x \in X$ admits an open neighbourhood U such that the induced map $U \twoheadrightarrow f(U)$ is open, and $f(U)$ is a convex subspace of some $V \in \mathfrak{V}$.

Remarks. 1. A subset $A \subset Y$ is locally convex if and only if the inclusion map $A \hookrightarrow Y$ is locally convex.

2. The open neighbourhood U of x in Definition 5 can be chosen arbitrarily small. In fact, let $U' \subset U$ be any smaller open neighbourhood of x. Then $f(U')$ is an open subset of $f(U)$. Hence there exists some $V' \in \mathfrak{V}$ with $f(x) \in V' \cap f(U) \subset f(U')$. Thus $U'' := U' \cap f^{-1}(V')$ is an open neighbourhood of x with $f(U'') = V' \cap f(U') = V' \cap f(U)$, which is a convex subspace of V'.

3. If X is a connected Hausdorff space and Y a length metric space [9, 13], a continuous map $f: X \to Y$ is locally convex if and only if f is locally open onto its image and has local convexity data in the sense of [6].

Proposition 7. *Let $f: X \to Y$ be a continuous map between topological spaces, where Y has a local convexity structure $\mathfrak{V}$. If f is locally convex, then $f^{\#}$ is locally convex.*

Proof. Assume that f is locally convex, and let U be an open neighbourhood of $x \in X$ such that the induced map $U \twoheadrightarrow f(U)$ is open onto a convex subspace of some $V \in \mathfrak{V}$. Since q^f is open by Proposition 5, this property of U carries over to the neighbourhood $q^f(U)$ of $q^f(x)$. Hence $f^{\#}$ is locally convex. □

4 Geodesic manifolds

In this section, we introduce a general concept of geodesic which does not refer to any kind of metric.

Definition 6. Let Y be a topological space with a local convexity structure $\mathfrak{V}$, and let $e: C \to Y$ be a continuous map with a connected topological space C. By $\mathfrak{V}_e$ we denote the set of all open sets U in C which are mapped homeomorphically onto a convex subspace of some $V \in \mathfrak{V}$. We call e *étale* if e is closed and $\mathfrak{V}_e$ covers C. We say that $e: C \to Y$ is *generated* by a subset $F \subset C$ if there is no closed connected subspace $A \subsetneq C$ with $F \subset A$ such that $e(U \cap A)$ is convex for all $U \in \mathfrak{V}_e$.

In particular, étale maps are locally convex. Furthermore, every étale map $e\colon C \to Y$ induces a local convexity structure $\mathfrak{V}_e$ on C. So the condition (Definition 6) that $e(U \cap A)$ is convex for all $U \in \mathfrak{V}_e$ just states that A is convex with respect to $\mathfrak{V}_e$. If $F \subset C$ is connected, then $C(F)$ is connected. Therefore, an étale map $e\colon C \to Y$ is generated by a connected set F if and only if $\overline{C(F)} = C$. Note that the composition of étale maps is étale.

Definition 7. Let Y be a Hausdorff space with a local convexity structure $\mathfrak{V}$. We call Y a *geodesic manifold* if the following are satisfied.

(G1) For a finite set $F \subset Y$, the closure of $C(F)$ is compact.

(G2) If an étale map $e\colon C \to Y$ with C compact is generated by $\{x, y\} \subset C$, then every connected set $A \subset C$ with $x, y \in A$ coincides with C.

If, in addition, every $V \in \mathfrak{V}$ is star-finite and regular (as a topological space), we call Y a *geodesic q-manifold*.

The letter "q" is reminiscent of Proposition 4. Since a geodesic manifold Y is locally connected, [8], chap. I, 11.6, Proposition 11, implies that Y is the topological sum of its connected components.

Definition 8. Let Y be a geodesic manifold. We define a *geodesic* in Y to be an étale map $e\colon C \to Y$, generated by $\{x, y\} \subset C$, where C is compact. The points $e(x)$ and $e(y)$ will be called the *end points* of the geodesic.

More generally, we define a *line path* in Y to be a continuous map $e\colon L \to Y$, where L is a linear continuum (Example 1) with end points x_0 and x_n and a sequence of intermediate points $x_0 < x_1 < \cdots < x_n$ such that for $i < n$, the restriction of e to the interval $[x_i, x_{i+1}]$ is an inclusion which identifies $[x_i, x_{i+1}]$ with $C(e(x_i), e(x_{i+1})) \subset U_i$ for some U_i in the atlas of Y. If e is an inclusion, we speak of a *simple* line path and identify it with the subset $L \subset Y$. A subset $A \subset Y$ will be called *line-connected* if every pair of points $x, y \in A$ is connected by a simple line path $L \subset A$.

Proposition 8. *Let Y be a geodesic manifold with atlas $\mathfrak{V}$, and let $e\colon C \to Y$ be an étale map. Then C is line-connected.*

Proof. Let $x, y \in C$ be given. By Lemma 2, there is a sequence $U_1, \ldots, U_n \in \mathfrak{V}_e$ with $x \in U_1, y \in U_n$, and $U_i \cap U_{i+1} \neq \varnothing$ for $i < n$. Choose $x_i \in U_i \cap U_{i+1}$ for $i < n$. With $x_0 := x$ and $x_n := y$, the $C(x_i, x_{i+1})$ constitute a line path $e\colon L \to Y$ in C which connects x and y. Assume that the interval $[x, x_i] \subset L$ maps onto a simple line path L'. If $C(x_i, x_{i+1})$ intersects L' in a point $\neq x_i$, there is a largest $z \in C(x_i, x_{i+1})$ with property. Thus, if z' denotes the corresponding point on L', we can replace the interval $[z', z]$ by $\{z\}$ and attach the segment $C(z, x_{i+1})$. After less than n modifications, we get a simple line path between x and y. □

By (G2), we have the following

Corollary. *Let Y be a geodesic manifold. Every geodesic with end points $x, y \in Y$ is a line path.*

In particular, a simple geodesic with end points $x, y \in Y$ is just a minimal connected set $C \subset Y$ with $x, y \in C$ which is locally convex.

Let Y be a geodesic manifold. For $x, y \in Y$, we define a *simple arc* between x and y to be a subspace $A \subset Y$ which is a linear continuum with end points x and y. We fix a linear order on A such that x becomes the smallest element and denote the set of all such A by $\Omega_Y(x, y)$. In particular, every simple line path between x and y belongs to $\Omega_Y(x, y)$. Clearly, every $A \in \Omega_Y(x, y)$ admits an inscribed line path L between x and y. Although there is no concept of length at our disposal, the intuition that L is "shorter" than A can be expressed by the inclusion $\overline{C(L)} \subset \overline{C(A)}$. Thus it is natural to define a preordering on $\Omega_Y(x, y)$ by

$$A \prec B :\Longleftrightarrow \overline{C(A)} \subset \overline{C(B)}. \tag{15}$$

If $A \prec B$ holds for a pair $A, B \in \Omega_Y(x, y)$, we say that A is a *straightening* of B. Define $B \in \Omega_Y(x, y)$ to be *minimal* if $A \prec B$ implies $B \prec A$ for all $A \in \Omega_Y(x, y)$. We have the following straightening theorem which justifies the term "geodesic" manifold in Definition 7.

Theorem 1. *Let Y be a geodesic manifold. Every simple arc $A \in \Omega_Y(x, y)$ in Y can be straightened to a minimal $C \in \Omega_Y(x, y)$. A simple arc $A \in \Omega_Y(x, y)$ is minimal if and only if A is a convex simple geodesic.*

Proof. Let $A \in \Omega_Y(x, y)$ be given. Since $C(A)$ is connected, $\overline{C(A)}$ is connected. Proposition 6 implies that $\overline{C(A)}$ is convex. So the inclusion $\overline{C(A)} \hookrightarrow Y$ is étale. By Proposition 8, there exists a simple line path $L \subset \overline{C(A)}$ between x and y. Hence $L \prec A$. As L belongs to the convex hull of a finite set, (G1) implies that $\overline{C(L)}$ is compact. We have to verify that $\overline{C(L)}$ contains a minimal $C \in \Omega_Y(x, y)$. Let $\mathscr{C}$ be a chain of compact convex connected sets $C \subset \overline{C(L)}$ with $x, y \in C$. Then $D := \bigcap \mathscr{C}$ is compact and convex, and $x, y \in D$. We show first that every open set V of Y with $D \subset V$ contains some $C \in \mathscr{C}$. In fact, the set $\overline{C(L)}$ is compact, and $\bigcap_{C \in \mathscr{C}} (C \smallsetminus V) = \varnothing$. Hence $C \smallsetminus V = \varnothing$ for some $C \in \mathscr{C}$.

Next we show that D is connected. Suppose that there is a disjoint union $D = D_1 \sqcup D_2$ with non-empty compact sets D_1 and D_2. Then we can find open sets U_1 and U_2 in Y with $D_i \subset U_i$ such that $U_1 \cap U_2 = \varnothing$ (see, e.g., [20], chap. V, Theorem 8). Hence $D \subset U_1 \sqcup U_2$, which yields $C \subset U_1 \sqcup U_2$ for some $C \in \mathscr{C}$. Since

C is connected, we can assume that $C \subset U_1$. This gives $D_2 \subset U_1 \cap U_2 = \varnothing$, a contradiction. Thus D is connected. By Zorn's Lemma, it follows that there exists a minimal compact convex connected set C with $x, y \in C$. Hence $C \hookrightarrow Y$ is an étale map generated by $\{x, y\}$. Therefore, (G2) implies that C admits no connected proper subset $C' \subset C$ with $x, y \in C'$. By Proposition 8, it follows that C is a simple line path, whence $C \in \Omega_Y(x, y)$, and C is minimal.

In particular, we have shown that if $A \in \Omega_Y(x, y)$ is minimal, then A is a convex simple geodesic between x and y. Conversely, if $A \in \Omega_Y(x, y)$ is a convex simple geodesic, then $A = \overline{C(A)}$, and thus A is minimal. □

We conclude this section with some typical examples.

Example 5. Let Y be a geodesic manifold with atlas $\mathfrak{V}$, and let Z be a closed locally convex subspace. Then $Z \hookrightarrow Y$ is étale. Every finite set F in Z is contained in a compact convex set C in Y. Hence $C \cap Z$ is compact and convex in Z. Thus Z satisfies (G1). As (G2) trivially carries over to Z, it follows that Z is a geodesic manifold. If Y is a geodesic q-manifold, then so is Z.

Example 6. Let Y be a complete locally compact length metric space [9, 13]. By the Hopf-Rinow Theorem ([9], Proposition I.3.7), the closed metric balls in Y are compact, and any two points in Y are connected by a shortest path. It is natural to assume that Y admits a basis of convex open sets where shortest paths are unique. This provides Y with a local convexity structure $\mathfrak{V}$ which satisfies (G1). Note that by [9], I.3.12, the map (2) is continuous where it is defined.

Now let $e \colon C \to Y$ be an étale map generated by $\{x, y\} \subset C$, where C is compact. Similar to the case of a covering of length metric spaces ([9], Proposition I.3.25), the length metric d_Y of Y can be lifted to a length metric d_C of C such that $d_C(u, v) \geqslant d_Y(e(u), e(v))$ for all $u, v \in C$. (If $d_C(u, v) = 0$ with $u \neq v$, a neighbourhood $U \in \mathfrak{V}_e$ of u cannot contain v. As U contains a closed neighbourhood of u in C, we get $d_C(u, v) > 0$.) Since C is compact, the Hopf-Rinow Theorem, applied to C, yields a shortest path $L \subset C$ between x and y. Hence $C = L$, which proves (G2). By Proposition 4, Y is a geodesic q-manifold.

Example 7. Let Y be a locally convex topological vector space. For $x, y \in Y$, we set $C(x, y) := \{\lambda x + (1 - \lambda)y \mid 0 \leqslant \lambda \leqslant 1\}$ to make Y into a convexity space. For a finite set $F \subset Y$, the closed convex hull $\overline{C(F)}$ of F is contained in a finite dimensional subspace of Y. Hence $\overline{C(F)}$ is compact. Thus Y satisfies (G1). Let $e \colon C \to Y$ be an étale map generated by $\{x, y\} \subset C$, where C is compact. By Proposition 8, e is generated by a simple line path in C. Hence $e(C)$ is contained in a finite dimensional subspace of Y. So Example 6 applies, which proves (G2). Thus Y is a geodesic manifold. Moreover, Example 4 shows that Y is a geodesic q-manifold if and only if Y does not contain a locally convex direct sum $\mathbb{R}^{(\aleph_0)}$ as a subspace.

5 The Lokal-global-Prinzip

With respect to convex neighbourhoods, étale maps have the following disjointness property.

Proposition 9. *Let Y be a geodesic manifold with atlas $\mathfrak{V}$, and let $e\colon C \to Y$ be an étale map. Assume that $U, U' \in \mathfrak{V}_e$. If $e|_{U\cup U'}$ is not injective, then $U \cap U' = \varnothing$.*

Proof. If $e|_{U\cup U'}$ is not injective, there exist $x \in U$ and $x' \in U'$ with $e(x) = e(x')$. Suppose that there is some $z \in U \cap U'$. Then $x \neq z$, and $U \cap U' \cap C(x,z)$ is a convex open subset of $C(x,z) \smallsetminus \{x\}$. Hence there is a point $t \in C(x,z)$ with $(U \smallsetminus U') \cap C(x,z) = C(x,t)$. So the homeomorphisms $C(x,z) \cong C(e(x),e(z)) \cong C(x',z)$ give rise to a point $t' \in U'$ with $e(t) = e(t')$ and $(U' \smallsetminus U) \cap C(x',z) = C(x',t')$. Moreover, $D := C(t,z) \cup C(t',z) = C(t,z) \cup \{t'\}$ since $e|_U$ is injective. Therefore, D is not a minimally connected superset of $\{t,z\}$. On the other hand, D is compact with open subsets $C(t,z)$ and $C(t',z)$. Hence $e|_D\colon D \to Y$ is an étale map generated by $\{t,z\}$, contrary to (G2). ☐

As an immediate consequence, the fibers of an étale map can be separated by pairwise disjoint neighbourhoods.

Corollary 1. *Let Y be a geodesic manifold, and let $e\colon C \to Y$ be an étale map. For a given $y \in Y$, choose a neighbourhood $U_x \in \mathfrak{V}_e$ of each $x \in f^{-1}(y)$. Then the U_x are pairwise disjoint.*

Corollary 2. *Let Y be a geodesic manifold, and let $e\colon C \to Y$ be an étale map. Then C is a Hausdorff space.*

Proof. Let $x, x' \in C$ be given. If $e(x) \neq e(x')$, there are disjoint neighbourhoods of $e(x)$ and $e(x')$, and their inverse images give disjoint neighbourhoods of x and x'. So we can assume that $e(x) = e(x')$. Choose $U, U' \in \mathfrak{V}_e$ with $x \in U$ and $x' \in U'$. By Proposition 9, $U \cap U' = \varnothing$. Thus C is Hausdorff. ☐

If the geodesic manifold is regular, the fibers are even discrete, which leads to the following finiteness result.

Proposition 10. *Let $e\colon C \to Y$ be an étale map into a geodesic q-manifold Y. Then the fibers of e are finite.*

Proof. Let $\mathfrak{V}$ denote the atlas of Y, and let $y \in Y$ be given. For each $x \in e^{-1}(y)$, we choose a neighbourhood $U_x \in \mathfrak{V}_e$ such that the images $e(U_x)$ are contained in a fixed $V' \in \mathfrak{V}$. By the Corollary 1, these neighbourhoods are pairwise disjoint.

Without loss of generality, we can assume that $|C| > 1$. Since C is a connected Hausdorff space by Corollary 2, this implies that C has no isolated points. As e is closed, the complement of $\bigcup\{U_x \mid x \in e^{-1}(y)\}$ is mapped to a closed set $A \subset Y$ with $y \notin A$. So there exists an open neighbourhood $W \subset V'$ of y with $e^{-1}(W) \subset \bigcup\{U_x \mid x \in e^{-1}(y)\}$. By the regularity of Y, we find a convex open neighbourhood V of y with $\overline{V} \subset W$.

For any $x \in e^{-1}(y)$, the set $U_x \cap e^{-1}(V)$ is an open neighbourhood of x, hence not a singleton. Therefore, the $V_x := e(U_x \cap e^{-1}(V))$ are convex subsets of V with $|V_x| > 1$ and $y \in V_x$. Choose arbitrary $z_x \in U_x \cap e^{-1}(V)$ with $y_x := e(z_x) \neq y$ for all $x \in e^{-1}(y)$. Now let $Z \subset \bigcup\{C(x, z_x) \,|\, x \in e^{-1}(y)\}$ be such that $Z \cap C(x, z_x)$ is closed in $U_x \cap e^{-1}(V)$ for every $x \in e^{-1}(y)$. We claim that Z is closed. Thus let $z \in \overline{Z}$ be given. Then $e(z) \in \overline{e(Z)} \subset \overline{V} \subset W$. Hence $z \in e^{-1}(W) \subset \bigcup\{U_x \mid x \in e^{-1}(y)\}$, which yields $z \in Z$. Thus Z is closed. Since e is closed, this implies that $S(y) := \bigcup\{C(y, y_x) \,|\, x \in e^{-1}(y)\}$ is closed and carries the finest topology such that the maps $C(y, y_x) \hookrightarrow S(y)$ are continuous for all $x \in e^{-1}(y)$.

Suppose that $e^{-1}(y)$ is infinite. By Ramsey's Theorem [32], there must be an infinite subset E of $e^{-1}(y)$ such that either $C(y, y_u) \cap C(y, y_v) = \{y\}$ for all pairs of different $u, v \in E$, or $C(y, y_u) \cap C(y, y_v) \neq \{y\}$ for different $u, v \in E$. The first case is impossible since V is star-finite by Definition 7. Otherwise, there is a point $y' \in V \smallsetminus \{y\}$ and a set $Z \subset \bigcup\{C(x, z_x) \mid x \in e^{-1}(y)\}$ with $|Z \cap C(x, z_x)| = 1$ for all $x \in E$ such that $e(Z)$ is an infinite non-closed subset of $C(y, y')$. Since Z is closed, this gives a contradiction. □

As a consequence, the geodesic structure of a geodesic q-manifold can be lifted along étale maps.

Theorem 2. *Let $e\colon C \to Y$ be an étale map into a geodesic q-manifold Y with atlas $\mathfrak{V}$. Then C is a geodesic q-manifold with atlas $\mathfrak{V}_e$.*

Proof. By Corollary 2 of Proposition 9, C is a Hausdorff space. We show first that C is regular. Let $U_x \in \mathfrak{V}_e$ be a neighbourhood of $x \in C$. We choose neighbourhoods $U_z \in \mathfrak{V}_e$ for all z in the fiber of $y := e(x)$. By Corollary 1 of Proposition 9, the U_z are pairwise disjoint. Since Y is regular and e closed, there is a closed neighbourhood V of y with $e^{-1}(V) \subset \bigcup\{U_z \mid z \in e^{-1}(y)\}$. Hence

$$U_x \cap e^{-1}(V) = e^{-1}(V) \smallsetminus \bigcup\{U_z \mid z \in e^{-1}(y) \smallsetminus \{x\}\}$$

is a closed neighbourhood of x. Thus C is regular.

Let $F \subset C$ be finite. Then $\overline{C(e(F))}$ is compact. By Proposition 10, the fibers of e are compact. Hence $e^{-1}(\overline{C(e(F))})$ is compact by [8], chap. I.10, Proposition 6. Furthermore, $e^{-1}(\overline{C(e(F))})$ is convex with respect to $\mathfrak{V}_e$. Therefore, the closed subset $\overline{C(F)}$ of $e^{-1}(\overline{C(e(F))})$ is compact. This proves (G1) for C.

Next let $e'\colon C' \to C$ be an étale map with C' compact which is generated by $\{x, y\} \subset C'$. Then ee' is étale and generated by $\{x, y\}$. Hence C' is minimal among the connected sets $B \subset C'$ with $x, y \in B$. Thus C satisfies (G2).

Finally, let $S(x, E) := \bigcup\{C(x, z) \mid z \in E\}$ be a closed star in some $U \in \mathfrak{V}_e$. Since C is regular, we find a closed convex neighbourhood $U' \subset U$ of x. By Proposition 3, this implies that $S(x, E) \cap U'$ is a star in U which is closed in C. Therefore, $e(S(x, E) \cap U')$ is a closed star in some $V \in \mathfrak{V}$. So E is finite, which proves that C is a geodesic q-manifold. □

Now we are ready to prove our main result which essentially states that the image of an étale map is weakly convex in the following sense (cf. [6], Definition 2.16).

Definition 9. Let Y be a geodesic manifold. We call a subset $A \subset Y$ *weakly convex* if every pair of points $x, y \in A$ can be connected by a geodesic.

The following theorem extends previous versions of the Lokal-global-Prinzip for convexity of maps (see [11, 17, 6, 7]).

Theorem 3. *Let $f\colon X \to Y$ be a locally convex continuous map from a connected topological space X to a geodesic q-manifold Y. Assume that $f^\#$ is closed. Then $f(X)$ is weakly convex.*

Proof. Let $\mathfrak{V}$ be the atlas of Y. By Proposition 7, the map $f^\#$ again is locally convex, and Proposition 5 implies that $f^\#$ is étale. By Theorem 2, it follows that X^f is a geodesic manifold. For $z, z' \in X^f$, Proposition 8 shows that there is a connecting simple line path L between z and z'. Theorem 1 shows that L can be straightened to a convex simple geodesic C. Thus $f^\#|_C\colon C \to Y$ is a geodesic between $f^\#(z)$ and $f^\#(z')$. Hence $f(X)$ is weakly convex. □

In the special case where f is an inclusion $X \hookrightarrow Y$, the preceding proof yields

Corollary. *Let C be a closed connected locally convex subset of a geodesic manifold Y. Then C is weakly convex.*

Proof. By Example 5, C is a geodesic manifold, and $C \hookrightarrow Y$ is étale. As in the proof of Theorem 3, this implies that C is weakly convex. □

Remarks. 1. If f is closed, then $f^\#$ is closed. However, the latter condition is much weaker. For example, if f is a local homeomorphism, then $f^\#$ is identical, but f need not be closed.

2. The preceding corollary extends Klee's generalization of a classical result due to Tietze [35] and Nakajima (Matsumura) [27]. Klee's Theorem [22] states that the above corollary holds in a locally convex topological vector space Y. Note that the usual proof of Klee's Theorem rests on the linear structure of Y, while the corollary of Theorem 3 merely depends on a local convexity structure in the sense of Definition 4.

Acknowledgements

The paper idea was born on the occasion of Tudor S. Ratiu's 60th birthday, because the second author wanted to thank Tudor S. Ratiu, Alan D. Weinstein, her parents, her brother Michael, Christiane De Paola, Brita and Milosch Meriac, Gosia and Michael Ederer, Beate Richter, Rita Haller-Haid, and Michael Haager for their tireless and outstanding support in good times as in bad, the German and Stuttgart authorities, the Baden-Württemberg ministry of interior and committees, and everyone who have helped to make her long-life stay in Stuttgart possible.

The second author thanks the first author for suggesting such a nice topic, his excellent ideas for this work and his outstanding support, the RatiuFest 2010 organizers (the late Jerrold E. Marsden and Juan-Pablo Ortega) for their promise and their invitation with grant in April 2010 (without that promise this paper would never have been born!), Karl-Hermann Neeb and his colleague, Tudor S. Ratiu, and Alan D. Weinstein for their advice and many useful comments, Ralph L. Cohen for his invitation to attend the Copenhagen Topology Conference (CTC) 2010 in Copenhagen, Denmark, Jan 8-10, 2010, that has opened new possibilities and also connection to Københavns Universitet, Aarhus Universitet, Københavns Universitet, and especially Dennis P. Sullivan for awakening and keeping interest especially in Topology, and Karl-Hermann Neeb for keeping her interested in doing research.

Several parts of this work were presented at the Young Topologists Meeting 2010 in København, Denmark, June 16-20, 2010, and at Geometry, Mechanics, and Dynamics: A workshop celebrating the 60th birthday of Tudor Ratiu at CIRM, Luminy, France, July 12-16, 2010. Kindly invitations, hospitality, and grants of the organizers of both events, Aarhus Universitet, and Københavns Universitet are highly acknowledged.

References

[1] Abraham, R., Marsden, J. E.: Foundations of mechanics, 2nd ed., AMS Chelsea Publishing, 2008

[2] Audin, M.: Torus actions on symplectic manifolds, 2nd rev. ed., Progress in Mathematics **93**, Birkhäuser Verlag, Basel, 2004

[3] Atiyah, M. F.: Convexity and commuting Hamiltonians, Bull. London Math. Soc. **14**, 1-15 (1982)

[4] Bangert, V.: Konvexe Mengen in Riemannschen Mannigfaltigkeiten, Math. Z. **162**, 263-286 (1978)

[5] Benoist, Y.: Actions symplectiques de groupes compacts, Geom. Dedicata **89**, 181-245 (2002)

[6] Birtea, P., Ortega, J.-P., Ratiu, T. S.: A local-to-global principle for convexity in metric spaces, J. Lie Theory **18**, 445-469 (2008)

[7] Birtea, P., Ortega, J.-P., Ratiu, T. S.: Openness and convexity for momentum maps, Trans. Amer. Math. Soc. **361**, 603-630 (2009)

[8] Bourbaki, N.: General topology, Addison-Wesley, Hermann, Paris, 1966

[9] Bridson, M. R., Haefliger, A.: Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften **319**, Springer-Verlag, Berlin, 1999

[10] Cassidy, C., Hébert, M., Kelly, G. M.: Reflective subcategories, localizations and factorization systems, J. Austral. Math. Soc. (Ser. A) **38**, 287-329 (1985)

[11] Condevaux, M., Dazord, P., Molino, P.: Géométrie du moment, In: Travaux du Séminaire Sud-Rhodanien de Géométrie, I, Publ. Dép. Math. Nouvelle Sér. B **88**-1, 131-160, Univ. Claude-Bernard, Lyon, 1988

[12] Freyd, P. J., Kelly, G. M.: Categories of continuous functors, I, J. Pure Appl. Algebra **2**, 169-191 (1972)

[13] Gromov, M.: Metric structures for Riemannian and non-Riemannian spaces, Progress in Mathematics **152**, Birkhäuser Boston, Inc., Boston, MA, 1999

[14] Guillemin, V., Sternberg, S.: Convexity properties of the moment mapping, Invent. Math. **67**, 491-513 (1982)

[15] Guillemin, V., Sternberg, S.: Convexity properties of the moment mapping, II, Invent. Math. **77**, 533-546 (1984)

[16] Herrlich, H., Salicrup, G., Strecker, G. E.: Factorizations, denseness, separation, and relatively compact objects, Topology Appl. **27**, 157-169 (1987)

[17] Hilgert, J., Neeb, K.-H., Plank, W.: Symplectic convexity theorems and coadjoint orbits, Compositio Math. **94**, 129-180 (1994)

[18] Iliadis, S. D., Fomin, S.: The method of centered systems in the theory of topological spaces, Uspehi Mat. Nauk **21**, 47-76 (1966)

[19] Karshon, Y., Bjorndahl, C.: Revisiting Tietze-Nakajima - local and global convexity for maps, `arXiv:math/0701745v2[math.CO]`

[20] Kelley, J. L.: General topology, D. Van Nostrand Company, Inc., Toronto-New York-London, 1955

[21] Kirwan, F.: Convexity properties of the moment mapping, III, Invent. Math. **77**, 547-552 (1984)

[22] Klee, V. L.: Convex sets in linear spaces, Duke Math. J. **18**, 443-466 (1951)

[23] Koethe, G.: Topologische lineare Räume, I (German), 2nd ed., Die Grundlehren der Mathematischen Wissenschaften, Band **107**, Springer-Verlag, Berlin-New York, 1966

[24] Michael, E.: Cuts, Acta Math. **111**, 1-36 (1964)

[25] Michael, E.: A note on closed maps and compact sets, Israel J. Math. **2**, 173-176 (1964)

[26] Marsden, J. E., Ratiu, T. S.: Introduction to mechanics and symmetry, current 2nd printing, Texts in Applied Mathematics **17**, Springer-Verlag, 2003

[27] Nakajima, S. (Matsumura): Über konvexe Kurven und Flächen, Tôhoku Math. J. **29**, 227-230 (1928)

[28] Nakamura, M.: Note on Banach spaces, III, A proof of Tietze-Matsumura's theorem, Proc. Imp. Acad. Tokyo **18**, 267-268 (1942)

[29] Ortega, J.-P., Ratiu, T. S.: Momentum maps and Hamiltonian reduction, Progress in Mathematics **222**, Birkhäuser Boston, Boston, MA, 2004.

[30] Ortega, J.-P., Ratiu, T. S.: The reduced spaces of a symplectic Lie group action, Ann. Global Anal. Geom. **30**, 335-381 (2006)

[31] Ortega, J.-P., Ratiu, T. S.: The stratified spaces of a symplectic Lie group action, Rep. Math. Phys. **58**, 51-75 (2006)

[32] Ramsey, F. P.: On a problem of formal logic, Proc. London Math. Soc. **30**, 264-285 (1930)

[33] Schaefer, H. H., Wolff, M. P.: Topological vector spaces, Springer-Verlag, New York, 1999

[34] da Silva, A. C.: Lectures on symplectic geometry, Lecture Notes in Mathematics **1764**, Springer-Verlag, 2000

[35] Tietze, H.: Über Konvexheit im Kleinen und im Großen und über gewisse den Punkten einer Menge zugeordnete Dimensionszahlen, Math. Z. **28**, 697-707 (1928)

[36] Weinstein, A.: Symplectic manifolds and their Lagrangian submanifolds, Adv. Math. **6**, 329-346 (1971)

[37] Weinstein, A.: Lectures on symplectic manifolds, CBMS Regional Conference Series in Mathematics **29**, Amer. Math. Soc., Providence, 1977

[38] Willard, S.: General topology, Reprint of the 1970 original (Addison-Wesley, Reading, MA), Dover Publications, Inc., Mineola, NY, 2004