On the canonical structure of regular pencil of singular matrix-functions

S.V. Gaidomak

Institute for System Dynamics and Control Theory SB RAS, Lermontov str., 134, Irkutsk, Russia 664033

Abstract

The work is devoted to investigation of the canonical structure of regular, in domain of definition $\mathcal{U} \subseteq \mathbb{R}^m$, the pencil of matrix-functions $A(x) + \lambda B(x)$. It is supposed that $\det A(x) \equiv 0$ and $\det B(x) \equiv 0 \;\; \forall \; x \in \mathcal{U}$, and all roots of the characteristic equation $\det(A(x) + \lambda B(x)) = 0$ with $\lambda = \lambda(x) \in C^p(\mathcal{U})$, are of constant multiplicity.

Keywords: matrix-function, regular pencil, canonical structure, p-smoothly similar matrix-functions, p-smoothly equivalent pencils

In the study of linear partial differential-algebraic equations of the form

$$A(x)u_t(x) + B(x) \sum_{i=1}^{n} u_{x_i}(x) + C(x)u(x) = f(x),$$

where coefficients $A(x)$, $B(x)$ and $C(x)$ are singular, at every point of the domain of definition $\mathcal{U}$, matrix of $C^p(\mathcal{U})$ functions depending on many variables, it remains an important question about the possibility of bringing the pencil $A(x) + \lambda B(x)$ to a canonical form with the help of suitable p times differentiable linear transformation (see, for example [1]-[3]). In the papers [4]-[6], we considered the linear partial differential-algebraic equations with the pencil $A(x) + \lambda B(x)$, which has the structure of the "rank-degree". This is perhaps the most simple structure of the pencil $A(x) + \lambda B(x)$, where it is supposed that $\det A(x) \equiv 0$ and $\det B(x) \equiv 0 \; \forall \; x \in \mathcal{U}$, with non-multiple finite and infinite elementary divisors of the pencil. The structure of the matrix-functions pencil known as "rank-degree" in the first time appears when studying of differential-algebraic systems [7] and later it was extended

by the author of [7] to the case, when the pencil of matrix-functions depend on the several variables.

In this paper, we consider some canonical srtucture of regular pencil $A(x) + \lambda B(x)$, where it is supposed that $\det A(x) \equiv 0$ and $\det B(x) \equiv 0$ in $\mathcal{U}$, with finite and infinite elementary divisors, which has constant multiplicity equal no less than one in $\mathcal{U}$. As a result, we obtain sufficient conditions for the p-smooth equivalence of the matrix-functions pencil to this canonical form.

Throughout the paper, we use the following notations. Let $A(x)$ and $B(x)$ be an $n \times n$ matrices of $C^p(\mathcal{U})$ real-valued functions of m real variables $x \equiv (x^1, x^2, \ldots, x^m) \in \mathcal{U} \subseteq \mathbb{R}^m$, where $\mathcal{U}$ is supposed to be a compact and simply connected domain in the $\mathbb{R}^m$. The expression of the form $A(x) + \lambda B(x)$, with $\lambda = \lambda(x) \in C^p(\mathcal{U})$ is called the matrix-functions pencil. One says, that the pencil under consideration is regular in $\mathcal{U}$, if exists a function $c = c(x) \in C^p(\mathcal{U})$, for which the condition $\det(A(x) + cB(x)) \neq 0 \ \forall \ x \in \mathcal{U}$ holds [8]. In what follows, we need in the following two definitions.

Definition 1. [9], [10], [11] Two square matrices of $C^p(\mathcal{U})$ functions $A(x)$ and $\tilde{A}(x)$ of order n are called p-smoothly similar if there exists some matrix-function $T(x)$ satisfying the following conditions:

(i) the elements of $T(x)$ belong to $C^p(\mathcal{U})$;
(ii) $T(x)$ is nonsingular in the domain of definition $\mathcal{U}$;
(iii) $T^{-1}(x)A(x)T(x) = \tilde{A}(x)$ in $\mathcal{U}$.

Moreover, if $T(x)$ is the unitary matrix-function, then one says that the matrix-functions $A(x)$ and $\tilde{A}(x)$ are p-smoothly unitarily similar in $\mathcal{U}$ [8].

Definition 2. Two pencils of square $C^p(\mathcal{U})$ matrix-functions of order n, say, $A(x)+\lambda B(x)$ and $\tilde{A}(x)+\lambda\tilde{B}(x)$ is called p-smoothly equivalent, if there exists a pair of matrix-functions $P(x)$ and $Q(x)$, independing on λ and satisfying the following conditions:

(i) the elements of $P(x)$ and $Q(x)$ belong to $C^p(\mathcal{U})$;
(ii) $P(x)$ and $Q(x)$ are nonsingular in $\mathcal{U}$;
(iii) $P(x)(A(x) + \lambda B(x))Q(x) = \tilde{A}(x) + \lambda\tilde{B}(x)$ in $\mathcal{U}$.

The investigation of the structure of pencils in the classical theory of matrices is commonly based on the properties of their similarity to the canonical forms. When studying the matrix-functions pencils, we keep this trend.

Our paper is organized as follows. In the second section, we prove an auxiliary result concerning p-smooth similarity of matrix-functions with a single eigenvalue identically equal to zero to some nilpotent matrix. In the third section, we prove a theorem on the p-smooth equivalence of pencil $A(x) + \lambda B(x)$ to some specified canonical form. This theorem is the main result of our paper. In conclusion of the paper, we give two examples of pencils to illustrate our theorem.

1. Lemma on the similarity of matrix-functions

A well-known fundamental theorem of Shur and Toeplitz [12] guarantees that every constant matrix A is unitarily similar to a triangular matrix. Let us put the following question: whether this statement holds for matrix-functions? It turns out that for analytic matrix-functions of one variable the theorem of Shur and Toeplitz remains valid [13], but for analytic matrix-functions of several variables this theorem, in general, is not true. For example, consider the matrix-function with single eigenvalue identically equal to zero in the domain of definition, namely,

$$A(x) = \begin{pmatrix} x_1(x_1 + x_2) & -(x_1 + x_2)^2 \\ x_1^2 & -x_1(x_1 + x_2) \end{pmatrix}, \quad \mathcal{U} = \mathbb{R}^2.$$

It is evident that all elements of $A(x)$ are analytic (polynomial) functions at every point of $\mathcal{U}$, but nevertheless the matrix-function $A(x)$ can not be cast to triangular form with the aid of a nonsingular analytic linear transformation, since the functions $\alpha(x)$ and $\beta(x)$ from lemma 1 of the paper [14], which must satisfy the conditions:

$$\alpha(x)x_1(x_1 + x_2) - \beta(x)x_1^2 = 0, \quad \alpha^2(x) + \beta^2(x) = 1$$

are evidently not analytic at the point $(0,0)$.

This example shows that p-smoothly unitary similar matrix-functions of several variables should satisfy additional conditions. In the case when the matrix-function have single eigenvalue being equal identically zero in $\mathcal{U}$, additional condition to be required is the one of constant rank of the matrix-function in $\mathcal{U}$. Let us prove the following auxiliary statement.

Lemma 1. *Let $A(x)$ be an $n \times n$ matrix-function defined in $\mathcal{U} \subseteq \mathbb{R}^m$ satisfying the following conditions:*

(i) *the elements of $A(x)$ belong to the space $C^p(\mathcal{U})$;*

(ii) *the $A(x)$ has single eigenvalue being equal identically to zero in $\mathcal{U}$;*

(iii) *$A(x)$ has constant rank in $\mathcal{U}$. Then $A(x)$ is p-smoothly unitarily similar to some nilpotent matrix-function $\mathcal{N}(x)$.*

PROOF. The method of proof, in general, follows the line suggested in the papers [12] and [13]. Thus, there is no need to write down in detail all the proof of the lemma. Let us consider only some important aspects of the proof.

Let $X(x)$ be a right eigenvector of $A(x)$, corresponding to the eigenvalue $\lambda \equiv 0$ in $\mathcal{U}$. Let us prove that the elements of eigenvector $X(x)$ belong to the space $C^p(\mathcal{U})$. We write down the equation for the right eigenvector

$$A(x)X(x) = 0. \tag{1}$$

Since, by assumption, the rank of $A(x)$ does not depend on x in $\mathcal{U}$, then (see lemma 2, [11]) there exists a pair of $C^p(\mathcal{U})$ matrix-functions $P(x)$ and $Q(x)$ nonsingular in $\mathcal{U}$, such that

$$P(x)A(x)Q(x) = \mathrm{diag}\{E_r, \mathcal{O}\}, \quad \text{where} \quad r \equiv \mathrm{rank}(A(x)).$$

It is obvious that $r < n$. Let $\{e_i, \ i = \overline{1,n}\}$ be standard orthonormal basis of linear space $\mathbb{R}^n$. It is easily seen that vector-function $X(x) = Q(x)e_n$ is the solution of equation (1). Furthermore, the all components of vector $X(x)$ belong to the space $C^p(\mathcal{U})$. Let us show now that it is possible to construct, attached to $X(x)$, the orthonormal system of linearly independent vectors

$$\hat{X}_1(x), \ \hat{X}_2(x), \ \ldots, \ \hat{X}_n(x), \tag{2}$$

satisfying the following conditions:[1]

$$\hat{X}_1(x) = X(x)/\|X(x)\|, \tag{3}$$

$$\hat{X}_i(x)^\top \cdot \hat{X}_j(x) = 0, \quad \forall \ i \neq j, \ \ i,j = \overline{1,n} \quad \forall \ x \in \mathcal{U}, \tag{4}$$

$$X_i(x) \in C^p(\mathcal{U}), \quad \|\hat{X}_i(x)\| \equiv 1, \quad \forall \ i = \overline{1,n}. \tag{5}$$

---

[1] Here $\|\cdot\|$ denotes euclidean norm.

4

Consider following set of vector-functions [13]

$$Z_1(x) = Q(x)e_n, \ Z_2(x) = Q(x)e_{n-1}, \ \ldots, \ Z_n(x) = Q(x)e_1. \qquad (6)$$

Since the matrix-function $Q(x)$ is nonsingular in $\mathcal{U}$, then (6) gives the set of linearly independent vector-functions $\mathcal{U}$. Applying the Gram-Schmidt procedure to (6), we obtain the orthogonal set of vectors

$$X_1(x) = Z_1(x),$$

$$X_2(x) = Z_2(x) - \frac{X_1^\top(x) \cdot Z_2(x)}{X_1^\top(x) \cdot X_1(x)} X_1(x),$$

$$X_3(x) = Z_3(x) - \frac{X_1^\top(x) \cdot Z_3(x)}{X_1^\top(x) \cdot X_1(x)} X_1(x) - \frac{X_2^\top(x) \cdot Z_3(x)}{X_2^\top(x) \cdot X_2(x)} X_2(x), \qquad (7)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$X_n(x) = Z_n(x) - \frac{X_1^\top(x) \cdot Z_n(x)}{X_1^\top(x) \cdot X_1(x)} X_1(x) - \cdots - \frac{X_{n-1}^\top(x) \cdot Z_n(x)}{X_{n-1}^\top(x) \cdot X_{n-1}(x)} X_{n-1}(x).$$

By construction, each of vector-functions $X_i(x)$ is not equal to zero vector in $\mathcal{U}$. This means that

$$X_i^\top(x) \cdot X_i(x) \neq 0, \quad i = \overline{1,n}, \ \ \forall \, x \in \mathcal{U}$$

and the components of vector-functions (7) belong to the space $C^p(\mathcal{U})$. Moreover, the vector-functions $\hat{X}_i(x) = X_i(x)/\|X_i(x)\|$ satisfy required condition (3)-(5). Then, clearly, the matrix-function

$$U(x) = (\hat{X}_1(x), \hat{X}_2(x), \ldots, \hat{X}_n(x))$$

is nonsingular in $\mathcal{U}$. Moreover, it satisfies the condition $U^\top(x) = U^{-1}(x)$, that is, $U(x)$ is the unitary matrix-functions. It remains to prove that

$$U(x)^{-1}A(x)U(x) = \mathcal{N}(x),$$

where $\mathcal{N}(x)$ is some nilpotent matrix-function. This part of the proof goes by induction and completely coincides with the proof of theorem Shur and Toeplitz [12]. Therefore, we will not dwell on this part of the proof. So, the lemma is proved.

5

It is worth to remark the third condition of the lemma is sufficient, but not necessary. We can give an example, when the rank of $A(x)$ is variable in the domain of definition $\mathcal{U}$, but nevertheless $A(x)$ is p-smoothly similar to some nilpotent matrix-function. For example, consider the matrix-function

$$A(x) = \begin{pmatrix} x_1 & -1 & x_1(x_1 + x_2) \\ x_1^2 & -x_1 & -x_1 - x_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{U} = \mathbb{R}^2.$$

On the line given by the equation $x_2 = -x_1$, the rank of $A(x)$ is equal to one, while outside this line the rank of $A(x)$ is two. Remark, that in this example, there are not isolated points of change of the rank. In this case, we are able construct the unitary matrix-function $U(x)$ in $\mathcal{U}$. It is given by

$$U(x) = \frac{1}{\sqrt{1 + x_1^2}} \begin{pmatrix} 1 & x_1 & 0 \\ x_1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

while the nilpotent matrix-function $\mathcal{N}(x)$ takes the following form:

$$\mathcal{N}(x) = (1 + x_1^2) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & x_1 + x_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. The theorem on canonical structure of regular matrix-functions pencil

Let us prove the theorem on canonical form of regular matrix-functions pencil.

Theorem 2. *Let the following conditions be satisfied:*

(i) *the all roots of characteristic polynomial $\det(A(x) + \lambda B(x))$ are real and of constant multiplicity in the domain of definition $\mathcal{U}$;*

(ii) *the leading coefficient of polynomial $\det(A(x) + \lambda B(x))$ is not identically equal to zero on $\mathcal{U}$;*

(iii) *the ranks of $A(x)$ and $B(x)$ are independent of $x \in \mathcal{U}$ and less than dimension n.*

Then the pencil $A(x) + \lambda B(x)$ is p-smoothly equivalent to the following canonical form:

$$\text{diag}\{E_d, \mathcal{M}(x), E_{\hat{l}}\} + \lambda \, \text{diag}\{\mathcal{J}(x), E_l, \mathcal{N}(x)\}, \qquad (8)$$

where E_d denotes identity matrix of order d; $\mathcal{M}(x)$ and $\mathcal{N}(x)$ are some nilpotent matrices of orders l and $\hat{l}$, respectively, $\hat{l} = n - d - l$; $\mathcal{J}(x) = \text{diag}\{\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_k\}$, where $\mathcal{J}_i$ for $i = \overline{1, k}$ are nonsingular matrix-functions of orders p_i, respectively; $\sum_{i=1}^{k} p_i = d$; every block $\mathcal{J}_i$ has single eigenvalue $-1/\lambda_i(x)$ in $\mathcal{U}$; $\lambda_i(x)$ for $i = \overline{1, k}$ being eigenvalues of the characteristic polynomial $\det(A(x) + \lambda B(x))$ are not equal to zero in $\mathcal{U}$.

PROOF. It is obvious, that the coefficients of characteristic polynomial $\Delta(\lambda, x) = \det(A(x) + \lambda B(x))$ belong to the space $C^p(\mathcal{U})$. Moreover, it is known, that the roots $\lambda_i(x)$ for $i = \overline{1, k+1}$ belong to the space $C^p(\mathcal{U})$ (see lemma 1, [10]). First condition of the theorem, require that the roots of $\Delta(\lambda, x)$ do not coincide in $\mathcal{U}$, that is, $\lambda_i(x) \neq \lambda_j(x) \;\; \forall \; i \neq j$ and $\forall \; x \in \mathcal{U}$. The second condition of the theorem excludes the case, when $\Delta(\lambda, x) \equiv 0$ in $\mathcal{U}$, that is, the case of singular pencil. In these circumstances, there exists some function $c = c(x) \in C^p(\mathcal{U})$, such that $c(x) \neq 0 \; \forall \; x \in \mathcal{U}$ for which the condition

$$\det(A(x) + cB(x)) \neq 0 \;\; \forall \; x \in \mathcal{U}$$

holds. The function $c(x)$ can be constructed, for instance, as the arithmetic mean of the two neighboring roots of characteristic polynomial $\Delta(\lambda, x)$.

Let $A_1(x) \equiv A(x) + cB(x)$, then

$$A(x) + \lambda B(x) = A_1(x) + (\lambda - c)B(x). \qquad (9)$$

Multiplying the equation (9) by the matrix $A_1^{-1}(x)$ on the left, yields

$$A_1^{-1}(x)\left[A(x) + \lambda B(x)\right] = E_n + (\lambda - c)A_1^{-1}(x)B(x). \qquad (10)$$

Let us consider for matrix-function $A_1^{-1}(x)B(x)$ its characteristic polynomial

$$\tilde{\Delta}(\xi, x) = \det(A_1^{-1}(x)B(x) - \xi E_n),$$

with $\xi = \xi(x)$ being some unknown function. Making use of elemetary properties of the determinant, we can write

$$\tilde{\Delta}(\xi, x) = (-1)^n \det A_1^{-1}(x) \det(\xi A(x) + (\xi c - 1)B(x)). \qquad (11)$$

7

Let us write down the characteristic polynomial $\Delta(\lambda, x)$ in the form

$$\Delta(\lambda, x) = \sum_{i=0}^{n} S_i(x)\lambda^i, \tag{12}$$

where the coefficients $S_i(x)$ are the sums of the all minors of order n, composed of the $n - i$ rows of $A(x)$ and the i rows of $B(x)$. Clearly, $S_0(x)$ and $S_n(x)$ are determinants of $A(x)$ and $B(x)$, respectively.

In virtue of the third condition the theorem, the ranks of $A(x)$ and $B(x)$ are less than dimension n. Hence, $S_0(x) \equiv 0$ and $S_n(x) \equiv 0$ in $\mathcal{U}$. Then, among the roots of the polynomial $\Delta(\lambda, x)$ we can always find the root which is identically equal to zero in $\mathcal{U}$. It can be assumed, without loss of generality, that, say, $\lambda_{k+1}(x)$ be a zero root of multiplicity l, where $l \geq 1$. From what we said, it follows that (12) takes the form

$$\Delta(\lambda, x) = \lambda^l \sum_{i=l}^{l+d} S_i(x)\lambda^{i-l}, \quad l + d \leq n, \tag{13}$$

where l and d is independent of x in $\mathcal{U}$. Since the multiplicities of the roots of polynomial $\Delta(\lambda, x)$ are constant in $\mathcal{U}$, then coefficients $S_l(x)$ and $S_{l+d}(x)$ are not equal to zero in $\mathcal{U}$. In virtue of conditions the theorem, (13) is also specified as

$$\Delta(\lambda, x) = S_{l+d}(x)\lambda^l \prod_{i=1}^{k}(\lambda - \lambda_i(x))^{p_i}, \quad \text{where} \quad \sum_{i=1}^{k} p_i = d. \tag{14}$$

Together with $\Delta(\lambda, x)$, we consider the polynomial

$$\hat{\Delta}(\mu, \lambda, x) = \det(\mu A(x) + \lambda B(x))$$

with some μ being some unknown function $\mu = \mu(x)$. Let us write down the latter as

$$\hat{\Delta}(\lambda, x) = \sum_{i=0}^{n} S_i(x)\lambda^i \mu^{n-i}.$$

Making use of (13) and (14) we get

$$\tilde{\Delta}(\lambda, x) = (-1)^n \epsilon(x)\xi^{\hat{l}} \left(\xi - \frac{1}{c}\right)^l \prod_{i=1}^{k} \left(\xi - \frac{1}{c - \lambda_i(x)}\right)^{p_i}, \tag{15}$$

8

with
$$\epsilon(x) = \det A_1^{-1}(x)S_{l+d}(x)c^l \prod_{i=1}^{k}(c - \lambda_i(x))^{p_i}.$$

It is worth to remember that in the relation (15) $c \neq \lambda_i(x)$ for all $x \in \mathcal{U}$ and for all $i = \overline{1, n}$. On the other hand, we can write

$$\tilde{\Delta}(\lambda, x) = (-1)^n \xi^n + O(\xi^{n-1}), \tag{16}$$

where $O(\xi^{n-1})$ stands for the sum of the terms containing degrees of ξ less, than n. Comparing relations (15) and (16), we conclude that $\epsilon(x) \equiv 1$. Thus,

$$\tilde{\Delta}(\lambda, x) = (-1)^n \xi^{\hat{l}} \left(\xi - \frac{1}{c}\right)^l \prod_{i=1}^{k}(\xi - \xi_i(x))^{p_i} \text{ with } \xi_i(x) = \frac{1}{c - \lambda_i(x)}.$$

Since $\lambda_i(x) \neq \lambda_j(x)$ for all $i, j = \overline{1, n}$ and $c \neq 0$ in $\mathcal{U}$, then, according to the theorem from [15] there exists the matrix-function $T(x)$, which satisfies all the conditions of the definition 1 and the following relation:

$$T^{-1}(x)A_1^{-1}(x)B(x)T(x) = \text{diag}\{J_1(x), J_2(x), \dots, J_k(x), M(x), N(x)\},$$

where $J_i(x)$ for $i = \overline{1, k}$ are nonsingular in $\mathcal{U}$ and square matrix-functions with pair-wise unequal eigenvalues ξ_i for $i = \overline{1, n}$. According to the theorem from [15], $M(x)$ is the nonsingular in $\mathcal{U}$ matrix-function of order l with one eigenvalue equal to $1/c$, while $N(x)$ is the matrix-function of order $\hat{l}$ with one eigenvalue being identically equal to zero in $\mathcal{U}$.

Multiplying the pencil from (10) by the $T^{-1}(x)$ and $T(x)$ on the left and on the right, respectively, we obtain

$$E_n + (\lambda - c)\text{diag}\{J_1(x), J_2(x), \dots, J_k(x), M(x), N(x)\}$$

$$= \text{diag}\{E_{p_1} - cJ_1(x), E_{p_2} - cJ_2(x), \dots, E_{p_k} - cJ_k(x), E_l - cM(x), E_{\hat{l}} - cN(x)\}$$

$$+ \lambda \, \text{diag}\{J_1(x), J_2(x), \dots, J_k(x), M(x), N(x)\}. \tag{17}$$

Since, the ranks of matrix-functions $B(x)$, $J(x)$ and $M(x)$ are constants in $\mathcal{U}$, then, in virtue of the known property of rank [8], the rank $N(x)$ is independent of x in $\mathcal{U}$. Lemma 1 says, that $N(x)$ is p-smoothly unitary similar to some nilpotent matrix-function $\hat{N}(x)$. This means that there exists the nonsingular unitary matrix-functions $U(x)$, satisfying the relation

$$U^{-1}(x)N(x)U(x) = \hat{N}(x).$$

Take the following matrix-function:

$$\tilde{U}(x) = \operatorname{diag}\{E_d, E_l, U(x)\}.$$

Multiplying the pencil (17) by the $\tilde{U}^{-1}(x)$ and $\tilde{U}(x)$ on the left and on the right, respectively, we bring this pencil to the form

$$\operatorname{diag}\{E_{p_1} - cJ_1(x), E_{p_2} - cJ_2(x), \ldots, E_{p_k} - cJ_k(x), E_l - cM(x), E_{\hat{l}} - c\hat{N}(x)\}$$

$$+ \lambda \operatorname{diag}\{J_1(x), J_2(x), \ldots, J_k(x), M(x), \hat{N}(x)\}. \tag{18}$$

Let us consider now the following blocks $E_{p_i} - cJ_i(x)$ for $i = \overline{1,k}$ and the characteristic polynomials for each of them:

$$\det(E_{p_i} - cJ_i(x) - \nu_i E_{p_i}) = (-1)^{p_i} c^{p^i} \det\left(J_i(x) - \frac{1 - \nu_i}{c} E_{p_i}\right) \tag{19}$$

By assumption, the function c must not coincide with any root of characteristic polynomial $\Delta(\lambda, x)$ in $\mathcal{U}$. In particular, this means that $c \neq 0$ in $\mathcal{U}$. Since, the roots of (19), namely, $\nu_i = -\lambda_i(x)/(c - \lambda_i(x))$ are not equal to zero in $\mathcal{U}$, then the matrix-functions $E_{p_i} - cJ_i(x)$ are nonsingular in $\mathcal{U}$. Multiplying the pencil (18) by the matrix-function

$$\bar{J}(x) = \operatorname{diag}\{\hat{J}(x), E_l, (E_{\hat{l}} - c\hat{N}(x))^{-1}\},$$

on the left whith

$$\hat{J}(x) = \operatorname{diag}\{(E_{p_1} - cJ_1(x))^{-1}, (E_{p_2} - cJ_2(x))^{-1}, \ldots, (E_{p_k} - cJ_k(x))^{-1}\},$$

we obtain the pencil

$$\operatorname{diag}\{E_d, E_l - cM(x), E_{\hat{l}}\} + \lambda \operatorname{diag}\{\mathcal{J}(x), M(x), \mathcal{N}(x)\}, \tag{20}$$

where

$$\mathcal{J}(x) = \operatorname{diag}\{\mathcal{J}_1(x), \mathcal{J}_2(x), \ldots, \mathcal{J}_k(x)\}$$

with blocks $\mathcal{J}_i(x) = (E_{p_i} - cJ_i(x))^{-1} J_i(x)$ and $\mathcal{N}(x) = (E_{\hat{l}} - c\hat{N}(x))^{-1}\hat{N}(x)$.

The matrix-function $\mathcal{N}(x)$ is nilpotent, since it is constructed as a product of triangular and nilpotent matrix-functions. Each block $\mathcal{J}_i(x)$ in (20) is nonsingular in $\mathcal{U}$ and has unique eigenvalue $-1/\lambda_i(x) \ \forall \ i$. Multiplying the pencil (20) by the $\bar{M}(x) = \operatorname{diag}\{E_d, M^{-1}(x), E_{\hat{l}}\}$ on the left, gives

$$\operatorname{diag}\{E_d, \hat{M}(x), E_{\hat{l}}\} + \lambda \operatorname{diag}\{\mathcal{J}(x), E_l, \mathcal{N}(x)\}, \tag{21}$$

10

with $\hat{M}(x) \equiv M^{-1}(x) - cE_l$. Consider the block $\hat{M}(x)$. Its characteristic equation takes the form

$$\det\left(M(x) - \frac{1}{c + \zeta} E_l \right) = 0, \quad \zeta \neq -c \; \forall \; x \in \mathcal{U},$$

where $\zeta \equiv \zeta(x)$ is some unknown function. From the latter it follows, that all eigenvalues of $\hat{M}(x)$ are identically equal to zero, because all eigenvalues of $M(x)$ are equal to $1/c$. Furthermore, the rank of $A(x)$ is constant in $\mathcal{U}$. Taking into account the property of the rank, we conclude that the rank of $\hat{M}(x)$ is independent of $x \in \mathcal{U}$. According to the lemma 1, there exists the nonsingular in $\mathcal{U}$ unitary matrix-function , satisfying the relation

$$\bar{U}^{-1}(x)\hat{M}(x)\bar{U}(x) = \mathcal{M}(x),$$

where $\mathcal{M}(x)$ is some nilpotent matrix-function.

Let $\hat{U}(x) \equiv \mathrm{diag}\{E_d, \bar{U}(x), E_{\hat{\imath}}\}$. Multiplying the pencil (21) by the matrix-functions $\hat{U}^{-1}(x)$ and $\hat{U}(x)$ on the left and on the right, respectively, we obtain the pencil (8). Thus, we have proved the existence of the following nonsingular matrix-functions in $\mathcal{U}$:

$$P(x) = \hat{U}^{-1}(x)\bar{M}(x)\bar{J}(x)\tilde{U}^{-1}(x)T^{-1}(x)A_1^{-1}(x),$$

$$Q(x) = T(x)\tilde{U}(x)\hat{U}(x),$$

which bring the pencil $A(x) + \lambda B(x)$ to the canonical form (8). The elements of $P(x)$ and $Q(x)$ belong to the spase $C^p(\mathcal{U})$. Therefore the theorem is proved.

In conclusion of this section, let us spend some lines to give a pair of remarks.

Remark 1. Let us require, in circumstances of the theorem 2, the implementation of the following relations

$$\mathrm{rank}\; B(x) = \deg\left[\det(A(x) + \lambda B(x))\right] \tag{22}$$

and

$$\mathrm{rank}\; B(x) = \deg\left[\det(\mu A(x) + B(x))\right], \tag{23}$$

where $\mu \equiv \mu(x) \in C^p(\mathcal{U})$ is some unknown function, then the pencil $A(x) + \lambda B(x)$ is specified to be p-smoothly equivalent to the canonical form (8), in which

$$\mathcal{N}(x) \equiv \mathcal{O}_l \; \text{and} \; \mathcal{M}(x) \equiv \mathcal{O}_{\hat{\imath}},$$

11

where $\mathcal{O}_l$ is the zero block of order l. In this case $A(x) + \lambda B(x)$ is called the pencil satisfying the criterion "rank-degree". The structure of this pencil was investigated in [7].

Remark 2. If, in circumstances of the theorem 2 and conditions (22) and (23), we additionally require that all the roots of the characteristic polynomial $\det(A(x)+\lambda B(x))$ are simple, then the pencil $A(x)+\lambda B(x)$ is p-smoothly equivalent to a canonical form (8), in which

$$\mathcal{N}(x) \equiv \mathcal{O}_l, \ \ \mathcal{M}(x) \equiv \mathcal{O}_{\hat{l}}, \ \ \mathcal{J}(x) = \mathrm{diag}\left\{-1/\lambda_1(x), -1/\lambda_2(x), \ldots, -1/\lambda_k(x)\right\},$$

where $\lambda_i(x)$ are roots of the characteristic polynomial $\det(A(x) + \lambda B(x))$.

3. Examples

The goal of this section is to show the pair of simple examples to illustrate our theorem 2.

EXAMPLE 1. Consider the pencil

$$A(x) + \lambda B(x) = \begin{pmatrix} x_1 + x_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_1 x_2 & -x_2^2 \\ 0 & x_1^2 & -x_1 x_2, \end{pmatrix} \quad (24)$$

where $x \in \mathcal{U} = [a, b] \times [a, b] \subseteq \mathbb{R}^2$, and $a > 0$. Clearly, rank $A(x) =$ rank $B(x) = 2$ for $x \in \mathcal{U}$ and elements of $A(x)$ and $B(x)$ are the analytic (polynomial) functions. Characteristic polynomial for the pencil (24) is specified as

$$\det(A(x) + \lambda B(x)) = x_1 x_2 \lambda(\lambda + x_1 + x_2). \quad (25)$$

The roots of polynomial (25) are $\lambda_1 \equiv 0$ and $\lambda_2(x) = -x_1 - x_2$. They do not coincides in $\mathcal{U}$ and their multiplicity are constants in this domain. Furthermore, the leading coefficient of (25) is not equal to zero in $\mathcal{U}$. According to the theorem 2, the pencil (24) in $\mathcal{U}$ is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 1/(x_1 + x_2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix-functions $P(x)$ and $Q(x)$ can be calculated step-by-step using proof of theorem 2. To do this, it is enough to put $c \equiv 1$. We obtain these

transforming matrices in the form

$$P(x) = \begin{pmatrix} 1/(x_1 + x_2) & 0 & 0 \\ 0 & 1/(x_1 x_2) & 0 \\ 0 & -x_1/x_2 & 1 \end{pmatrix} \text{ and } Q(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x_2/x_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

One sees that the matrix-functions $P(x)$ and $Q(x)$ are nonsingular and analytic in $\mathcal{U}$.

EXAMPLE 2. Consider the pencil

$$A(x) + \lambda\ B(x) = \begin{pmatrix} 0 & 0 & \gamma(x)\sigma(x) \\ 0 & \gamma(x) & 0 \\ 0 & -\gamma(x)x_2 & \gamma(x) \end{pmatrix}$$

$$+ \lambda \begin{pmatrix} 0 & \sigma(x)x_2 & -\upsilon(x)\sigma(x) \\ x_1^2 & \upsilon(x) & -\upsilon(x)\sin(x_1) \\ 0 & -x_1(\upsilon(x) - 1) & \upsilon(x)(\upsilon(x) - 1) \end{pmatrix}, \tag{26}$$

$$x \in \mathcal{U} = [1, b] \times [1, b] \subseteq \mathbb{R}^2,$$

$$\gamma(x) = x_1 x_2, \quad \sigma(x) = x_1 + x_2, \quad \upsilon(x) = x_2 \sin(x_1).$$

Observe, that there exists the minors of second order

$$A_{23} = \begin{vmatrix} \gamma(x) & 0 \\ -\gamma(x)x_2 & \gamma(x) \end{vmatrix} \neq 0 \text{ and } B_{23} = \begin{vmatrix} 0 & \sigma(x)x_2 \\ x_1^2 & \upsilon(x) \end{vmatrix} \neq 0 \ \forall\ x \in \mathcal{U}.$$

Thus, $\text{rank}(A(x)) = \text{rank}(B(x)) = 2\ \forall\ x \in \mathcal{U}$. One sees that the characteristic polynomial for the pencil (26), that is

$$\det(A(x) + \lambda B(x)) = -x_1^4 x_2^3 (x_1 + x_2)\lambda$$

has single root $\lambda \equiv 0\ \forall\ x \in \mathcal{U}$. Moreover, the leading coefficient of the characteristic polynomial is not equal zero in $\mathcal{U}$. According to our theorem 2, the pencil (26) in $\mathcal{U}$ is smoothly equivalent to

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \varphi(x) \\ 0 & 0 & 0 \end{pmatrix}. \tag{27}$$

The matrix-functions $P(x)$ and $Q(x)$ are of the form

$$P(x) = \frac{x_1^2}{\upsilon(x)x_2} \begin{pmatrix} -\rho(x)\gamma(x) & \nu(x)x_2 & \nu(x) \\ (\sin(x_1) + x_2)x_1^2 & 0 & -\sin(x_1)\sigma(x)x_1^2 \\ (1 - \upsilon(x))x_1^2 & 0 & -\sigma(x)x_1^2 \end{pmatrix},$$

13

$$Q(x) = \frac{1}{\rho(x)} \begin{pmatrix} \rho(x) & 0 & 0 \\ 0 & \sin(x_1) & 1 \\ 0 & 1 & -\sin(x_1) \end{pmatrix}$$

with $\nu(x) = \rho(x)\gamma(x)\sigma(x)$ and $\rho(x) = (1 + \sin^2(x_1))^{1/2}$. Multiplying the pencil (26) by the matrix-functions $P(x)$ and $Q(x)$ on the left and on the right, respectively, we obtain the function $\varphi(x) = (1+\sin^2(x_1))/x_1$. One sees that the elements of matrices $P(x)$ and $Q(x)$ are analytic in $\mathcal{U}$.

In conclusion, let us observe, that the blocks $\mathcal{J}_i(x)$ in the canonical structure (8) are nonsingular matrix-functions of orders p_i, respectively. They do not have, generally speaking, Jordan structure. To make the blocks $\mathcal{J}_i(x)$ entirely coinciding with the Jordan blocks we must require additional conditions on the matrix-functions $A(x)$ and $B(x)$. In this paper we would not want to do this. Obtained the canonical form of the pencil (8) is quite sufficient to start the study of some class of the linear partial differential-algebraic equations and constructing numerical methods for them.

References

[1] *W. Lucht, K. Strehmel and C. Eichler-Liebenow* Indexes and special discretization methods for linear partial differential algebraic equations, BIT 39 (1999) 484–512.

[2] *K. Debrabant, K. Strehmel* Convergence of Runge-Kutta methods applied to linear partial differential-algebraic equations, Applied Numerical Mathematics 53 (2005) 213–219.

[3] *C. K. Godunov* Equations of the mathematical physics. Nauka, Moscow, 1971 (in Russian).

[4] *S. V. Gaidomak* Spline collocation method for linear singular hyperbolic systems, Comput. Math. Math. Phys. 48 (2008), 1161–1180.

[5] *S. V. Gaidomak* Three-layer finite-difference method for linear partial differential-algebraic systems, Differential Equations 46 (2010) 586–597.

[6] *S. V. Gaidomak* Stability of an implicit difference scheme for a linear differential-algebraic system of partial differential equations, Comput. Math. Math. Phys. 50 (2010) 673–683.

[7] *V. F. Chistyakov* Algebraic-differential operators with finite-dimensional kernel. Nauka, Novosibirsk, 1996 (in Russian).

[8] *F. R. Gantmacher* The Theory of Matrices. Chelsea Publishing Company, New York 68, 1959.

[9] *B. V. Verbitskii* Some global property of matrix-functions that depend on several variables, Russian Math (Iz. Vuz) 22 (1978) 5–12.

[10] *B. V. Verbitskii* A certain global property of matrix-valued functions that depend on several variables, Uspekhi Mat. Nauk 28 (1973) 233-234 (in Russian).

[11] *B. V. Verbickii* On a global property of a matrix-valued function of one variable, Math. USSR Sb. 20 (1973) 53–65.

[12] P. Lankaster Theory of Matrices. Academic Press, New York-London, 1969.

[13] *H. Gingold and P. F. Hsieh* Globally analytic triangularization of a matrix function, Linear Algebra Appl. 169 (1992) 75-101.

[14] *L. M. Silverman and R. S. Bucy* Generalizations of a theorem of Dolezal, Theory of Computing Systems 4 (1969), 334–339.

[15] *P. F. Hsieh and Y. Sibuya* A global analysis of matrices of functions of several variables, J. Math. Anal. Appl. 14 (1966) 332-340.