

# HIGHER WHITEHEAD PRODUCTS IN TORIC TOPOLOGY

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ABSTRACT. In this paper we study a relation between the moment-angle complex  $\mathcal{Z}_K$  and the Davis-Januskiewicz space  $DJ_K$  for MF-complexes  $K$  by considering the homotopy fibration sequence  $\mathcal{Z}_K \xrightarrow{\tilde{w}} DJ_K \rightarrow \prod_{i=1}^n \mathbb{C}P^\infty$ . After observing that the homotopy type of  $\mathcal{Z}_K$  is a wedge of spheres, we describe  $\tilde{w}$  as a sum of higher and iterated Whitehead products.

## 1. INTRODUCTION

To a simplicial complex  $K$  on  $n$  vertices, Davis and Januszkiewicz [DJ] associated two fundamental objects of toric topology: the moment-angle complex  $\mathcal{Z}_K$  and the Davis-Januszkiewicz space  $DJ_K$ , whose study connects algebraic geometry, topology, combinatorics, and commutative algebra. Algebraic topologists, on their side, have tried to understand the homotopy theory of these spaces and other related topological spaces with a torus action. Recent developments [BP1] have shown that from the homotopy theoretical point of view both spaces  $\mathcal{Z}_K$  and  $DJ_K$  could be considered as polyhedral products of the topological pairs  $(D^2, S^1)$  and  $(\mathbb{C}P^\infty, *)$ , respectively. That lead to a wide generalisation [BP2, S] to polyhedral products  $(\underline{X}, \underline{A})^K$  of  $n$ -tuples of topological pairs  $(\underline{X}, \underline{A}) = (X_i, A_i)_{i=1}^n$ . These spaces are still very closely related to combinatorics, and commutative algebra but as yet are not known to have a strong connection with algebraic geometry.

Let  $X_1, \dots, X_n$  be path-connected spaces and let  $\underline{X} = \{X_1, \dots, X_n\}$ . Following [BP2, BBCG], for  $\sigma = (i_1, \dots, i_k)$  let  $X^\sigma = \prod_{j=1}^k X_{i_j}$ , and let  $DJ_K(\underline{X}) = \bigcup_{\sigma \in K} X^\sigma$ . Notice that there is an inclusion  $DJ_K(\underline{X}) \rightarrow \prod_{i=1}^n X_i$ . Define  $\mathcal{Z}_K(\underline{X})$  by the homotopy fibration

$$(1) \quad \mathcal{Z}_K(\underline{X}) \longrightarrow DJ_K(\underline{X}) \longrightarrow \prod_{i=1}^n X_i.$$

In this paper we will first consider the case when each  $X_i$  is a sphere, writing  $\underline{S} = (S^{m_1+1}, \dots, S^{m_n+1})$ . If  $X_1, \dots, X_n$  all equal a common space  $X$ , we instead write  $\mathcal{Z}_K(X)$  and  $DJ_K(X)$ . When  $X_i = \mathbb{C}P^\infty$  for each  $1 \leq i \leq n$ , the homotopy fibration (1) specializes to the case of primary interest in toric topology, that is, to the homotopy fibration  $\mathcal{Z}_K \rightarrow DJ_K \rightarrow \prod_{i=1}^n \mathbb{C}P^\infty$ . For the sake of clarity, we remark that in the terms of polyhedral products our  $\mathcal{Z}_K(X)$  is actually  $(\text{Cone } \Omega X, \Omega X)^K$ , whereas  $DJ_K(X) = (X, *)^K$ .

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To date, there has been considerable success in studying the homotopy type of  $\mathcal{Z}_K$  [GT] or its suspension [BBCG]. However, no attempt has been made to study the map  $\mathcal{Z}_K \rightarrow DJ_K$ . This means that the interesting information that is known about the moment-angle complex  $\mathcal{Z}_K$  cannot be related to the Davis-Januszkiewicz space  $DJ_K$ . The purpose of this paper is to remedy this deficiency. We show that for a certain family of simplicial complexes  $K$ ,  $\mathcal{Z}_K$  is homotopy equivalent to a wedge of spheres and the homotopy equivalence may be chosen so that the map  $\mathcal{Z}_K \rightarrow DJ_K$  consists of a specified collection of higher Whitehead products and iterated Whitehead products. The defining property of this family is that the complex  $K$  must be the union of the boundaries of its missing faces. For such a complex  $K$ , we show that each missing face corresponds to the existence of a nontrivial higher Whitehead product whose adjoint has a nonzero Hurewicz image in  $H_*(\Omega DJ_K; \mathbb{Q})$ .

As mentioned, polyhedral products currently enjoy great popularity; in particular their loop homology with various coefficients and for different families of simplicial complexes has been calculated. Some simple but important examples of the homology of  $\Omega DJ_K(\underline{X})$  were calculated by Lemaire [L] in 1974 before the notion of  $\mathcal{Z}_K(\underline{X})$  and  $DJ_K(\underline{X})$  were introduced. Panov and Ray [PR] introduced categorical formalism to study the loop homology of  $DJ_K$  and gave explicit calculations when  $K$  is a flag complex. Dobrinskaya [D] has a general approach for calculating the homology of  $\Omega DJ_K(\underline{X})$  for an arbitrary simplicial complex  $K$  in terms of the homology of  $\Omega(\underline{X})$  and some special relations coming from the homology of  $\Omega DJ_K(S^2)$ . As an intermediate goal towards understanding the map  $\mathcal{Z}_K \rightarrow DJ_K$  we need to calculate the rational homology of  $\Omega DJ_K(\underline{S})$  and  $\Omega DJ_K$ . However, it is important to emphasize that we do this in such a way as to remember the geometry of the space, that is, in such a way as to keep track of specific Hurewicz images. The existing models for rational loop homology are not known to do this, so we have to produce our own model which does.

The methods we use lend themselves well to concrete calculations. We include several examples to illustrate this.

In what follows  $K$  will be a simplicial complex on  $n$  vertices whose simplices are subsets of the vertex set  $[n] = \{1, \dots, n\}$ . That is, a simplex  $\sigma \in K$  corresponds to a sequence  $(i_1, \dots, i_k)$  where  $1 \leq i_1 < \dots < i_k \leq n$  and the integers  $i_j$  are the vertices of  $K$  which are in  $\sigma$ . Let  $|\sigma| = k - 1$  be the dimension of  $\sigma$ . We concentrate on the collection  $MF(K)$  of *missing faces*. To be precise, a sequence  $(i_1, \dots, i_k)$  is in  $MF(K)$  if: (i)  $(i_1, \dots, i_k) \notin K$ , and (ii) every proper subsequence of  $(i_1, \dots, i_k)$  is in  $K$ . For example, let  $K$  be the simplicial complex on 4 vertices with edges  $(1, 2), (1, 3), (1, 4), (2, 3), (2, 4)$ . Then  $MF(K) = \{(3, 4), (1, 2, 3), (1, 2, 4)\}$ .

**Definition 1.1.** Let  $K$  be a simplicial complex on  $n$  vertices. We say that  $K$  is an *MF-complex* if

$$|K| = \bigcup_{\sigma \in MF(K)} |\partial\sigma|$$

where  $|K|$  and  $|\partial\sigma|$  denote the geometrical realisations of  $K$  and  $\partial\sigma$ , respectively.

If  $\sigma = (i_1, \dots, i_k) \in K$ , let  $FW(\sigma)$  be the fat wedge of the product  $X_{i_1} \times \dots \times X_{i_k}$ . If  $K$  is an  $MF$ -complex then  $DJ_K(\underline{X})$  can be written as

$$DJ_K(\underline{X}) = \operatorname{colim}_{\sigma \in MF(K)} FW(\sigma).$$

Our first result shows that colimits of fat wedges behave nicely when included to the product  $X_1 \times \dots \times X_n$ , allowing us to determine the homotopy fibre of the inclusion.

**Theorem 1.2.** *Let  $K$  be an  $MF$ -complex on  $n$  vertices. Let  $\underline{X} = \{X_1, \dots, X_n\}$  where each  $X_i$  is a path-connected topological space. Then  $\mathcal{Z}_K(\underline{X})$  is homotopy equivalent to a wedge of spaces of the form  $\Sigma^t \Omega X_{i_1} \wedge \dots \wedge \Omega X_{i_k}$  for various  $1 \leq t < n$  and sequences  $(i_1, \dots, i_k)$  where  $1 \leq i_1 < \dots < i_k \leq n$ .*

**Corollary 1.3.** *Let  $K$  be an  $MF$ -complex on  $n$  vertices. Then each of  $\mathcal{Z}_K(\underline{S})$  and  $\mathcal{Z}_K$  is homotopy equivalent to a wedge of simply-connected spheres.*

Next, as an intermediate step, we calculate  $H_*(\Omega DJ_K(\underline{S}); \mathbb{Q})$  and  $H_*(\Omega DJ_K; \mathbb{Q})$  using an Adams-Hilton model, with the emphasis on detecting Hurewicz images. To state this we introduce some notation. If  $V$  is a graded  $\mathbb{Q}$ -vector space, let  $L\langle V \rangle$  be the free graded Lie algebra generated by  $V$ , and let  $UL\langle V \rangle$  be its universal enveloping algebra. Let  $L_{ab}\langle V \rangle$  be the free abelian Lie algebra generated by  $V$ , whose bracket is identically zero. If  $L$  is a Lie algebra and  $x_1, \dots, x_k \in L$ , let  $[[x_1, x_2], \dots, x_k]$  denote the iterated bracket  $[\dots [[x_1, x_2], x_3], \dots, x_k]$ .

Let  $b_i$  be the Hurewicz image of the adjoint of the coordinate inclusion  $S^{m_i+1} \rightarrow DJ_K(\underline{S})$ . Abusing notation, let  $b_i$  also be the Hurewicz image of the adjoint of the composite  $S^2 \rightarrow \mathbb{C}P^\infty \rightarrow DJ_K$ , where the left map is the inclusion of the bottom cell and the right map is the inclusion of the  $i^{\text{th}}$ -coordinate. By  $u_\sigma$  we denote the Hurewicz image of the adjoint of the Whitehead product corresponding to a missing face  $\sigma \in MF(K)$ . We will phrase  $H_*(\Omega DJ_K(\underline{S}); \mathbb{Q})$  and  $H_*(\Omega DJ_K; \mathbb{Q})$  as quotients of  $U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle)$ . A distinction needs to be made between the elements  $u_\sigma$  where  $|\sigma| = 1$  and  $|\sigma| > 1$ . The latter elements are independent from  $b_1, \dots, b_n$ . On the other hand, if  $|\sigma| = 1$  then  $\sigma = (i_1, i_2)$  and  $u_\sigma = [b_{i_1}, b_{i_2}]$  which are not independent from  $b_1, \dots, b_n$ . This leads to additional relations determined by the graded Jacobi identity and face relations. Specifically, we have  $[u_\sigma, b_j] = [[b_{i_1}, b_{i_2}], b_j] = [b_{i_1}, [b_{i_2}, b_j]] - (-1)^{|b_{i_1}||b_{i_2}|} [b_{i_2}, [b_{i_1}, b_j]]$  and if  $(i, j) \in K$  then  $[b_i, b_j] = 0$ . The collection of such relations forms an ideal in  $U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle)$  which we label as  $J$ . Note that if every missing face  $\sigma \in MF(K)$  is of dimension greater than 1, then  $J$  is trivial.

**Theorem 1.4.** *Let  $K$  be an  $MF$ -complex. There is an algebra isomorphism*

$$H_*(\Omega DJ_K(\underline{S}); \mathbb{Q}) \cong U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle) / J$$

where each  $u_\sigma$  is the Hurewicz image of the adjoint of a higher Whitehead product. Further, the looped inclusion  $\Omega DJ_K(\underline{S}) \rightarrow \prod_{i=1}^n \Omega S^{m_i+1}$  is modelled by the map

$$U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle) / J \xrightarrow{U(\pi)} UL_{ab}\langle b_1, \dots, b_n \rangle$$

where  $\pi$  is the projection.

**Theorem 1.5.** *Let  $K$  be an MF-complex. There is an algebra isomorphism*

$$H_*(\Omega DJ_K; \mathbb{Q}) \cong U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle) / (I + J)$$

where  $u_\sigma$  is the Hurewicz image of the adjoint of a higher Whitehead product,  $J$  is as in Theorem 1.4, and  $I$  is the ideal

$$I = (b_i^2, [u_\sigma, b_{j_\sigma}] \mid 1 \leq i \leq n, \sigma = (i_1, \dots, i_k) \in MF(K), j_\sigma \in \{i_1, \dots, i_k\}).$$

Further, there is a commutative diagram of algebras

$$\begin{array}{ccc} H_*(\Omega DJ_K(S^2); \mathbb{Q}) & \xrightarrow{\cong} & U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle) / J \\ \downarrow (\Omega DJ_K(i))_* & & \downarrow q \\ H_*(\Omega DJ_K; \mathbb{Q}) & \xrightarrow{\cong} & U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle) / (I + J) \end{array}$$

where  $q$  is the quotient map.

Our main theorems are homotopy theoretic. For  $1 \leq i \leq n$ , let  $a_i: S^{m_i+1} \rightarrow DJ_K(\underline{S})$  be the inclusion of the  $i^{\text{th}}$ -coordinate. Let  $\iota: S^2 \rightarrow \mathbb{C}P^\infty$  be the inclusion of the bottom cell. Let  $\tilde{a}_i$  be the composite  $\tilde{a}_i: S^2 \xrightarrow{\iota} \mathbb{C}P^\infty \rightarrow DJ_K$  where the right map is the inclusion of the  $i^{\text{th}}$ -coordinate. The analogue of the algebraic ideal  $J$  occurs when  $\sigma = (i_1, i_2)$ , in which case there is a Whitehead product  $w_\sigma = [a_{i_1}, a_{i_2}]$ ; as in the algebraic case, there are additional relations determined by the Jacobi identity and face relations in cases of the form  $[w_\sigma, a_j]$  when  $(i_1, j) \in K$  or  $(i_2, j) \in K$ . For  $|\sigma| = 1$ , let  $W_\sigma$  be the collection of all independent Whitehead products of the form  $[[w_\sigma, a_{j_1}] \dots, a_{j_l}]$  where  $1 \leq j_1 \leq \dots \leq j_l \leq l$  and  $1 \leq l < \infty$ .

**Theorem 1.6.** *Let  $K$  be an MF-complex on  $n$  vertices, so that there is a homotopy equivalence  $Z_K(\underline{S}) \simeq \bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha}$ . The equivalence can be chosen so that the composite*

$$\bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha} \rightarrow Z_K(\underline{S}) \rightarrow DJ_K(\underline{S})$$

is a wedge sum of the following maps:

- (a) a higher Whitehead product  $w_\sigma: S^{t_\sigma} \rightarrow DJ_K(\underline{S})$  for each missing face  $\sigma = (i_1, \dots, i_k) \in MF(K)$ , where  $t_\sigma = k - 1 + (\sum_{j=1}^k m_{i_j})$ ;

(b) *an iterated Whitehead product*

$$[[w_\sigma, a_{j_1}] \dots, a_{j_l}]: S^{t_\sigma+l} \longrightarrow DJ_K(\underline{S})$$

for each  $\sigma \in MF(K)$  of dimension greater than 1 and each list  $1 \leq j_1 \leq \dots \leq j_l \leq l$ , where  $1 \leq l < \infty$ ;

(c) *the collection of independent iterated Whitehead products  $W_\sigma$  for each  $\sigma \in MF(K)$  of dimension 1.*

Given  $\sigma = (i_1, \dots, i_k)$ , let  $J_\sigma = \{1, \dots, n\} - \{i_1, \dots, i_k\}$ . If  $\sigma = (i_1, i_2)$ , let  $\tilde{w}_\sigma$  be the Whitehead product  $[\tilde{a}_{i_1}, \tilde{a}_{i_2}]$ . As above, for  $|\sigma| = 1$ , let  $\tilde{W}_\sigma$  be the collection of all independent Whitehead products of the form  $[[\tilde{w}_\sigma, \tilde{a}_{j_1}] \dots, \tilde{a}_{j_l}]$  where  $1 \leq j_1 \leq \dots \leq j_l \leq l$  and  $1 \leq l < \infty$ .

**Theorem 1.7.** *Let  $K$  be an  $MF$ -complex on  $n$  vertices, so that there is a homotopy equivalence  $\mathcal{Z}_K \simeq \bigvee_{\tilde{\alpha} \in \tilde{\mathcal{I}}} S^{t_{\tilde{\alpha}}}$ . The equivalence can be chosen so that the composite*

$$\bigvee_{\tilde{\alpha} \in \tilde{\mathcal{I}}} S^{t_{\tilde{\alpha}}} \longrightarrow \mathcal{Z}_K \longrightarrow DJ_K$$

is a wedge sum of the following maps:

(a) *a higher Whitehead product  $\tilde{w}_\sigma: S^{2|\sigma|+1} \longrightarrow DJ_K$  for each missing face  $\sigma \in MF(K)$ ;*

(b) *an iterated Whitehead product*

$$[[\tilde{w}_\sigma, \tilde{a}_{j_1}] \dots, \tilde{a}_{j_l}]: S^{2|\sigma|+l+1} \longrightarrow DJ_K$$

for each  $\sigma \in MF(K)$  of dimension greater than 1 and each list  $j_1 < \dots < j_l$  in  $J_\sigma$ , where  $1 \leq l \leq n$ ;

(c) *the collection of independent iterated Whitehead products  $\tilde{W}_\sigma$  for each  $\sigma \in MF(K)$  of dimension 1.*

Although in this paper our goal is to identify the map  $\mathcal{Z}_K \longrightarrow DJ_K$  for  $K$  an  $MF$ -complex, we are hoping to generalise this first to a much larger family of simplicial complexes and second to the map between polyhedral products of any  $n$  tuple of topological pairs. As a consequence we should obtain a homotopy wedge decomposition of  $\mathcal{Z}_K(\underline{X}, \underline{A})$ , which will reduce to a wedge of spheres in the case of  $\mathcal{Z}_K$ .

## 2. THE OBJECTS OF STUDY

This section gives an initial analysis of  $MF$ -complexes. First, we compare  $MF$ -complexes to another family of simplicial complexes that has received considerable attention for its role in producing wedge decompositions of  $\mathcal{Z}_K$ . Then we show that both  $\mathcal{Z}_K(\underline{S})$  and  $\mathcal{Z}_K$  are homotopy equivalent to a wedge of simply-connected spheres if  $K$  is an  $MF$ -complex, proving Theorem 1.3.

We compare  $MF$ -complexes to shifted complexes. A simplicial complex  $K$  on  $n$  vertices is *shifted* if there is an ordering on the vertex set such that whenever  $\sigma$  is a simplex of  $K$  and  $v' < v$ , then  $(\sigma - v) \cup v'$  is a simplex of  $K$ . In [GT] it was shown that if  $K$  is a shifted complex then  $\mathcal{Z}_K$  is homotopy equivalent to a wedge of spheres. More properties of  $\mathcal{Z}_K$  for shifted complexes  $K$  were considered in [BBCG, D].

We show that  $MF$ -complexes and shifted complexes form distinct families, with nontrivial intersection. Consider the three examples:

- (1)  $K_1$  is the simplicial complex on 4 vertices with edges  $(1, 2), (1, 3), (1, 4), (2, 3), (2, 4)$ ;
- (2)  $K_2$  is the simplicial complex on 4 vertices with edges  $(1, 2), (1, 3), (1, 4), (2, 3)$ ;
- (3)  $K_3$  is the simplicial complex on 5 vertices with edges  $(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (3, 5)$ .

Observe that  $K_1$  and  $K_2$  are shifted but  $K_3$  is not. The list of minimal missing faces in each case is:

- (1)  $MF(K_1) = \{(3, 4), (1, 2, 3), (1, 2, 4)\}$ ;
- (2)  $MF(K_2) = \{(2, 4), (3, 4), (1, 2, 3)\}$ ;
- (3)  $MF(K_3) = \{(2, 5), (3, 4), (4, 5), (1, 2, 3), (1, 2, 4), (1, 3, 5)\}$ .

Observe that  $|K_1| = \bigcup_{\sigma \in M(K_1)} |\partial\sigma|$  and  $|K_3| = \bigcup_{\sigma \in M(K_3)} |\partial\sigma|$ , but  $\bigcup_{\sigma \in M(K_2)} |\partial\sigma| = |K_2 - (1, 4)|$ . Thus  $K_1$  is a shifted complex which is also an  $MF$ -complex, while  $K_2$  is a shifted complex which is not an  $MF$ -complex, and  $K_3$  is an  $MF$ -complex which is not shifted.

Now we turn to the homotopy type of  $\mathcal{Z}_K(\underline{S})$  and  $\mathcal{Z}_K$  when  $K$  is an  $MF$ -complex. We begin with proving a much more general result.

**Proof of Theorem 1.2:** Since  $K$  is an  $MF$ -complex, we have  $|K| = \bigcup_{\sigma \in MF(K)} |\partial\sigma|$ . This union can be realized by iteratively gluing on one  $\partial\sigma$  at a time. That is, if  $l$  is the cardinality of  $MF(K)$  (note  $l$  is finite) then the elements of  $MF(K)$  can be ordered as  $\sigma_1, \dots, \sigma_l$  in such a way that for  $1 \leq i \leq l$  there is a sequence of subcomplexes  $K_i = \bigcup_{j=1}^i \partial\sigma_j$  of  $K$  and pushouts

$$\begin{array}{ccc} \partial\sigma_i \cap K_{i-1} & \longrightarrow & K_{i-1} \\ \downarrow & & \downarrow \\ \partial\sigma_i & \longrightarrow & K_i. \end{array}$$

Observe that the intersection  $\partial\sigma \cap K_{i-1}$  is a face of both  $\partial\sigma$  and  $K_{i-1}$ . That is,  $\partial\sigma$  and  $K_{i-1}$  have been glued along a common face.

Write  $Y \in \mathcal{W}$  if  $Y$  is a space that is homotopy equivalent to a wedge of spaces of the form  $\Sigma^t \Omega X_{i_1} \wedge \dots \wedge \Omega X_{i_k}$  for various  $1 \leq t < n$  and sequences  $(i_1, \dots, i_k)$  where  $1 \leq i_1 < \dots < i_k \leq n$ . For any such sequence, let  $FW(i_1, \dots, i_k)$  be the fat wedge of the product  $\prod_{j=1}^k X_{i_j}$ . By definition, the homotopy fibre of the inclusion  $FW(i_1, \dots, i_k) \longrightarrow \prod_{j=1}^k X_{i_j}$  is  $\mathcal{Z}_K$  for  $K = (\Delta^k)_{k-1} = \partial\sigma$ . On the other hand, by [P2], this homotopy fibre is homotopy equivalent to  $\Sigma^{k-1} \Omega X_{i_1} \wedge \dots \wedge \Omega X_{i_k}$ . In particular, in an  $MF$ -complex  $|K| = \bigcup_{\sigma \in MF(K)} |\partial\sigma|$ , we have  $\mathcal{Z}_{\partial\sigma} \in \mathcal{W}$  for every  $\sigma \in MF(K)$ .

Now we proceed with the proof by induction. Since  $K_1 = \partial\sigma_1$ , the previous paragraph shows that  $\mathcal{Z}_{K_1} \in \mathcal{W}$ . Now suppose that  $\mathcal{Z}_{K_{i-1}} \in \mathcal{W}$ . We have  $K_i$  constructed by gluing  $\partial\sigma_i$  and  $K_{i-1}$  along a common face,  $\mathcal{Z}_{\partial\sigma_i} \in \mathcal{W}$  by the preceding paragraph, and  $\mathcal{Z}_{K_{i-1}} \in \mathcal{W}$  by assumption. Under these circumstances, [GT, Theorem 1.3] implies that  $\mathcal{Z}_{K_i} \in \mathcal{W}$ . (Actually, [GT, Theorem 1.3] is stated for the special case when  $X_i = \mathbb{C}P^\infty$  for  $1 \leq i \leq n$ , but the proof goes through without change in the general case.) Hence, by induction,  $\mathcal{Z}_K(\underline{X}) = \mathcal{Z}_{K_i}$  is in  $\mathcal{W}$ .  $\square$

**Proof of Theorem 1.3:** In Theorem 1.2, suppose that  $X_i = S^{m_i+1}$  for each  $1 \leq i \leq n$ . Then  $\mathcal{Z}_K(\underline{S})$  is homotopy equivalent to a wedge of spaces of the form  $\Sigma^t \Omega S^{m_{i_1}+1} \wedge \cdots \wedge \Omega S^{m_{i_k}+1}$  for various  $1 \leq t < n$  and sequences  $(i_1, \dots, i_k)$ . By [J], there is a homotopy equivalence  $\Sigma \Omega S^{m_i+1} \simeq \bigvee_{j=1}^\infty S^{j m_i+1}$ . Since  $S^{j m_i+1}$  is a suspension, we can iterate this to show that each wedge summand  $\Sigma^t \Omega S^{m_{i_1}+1} \wedge \cdots \wedge \Omega S^{m_{i_k}+1}$  is homotopy equivalent to a wedge of simply-connected spheres. Thus  $\mathcal{Z}_K(\underline{S})$  is homotopy equivalent to a wedge of simply-connected spheres.

Next, in Theorem 1.2, suppose that  $X_i = \mathbb{C}P^\infty$  for each  $1 \leq i \leq n$ . Then  $\mathcal{Z}_K$  is homotopy equivalent to a wedge of spaces of the form  $\Sigma^t \Omega \mathbb{C}P_{i_1}^\infty \wedge \cdots \wedge \Omega \mathbb{C}P_{i_k}^\infty$  for various  $1 \leq t < n$  and sequences  $(i_1, \dots, i_k)$ . Since  $\Omega \mathbb{C}P^\infty \simeq S^1$ , each wedge summand  $\Sigma^t \Omega \mathbb{C}P_{i_1}^\infty \wedge \cdots \wedge \Omega \mathbb{C}P_{i_k}^\infty$  is homotopy equivalent to  $S^{k+t}$ . Thus  $\mathcal{Z}_K$  is homotopy equivalent to a wedge of simply-connected spheres.  $\square$

### 3. HIGHER WHITEHEAD PRODUCTS AND FAT WEDGES

In this section we define a higher Whitehead product by means of a fat wedge, and relate the existence of a missing face in  $K$  to the existence of a nontrivial higher Whitehead in  $DJ_K(\underline{X})$ . Let  $X_1, \dots, X_n$  be path-connected spaces and let  $\underline{X} = \{X_1, \dots, X_n\}$ . The fat wedge is the space

$$FW(\underline{X}) = \{(x_1, \dots, x_n) \in X_1 \times \cdots \times X_n \mid \text{at least one } x_i = *\}.$$

Consider the homotopy fibration obtained by including  $FW(\underline{X})$  into the product  $X_1 \times \cdots \times X_n$ . The homotopy type of the fibre was first identified by Porter [P2], who showed that there is a homotopy fibration

$$\Sigma^{n-1} \Omega X_1 \wedge \cdots \wedge \Omega X_n \longrightarrow FW(\underline{X}) \longrightarrow X_1 \times \cdots \times X_n.$$

If each  $X_i$  is a suspension,  $X_i = \Sigma Y_i$ , then the standard suspension map  $E: Y \longrightarrow \Omega \Sigma Y$  induces a composite

$$\phi_n: \Sigma^{n-1} Y_1 \wedge \cdots \wedge Y_n \longrightarrow \Sigma^{n-1} \Omega \Sigma Y_1 \wedge \cdots \wedge \Omega \Sigma Y_n \longrightarrow FW(\underline{\Sigma Y}).$$

The map  $\phi_n$  is the attaching map that yields the product. That is, there is a homotopy cofibration

$$\Sigma^{n-1} Y_1 \wedge \cdots \wedge Y_n \xrightarrow{\phi_n} FW(\underline{\Sigma Y}) \longrightarrow \Sigma Y_1 \times \cdots \times \Sigma Y_n.$$

In the case  $n = 2$ , we have  $FW(\underline{\Sigma Y}) = \Sigma Y_1 \vee \Sigma Y_2$  and  $\phi_2$  is the Whitehead product  $[i_1, i_2]$ , where  $i_1$  and  $i_2$  are the inclusions of  $\Sigma Y_1$  and  $\Sigma Y_2$  respectively into  $\Sigma Y_1 \vee \Sigma Y_2$ . This is the universal example for Whitehead products. Given a space  $Z$  and maps  $f: \Sigma Y_1 \longrightarrow Z$  and  $g: \Sigma Y_2 \longrightarrow Z$ , the Whitehead product  $[f, g]$  of  $f$  and  $g$  is the composite  $\Sigma Y_1 \wedge Y_2 \xrightarrow{\phi_2} \Sigma Y_1 \vee \Sigma Y_2 \xrightarrow{f \perp g} Z$ , where  $\perp$

denotes the wedge sum. Porter [P1] used the maps  $\phi_n$  for  $n > 2$  as universal examples to define higher Whitehead products.

**Definition 3.1.** For  $n \geq 2$ , let  $Y_1, \dots, Y_n$  and  $Z$  be path-connected spaces, and let  $f_i: \Sigma Y_i \rightarrow Z$  be maps. Suppose that the wedge sum  $\bigvee_{i=1}^n \Sigma Y_i \rightarrow Z$  of the maps  $f_i$  extends to a map  $f: FW(\underline{\Sigma Y}) \rightarrow Z$ . The  $n^{\text{th}}$ -higher Whitehead product of the maps  $f_1, \dots, f_n$  is the composite

$$[f_1, \dots, f_n]: \Sigma^{n-1} Y_1 \wedge \dots \wedge Y_n \xrightarrow{\phi_n} FW(\underline{\Sigma Y}) \xrightarrow{f} Z.$$

If  $n = 2$ , the Whitehead product of two maps  $f_1$  and  $f_2$  is always defined, and the homotopy class of  $[f_1, f_2]$  is uniquely determined by the homotopy classes of  $f_1$  and  $f_2$ . If  $n > 2$ , it may not be the case that the higher Whitehead product of  $n$  maps  $f_1, \dots, f_n$  exists, as there may be nontrivial obstructions to extending the given map  $\bigvee_{i=1}^n \Sigma Y_i \rightarrow Z$  to the fat wedge. Even if such an extension exists, there may be many inequivalent choices of an extension, implying that the homotopy class of  $[f_1, \dots, f_n]$  need not be uniquely determined by the homotopy classes of  $f_1, \dots, f_n$ .

When  $n = 2$ , the adjoint of the Whitehead product  $[f_1, f_2]$  is homotopic to a Samelson product. Its image in homology is given by commutators. We wish to have analogous information about higher Whitehead products. The universal example is given by the adjoint of  $\phi_n$ , which is a map  $\Sigma^{n-2} Y_1 \wedge \dots \wedge Y_n \rightarrow \Omega FW(\underline{\Sigma Y})$ . We want to know the Hurewicz image of this map. To do so we need a good model for  $H_*(\Omega FW(\underline{\Sigma Y}))$  which sees this Hurewicz image. Producing such a model in the case when each  $Y_i$  is a sphere is the subject of Section 4.

Before getting to this, we give a general result which identifies nontrivial higher Whitehead products in  $DJ_K(\underline{\Sigma Y})$  for any simplicial complex  $K$ . In short, there is a nontrivial higher Whitehead product for each missing face of  $K$  which, moreover, lifts to  $\mathcal{Z}_K(\underline{\Sigma Y})$ . In what follows we will consider sub-products  $\prod_{j=1}^k \Sigma Y_{i_j}$  of  $\prod_{i=1}^n \Sigma Y_i$ . If  $\sigma = (i_1, \dots, i_k)$ , let  $FW(\underline{\Sigma Y}, \sigma)$  be the fat wedge of the sub-product  $\prod_{j=1}^k \Sigma Y_{i_j}$ .

**Lemma 3.2.** *Let  $K$  be a simplicial complex on  $n$  vertices. If  $\sigma = (i_1, \dots, i_k) \in MF(K)$  then there is a map  $FW(\underline{\Sigma Y}, \sigma) \rightarrow DJ_K(\underline{\Sigma Y})$  and the composite*

$$\Sigma^{k-1} Y_{i_1} \wedge \dots \wedge Y_{i_k} \xrightarrow{\phi_k} FW(\underline{\Sigma Y}, \sigma) \rightarrow DJ_K(\underline{\Sigma Y})$$

*is nontrivial.*

*Proof.* Since  $\sigma = (i_1, \dots, i_k) \in MF(K)$ , every proper subsequence of it is a face of  $K$  but  $\sigma$  itself is not in  $K$ . This implies that there is a map  $FW(\underline{\Sigma Y}, \sigma) \rightarrow DJ_K(\underline{\Sigma Y})$ . Now suppose that the composite  $\Sigma^{k-1} Y_{i_1} \wedge \dots \wedge Y_{i_k} \xrightarrow{\phi_k} FW(\underline{\Sigma Y}, \sigma) \rightarrow DJ_K(\underline{\Sigma Y})$  is trivial. Then the map  $FW(\underline{\Sigma Y}, \sigma) \rightarrow DJ_K(\underline{\Sigma Y})$  extends to a map  $\prod_{j=1}^k \Sigma Y_{i_j} \rightarrow DJ_K(\underline{\Sigma Y})$ , which implies that  $\sigma = (i_1, \dots, i_k)$  is in  $K$ , a contradiction.  $\square$



**Lemma 3.3.** *The nontrivial map  $\Sigma^{k-1}Y_{i_1} \wedge \cdots \wedge Y_{i_k} \xrightarrow{\phi_k} FW(\underline{\Sigma Y}, \sigma) \rightarrow DJ_K(\underline{\Sigma Y})$  in Lemma 3.2 lifts to  $\mathcal{Z}_K(\underline{\Sigma Y})$ .*

*Proof.* The inclusion  $FW(\underline{\Sigma Y}, \sigma) \rightarrow DJ_K(\underline{\Sigma Y})$  is coordinate-wise, so the composite of inclusions  $FW(\underline{\Sigma Y}, \sigma) \rightarrow DJ_K(\underline{\Sigma Y}) \rightarrow \prod_{i=1}^n \Sigma Y_i$  is homotopic to the composite of inclusions  $FW(\underline{\Sigma Y}, \sigma) \rightarrow \prod_{j=1}^k \Sigma Y_{i_j} \rightarrow \prod_{i=1}^n \Sigma Y_i$ . Thus there is an induced homotopy fibration diagram

$$\begin{array}{ccccc} F & \longrightarrow & FW(\underline{\Sigma Y}, \sigma) & \longrightarrow & \prod_{j=1}^k \Sigma Y_{i_j} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Z}_K(\underline{\Sigma Y}) & \longrightarrow & DJ_K(\underline{\Sigma Y}) & \longrightarrow & \prod_{i=1}^n \Sigma Y_i \end{array}$$

where  $F \simeq \Sigma^{k-1}\Omega \Sigma Y_{i_1} \wedge \cdots \wedge \Omega \Sigma Y_{i_k}$ . By definition,  $\phi_k$  factors through  $F$ , and so the lemma follows.  $\square$

#### 4. ADAMS-HILTON MODELS

Let  $X$  be a simply-connected  $CW$ -complex of finite type and  $R$  a commutative ring. The Adams-Hilton model [AH] is a means of calculating  $H_*(\Omega X; R)$ . One advantage it has over other models for  $H_*(\Omega X; R)$  is its relative simplicity, which allows for concrete calculations in certain cases. Its presentation is stated in Theorem 4.1.

Let  $V$  be a graded  $R$ -module, and let  $T(V)$  be the free tensor algebra on  $V$ . For a space  $X$ , let  $CU_*(X)$  be the cubical singular chain complex on  $X$  with coefficients in  $R$ . Note that  $CU_*(X)$  is naturally chain equivalent to the simplicial singular chain complex on  $X$ . If  $X$  is a homotopy associative  $H$ -space then the multiplication on  $X$  induces a multiplication on  $CU_*(X)$ , giving it the structure of a differential graded algebra. A map  $A \rightarrow B$  of differential graded algebras is a *quasi-isomorphism* if it induces an isomorphism in homology.

**Theorem 4.1.** *Let  $R$  be a commutative ring and let  $X$  be a simply-connected  $CW$ -complex of finite type. The Adams-Hilton model for  $X$  is a differential graded  $R$ -algebra  $AH(X)$  satisfying:*

- (a) *if  $X = pt \cup (\bigcup_{\alpha \in S} e_\alpha)$  is a  $CW$ -decomposition of  $X$  then  $AH(X) = T(V; d_V)$  where  $V = \{b_\alpha\}_{\alpha \in S}$  and  $|b_\alpha| = |e_\alpha| - 1$ ;*
- (b) *the differential  $d_V$  depends on the attaching maps of the  $CW$ -complex  $X$ ;*
- (c) *there is a map of differential graded algebras  $\theta_X: AH(X) \rightarrow CU_*(\Omega X)$  which induces an isomorphism  $H_*(AH(X)) \cong H_*(\Omega X; R)$ .*

$\square$

Notice that the generators of  $AH(X)$  are in one-to-one correspondence with the cells of  $X$ , shifted down by one dimension. However, the differential  $d_V$  and the quasi-isomorphism  $\theta_X$  are not uniquely determined by the  $CW$ -structure of  $X$ . There may be many inequivalent choices of both  $d_V$  and  $\theta_X$

which result in an isomorphism  $H_*(AH(X)) \cong H_*(\Omega X; R)$ . In that sense, there may be many Adams-Hilton models for  $H_*(\Omega X; R)$ . One would hope to choose a model which is particularly advantageous. This is what we aim to do for  $X = DJ_K(\underline{S})$  or  $DJ_K$  and  $R = \mathbb{Q}$ , by choosing a model which keeps track of the Hurewicz images of adjointed higher Whitehead products.

We start with some general constructions in the case of  $DJ_K(S^2)$  and  $DJ_K$ , producing Adams-Hilton models for both which are compatible with the inclusion  $S^2 \xrightarrow{\iota} \mathbb{C}P^\infty$  of the bottom cell. The model will then be generalized to  $DJ_K(\underline{S})$ , but without the need for an accompanying map.

By definition, for  $\sigma = (i_1, \dots, i_k)$ , let  $S^\sigma = \prod_{j=1}^k S_{i_j}^2$ , with the lower index recording coordinate position, and let  $DJ_K(S^2) = \bigcup_{\sigma \in K} S^\sigma$ . Similarly, regarding  $\mathbb{C}P^\infty$  as  $BT$  where  $T = S^1$ , by definition  $DJ_K = \bigcup_{\sigma \in K} BT^\sigma$ , where  $BT^\sigma = BT_{i_1} \times \dots \times BT_{i_k}$ , again with the lower index recording coordinate position. Let  $\iota^\sigma : S^\sigma \rightarrow BT^\sigma$  be the product map  $\prod_{j=1}^k \iota$ . Then the map  $DJ_K(\underline{S}) \xrightarrow{DJ_K(\iota)} DJ_K$  is, by definition,  $\bigcup_{\sigma \in K} \iota^\sigma$ .

Many useful properties of Adams-Hilton models were proved in [AH]; a nice summary can be found in [An, 8.1]. First, an Adams-Hilton model of a  $CW$ -subspace can be extended to one for the whole space. Start with the inclusion of the bottom cell  $S^2 \xrightarrow{\iota} \mathbb{C}P^\infty$ . Then an Adams-Hilton model for  $S^2$  can be extended to one for  $\mathbb{C}P^\infty$ . Second, an Adams-Hilton model for a product  $AH(X \times Y)$  can be chosen so that it is quasi-isomorphic to  $A(X) \otimes A(Y)$ , and this respects the quasi-isomorphisms  $\theta_{X \times Y}$  and  $\theta_X \otimes \theta_Y$  to the respective cubical singular chain complexes. In our case, this lets us take the given model  $AH(S^2)$  for  $S^2$  and its extension  $AH(\mathbb{C}P^\infty)$  for  $\mathbb{C}P^\infty$  and produce a model for  $S^\sigma$  mapping to  $BT^\sigma$  which, up to quasi-isomorphisms, is  $AH(S^2)^{\otimes \sigma}$  mapping factor-wise to  $AH(\mathbb{C}P^\infty)^{\otimes \sigma}$ . Third, Adams-Hilton models preserve colimits, given coherency conditions. That is, if  $\{X_\alpha\}$  is a family of  $CW$ -subcomplexes of  $X$  and  $X = \bigcup_\alpha X_\alpha$ , and there are models  $AH(X_\alpha)$  satisfying the coherency conditions  $d_{V_\alpha}|_{AH(X_\alpha \cap X_\beta)} = d_{V_\beta}|_{AH(X_\alpha \cap X_\beta)}$  and  $\theta_{X_\alpha}|_{AH(X_\alpha \cap X_\beta)} = \theta_{X_\beta}|_{AH(X_\alpha \cap X_\beta)}$  for all pairs  $(\alpha, \beta)$ , then  $\text{colim}_\alpha AH(X_\alpha)$  is an Adams-Hilton model for  $X$ . In our case, we have  $DJ_K(\underline{S}) \xrightarrow{DJ_K(\iota)} DJ_K$  equalling, by definition,  $\bigcup_{\sigma \in K} S^\sigma \xrightarrow{\bigcup_{\sigma \in K} \iota^\sigma} \bigcup_{\sigma \in K} BT^\sigma$ . Notice that intersection  $S^{\sigma_1} \cap S^{\sigma_2}$  is again a sub-product, namely  $S^{\sigma_1 \cap \sigma_2}$ . Similarly,  $BT^{\sigma_1} \cap BT^{\sigma_2} = BT^{\sigma_1 \cap \sigma_2}$ . Thus the compatibility of Adams-Hilton models with products implies that the coherency conditions will be satisfied for  $\bigcup_{\sigma \in K} S^\sigma$  and  $\bigcup_{\sigma \in K} BT^\sigma$ , and for the map  $\bigcup_{\sigma \in K} \iota^\sigma$ . Hence we have

$$\begin{array}{ccc} AH(DJ_K(\underline{S})) & \xrightarrow{=} & \text{colim}_{\sigma \in K} AH(S^\sigma) \\ \downarrow AH(DJ_K(\iota)) & & \downarrow \text{colim}_{\sigma \in K} AH(\iota^\sigma) \\ AH(DJ_K) & \xrightarrow{=} & \text{colim}_{\sigma \in K} AH(BT^\sigma). \end{array}$$

Since homology commutes with colimits, after taking homology we obtain the following.

**Proposition 4.2.** *Let  $K$  be a simplicial complex on  $n$  vertices. There is a commutative diagram of algebras*

$$\begin{array}{ccc} H_*(\Omega DJ_K(S^2); \mathbb{Q}) & \xrightarrow{\cong} & \text{colim}_{\sigma \in K} H_*(\Omega S^\sigma; \mathbb{Q}) \\ \downarrow (\Omega DJ_K(\iota))_* & & \downarrow \text{colim}_{\sigma \in K} (\Omega \iota^\sigma)_* \\ H_*(\Omega DJ_K; \mathbb{Q}) & \xrightarrow{\cong} & \text{colim}_{\sigma \in K} H_*(\Omega BT^\sigma; \mathbb{Q}). \end{array}$$

□

Next, we consider an analogue of Proposition 4.2 with respect to  $MF$ -complexes and fat wedges. To distinguish fat wedges, given  $\sigma = (i_1, \dots, i_k)$ , let  $FW(S^2, \sigma)$  be the fat wedge of  $\prod_{j=1}^k S_{i_j}^2$ , where the lower index refers to coordinate position. Let  $FW(\sigma)$  be the fat wedge of  $\prod_{j=1}^k \mathbb{C}P_{i_j}^\infty$ . Let  $\iota^\sigma: FW(S^2, \sigma) \rightarrow FW(\sigma)$  be the map of fat wedges induced by  $\iota$ . Note that  $FW(S^2, \sigma) = \bigcup_{\tau \in (\Delta^k)_{k-1}} S^\tau$  and  $FW(\sigma) = \bigcup_{\tau \in (\Delta^k)_{k-1}} BT^\tau$ . Suppose  $K$  is an  $MF$ -complex on  $n$  vertices. Then

$$DJ_K(\underline{S}) = \bigcup_{\sigma \in MF(K)} FW(S^2, \sigma) = \bigcup_{\sigma \in MF(K)} \bigcup_{\tau \in (\Delta^k)_{k-1}} S^\tau$$

is simply a reorganization of the union in  $DJ_K(\underline{S}) = \bigcup_{\tau \in K} S^\tau$ . Crucially, observe that if  $\sigma_1 \neq \sigma_2$  then the intersection  $FW(S^2, \sigma_1) \cap FW(S^2, \sigma_2)$  is a sub-product of  $\prod_{i=1}^n S^2$ . Thus, under the reorganization of the union, the coherency conditions for the Adams-Hilton model will be satisfied. The same is true for  $DJ_K$  and the map  $DJ_K(\iota)$ . Thus, in this case, Proposition 4.2 can be reformulated as follows.

**Proposition 4.3.** *Let  $K$  be an  $MF$ -complex. There is a commutative diagram of algebras*

$$\begin{array}{ccc} H_*(\Omega DJ_K(S^2); \mathbb{Q}) & \xrightarrow{\cong} & \text{colim}_{\sigma \in MF(K)} H_*(\Omega FW(S^2, \sigma); \mathbb{Q}) \\ \downarrow (\Omega DJ_K(\iota))_* & & \downarrow \text{colim}_{\sigma \in MF(K)} (\Omega \iota^\sigma)_* \\ H_*(\Omega DJ_K; \mathbb{Q}) & \xrightarrow{\cong} & \text{colim}_{\sigma \in MF(K)} H_*(\Omega FW(\sigma); \mathbb{Q}) \end{array}$$

□

Proposition 4.3 reduces the problem of calculating  $H_*(\Omega DJ_K(S^2); \mathbb{Q})$  and  $H_*(\Omega DJ_K; \mathbb{Q})$  to that of calculating the rational homology of looped fat wedges – with the proviso that the underlying model for the fat wedges must be compatible with the inclusion of sub-products. In our case we want more, that the homology of the looped fat wedges also keeps track of the Hurewicz images of adjointed higher Whitehead products. We will discuss this further in the next section.

Observe that the argument for producing an Adams-Hilton model of  $DJ_K(S^2)$  above is equally valid for  $DJ_K(\underline{X})$ , where  $\underline{X} = \{X_1, \dots, X_n\}$ . The cases we particularly care about are  $\underline{S} = \{S^{m_1+1}, \dots, S^{m_n+1}\}$ , with the special case of  $DJ_K(S^2)$ , and the case of  $DJ_K$ . The focus is on these cases as they give models which can be explicitly calculated. We do so for  $DJ_K(\underline{S})$  in Section 6 and  $DJ_K$  in Section 7. For future reference, we state the case for  $\underline{S}$ .

**Proposition 4.4.** *Let  $K$  be an MF-complex. An Adams-Hilton model for  $DJ_K(\underline{S})$  is*

$$AH(DJ_K(\underline{S})) = \operatorname{colim}_{\sigma \in MF(K)} AH(FW(\sigma)).$$

□

## 5. AN ADAMS-HILTON MODEL FOR $FW(\underline{S})$

We are aiming for an Adams-Hilton model for  $FW(\underline{S})$  over  $\mathbb{Q}$  which is compatible with the inclusion of sub-products, and which keeps track of the Hurewicz images of adjointed higher Whitehead products. We will obtain one by using Allday's construction of a minimal Quillen model for  $\pi_*(\Omega FW(\underline{S})) \otimes \mathbb{Q}$  and then using this to produce an Adams-Hilton model for  $H_*(\Omega FW(\underline{S}); \mathbb{Q})$ .

We begin with some general statements which hold for any path-connected space  $X$ . Assume from now on that the ground ring  $R$  is  $\mathbb{Q}$ . Observe that  $\pi_*(X)$  can be given the structure of a graded Lie algebra by using the Whitehead product to define the bracket. Equivalently, by adjoining,  $\pi_*(\Omega X)$  may be given the structure of a graded Lie algebra by using the Samelson product. Quillen [Q] associated to  $X$  a free differential graded Lie algebra  $\lambda(X)$  over  $\mathbb{Q}$  with the property that there is an isomorphism  $H_*(\lambda(X)) \rightarrow \pi_*(\Omega X) \otimes \mathbb{Q}$ . The free property of  $\lambda(X)$  lets us write it as  $L\langle V; d_V \rangle$  for some graded  $\mathbb{Q}$ -module  $V$  and differential  $d_V$  on  $V$ . A Quillen model  $MQ(X)$  is *minimal* if the differential has the property that  $d(L\langle V \rangle) \subseteq [L\langle V \rangle, L\langle V \rangle]$ .

Allday [Al] gave an explicit construction of a minimal Quillen model for  $\pi_*(\Omega FW(\underline{S})) \otimes \mathbb{Q}$ . This is stated in Theorem 5.1 once some notation has been introduced. The cells of  $FW(\underline{S})$  are in one-to-one correspondence with sequences  $(i_1, \dots, i_k)$  where  $1 \leq i_1 < \dots < i_k \leq n$  and  $k < n$ . The sequence  $(i_1, \dots, i_k)$  corresponds to the top cell of the coordinate subspace  $S^{m_{i_1}+1} \times \dots \times S^{m_{i_k}+1}$  within  $FW(\underline{S})$ . This cell has dimension  $\sum_{s=1}^k (m_{i_s} + 1)$ . Note that the condition  $k < n$  excludes only one sequence,  $(1, 2, \dots, n)$ , corresponding to the top cell of the product  $S^{m_1+1} \times \dots \times S^{m_n+1}$ . Allday's minimal Quillen model for  $\pi_*(\Omega FW(\underline{S})) \otimes \mathbb{Q}$  is of the form

$$MQ(FW(\underline{S})) = L\langle V; d_V \rangle$$

where  $V$  has one generator  $b_I$  for each sequence  $I = (i_1, \dots, i_k)$  with  $1 \leq i_1 < \dots < i_k \leq n$  and  $k < n$ , and the degree of  $b_I$  is  $(\sum_{s=1}^k (m_{i_s} + 1)) - 1$ . Similarly, his minimal Quillen model for  $\pi_*(\Omega \prod_{i=1}^n S^{m_i+1}) \otimes \mathbb{Q}$  is of the form

$$MQ\left(\prod_{i=1}^n S^{m_i+1}\right) = L\langle W; d_W \rangle$$

where  $W$  has one generator  $b_I$  for each sequence  $I = (i_1, \dots, i_k)$  with  $1 \leq i_1 < \dots < i_k \leq n$ , and the degree of  $b_I$  is  $(\sum_{s=1}^k (m_{i_s} + 1)) - 1$ .

To describe the differentials  $d_V$  and  $d_W$ , fix a sequence  $I = (i_1, \dots, i_k)$  where  $1 \leq i_1 < \dots < i_k \leq n$  and  $k \geq 2$ . If  $k < n$  this corresponds to a generator  $b_I$  of  $V$ , and if  $k = n$  this corresponds to a generator  $b_I$  of  $W$ . In either case, the degree of  $b_I$  is  $|b_I| = (\sum_{s=1}^k m_{i_s} + 1) - 1$ . Let  $\mathcal{S}_I$  be the collection

of all shuffles  $(J, J')$  of  $\{i_1, \dots, i_k\}$  with the property that  $j_1 = 1$  (known as a type II shuffle relative to 1). If  $(J, J')$  is an  $(r, s)$ -shuffle of  $\{1, \dots, k\}$ , let  $\epsilon(J, J') \in \{0, 1\}$  be the number determined by the equation  $z_{i_1} \cdots z_{i_k} = (-1)^{\epsilon(J, J')} z_{j_1} \cdots z_{j_r} z_{j'_1} \cdots z_{j'_s}$  in the graded rational symmetric algebra generated by  $z_{i_1}, \dots, z_{i_k}$  with  $|z_{i_t}| = m_{i_t} + 1$  for  $1 \leq t \leq k$ . Let

$$a_I = -\sum_{(J, J') \in \mathcal{S}_I} (-1)^{|b_I| + \epsilon(J, J')} [b_J, b_{J'}].$$

As special cases, let  $b = b_{(1, \dots, n)}$  and  $a = a_{(1, \dots, n)}$ .

**Theorem 5.1.** *With  $V$  and  $W$  as defined above, minimal Quillen models  $L\langle V; d_V \rangle$  and  $L\langle W; d_W \rangle$  for  $\pi_*(\Omega FW(\underline{\mathcal{S}})) \otimes \mathbb{Q}$  and  $\pi_*(\prod_{i=1}^n \Omega S^{m_i+1}) \otimes \mathbb{Q}$  can be chosen to satisfy the following properties:*

- (a)  $W = V \oplus \{b\}$ ;
- (b)  $d_V(b_I) = 0$  if  $I = (i)$  for  $1 \leq i \leq n$ ;
- (c)  $d_V(b_I) = a_I$  for  $I = (i_1, \dots, i_k)$  with  $2 \leq k < n$ ;
- (d)  $d_W$  restricted to  $V$  is  $d_V$ ;
- (e)  $d_W(b) = a$ ;
- (f) the adjoint of the higher order Whitehead product  $S^{|b|-1} \xrightarrow{\phi_n} FW(\underline{\mathcal{S}})$  which attaches the top cell to the product  $\prod_{i=1}^n S^{m_i+1}$  is homotopic to  $a$ .

□

There is an explicit map  $\alpha: L\langle V \rangle \rightarrow \pi_*(\Omega FW(\underline{\mathcal{S}}))$ . Let  $b_I \in V$  for  $I = (i_1, \dots, i_k)$ . This corresponds to the top cell of the coordinate subspace  $S^{m_{i_1}+1} \times \cdots \times S^{m_{i_k}+1}$  in  $FW(\underline{\mathcal{S}})$ . Let  $FW(i_1, \dots, i_k)$  be the fat wedge in  $S^{m_{i_1}+1} \times \cdots \times S^{m_{i_k}+1}$ . Let  $\alpha(b_I)$  be the adjoint of the composite  $S^{|b_I|-1} \xrightarrow{\phi_k} FW(i_1, \dots, i_k) \rightarrow \prod_{j=1}^k S^{m_{i_j}+1} \rightarrow FW(\underline{\mathcal{S}})$ , where the latter two maps are the inclusions. Now extend  $\alpha$  to  $L\langle V \rangle$  by using the fact that  $\pi_*(\Omega FW(\underline{\mathcal{S}})) \otimes \mathbb{Q}$  is a Lie algebra under the Samelson product. Allday's statement that  $L\langle V; d_V \rangle$  is a minimal Quillen model for  $\pi_*(\Omega FW(\underline{\mathcal{S}})) \otimes \mathbb{Q}$  says two things: first, that  $\alpha$  can be upgraded from a map of Lie algebras to a map of differential graded Lie algebras, where the differential on  $\pi_*(\Omega FW(\underline{\mathcal{S}})) \otimes \mathbb{Q}$  is zero, and second, that this upgraded map induces an isomorphism in homology. A similar construction can be made with respect to  $\prod_{i=1}^n S^{m_i+1}$ .

We now pass from a minimal Quillen model to an Adams-Hilton model. In general, observe that the Hurewicz homomorphism  $\pi_*(\Omega X) \otimes \mathbb{Q} \rightarrow H_*(\Omega X; \mathbb{Q})$  factors as the composite  $\pi_*(\Omega X) \otimes \mathbb{Q} \xrightarrow{c} CU_*(\Omega X) \xrightarrow{h} H_*(\Omega X; \mathbb{Q})$ , where  $CU_*(\Omega X)$  is the cubical singular chain complex with coefficients in  $\mathbb{Q}$ ,  $c$  is the canonical map to the cubical chains, and  $h$  is the quotient map to the homology of the chain complex. Let  $MQ(X)$  be a minimal Quillen model for  $\pi_*(\Omega X) \otimes \mathbb{Q}$ , and suppose there is an associated map of differential graded Lie algebras  $\alpha: MQ(X) = L\langle V_X; d_{V_X} \rangle \rightarrow \pi_*(\Omega X) \otimes \mathbb{Q}$  which induces an isomorphism in homology. Since  $CU_*(\Omega X)$  is a differential graded algebra, the composite  $c \circ \alpha$  extends to a map  $\theta_X: UL\langle V_X; d_{V_X} \rangle \rightarrow CU_*(\Omega X)$  of differential graded algebras.

Thus there is a commutative diagram

$$(2) \quad \begin{array}{ccccc} L\langle V_X; d_{V_X} \rangle & \xrightarrow{i} & UL\langle V_X; d_{V_X} \rangle & & \\ \downarrow \alpha & & \downarrow \theta_X & & \\ \pi_*(\Omega X) \otimes \mathbb{Q} & \xrightarrow{c} & CU_*(\Omega X) & \xrightarrow{h} & H_*(\Omega X; \mathbb{Q}) \end{array}$$

where  $i$  is the inclusion. By Milnor-Moore [MM], regarding  $\pi_*(\Omega X) \otimes \mathbb{Q}$  as a Lie algebra, we have  $H_*(\Omega X; \mathbb{Q}) \cong U(\pi_*(\Omega X) \otimes \mathbb{Q})$ , with the isomorphism induced by the Hurewicz homomorphism. On the other hand,  $h$  is a map of differential graded algebras once  $H_*(\Omega X; \mathbb{Q})$  has been given the zero differential. Thus  $h \circ \theta_X$  is a map of differential graded algebras, and therefore it is determined by its restriction to the generating set  $V$ . The commutativity of the diagram implies that  $h \circ \theta_X|_V = h \circ c \circ \alpha|_V =: q$ . Thus  $h \circ \theta_X = U(q)$ , implying that  $h \circ \theta_X$  induces an isomorphism in rational homology. Hence  $UL\langle V_X; d_{V_X} \rangle$  together with the quasi-isomorphism  $\theta_X$  is an Adams-Hilton model for  $X$ .

In our case, we obtain Adams-Hilton models  $(UL\langle V; d_V \rangle, \theta_{FW})$  and  $(UL\langle W; d_W \rangle, \theta_{\Pi})$  for  $FW(\underline{S})$  and  $\prod_{i=1}^n S^{m_i+1}$ , respectively. Theorem 5.1 therefore implies the following.

**Theorem 5.2.** *The Adams-Hilton models  $AH(FW(\underline{S})) = (UL\langle V; d_V \rangle, \theta_{FW})$  and  $AH(\prod_{i=1}^n S^{m_i+1}) = (UL\langle W; d_W \rangle, \theta_{\Pi})$  have the following properties:*

- (a)  $W = V \oplus \{b\}$ ;
- (b)  $d_V(b_I) = 0$  if  $I = (i)$  for  $1 \leq i \leq n$ ;
- (c)  $d_V(b_I) = a_I$  for  $I = (i_1, \dots, i_k)$  with  $2 \leq k < n$ ;
- (d)  $d_W$  restricted to  $AH(FW(\underline{S}))$  is  $d_V$ ;
- (e)  $d_W(b) = a$ ;
- (f) the adjoint of the higher order Whitehead product  $S^{|b|-1} \xrightarrow{\phi_n} FW(\underline{S})$  which attaches the top cell to the product  $\prod_{i=1}^n S^{m_i+1}$  has “Hurewicz” image  $a$ .

□

**Remark 5.3.** The inductive definition of the differential  $d_V$  in the minimal Quillen model  $L\langle V; d_V \rangle$  for  $FW(\underline{S})$  in Theorem 5.2 implies that the differential is compatible with the inclusion of sub-products. The same is therefore true in  $UL\langle V; d_V \rangle$ . Moreover, the differential  $d_V$  is what turns the map  $\alpha$  into a map of differential graded Lie algebras, and so both  $\alpha$  and its extension to the quasi-isomorphism  $\theta_{FW}$  are compatible with the inclusion of sub-products.

As well as the inductive nature of the Adams-Hilton model, Theorem 5.2 also explicitly identifies the “Hurewicz” image of the adjoint of the higher order Whitehead product  $\phi_n$ . We put Hurewicz in quotes as this image is an element in an Adams-Hilton model, whereas the honest Hurewicz image is obtained after taking homology. That is,  $a$  is a cycle in  $AH(FW(\underline{S}))$ , which could also be a boundary. This turns out not to be the case. Observe that there is a sequence of isomorphisms

$H_*(AH(FW(\underline{S}))) \cong H_*(UL\langle V; d_V \rangle) \cong U(H_*(L\langle V; d_V \rangle))$ , since homology commutes with the universal enveloping algebra functor. To calculate  $H_*(L\langle V; d_V \rangle)$  we proceed exactly as in [B], where Bubenik used separated Lie models to elegantly obtain the answer. This is stated in Theorem 5.4 in terms of the universal enveloping algebra rather than the Lie algebra as we are ultimately after  $H_*(\Omega FW(\underline{S}); \mathbb{Q})$ . To state the result we need to introduce more notation. For  $1 \leq i \leq n$ , let  $b_i$  be the generator in  $V$  (or  $W$ ) corresponding to the sequence  $I = (i)$ . That is,  $b_i$  corresponds to the cell  $S^{m_i+1}$  in  $S^{m_1+1} \times \dots \times S^{m_n+1}$ . Let  $N = (\sum_{i=1}^k m_i + 1) - 2$ . In general, for a  $\mathbb{Q}$ -module  $M$ , let  $L_{ab}\langle M \rangle$  be the free abelian Lie algebra generated by  $M$ . That is, the bracket is identically zero in  $L_{ab}\langle M \rangle$ . Note that  $H_*(AH(\prod_{j=1}^k S^{m_j+1})) \cong UL_{ab}\langle b_1, \dots, b_n \rangle$ .

**Theorem 5.4.** *For  $n \geq 3$ , there are algebra isomorphisms*

$$H_*(\Omega FW(\underline{S}); \mathbb{Q}) \cong H_*(AH(FW(\underline{S}))) \cong U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u \rangle)$$

where  $u$ , of degree  $N$ , is the Hurewicz image of the adjoint of the higher Whitehead product  $\phi_n$ . Further, the looped inclusion  $\Omega FW(\underline{S}) \rightarrow \prod_{i=1}^n \Omega S^{m_i+1}$  is modelled by the map

$$U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u \rangle) \xrightarrow{U(\pi)} UL_{ab}\langle b_1, \dots, b_n \rangle$$

where  $\pi$  is the projection. □

Note that the calculation of  $H_*(\Omega FW(\underline{S}); \mathbb{Q})$  is not new, it was originally done by Lemaire [L]. What is important to keep in mind about Theorem 5.4 is that the calculation also keeps track of the Hurewicz image of the adjointed higher Whitehead product  $\phi_n$ .

**Remark 5.5.** When  $n = 2$ , we have  $FW(\underline{S}) = S^{m_1+1} \vee S^{m_2+1}$  and then it is well known that  $H_*(\Omega FW(\underline{S}); \mathbb{Q}) \cong H_*(\Omega(S^{m_1+1} \vee S^{m_2+1}); \mathbb{Q}) \cong UL\langle b_1, b_2 \rangle$ . In this case  $\phi_2$  is the ordinary Whitehead product and its adjoint has Hurewicz image  $u = [b_1, b_2]$ . In this case we can regard  $L\langle b_1, b_2 \rangle$  as  $L_{ab}\langle b_1, b_2 \rangle \amalg L\langle u \rangle$ , modulo Jacobi identities on brackets of the form  $[u, -] = [[b_1, b_2], -]$ .

## 6. PROPERTIES OF $\Omega DJ_K(\underline{S})$ AND $\Omega Z_K(\underline{S})$ FOR $MF$ -COMPLEXES

In this section we explicitly calculate  $H_*(\Omega DJ_K(\underline{S}); \mathbb{Q})$  when  $K$  is an  $MF$ -complex, proving Theorem 1.4. This is then used in tandem with the loops on the homotopy fibration  $Z_K(\underline{S}) \xrightarrow{f} DJ_K(\underline{S}) \xrightarrow{g} \prod_{i=1}^n S^{m_i+1}$  to calculate  $H_*(\Omega Z_K(\underline{S}); \mathbb{Q})$ . We then give a homotopy decomposition of  $Z_K(\underline{S})$  as a wedge of spheres and describe the map  $Z_K(\underline{S}) \rightarrow DJ_K(\underline{S})$  in terms of higher Whitehead products and iterated Whitehead products, proving Theorem 1.6.

**Remark 6.1.** To simplify the presentation, for the remainder of Sections 6 and 7 we will assume that the given  $MF$ -complex  $K$  has the property that  $|\sigma| > 1$  for every  $\sigma \in MF(K)$ . This is to appeal directly to Theorem 5.4. If  $|\sigma| = 1$  for some  $\sigma = (i_1, i_2) \in MF(K)$ , then the calculations can be modified by regarding  $L\langle b_{i_1}, b_{i_2} \rangle$  as  $L_{ab}\langle b_{i_1}, b_{i_2} \rangle \amalg L\langle u \rangle$  for  $u = [b_{i_1}, b_{i_2}]$  as in Remark 5.5, and by introducing the ideal  $J$  discussed in the Introduction.

We begin by calculating  $H_*(\Omega DJ_K(\underline{S}); \mathbb{Q})$  using the Adams-Hilton model

$$AH(DJ_K(\underline{S})) = \operatorname{colim}_{\sigma \in MF(K)} AH(FW(\sigma))$$

in Proposition 4.4. Let  $b_1, \dots, b_n$  be the generators in  $AH(DJ_K(\underline{S}))$  corresponding to the cells  $S^{m_1+1}, \dots, S^{m_n+1}$  respectively. For  $\sigma = (i_1, \dots, i_k) \in K$ , observe that  $\{b_{i_1}, \dots, b_{i_k}\}$  corresponds to the cells  $S^{m_{i_j}+1}$  which are in  $FW(\sigma)$ . Let  $N_\sigma = (\sum_{j=1}^k m_{i_j} + 1) - 2$ . By Theorem 5.4, we have  $H_*(AH(FW(\sigma))) \cong U(L_{ab}\langle b_{i_1}, \dots, b_{i_k} \rangle \amalg L\langle u_\sigma \rangle)$  where  $u_\sigma$  is the Hurewicz image of the adjoint of a higher Whitehead product  $S^{N_\sigma+1} \rightarrow FW(\sigma)$ .

**Proof of Theorem 1.4:** Consider the string of isomorphisms

$$\begin{aligned} H_*(\Omega DJ_K(\underline{S}); \mathbb{Q}) &\cong H_*(AH(DJ_K(\underline{S}))) \\ &\cong \operatorname{colim}_{\sigma \in MF(K)} H_*(AH(FW(\sigma))) \\ &\cong \operatorname{colim}_{\sigma \in MF(K)} U(L_{ab}\langle b_{i_1}, \dots, b_{i_k} \rangle \amalg L\langle u_\sigma \rangle) \\ &\cong U(\operatorname{colim}_{\sigma \in MF(K)} L_{ab}\langle b_{i_1}, \dots, b_{i_k} \rangle \amalg L\langle u_\sigma \rangle) \\ &\cong U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle). \end{aligned}$$

The first isomorphism exists because  $AH(DJ_K(\underline{S}))$  is an Adams-Hilton model. The second isomorphism exists because  $AH(DJ_K(\underline{S})) \cong \operatorname{colim}_{\sigma \in MF(K)} AH(FW(\sigma))$  by Proposition 4.4, and because homology commutes with colimits. The third isomorphism exists by Theorem 5.4. For the fourth isomorphism, Remark 5.3 implies that the calculation of  $H_*(AH(FW(\sigma))) \cong U(L_{ab}\langle b_{i_1}, \dots, b_{i_k} \rangle)$  is compatible with the inclusion of sub-products. Therefore both the underlying Lie algebra and its universal enveloping algebra respect the colimit over  $MF(K)$ . Hence the fourth isomorphism holds, and the fifth isomorphism is obtained by evaluating the colimit. This establishes the asserted isomorphism. The statement regarding Hurewicz images now follows from that in Theorem 5.4. The statement regarding the model for the looped inclusion follows again from Remark 5.3 regarding the compatibility of the colimit with the inclusion of sub-products.  $\square$

Theorem 1.4 is the crucial algebraic result. We first use it to determine  $H_*(\Omega \mathcal{Z}_K(\underline{S}); \mathbb{Q})$ , and then to determine a more detailed description of the Hurewicz homomorphism.

Since  $L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle$  is a coproduct, there is a short exact sequence of graded Lie algebras

$$(3) \quad L\langle R \rangle \xrightarrow{i} L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle \xrightarrow{\pi} L_{ab}\langle b_1, \dots, b_n \rangle$$

where  $i$  is the inclusion,  $\pi$  is the projection, and

$$R = \{[[u_\sigma, b_{j_1}], \dots, b_{j_l}] \mid \sigma \in MF(K), 1 \leq j_1 \leq \dots \leq j_l \leq n, 0 \leq l < \infty\}.$$

Here, when  $l = 0$  we interpret the bracket as simply being  $u_\sigma$ .



**Proposition 6.2.** *There is a commutative diagram of algebras*

$$\begin{array}{ccc} H_*(\Omega\mathcal{Z}_K(\underline{S}); \mathbb{Q}) & \xrightarrow{(\Omega f)_*} & H_*(\Omega DJ_K(\underline{S}); \mathbb{Q}) \\ \downarrow \cong & & \downarrow \cong \\ UL\langle R \rangle & \xrightarrow{U(i)} & U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle). \end{array}$$

*Proof.* By [CMN, 3.7], a short exact sequence of graded Lie algebras induces a short exact sequence of Hopf algebras. In our case, (3) induces a short exact sequence

$$UL\langle R \rangle \xrightarrow{U(i)} U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle) \xrightarrow{U(\pi)} UL_{ab}\langle b_1, \dots, b_n \rangle.$$

Here, by a short exact sequence of Hopf algebras, we mean that there is an isomorphism

$$U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle) \cong UL_{ab}\langle b_1, \dots, b_n \rangle \otimes UL\langle R \rangle$$

as right  $UL\langle R \rangle$ -modules and left  $UL_{ab}\langle b_1, \dots, b_n \rangle$ -comodules. In particular,  $U(i)$  is the algebra kernel of  $U(\pi)$ . On the other hand, Theorem 1.4 implies that  $U(\pi)$  is a model for the looped inclusion  $\Omega DJ_K(\underline{S}) \xrightarrow{\Omega g} \prod_{i=1}^n \Omega S^{m_i+1}$ . Since  $\Omega g$  has a right homotopy inverse, the homotopy decomposition  $\Omega DJ_K(\underline{S}) \simeq (\prod_{i=1}^n \Omega S^{m_i+1}) \times \Omega\mathcal{Z}_K(\underline{S})$  implies that there is a short exact sequence of Hopf algebras  $H_*(\Omega\mathcal{Z}_K(\underline{S}); \mathbb{Q}) \longrightarrow H_*(\Omega DJ_K(\underline{S}); \mathbb{Q}) \xrightarrow{U(\pi)} H_*(\prod_{i=1}^n \Omega S^{m_i+1}; \mathbb{Q})$ . Thus  $H_*(\Omega\mathcal{Z}_K(\underline{S}); \mathbb{Q})$  is also the algebra kernel of  $U(\pi)$ , and the proposition follows.  $\square$

Theorem 1.4 implies that the rational homology of  $\Omega DJ_K(\underline{S})$  is generated by Hurewicz images. Specifically, for  $\sigma \in MF(K)$ , let

$$w_\sigma: S^{N_\sigma+1} \xrightarrow{\phi_k} FW(\sigma) \longrightarrow DJ_K(\underline{S})$$

be the higher Whitehead product. Let

$$s_\sigma: S^{N_\sigma} \longrightarrow \Omega DJ_K(\underline{S})$$

be the adjoint of  $w_\sigma$ . As stated in Theorem 1.4, the element  $u_\sigma \in H_*(\Omega DJ_K(\underline{S}); \mathbb{Q})$  is the Hurewicz image of  $s_\sigma$ . For  $1 \leq i \leq n$ , let

$$a_i: S^{m_i+1} \longrightarrow DJ_K(\underline{S})$$

be the coordinate inclusion and let

$$\bar{a}_i: S^{m_i} \longrightarrow \Omega DJ_K(\underline{S})$$

be the adjoint of  $a_i$ . The Hurewicz image of  $\bar{a}_i$  is  $b_i$ . Let  $\mathcal{I}$  be the index set for  $R$ . Then  $\alpha \in \mathcal{I}$  corresponds to a face  $\sigma \in MF(K)$  and a sequence  $(j_1, \dots, j_l)$  where  $1 \leq j_1 \leq \dots \leq j_l \leq n$  and  $0 \leq l < \infty$ . Given such an  $\alpha$ , let  $t_\alpha = (\sum_{t=1}^l m_{j_t}) + (N_\sigma - 2)$ . Then there is a Samelson product

$$[[s_\sigma, \bar{a}_{j_1}], \dots, \bar{a}_{j_l}]: S^{t_\alpha} \longrightarrow \Omega DJ_K(\underline{S}).$$

Since Samelson products commute with Hurewicz images, the Hurewicz image of  $[[s_\sigma, \bar{a}_{j_1}], \dots, \bar{a}_{j_i}]$  is  $[[u_\sigma, b_{j_1}], \dots, b_{j_i}]$ . Adjoining, we have a Whitehead product

$$[[w_\sigma, a_{j_1}], \dots, a_{j_i}]: S^{t_\alpha+1} \longrightarrow DJ_K(\underline{\mathcal{S}}).$$

Taking the wedge sum over all possible  $\alpha$ , we obtain a map

$$W: \bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha+1} \longrightarrow DJ_K(\underline{\mathcal{S}}).$$

**Corollary 6.3.** *The map  $\Omega(\bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha+1}) \xrightarrow{\Omega W} \Omega DJ_K(\underline{\mathcal{S}})$  induces in rational homology the map  $UL\langle R \rangle \xrightarrow{U(i)} U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle)$ .*

*Proof.* Let  $S$  be the composite

$$S: \bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha} \xrightarrow{E} \Omega\left(\bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha+1}\right) \xrightarrow{\Omega W} \Omega DJ_K(\underline{\mathcal{S}}).$$

Then  $S$  is homotopic to the adjoint of  $W$ . In particular, the wedge summands of  $S$  are the Samelson products  $[[s_\sigma, \bar{a}_{j_1}], \dots, \bar{a}_{j_i}]$  for  $\alpha \in \mathcal{I}$ . Thus, taking Hurewicz images,  $S_*$  is the composite

$$\bar{i}: R \hookrightarrow UL\langle R \rangle \xrightarrow{U(i)} U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle) \xrightarrow{\cong} H_*(\Omega DJ_K(\underline{\mathcal{S}})).$$

By the Bott-Samelson Theorem,  $H_*(\Omega(\bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha+1}); \mathbb{Q}) \cong T(\tilde{H}_*(\bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha}; \mathbb{Q}))$ , and the latter algebra is isomorphic to  $UL\langle R \rangle$ . Therefore, as  $(\Omega W)_*$  is the multiplicative extension of  $S_*$ , it induces the multiplicative extension  $U(i)$  of  $\bar{i}$ .  $\square$

Finally, we bring  $\mathcal{Z}_K(\underline{\mathcal{S}})$  back into the picture.

**Theorem 6.4.** *The map  $\bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha+1} \xrightarrow{W} DJ_K(\underline{\mathcal{S}})$  lifts to  $\mathcal{Z}_K(\underline{\mathcal{S}})$ , and induces a homotopy equivalence  $\bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha+1} \longrightarrow \mathcal{Z}_K(\underline{\mathcal{S}})$ .*

*Proof.* By Lemma 3.3, each higher Whitehead product  $w_\sigma$  lifts to  $\mathcal{Z}_K(\underline{\mathcal{S}})$ . Therefore each iterated Whitehead product  $[[w_\sigma, a_{j_1}], \dots, a_{j_i}]$  into  $DJ_K(\underline{\mathcal{S}})$  composes trivially to  $\prod_{i=1}^n S^{m_i+1}$  and so lifts to  $\mathcal{Z}_K(\underline{\mathcal{S}})$ . Hence there is a lift

$$\begin{array}{ccc} & \bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha+1} & \\ \lambda \swarrow & & \downarrow W \\ \mathcal{Z}_K(\underline{\mathcal{S}}) & \longrightarrow & DJ_K(\underline{\mathcal{S}}) \end{array}$$

for some map  $\lambda$ .

After looping, Corollary 6.3 implies that the map  $\Omega(\bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha+1}) \xrightarrow{\Omega \lambda} \Omega \mathcal{Z}_K(\underline{\mathcal{S}})$  induces an inclusion  $UL\langle R \rangle \xrightarrow{(\Omega \lambda)_*} H_*(\Omega \mathcal{Z}_K(\underline{\mathcal{S}}); \mathbb{Q})$ . By Proposition 6.2, there is an isomorphism  $H_*(\Omega \mathcal{Z}_K(\underline{\mathcal{S}}); \mathbb{Q}) \cong UL\langle R \rangle$ . Therefore a counting argument implies that the inclusion  $(\Omega \lambda)_*$  must be an isomorphism. Hence  $\Omega \lambda$  is a rational homotopy equivalence. That is,  $\Omega \lambda$  induces an isomorphism of rational homotopy groups. Therefore, so does  $\lambda$ , and so  $\lambda$  is a rational homotopy equivalence.

To upgrade this to an integral homotopy equivalence, observe that  $\mathcal{Z}_K(\underline{S})$  is homotopy equivalent to a wedge of simply-connected spheres by Corollary 1.3. Therefore  $\lambda$  is a map between two wedges of simply-connected spheres which is a rational homotopy equivalence. In particular,  $\lambda$  induces an isomorphism in rational homology. But as the integral homology of a wedge of spheres is torsion free, a map inducing an isomorphism in rational homology also induces an isomorphism in integral homology. Therefore, by Whitehead's theorem,  $\lambda$  is an integral homotopy equivalence.  $\square$

*Proof of Theorem 1.6:* This is simply a rephrasing of Theorem 6.4.  $\square$

### 7. PROPERTIES OF $\Omega DJ_K$ AND $\Omega \mathcal{Z}_K$ FOR $MF$ -COMPLEXES

Let  $\iota: S^2 \rightarrow \mathbb{C}P^\infty$  be the inclusion of the bottom cell. By naturality, there is a homotopy fibration diagram

$$\begin{array}{ccccc} \mathcal{Z}_K(S^2) & \longrightarrow & DJ_K(S^2) & \longrightarrow & \prod_{i=1}^r S^2 \\ \downarrow \mathcal{Z}_K(\iota) & & \downarrow DJ_K(\iota) & & \downarrow \prod_{i=1}^r \iota \\ \mathcal{Z}_K & \longrightarrow & DJ_K & \longrightarrow & \prod_{i=1}^r \mathbb{C}P^\infty. \end{array}$$

In this section we will use the calculations of  $H_*(\Omega DJ_K(S^2); \mathbb{Q})$  and  $H_*(\Omega \mathcal{Z}_K(S^2); \mathbb{Q})$  in Section 6 to calculate  $H_*(\Omega DJ_K; \mathbb{Q})$ , proving Theorem 1.5, and  $H_*(\Omega \mathcal{Z}_K; \mathbb{Q})$ . We then give a homotopy decomposition of  $\mathcal{Z}_K$  as a wedge of spheres and describe the map  $\mathcal{Z}_K \rightarrow DJ_K$  in terms of higher Whitehead products and iterated Whitehead products, proving Theorem 1.7.

By Proposition 4.3,  $H_*(\Omega DJ_K) \cong \operatorname{colim}_{\sigma \in MF(K)} H_*(\Omega FW(\sigma); \mathbb{Q})$ , so we first need to calculate  $H_*(\Omega FW(\sigma); \mathbb{Q})$  and then take a colimit to put the pieces together. We do this in Lemma 7.3 and Proposition 1.5 after two preliminary lemmas. In general, let  $X_1, \dots, X_n$  be path-connected spaces and consider the fat wedge  $FW(\underline{X})$  in  $\prod_{i=1}^n X_i$ . Let  $j$  be the inclusion  $j: FW(\underline{X}) \rightarrow \prod_{i=1}^n X_i$ .

**Lemma 7.1.** *The map  $\Omega FW(\underline{X}) \xrightarrow{\Omega j} \prod_{i=1}^n \Omega X_i$  has a right homotopy inverse, which can be chosen to be natural for maps  $X_i \rightarrow Y_i$ .*

*Proof.* The inclusion  $\bigvee_{i=1}^n X_i \rightarrow FW(\underline{X})$  is natural, as are the inclusions  $X_i \rightarrow \bigvee_{i=1}^n X_i$  for  $1 \leq i \leq n$ . Now loop and consider the composite

$$m: \prod_{i=1}^n \Omega X_i \longrightarrow \prod_{i=1}^n \Omega \left( \bigvee_{i=1}^n X_i \right) \xrightarrow{\mu} \Omega \left( \bigvee_{i=1}^n X_i \right) \longrightarrow \Omega FW(\underline{X}),$$

where  $\mu$  is the loop multiplication. All three maps in the composite are natural, and  $m$  is a right homotopy inverse of  $\Omega j$ .  $\square$

Let  $F$  be the homotopy fibre of  $j$ . As mentioned earlier, Porter [P2] showed that there is a homotopy equivalence  $F \simeq \Sigma^{n-1} \Omega X_1 \wedge \dots \wedge \Omega X_n$ . Further, in [P1, 1.2] he showed that this homotopy equivalence is natural for maps  $X_i \rightarrow Y_i$ . We record this as follows.

**Lemma 7.2.** *Let  $f_i: X_i \rightarrow Y_i$  be maps between simply-connected spaces. There is a homotopy commutative diagram between fibrations*

$$\begin{array}{ccccc} \Sigma^{n-1}\Omega X_1 \wedge \cdots \wedge \Omega X_n & \longrightarrow & FW(\underline{X}) & \xrightarrow{j} & \prod_{i=1}^n X_i \\ \downarrow \Sigma^{n-1}\Omega f_1 \wedge \cdots \wedge \Omega f_n & & \downarrow FW(f_1, \dots, f_n) & & \downarrow \prod_{i=1}^n f_i \\ \Sigma^{n-1}\Omega Y_1 \wedge \cdots \wedge \Omega Y_n & \longrightarrow & FW(\underline{Y}) & \xrightarrow{j} & \prod_{i=1}^n Y_i \end{array}$$

□

Let  $FW(S^2)$  and  $FW(\mathbb{C}P^\infty)$  be the fat wedges of  $\prod_{i=1}^n S^2$  and  $\prod_{i=1}^n \mathbb{C}P^\infty$  respectively. Let  $FW(\iota): FW(S^2) \rightarrow FW(\mathbb{C}P^\infty)$  be the map induced by the inclusion  $S^2 \xrightarrow{\iota} \mathbb{C}P^\infty$ .

**Lemma 7.3.** *There is a commutative diagram of algebras*

$$\begin{array}{ccc} H_*(\Omega FW(S^2); \mathbb{Q}) & \xrightarrow{\cong} & U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u \rangle) \\ \downarrow (\Omega FW(\iota))_* & & \downarrow q \\ H_*(\Omega FW(\mathbb{C}P^\infty); \mathbb{Q}) & \xrightarrow{\cong} & U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u \rangle) / I \end{array}$$

where  $u$ , of degree  $n-2$ , is the Hurewicz image of the adjoint of a higher Whitehead product,  $I$  is the ideal  $(b_i^2, [u, b_i] \mid 1 \leq i \leq n)$ , and  $q$  is the quotient map.

*Proof.* The isomorphism for  $H_*(\Omega FW(S^2); \mathbb{Q})$  holds by Theorem 5.4. To obtain the compatible isomorphism for  $H_*(\Omega FW(\mathbb{C}P^\infty); \mathbb{Q})$  we first consider what happens on the level of spaces. By Lemma 7.2, the map  $\iota$  induces a homotopy commutative diagram

$$\begin{array}{ccccc} \Sigma^{n-1}(\Omega S^2)^{(n)} & \longrightarrow & FW(S^2) & \xrightarrow{j} & \prod_{i=1}^n S^2 \\ \downarrow \Sigma^{n-1}(\Omega \iota)^{(n)} & & \downarrow FW(\iota) & & \downarrow \prod_{i=1}^n \iota \\ \Sigma^{n-1}(\Omega \mathbb{C}P^\infty)^{(n)} & \longrightarrow & FW(\mathbb{C}P^\infty) & \xrightarrow{j} & \prod_{i=1}^n \mathbb{C}P^\infty \end{array}$$

Note that  $\Omega \mathbb{C}P^\infty \simeq S^1$  so  $\Sigma^{n-1}(\Omega \mathbb{C}P^\infty)^{(n)} \simeq S^{2n-1}$ . Also, since  $S^1 \xrightarrow{E} \Omega S^2$  is a right homotopy inverse for  $\Omega \iota$ , if we let  $s = \Sigma^{n-1}E^{(n)}$  and  $t = \Sigma^{n-1}(\Omega \iota)^{(n)}$ , then the composite  $S^{2n-1} \xrightarrow{s} \Sigma^{n-1}(\Omega S^2)^{(n)} \xrightarrow{t} S^{2n-1}$  is homotopic to the identity map.

After looping we obtain a homotopy commutative diagram

$$(4) \quad \begin{array}{ccccc} \Omega(\Sigma^{n-1}(\Omega S^2)^{(n)}) & \longrightarrow & \Omega FW(S^2) & \xrightarrow{\Omega j} & \prod_{i=1}^n \Omega S^2 \\ \downarrow \Omega t & & \downarrow \Omega FW(\iota) & & \downarrow \prod_{i=1}^n \Omega \iota \\ \Omega S^{2n-1} & \longrightarrow & \Omega FW(\mathbb{C}P^\infty) & \xrightarrow{\Omega j} & \prod_{i=1}^n S^1 \end{array}$$

By Lemma 7.1,  $\Omega j$  has a natural right homotopy inverse, so there is a homotopy commutative diagram of sections

$$(5) \quad \begin{array}{ccc} \prod_{i=1}^n \Omega S^2 & \xrightarrow{m} & \Omega FW(S^2) \\ \downarrow \Pi_{i=1}^n \Omega i & & \downarrow \Omega FW(i) \\ \prod_{i=1}^n S^1 & \xrightarrow{m} & \Omega FW(\mathbb{C}P^\infty). \end{array}$$

Now we examine the effect of (5) in homology. By Theorem 1.4 and Proposition 6.2, a model for the homology of the homotopy fibration along the top row of (4) is

$$(6) \quad UL\langle R \rangle \xrightarrow{U(i)} U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u \rangle) \xrightarrow{U(\pi)} UL_{ab}\langle b_1, \dots, b_n \rangle,$$

where  $R = \{[u, b_{j_1}], \dots, b_{j_l} \mid 1 \leq j_1 \leq \dots \leq j_l \leq n, 0 \leq l < \infty\}$ . From (5), we obtain a right inverse  $m_*$  of  $U(\pi)$ . In particular, if  $H_*(\prod_{i=1}^n \Omega S^2; \mathbb{Q}) \cong \mathbb{Q}[c_1, \dots, c_n]$ , then  $m_*(c_i) = b_i + \gamma_i$  for some  $\gamma_i \in UL\langle R \rangle$ . However, the least degree of  $UL\langle R \rangle$  which is nontrivial is  $2n - 1$ , while  $c_i$  has degree 1. As  $n \geq 3$ , for degree reasons we must have  $\gamma_i = 0$ . Thus  $m_*(c_i) = b_i$ . For similar degree reasons, we have  $m_*(c_i^2) = b_i^2$  (even though  $m_*$  may not be multiplicative). On the other hand,  $(\Omega i)_*$  is an isomorphism on the first homology group and the same is true after composing with  $m_*$ , while  $(\Omega i)_*(c_i^2) = 0$ . Thus the commutativity of (5) after taking homology implies that for  $1 \leq i \leq n$ ,  $(\Omega FW(i))_*$  is degree one on  $b_i$  while  $(\Omega FW(i))_*(b_i^2) = 0$ . The latter implies by multiplicativity that  $(\Omega FW(i))_*$  sends the ideal  $(b_1^2, \dots, b_n^2)$  to 0.

Next, consider the commutator  $[u, b_i] \in H_*(\Omega FW(S^2); \mathbb{Q})$ . In terms of (6),  $[u, b_i]$  composes trivially with  $U(\pi)$  and so is the image of an element  $\delta_i \in UL\langle R \rangle$ . Note that  $\delta_i$  has degree  $2n$ . Taking homology in (4), we see that  $(\Omega t)_*(\delta_i) = 0$  for degree reasons. Thus the commutativity of the left square in (4) implies that  $(\Omega FW(i))_*([u, b_i]) = 0$ . By multiplicativity,  $(\Omega FW(i))_*$  therefore sends the ideal  $I = (b_i^2, [u, b_i] \mid 1 \leq i \leq n)$  to 0.

Thus there is a factorization

$$(7) \quad \begin{array}{ccc} H_*(\Omega FW(S^2); \mathbb{Q}) & \xrightarrow{\cong} & U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u \rangle) \\ \downarrow (\Omega FW(i))_* & & \downarrow q \\ H_*(\Omega FW(\mathbb{C}P^\infty); \mathbb{Q}) & \xleftarrow{h} & U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u \rangle)/I \end{array}$$

for some algebra map  $h$ , which is degree one on  $b_i$  for each  $1 \leq i \leq n$ . In addition, the fact that  $\Omega t$  has a right homotopy inverse implies that  $h$  is degree one on  $u$ .

We claim that  $h$  is an isomorphism, from which the lemma would follow. To see the isomorphism, let  $I'$  be the ideal  $([u, b_i] \mid 1 \leq i \leq n)$ . Observe that  $U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u \rangle)/I'$  is isomorphic to  $UL_{ab}\langle b_1, \dots, b_n, u \rangle \cong \mathbb{Q}[b_1, \dots, b_n, u]$ . Thus  $U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u \rangle)/I$  is isomorphic to  $\Lambda(b_1, \dots, b_n) \otimes \mathbb{Q}[u]$ . On the other hand, the section  $m$  in (5) implies that there is a homotopy decomposition  $\Omega FW(\mathbb{C}P^\infty) \simeq (\prod_{i=1}^n S^1) \times \Omega S^{2n-1}$ . Thus there is a coalgebra isomorphism

$H_*(\Omega FW(\mathbb{C}P^\infty); \mathbb{Q}) \cong \Lambda(c_1, \dots, c_n) \otimes \mathbb{Q}[v]$  where  $v$  has degree  $2n - 1$ . From the use of  $m$  and  $t$  in both the homotopy decomposition of  $\Omega FW(\mathbb{C}P^\infty)$  and the factorization of  $(\Omega FW(i))_*$  through  $h$ , we see that  $h(b_i) = c_i$  for  $1 \leq i \leq n$  and  $h(u) = v$ . As  $h$  is an algebra map, it therefore induces an isomorphism.  $\square$

Now we pass to a colimit of fat wedges to prove Theorem 1.5.

*Proof of Theorem 1.5:* With  $\sigma = (i_1, \dots, i_k)$ , consider the diagram

$$\begin{array}{ccc}
H_*(\Omega DJ_K(S^2); \mathbb{Q}) & \xrightarrow{(\Omega DJ_K(i))_*} & H_*(\Omega DJ_K; \mathbb{Q}) \\
\downarrow \cong & & \downarrow \cong \\
\operatorname{colim}_{\sigma \in MF(K)} U(L_{ab}\langle b_{i_1}, \dots, b_{i_k} \rangle \amalg L\langle u_\sigma \rangle) & \xrightarrow{\operatorname{colim}_{\sigma \in MF(K)} q_\sigma} & \operatorname{colim}_{\sigma \in MF(K)} U(L_{ab}\langle b_{i_1}, \dots, b_{i_k} \rangle \amalg L\langle u_\sigma \rangle) / I_\sigma \\
\downarrow \cong & & \downarrow \cong \\
U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle) & \xrightarrow{q} & U(L_{ab}\langle W \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle) / I
\end{array}$$

where  $I_\sigma$  is the ideal generated by  $(b_{\sigma_i}^2, [u_\sigma, b_{j_\sigma}] \mid j_\sigma \in \{i_1, \dots, i_k\})$ . The upper square commutes by combining Proposition 4.3 and Lemma 7.3. The lower square is the result of evaluating the colimit, and so commutes. Note that both squares commute as maps of algebras. The lower row is the isomorphism asserted by the theorem, and the outer rectangle is the asserted commutative diagram.  $\square$

Next, we use Theorem 1.5 to calculate  $H_*(\Omega Z_K; \mathbb{Q})$  in Proposition 7.5 as the universal enveloping algebra of a certain free graded Lie algebra. This will involve some explicit calculations involving graded Lie algebra identities, which we recall now. In general, if  $L$  is a graded Lie algebra over  $\mathbb{Q}$  with bracket  $[\ , \ ]$ , there is a graded anti-symmetry identity  $[x, y] = -(-1)^{|x||y|}[y, x]$  for all  $x, y \in L$  and a graded Jacobi identity  $[[x, y], z] = [x, [y, z]] - (-1)^{|x||y|}[y, [x, z]]$  for all  $x, y, z \in L$ .

The ideal in Theorem 1.5 involves brackets of the form  $[u_\sigma, b_j]$  where  $j \in \{i_1, \dots, i_k\}$ , where  $\sigma = (i_1, \dots, i_k)$ . Thus in the quotient we need to keep track of brackets of the form  $[u_\sigma, b_j]$  where  $j$  is in the complement of  $\{i_1, \dots, i_k\}$ . Let  $J_\sigma = \{1, \dots, n\} - \{i_1, \dots, i_k\}$ . Consider the free graded Lie algebra generated by

$$\tilde{R} = \{[[u_\sigma, b_{j_1}], \dots, b_{j_l}] \mid \sigma \in MF(K), \{j_1, \dots, j_l\} \subseteq J_\sigma, 1 \leq j_1 < \dots < j_l \leq n, 0 \leq l \leq n\}.$$

Note that each  $j_t$  can appear at most once in any given bracket. This should be compared to the  $\mathbb{Q}$ -module  $R$ , where each  $j_t$  can appear arbitrarily many times in a given bracket. Let  $i_R: \tilde{R} \rightarrow R$  be the inclusion and  $\pi_R: R \rightarrow \tilde{R}$  be the projection.

**Lemma 7.4.** *There is a short exact sequence of Lie algebras*

$$L(\tilde{R}) \xrightarrow{\tilde{i}} (L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle) / \tilde{I} \xrightarrow{\tilde{\pi}} L_{ab}\langle b_1, \dots, b_n \rangle / \tilde{I}'$$

where  $\tilde{I}$  is the ideal  $\tilde{I} = ([b_i, b_i], [u_\sigma, b_{j_\sigma}] \mid 1 \leq i \leq n, \sigma \in MF(K), j_\sigma \in \{i_1, \dots, i_k\})$ ,  $\tilde{I}'$  is the ideal  $([b_i, b_i] \mid 1 \leq i \leq n)$ ,  $\tilde{i}$  is the inclusion, and  $\tilde{\pi}$  is the projection.

*Proof.* To simplify notation, let  $L = L_{ab}\langle b_1, \dots, b_n \rangle \coprod L\langle u_\sigma \mid \sigma \in MF(K) \rangle$ . Observe from the definitions of  $\tilde{I}$  and  $\tilde{I}'$  that there is a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\pi} & L_{ab}\langle b_1, \dots, b_n \rangle \\ \downarrow q & & \downarrow q' \\ L/\tilde{I} & \xrightarrow{\tilde{\pi}} & L_{ab}\langle b_1, \dots, b_n \rangle/\tilde{I}' \end{array}$$

where  $q$  and  $q'$  are the quotient maps. By (3), the kernel of  $\pi$  is  $L\langle R \rangle$ . Let  $\tilde{L}$  be the kernel of  $\tilde{\pi}$ . The commutativity of the diagram implies that there is an induced map  $\tilde{q}: L\langle R \rangle \rightarrow \tilde{L}$ .

We claim that  $\tilde{q}$  is a surjection. Let  $x \in \tilde{L}$  and let  $x$  also denote its image in  $L/\tilde{I}$ . As  $q$  is onto there is an element  $y \in L$  such that  $q(y) = x$ . Let  $z = \pi(y)$ . If  $z = 0$  then by exactness  $y$  lifts to  $\tilde{y} \in L\langle R \rangle$  and so  $\tilde{q}(\tilde{y}) = x$ . If  $z \neq 0$ , then  $q'(z) = 0$  by exactness. Since  $L$  is a coproduct, the projection  $\pi$  has a right inverse  $r: L_{ab}\langle b_1, \dots, b_n \rangle \rightarrow L$  which is a map of Lie algebras. As the generators of the ideal  $\tilde{I}$  are all generators of the ideal  $\tilde{I}'$ , we have  $(q \circ r)(m) = 0$  if and only if  $q'(m) = 0$  for any  $m \in L_{ab}\langle b_1, \dots, b_n \rangle$ . Thus  $r(z)$  has the property that  $(q \circ r)(z) = 0$ . Therefore  $\tilde{y} = y - r(z)$  lifts to  $L\langle R \rangle$  and  $q(y - r(z)) = q(y) = x$ , so  $\tilde{q}(\tilde{y}) = x$ . Hence  $\tilde{q}$  is a surjection.

Now  $\tilde{q}$  is a surjection and  $\tilde{L}$  injects into  $L/\tilde{I}$ . Therefore  $\tilde{L}$  is isomorphic to the image of  $L\langle R \rangle$  under  $q$ . We next show that this image is  $L\langle \tilde{R} \rangle$ . We first perform two short calculations.

*Calculation 1:* The Jacobi identity states that  $[[a, b_i], b_j] = [a, [b_i, b_j]] - (-1)^{|a||b_i|}[b_i, [a, b_j]]$  for any  $a \in L$  and any  $1 \leq i, j \leq n$ . The abelian property of  $L_{ab}\langle b_1, \dots, b_n \rangle$  implies that  $[b_i, b_j] = 0$  and so  $[a, [b_i, b_j]] = 0$ . By the anti-symmetry identity,  $-(-1)^{|a||b_i|}[b_i, [a, b_j]] = (-1)^{|a||b_i|+|b_i||[a, b_j]|}[[a, b_j], b_i]$ . Since  $|b_i| = 1$  for  $1 \leq i \leq n$ , the sign on the right side of this equation equals  $(-1)^{2|a|+1}$ , which is  $-1$ . Therefore  $[[a, b_i], b_j] = -[[a, b_j], b_i]$ .

*Calculation 2:* The Jacobi identity states that  $[[a, b_i], b_i] = [a, [b_i, b_i]] - (-1)^{|a||b_i|}[b_i, [a, b_i]]$  for any  $a \in L$  and  $1 \leq i \leq n$ . Since  $[b_i, b_i] = 0$  in  $L$ , we have  $[a, [b_i, b_i]] = 0$ . As in Calculation 1, the anti-symmetry identity shows that  $-(-1)^{|a||b_i|}[b_i, [a, b_i]] = -[[a, b_i], b_i]$ . Thus  $[[a, b_i], b_i] = -[[a, b_i], b_i]$ , and so  $2[[a, b_i], b_i] = 0$ . As  $L$  is a Lie algebra over  $\mathbb{Q}$ , 2 is invertible and so  $[[a, b_i], b_i] = 0$ .

By Calculation 1, up to sign change, whenever consecutive  $b$ 's appear in a bracket of  $L\langle R \rangle$  or  $L$  their order can be interchanged. By Calculation 2, the effect of taking the quotient in  $L\langle R \rangle$  and  $L$  by the ideal  $I' = ([b_i, b_i] \mid 1 \leq i \leq n)$  is to annihilate all brackets in which appears a copy of  $[a, [b_i, b_i]]$ . Together with Calculation 1 which lets us freely interchange consecutive  $b$ 's, any bracket of the form  $[[u_\sigma, b_{j_1}], \dots, b_{j_l}]$  is zero if any  $b_{j_t}$  appears more than once. Thus, the only such nontrivial brackets must have  $1 \leq j_1 < \dots < j_l \leq n$ ,  $0 \leq l < n$ , as in the definition of  $\tilde{R}$ . The effect of then taking the quotient by the ideal generated by  $[u_\sigma, b_{j_\sigma}]$  for  $j_\sigma \in \{i_1, \dots, i_k\}$  is to annihilate those brackets in

$\{[u_\sigma, b_{j_1}], \dots, b_{j_l}] \mid \sigma \in MF(K), 1 \leq j_1 < \dots < j_l \leq n, 0 \leq l < n\}$  which do not have  $j_1, \dots, j_l \in J_\sigma$ . Thus the image of  $L\langle \tilde{R} \rangle$  under  $q$  is  $L\langle \tilde{R} \rangle$ .  $\square$

In general, the image of a graded Lie algebra  $L$  in its universal enveloping algebra  $UL$  has the property that  $[x, y] = xy - (-1)^{|x||y|}yx$ , where the multiplication is taking place in  $UL$ . In particular, the anti-symmetry identity implies that  $[x, x] = 2x^2$  if the degree of  $x$  is odd. Thus if 2 has been inverted in the ground ring, then the ideal in  $UL$  generated by  $[x, x]$  is identical to the ideal generated by  $x^2$ . In our case, the short exact sequence of Lie algebras in Lemma 7.4 implies that there is a short exact sequence of Hopf algebras

$$(8) \quad L\langle \tilde{R} \rangle \longrightarrow (L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle)/I \xrightarrow{\pi} L_{ab}\langle b_1, \dots, b_n \rangle/I'$$

where  $I$  is the ideal in Theorem 1.5 and  $I' = (b_i^2 \mid 1 \leq i \leq n)$ .

**Proposition 7.5.** *There is a commutative diagram of algebras*

$$\begin{array}{ccc} H_*(\Omega Z_K; \mathbb{Q}) & \longrightarrow & H_*(\Omega DJ_K; \mathbb{Q}) \\ \downarrow \cong & & \downarrow \cong \\ UL\langle \tilde{R} \rangle & \xrightarrow{U(\tilde{i})} & U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle)/I. \end{array}$$

*Proof.* Argue as in Proposition 6.2, replacing the short exact sequence of Hopf algebras appearing there with that in (8), and replacing Theorem 1.4 with Theorem 1.5.  $\square$

Now we use the description of  $H_*(\Omega DJ_K; \mathbb{Q})$  in Theorem 1.5 to produce maps as was done in the case of  $\Omega DJ_K(\underline{S})$ . For  $\sigma \in MF(K)$ , let

$$\tilde{w}_\sigma: S^{2|\sigma|+1} \xrightarrow{\phi_k} FW(S^2, \sigma) \xrightarrow{FW(\tilde{i})} FW(\sigma) \longrightarrow DJ_K$$

be the higher Whitehead product. By Theorem 1.5, the element  $u_\sigma \in H_*(\Omega DJ_K(\underline{S}); \mathbb{Q})$  is the Hurewicz image of the adjoint of  $\tilde{w}_\sigma$ . For  $1 \leq i \leq n$ , let  $\tilde{a}_i$  be the composite

$$\tilde{a}_i: S^2 \xrightarrow{i} \mathbb{C}P^\infty \longrightarrow DJ_K$$

where the right map is the  $i^{\text{th}}$ -coordinate inclusion. Let  $\tilde{\mathcal{I}}$  be the index set for  $\tilde{R}$ . Then  $\tilde{\alpha} \in \tilde{\mathcal{I}}$  corresponds to a face  $\sigma \in MF(K)$  and a sequence  $(j_1, \dots, j_l)$  where  $1 \leq j_1 < \dots < j_l \leq n$  and  $0 \leq l \leq n$ . Given such an  $\tilde{\alpha}$ , let  $t_{\tilde{\alpha}} = N_\sigma + l - 2$ . The inclusion  $i_R$  induces a map  $\tilde{i}_R: \bigvee_{\tilde{\alpha} \in \tilde{\mathcal{I}}} S^{t_{\tilde{\alpha}}+1} \longrightarrow \bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha+1}$ . Note that  $(\tilde{\Omega i}_R)_*$  can be identified with  $U(i_R)$ . Consider the composite

$$\tilde{W}: \bigvee_{\tilde{\alpha} \in \tilde{\mathcal{I}}} S^{t_{\tilde{\alpha}}+1} \xrightarrow{\tilde{i}_R} \bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha+1} \xrightarrow{W} DJ_K(S^2) \xrightarrow{DJ_K(\tilde{i})} DJ_K.$$

If  $\tilde{\alpha}$  indexes  $\sigma \in MF(K)$ , then the restriction of  $\tilde{W}$  to  $S^{t_{\tilde{\alpha}}+1}$  is the higher Whitehead product  $\tilde{w}_\sigma$ . Otherwise, the restriction of  $\tilde{W}$  to  $S^{t_{\tilde{\alpha}}+1}$  is an iterated Whitehead product of a single  $\tilde{w}_\sigma$  with some selection of the coordinate inclusions  $\tilde{a}_1, \dots, \tilde{a}_n$ , where each  $\tilde{a}_i$  appears at most once.



**Corollary 7.6.** *The map  $\Omega(\bigvee_{\tilde{\alpha} \in \tilde{\mathcal{I}}} S^{t_{\tilde{\alpha}+1}}) \xrightarrow{\Omega\tilde{W}} \Omega DJ_K$  induces in homology the map  $UL\langle\tilde{R}\rangle \xrightarrow{U(\tilde{i})} U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle)/I$ .*

*Proof.* Let  $\tilde{S}$  be the composite

$$\tilde{S}: \bigvee_{\tilde{\alpha} \in \tilde{\mathcal{I}}} S^{t_{\tilde{\alpha}+1}} \xrightarrow{E} \Omega\left(\bigvee_{\tilde{\alpha} \in \tilde{\mathcal{I}}} S^{t_{\tilde{\alpha}+1}}\right) \xrightarrow{\Omega\tilde{W}} \Omega DJ_K.$$

The definition of  $\tilde{W}$  implies that  $\tilde{S}_*$  induces the composite

$$\tilde{R} \hookrightarrow UL\langle\tilde{R}\rangle \xrightarrow{UL(i_R)} UL\langle R \rangle \xrightarrow{(\Omega W)_*} H_*(\Omega DJ_K(S^2)) \xrightarrow{(\Omega DJ_K(i))_*} H_*(\Omega DJ_K).$$

Now argue as in Corollary 6.3, using the description of  $(\Omega DJ_K(i))_*$  in Theorem 1.5, to obtain the asserted inclusion in homology.  $\square$

We finish by bringing  $\mathcal{Z}_K$  back into the picture.

**Theorem 7.7.** *The map  $\bigvee_{\tilde{\alpha} \in \tilde{\mathcal{I}}} S^{t_{\tilde{\alpha}+1}} \xrightarrow{\tilde{W}} DJ_K$  lifts to  $\mathcal{Z}_K$ , and induces a homotopy equivalence  $\bigvee_{\tilde{\alpha} \in \tilde{\mathcal{I}}} S^{t_{\tilde{\alpha}+1}} \rightarrow \mathcal{Z}_K$ .*

*Proof.* Argue as in Theorem 6.4, using Proposition 7.5 and Corollary 7.6 in place of Proposition 6.2 and Corollary 6.3 respectively.  $\square$

*Proof of Theorem 1.7:* This is simply a rephrasing of Theorem 7.7.  $\square$

## 8. EXAMPLES

We now discuss a useful family of examples. Let  $X_1, \dots, X_n$  be simply-connected  $CW$ -complexes of finite type. For  $1 \leq k \leq n$ , define the space  $T_k^n$  as

$$T_k^n = \{(x_1, \dots, x_n) \in X_1 \times \dots \times X_n \mid \text{at least } k \text{ of } x_1, \dots, x_n \text{ are } *\}.$$

For example,  $T_0^n = X_1 \times \dots \times X_n$ ,  $T_1^n$  is the fat wedge, and  $T_{n-1}^n = X_1 \vee \dots \vee X_n$ . Define the space  $F_k^n$  by the homotopy fibration

$$F_k^n \longrightarrow T_k^n \longrightarrow \prod_{i=1}^n X_i$$

where the right map is the inclusion. Note that  $T_k^n$  is  $DJ_K(\underline{X})$  for  $K = (\Delta^n)_{n-k-1}$ , the full  $k$ -skeleton of  $\Delta^n$ , and so  $F_k^n = \mathcal{Z}_K(\underline{X})$ .

In this case, the missing faces  $MF(K)$  of  $K = (\Delta^n)_{n-k-1}$  is precisely the set of  $\binom{n}{n-k}$  faces of  $\Delta^n$  of dimension  $n - k$ . In terms of sequences,

$$MF(K) = \{(i_1, \dots, i_{n-k}) \mid 1 \leq i_1 < \dots < i_{n-k} \leq n\}.$$

Observe that  $|K| = \bigcup_{\sigma \in MF(K)} |\partial\sigma|$ . Thus  $K$  is an  $MF$ -complex. Therefore, by Proposition 1.2,  $F_k^n$  is homotopy equivalent to a wedge of spaces of the form  $\Sigma^t \Omega X_{i_1} \wedge \dots \wedge \Omega X_{i_t}$  for various  $1 \leq t < n$  and sequences  $(i_1, \dots, i_t)$  where  $1 \leq i_1 < \dots < i_t \leq n$ . The precise homotopy type was given by

Porter [P2], and later the authors [GT] by the methods used to prove Proposition 1.2. There is a homotopy equivalence

$$(9) \quad F_k^n \simeq \bigvee_{j=n-k+1}^n \left( \bigvee_{1 \leq i_1 < \dots < i_j \leq n} \binom{j-1}{n-k} \Sigma^{n-k} \Omega X_{i_1} \wedge \dots \wedge \Omega X_{i_j} \right).$$

We will consider the special cases  $T_k^n(\underline{S})$  and  $F_k^n(\underline{S})$  when  $X_i = S^{m_i+1}$ ;  $T_k^n(S^2)$  and  $F_k^n(S^2)$  when  $X_i = S^2$  for  $1 \leq i \leq n$ ; and  $T_k^n(\mathbb{C}P^\infty)$  and  $F_k^n(\mathbb{C}P^\infty)$  when  $X_i = \mathbb{C}P^\infty$  for  $1 \leq i \leq n$ . Also, we will restrict to  $k < n-1$ , eliminating the product  $T_n^n(\ )$  and the wedge  $T_{n-1}^n(\ )$ , as both of these cases are well understood by other means. Note that with this restriction, as  $K = (\Delta^n)_{n-k-1}$ , we have  $|\sigma| > 1$  for every  $\sigma \in MF(K)$ . In particular, the ideal  $J$  in Theorems 1.4 and 1.5 is trivial.

Since  $K$  is an  $MF$ -complex, we have

$$T_k^n(\underline{S}) = \text{colim}_{\sigma \in MF(K)} FW(\sigma).$$

By Theorem 1.4, there is an algebra isomorphism

$$H_*(\Omega T_k^n(\underline{S}); \mathbb{Q}) \cong U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle)$$

where  $b_i$  and  $u_\sigma$  respectively are the Hurewicz images of the adjoints of the coordinate inclusion  $a_i: S^{m_i} \rightarrow T_k^n(\underline{S})$  and the higher Whitehead product  $w_\sigma: S^{N_\sigma} \rightarrow T_k^n(\underline{S})$ . By Theorem 6.4 there is a homotopy decomposition

$$\Omega F_k^n(\underline{S}) \simeq \bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha+1}$$

such that the composition  $\bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha+1} \rightarrow F_k^n(\underline{S}) \rightarrow T_k^n(\underline{S})$  is homotopic to the wedge sum of higher Whitehead products and iterated Whitehead products specified by

$$\{[[w_\sigma, a_{j_1}] \dots, a_{j_l}] \mid \sigma \in MF(K), 1 \leq j_1 \leq \dots \leq j_l \leq n, 0 < l < \infty\}.$$

Similarly, by Theorem 7.5, there is an algebra isomorphism

$$H_*(\Omega T_k^n(\mathbb{C}P^\infty); \mathbb{Q}) \cong U(L_{ab}\langle b_1, \dots, b_n \rangle \amalg L\langle u_\sigma \mid \sigma \in MF(K) \rangle)/I$$

where  $I$  is the ideal  $(b_i^2, [u_\sigma, b_{i\sigma}] \mid 1 \leq i \leq n, i_\sigma \in \sigma)$ , and  $b_i$  is the Hurewicz images of the adjoint of the composite  $\tilde{a}_i: S^2 \xrightarrow{i} \mathbb{C}P^\infty \rightarrow T_k^n(\mathbb{C}P^\infty)$  where the right map is the inclusion of the  $i^{\text{th}}$ -coordinate, and  $u_\sigma$  is the Hurewicz image of the adjoint of the higher Whitehead product  $\tilde{w}_\sigma: S^{N_\sigma} \rightarrow T_k^n(\mathbb{C}P^\infty)$ . By Theorem 7.7 there is a homotopy decomposition

$$\Omega F_k^n(\mathbb{C}P^\infty) \simeq \bigvee_{\tilde{\alpha} \in \tilde{\mathcal{I}}} S^{t_{\tilde{\alpha}}+1}$$

such that the composition  $\bigvee_{\tilde{\alpha} \in \tilde{\mathcal{I}}} S^{t_{\tilde{\alpha}}+1} \rightarrow F_k^n(\mathbb{C}P^\infty) \rightarrow T_k^n(\mathbb{C}P^\infty)$  is homotopic to the wedge sum of higher Whitehead products and iterated Whitehead products specified by

$$\{[[w_\sigma, a_{j_1}] \dots, a_{j_l}] \mid \sigma \in MF(K), 1 \leq j_1 < \dots < j_l \leq n, 0 < l \leq n\}.$$

It is useful to say more about how each Whitehead product corresponds to summands in the homotopy decompositions of  $F_k^n(\underline{S})$  and  $F_k^n(\mathbb{C}P^\infty)$ . We begin with the case of the fat wedge  $F_1^n(\underline{S})$ .

**Example 8.1.** Consider the homotopy fibration  $F_1^n(\underline{S}) \rightarrow T_1^n(\underline{S}) \rightarrow \prod_{i=1}^n S^{m_i+1}$ . By (9), there is a homotopy equivalence

$$F_1^n(\underline{S}) \simeq \Sigma^{n-1} \Omega S^{m_1+1} \wedge \dots \wedge \Omega S^{m_n+1}.$$

By [J], for any path-connected space  $X$  there is a homotopy decomposition  $\Sigma \Omega \Sigma X \simeq \bigvee_{n=1}^\infty \Sigma X^{(n)}$ . Taking  $X = S^{m_i}$  and iterating, we obtain a homotopy decomposition

$$\Sigma^{n-1} \Omega S^{m_1+1} \wedge \dots \wedge \Omega S^{m_n+1} \simeq \bigvee_{d_1, \dots, d_n=1}^\infty \Sigma^{n-1} S^{d_1 m_1 + \dots + d_n m_n}.$$

The lowest dimensional sphere in this wedge occurs when  $d_i = 1$  for every  $1 \leq i \leq n$ . Let  $t = (n-1) + (\sum_{i=1}^n m_i)$ . The composite  $S^t \rightarrow F_1^n(\underline{S}) \rightarrow T_1^n(\underline{S})$  is the map which attaches the top cell to the product  $\prod_{j=1}^{n-k+1} S^{m_j+1}$ . That is, this composite is the higher Whitehead product  $w_\sigma : S^t \xrightarrow{\phi_{n-1}} T_1^n(\underline{S})$ . Next, consider a sphere  $S^{d_1 m_1 + \dots + d_n m_n}$  where at least one  $d_i > 1$ . This sphere maps to  $T_1^n(\underline{S})$  by the iterated Whitehead product

$$[[w_\sigma, a_1], \dots, a_1], a_2], \dots, a_2] \dots a_n], \dots a_n]$$

where  $a_i$  appears  $d_i - 1$  times. Note that the collection of such brackets is in one-to-one correspondence with the set  $\{1 \leq j_1 \leq \dots \leq j_l \leq n, 0 < l < \infty\}$ .

Now we return to the more general case of  $F_k^n(\underline{S})$ .

**Example 8.2.** Consider the homotopy decomposition of  $F_k^n(\underline{S})$  in (9). When the index  $j$  is  $n-k+1$ , we have wedge summands  $(\bigvee_{1 \leq i_1 < \dots < i_{n-k+1} \leq n} \Sigma^{n-k} (\Omega S^{m_{i_1}+1} \wedge \dots \wedge \Omega S^{m_{i_{n-k+1}}+1}))$ . There are  $\binom{n}{n-k}$  such summands, and each corresponds to the homotopy fiber of  $FW(i_1, \dots, i_{n-k+1}) \rightarrow S^{m_{i_1}+1} \times \dots \times S^{m_{i_{n-k+1}}+1}$ , where each  $\sigma = (i_1, \dots, i_{n-k+1})$  is one of the missing faces in  $M((\Delta^n)_{n-k-1})$ . As in Example 8.1, the bottom cell of each wedge summand maps to  $T_k^n(\underline{S})$  by a higher Whitehead product  $w_\sigma$ , and the remaining summands map to  $T_k^n(\underline{S})$  by iterated Whitehead products consisting of one copy of  $w_\sigma$  and an appropriate number of coordinate inclusions.

Continuing, when the index  $j$  is larger than  $n-k+1$ , there are  $\binom{j-1}{n-k}$  wedge summands of the form  $\Sigma^{n-k} (\Omega S^{m_{i_1}+1} \wedge \dots \wedge \Omega S^{m_{i_j}+1})$  in  $F_k^n(\underline{S})$ . By [J], for each wedge summand there is a homotopy decomposition

$$(10) \quad \Sigma^{n-k} \Omega S^{m_{i_1}+1} \wedge \dots \wedge \Omega S^{m_{i_j}+1} \simeq \bigvee_{d_{i_1}, \dots, d_{i_j}=1}^\infty \Sigma^{n-k} S^{d_{i_1} m_{i_1} + \dots + d_{i_j} m_{i_j}}.$$

Note that  $i_1 < \dots < i_j$ . Partition the set  $\{i_1, \dots, i_j\}$  into two sets,  $I = \{i_1, \dots, i_{n-k}\}$  and  $J = \{i_{n-k+1}, \dots, i_j\}$ . The missing face in question is determined by  $I$ :  $\sigma = (i_1, \dots, i_{n-k}) \in M((\Delta^n)_{n-k-1})$ . The lowest dimensional sphere in (10) occurs when  $d_{i_t} = 1$  for all  $1 \leq t \leq j$ . This

maps to  $T_k^n(\underline{S})$  by the iterated Whitehead product  $[[w_\sigma, a_{i_{n-k+1}}], \dots, a_{i_j}]$ . Next, consider a wedge summand  $S^{d_{i_1}m_{i_1} + \dots + d_{i_j}m_{i_j}}$  in (10) where at least one of  $d_{i_1}, \dots, d_{i_{n-k+1}}$  is larger than 1. This sphere maps to  $T_k^n(\underline{S})$  by the iterated Whitehead product

$$[[w_\sigma, a_{i_1}], \dots, a_{i_1}, a_{i_2}], \dots, a_{i_2}] \dots a_{i_j}], \dots a_{i_j}]$$

where  $a_{i_t}$  appears  $d_{i_t} - 1$  times if  $1 \leq t \leq n - k$  and  $a_{i_t}$  appears  $d_{i_t}$  times if  $n - k + 1 \leq t \leq j$ . In particular, note that each  $a_{i_t}$  for  $n - k + 1 \leq t \leq j$  appears at least once in each bracket.

Next, we consider how Examples 8.1 and 8.2 modify when we pass from  $T_k^n(S^2)$  to  $T_k^n(\mathbb{C}P^\infty)$ .

**Example 8.3.** By Lemma 7.2, the inclusion  $S^2 \xrightarrow{\iota} \mathbb{C}P^\infty$  induces a homotopy commutative diagram

$$\begin{array}{ccccc} F_1^n(S^2) \simeq \Sigma^{n-1}(\Omega S^2)^{(j)} & \longrightarrow & T_1^n(S^2) & \longrightarrow & \prod_{i=1}^n S^2 \\ \downarrow \Sigma^{n-1}(\Omega \iota)^{(j)} & & \downarrow T_1^n(\iota) & & \downarrow \prod_{i=1}^n \iota \\ F_1^n(\mathbb{C}P^\infty) \simeq \Sigma^{n-1}S^j & \longrightarrow & T_1^n(\mathbb{C}P^\infty) & \longrightarrow & \prod_{i=1}^n \mathbb{C}P^\infty. \end{array}$$

Since  $\Sigma^{n-1}(\Omega \iota)^{(j)}$  has a right homotopy inverse, the map  $\Sigma^{n-1}S^j \rightarrow T_1^n(\mathbb{C}P^\infty)$  along the bottom row is homotopic to the composite  $S^{n-1+j} \rightarrow \Sigma^{n-1}(\Omega S^2)^{(j)} \rightarrow T_1^n(S^2) \rightarrow T_1^n(\mathbb{C}P^\infty)$ , and so is a higher Whitehead product. This higher Whitehead product  $\tilde{w}_\sigma$  corresponds to the one missing face  $\sigma \in M((\Delta_{n-2}^n))$ . Note that the remaining spheres in the homotopy decomposition of  $\Sigma^{n-1}(\Omega S^2)^{(j)}$  are not present in the homotopy decomposition of  $F_1^n(\mathbb{C}P^\infty)$ , and so there is no need to introduce iterated Whitehead products of  $\tilde{w}_\sigma$  with coordinate inclusions as in Example 8.1.

**Example 8.4.** By Lemma 7.2, the inclusion  $S^2 \xrightarrow{\iota} \mathbb{C}P^\infty$  induces a homotopy fibration diagram

$$\begin{array}{ccccc} F_k^n(S^2) \simeq \bigvee_{j=n-k+1}^n \left( \bigvee_{1 \leq i_1 < \dots < i_j \leq n} \binom{j-1}{n-k} \Sigma^{n-k}(\Omega S^2)^{(j)} \right) & \longrightarrow & T_k^n(S^2) & \longrightarrow & \prod_{i=1}^n S^2 \\ \downarrow & & \downarrow T_k^n(\iota) & & \downarrow \prod_{i=1}^n \iota \\ F_k^n(\mathbb{C}P^\infty) \simeq \bigvee_{j=n-k+1}^n \left( \bigvee_{1 \leq i_1 < \dots < i_j \leq n} \binom{j-1}{n-k} \Sigma^{n-k}S^j \right) & \longrightarrow & T_k^n(\mathbb{C}P^\infty) & \longrightarrow & \prod_{i=1}^n \mathbb{C}P^\infty. \end{array}$$

When the index  $j$  is  $n - k + 1$ ,  $F_k^n(\mathbb{C}P^\infty)$  has wedge summands  $\bigvee_{1 \leq i_1 < \dots < i_{n-k+1} \leq n} \Sigma^{n-k}S^j$ . There are  $\binom{n}{n-k}$  such summands, and each corresponds to the homotopy fiber of  $FW(i_1, \dots, i_{n-k+1}) \rightarrow \mathbb{C}P_{i_1}^\infty \times \dots \times \mathbb{C}P_{i_{n-k+1}}^\infty$ , where the indices on the product refer to their coordinate position within  $\prod_{i=1}^n \mathbb{C}P^\infty$ , and  $\sigma = (i_1, \dots, i_{n-k+1})$  is one of the missing faces in  $M((\Delta^n)_{n-k-1})$ . As in Example 8.3, each sphere maps to  $T_k^n(\mathbb{C}P^\infty)$  by a higher Whitehead product  $\tilde{w}_\sigma$ .

Continuing, when the index  $j$  is larger than  $n - k + 1$ , there are  $\binom{j-1}{n-k}$  wedge summands of the form  $\Sigma^{n-k}(S_{i_1}^1 \wedge \dots \wedge S_{i_j}^1)$  in  $F_k^n(\mathbb{C}P^\infty)$ , where the indices  $i_t$  refer to coordinate position. Partition the set  $\{i_1, \dots, i_j\}$  into two sets,  $I = \{i_1, \dots, i_{n-k}\}$  and  $J = \{i_{n-k+1}, \dots, i_j\}$ . The missing face in question is determined by  $I$ :  $\sigma = (i_1, \dots, i_{n-k+1}) \in M((\Delta^n)_{n-k-1})$ . As in Example 8.2, the sphere  $\Sigma^{n-k}(S_{i_1}^1 \wedge \dots \wedge S_{i_j}^1)$  maps to  $T_k^n(\mathbb{C}P^\infty)$  by the iterated Whitehead product  $(*) [[\tilde{w}_\sigma, \tilde{a}_{i_{n-k+1}}], \dots, \tilde{a}_{i_j}]$ .

The difference with Example 8.2 is that there are no other spheres arising from the decomposition of  $\Sigma\Omega S^2$  apart from the bottom  $\Sigma S^1$ , and so there are no additional iterated Whitehead products. In particular, each map  $\tilde{a}_{i_t}$  appears at most once in  $(*)$ .

Next, we consider the example of the shifted complex which appeared in the Introduction, and to speed up we consider only the case of  $\mathcal{Z}_K$  and  $DJ_K$ , omitting the case of  $\mathcal{Z}_K(\underline{S})$  and  $DJ_K(\underline{S})$ .

**Example 8.5.** Let  $K = \{(1), (2), (3), (4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}$ . Under this ordering of the vertices,  $K$  is shifted. The missing faces of  $K$  are given by  $MF(K) = \{(3, 4), (1, 2, 3), (1, 2, 4)\}$ . Observe that in this case  $|K| = \bigcup_{\sigma \in MF(K)} |\partial\sigma|$ . Now we can write  $DJ_K$  as  $DJ_K = \text{colim}_{\sigma \in MF(K)} FW(\sigma)$ . By Theorem 1.4, there is an algebra isomorphism

$$H_*(\Omega DJ_K(\underline{S})) \cong U(L_{ab}\langle b_1, b_2, b_3, b_4 \rangle \amalg L\langle u_1, u_2, u_3 \rangle) / J$$

where  $u_1, u_2, u_3$  are the Hurewicz images of the adjoint of the higher Whitehead products corresponding to the missing faces  $(3, 4), (1, 2, 3), (1, 2, 4)$ , respectively. The ideal  $J$  is determined by Jacobi identities and face relations based on the one missing face  $(3, 4)$  of dimension 1. Specifically, observe that the Jacobi identity gives  $[u_1, b_1] = [[b_3, b_4], b_1] = [b_3, [b_4, b_1]] + [b_4, [b_3, b_1]]$ . As both  $(1, 3)$  and  $(1, 4)$  are faces of  $K$ , we have  $[b_3, b_1] = 0$  and  $[b_4, b_1] = 0$ . Therefore  $[u_1, b_1] = 0$ . Similarly,  $[u_1, b_2] = 0$ . Thus  $J = ([u_1, b_1], [u_1, b_2])$ .

By Theorem 1.5, there is an algebra isomorphism

$$H_*(\Omega DJ_K) \cong U(L_{ab}\langle b_1, b_2, b_3, b_4 \rangle \amalg L\langle u_1, u_2, u_3 \rangle) / (I + J)$$

where  $u_1$  is the Hurewicz image of the adjoint of the Whitehead product  $\tilde{w}_1: S^3 \rightarrow \mathbb{C}P_3^\infty \vee \mathbb{C}P_4^\infty \rightarrow DJ_K$ , while  $u_2$  and  $u_3$  are the Hurewicz images of the adjoints of the higher Whitehead products  $\tilde{w}_2: S^5 \rightarrow FW(1, 2, 3) \rightarrow DJ_K$ , and  $\tilde{w}_3: S^5 \rightarrow FW(1, 2, 4) \rightarrow DJ_K$ , respectively;  $J$  is as above, and  $I = (b_i^2, [u_1, b_3], [u_2, b_4], [u_2, b_1], [u_2, b_2], [u_2, b_3], [u_3, b_1], [u_3, b_2], [u_3, b_4])$ .

By Theorem 1.7, the wedge summands of  $\mathcal{Z}_K$  and the maps to  $DJ_K$  are as follows. Part (a) gives the three summands  $S^3, S^5$  and  $S^5$  with maps  $\tilde{w}_1, \tilde{w}_2$  and  $\tilde{w}_3$  respectively. Part (b) gives two additional summands  $S^6$  and  $S^6$  from iterated Whitehead products

$$[\tilde{w}_2, \tilde{a}_4]: S^6 \rightarrow DJ_K \text{ and } [\tilde{w}_3, \tilde{a}_3]: S^6 \rightarrow DJ_K.$$

Part (c) is vacuous in this case. The possible Whitehead products are  $[\tilde{w}_1, \tilde{a}_1]$  and  $[\tilde{w}_1, \tilde{a}_2]$ . Note that this uses the fact that  $\tilde{w}_1 = [\tilde{a}_3, \tilde{a}_4]$  and so both  $[\tilde{w}_1, \tilde{a}_3]$  and  $[\tilde{w}_1, \tilde{a}_4]$  are ruled out as  $\tilde{a}_3$  appears twice in the first instance and  $\tilde{a}_4$  appears twice in the second. As for  $[\tilde{w}_1, \tilde{a}_1]$ , this corresponds to the algebraic element  $[u_1, b_1]$  which is in  $J$  and so is trivial. Similarly for  $[\tilde{w}_1, \tilde{a}_2]$ . Thus neither appears in  $\tilde{W}_{(3,4)}$ .

Collectively then, we obtain a homotopy equivalence  $\mathcal{Z}_K \simeq S^3 \vee 2S^5 \vee 2S^6$  and a map

$$S^3 \vee 2S^5 \vee 2S^6 \rightarrow DJ_K$$

which is the wedge sum of  $\tilde{w}_1$ ,  $\tilde{w}_2$ ,  $\tilde{w}_3$ ,  $[\tilde{w}_2, \tilde{a}_4]$  and  $[\tilde{w}_3, \tilde{a}_3]$ . Note that the homotopy equivalence matches that of [GT, Example 10.2], which was calculated by different means.

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