arXiv:1011.1609v1 [math.RT] 7 Nov 2010

A NOTE ON DERIVATIONS OF LIE ALGEBRAS

M. SHAHRYARI

Abstract. In this note, we will prove that a finite dimensional Lie algebra L of characteristic zero, admitting an abelian algebra of derivations $D \leq Der(L)$ with the property

$$L^n \subseteq \sum_{d\in D} d(L)$$

for some $n > 1$, is necessarily solvable. As a result, if L has a derivation $d : L \to L$, such that $L^n \subseteq d(L)$, for some $n > 1$, then L is solvable.

In [2], F. Ladisch proved that a finite group G, admitting an element a with the property $G' = [a, G]$, is solvable. Using this result, one can prove that a finite group is solvable, if it has a fixed point free automorphism. In this note, we prove a similar result for Lie algebras in a more general framework; we show that a finite dimensional Lie algebra L of characteristic zero, is solvable if it has an abelian subalgebra A with the property $L^n \subseteq [A, L]$, for some $n > 1$. Next, we use this result to prove that a finite dimensional Lie algebra L of characteristic zero, admitting an abelian algebra of derivations $D \leq Der(L)$ with the property

$$L^n \subseteq \sum_{d\in D} d(L)$$

for some $n > 1$, is necessarily solvable. As a special case, we conclude that if the Lie algebra L admits a derivation $d : L \to L$, such that $L^n \subseteq d(L)$, for some $n > 1$, then L is solvable. Note that a similar result was obtained by N. Jacobson in [1]: a finite dimensional Lie algebra of characteristic zero, admitting an invertible derivation, is nilpotent.

Our main theorem (Theorem 1 bellow) is also true for connected compact Lie groups and so, it may be also true for finite groups. Therefore, we ask the following question;

Let G be a finite group admitting an abelian subgroup A with the property $G^n \subseteq [A, G]$, for some $n > 1$. Is it true that G is solvable?

During this note, L is a finite dimensional Lie algebra over a field K of characteristic zero. By L^n and $L^{(n)}$, we will denote the n-th terms of the

Date: November 9, 2010.
MSC(2010): 17B40
Keywords: Lie algebras; Derivations; Solvable Lie algebras; Compact Lie groups.

lower central series and derived series of L, respectively. Also, $Der(L)$ will denote the algebra of derivations of L.

Theorem 1. Suppose there exists an abelian subalgebra $A \leq L$ and an integer $n > 1$, such that $L^n \subseteq [A, L]$. Then L is solvable.

Proof. Let $S = L^{n-1}$. First, we show that S is solvable. To do this, we use Cartan criterion. Let $x \in S$ and $y \in S'$. Since $S' = [L^{n-1}, L^{n-1}] \subseteq L^n \subseteq [A, L]$, so

$$y = \sum_i [a_i, u_i],$$

for some $a_1, \ldots, a_k \in A$ and $u_1, \ldots, u_k \in L$. Now, we have

$$Tr(ad_S x\ ad_S y) = \sum_i Tr(ad_S x\ ad_S [a_i, u_i]).$$

Since S is an ideal, we have $ad_S[a_i, u_i] = ad_S a_i\ ad_S u_i - ad_S u_i\ ad_S a_i$ and hence

$$\begin{aligned} Tr(ad_S x\ ad_S y) &= \sum_i Tr(ad_S x\ ad_S a_i\ ad_S u_i - ad_S x\ ad_S u_i\ ad_S a_i) \\ &= \sum_i Tr(ad_S a_i\ ad_S [u_i, x]). \end{aligned}$$

Now, $[u_i, x] \in [L, S] = L^n \subseteq [A, L]$, and so

$$[u_i, x] = \sum_j [b_{ij}, v_j],$$

for some $b_{i1}, \ldots, b_{il} \in A$ and $v_1, \ldots, v_l \in L$. Therefore

$$\begin{aligned} Tr(ad_S a_i\ ad_S [u_i, x]) &= \sum_j Tr\ (ad_S a_i\ ad_S [b_{ij}, v_j]) \\ &= \sum_j Tr\ (ad_S [a_i, b_{ij}] ad_S v_j) \\ &= 0. \end{aligned}$$

Therefore,

$$Tr(ad_S x\ ad_S y) = 0,$$

and hence S is solvable. We have $L^{(n-2)} \subseteq L^{n-1}$, so L is solvable.

As a result, we have;

Corollary 1. Suppose L is semisimple and A is an abelian subalgebra. Then $[A, L] \subsetneqq L$.

Using Lie functor, we can restate Theorem 1, for connected compact Lie groups;

Corollary 2. Suppose a connected compact Lie group G has an abelian Lie subgroup A, such that $G^n \subseteq [A, G]$, for some $n > 1$. Then G is solvable.

Theorem 2. Suppose there is an abelian subalgebra $D \leq Der(L)$ and an integer $n > 1$ such that

$$L^n \subseteq \sum_{d \in D} d(L).$$

Then L is solvable.

Proof. Suppose $\hat{L} = D \ltimes L$. Note that, elements of $\hat{L}$ are of the form (d, x), with $d \in D$ and $x \in L$. Also, we have

$$[(d, x), (d', y)] = (0, [x, y] + d(y) - d'(x)).$$

It is easy to see that for $d_1, \ldots, d_n \in D$ and $x_1, \ldots, x_n \in L$, we have

$$[(d_1, x_1), \ldots, (d_n, x_n)] = (0, [x_1, \ldots, x_n] + y),$$

for some $y \in \sum_{d \in D} d(L)$. Now,

$$[x_1, \ldots, x_n] \in \sum_{d \in D} d(L),$$

so there exists $\delta_1, \ldots, \delta_k \in D$ and $u_i, \ldots, u_k \in L$, such that

$$[x_1, \ldots, x_n] + y = \sum_i \delta_i(u_i).$$

We have

$$\begin{aligned} [(d_1, x_1), \ldots, (d_n, x_n)] &= (0, [x_1, \ldots, x_n] + y) \\ &= \sum_i (0, \delta_i(u_i)) \\ &= \sum_i [(\delta_i, 0), (0, u_i)] \\ &\in [D, \hat{L}]. \end{aligned}$$

Therefore, $\hat{L}^n \subseteq [D, \hat{L}]$, and hence $\hat{L}$ is solvable. So L is also solvable.

As a special case, if the Lie algebra L admits a derivation $d : L \to L$, such that $L^n \subseteq d(L)$, for some $n > 1$, then L is solvable.

Acknowledgment. The author would like to thank P. Shumyatiski and K. Ersoy for their comments and suggestions.

References

[1] N. Jacobson, *A note on automorphisms and derivations of Lie algebras*, Proc. of AMS. **6** (1955), 281 – 283.

[2] F. Ladisch, *Groups with anti-central elements*, Communications in Algebra **36** (2008), 2883 – 2894.

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran
E-mail address: mshahryari@tabrizu.ac.ir