

Using Hook Schur Functions to Compute Matrix Cocharacters

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Abstract

We develop a new integration method based on hook Schur functions instead of Schur functions to compute the cocharacters of matrices. We then use this method to compute some of the multiplicities in the cocharacter sequence of 3×3 matrices.

Let m_λ be the multiplicity of the irreducible S_n -character χ^λ in the pure trace cocharacter of $k \times k$ matrices over the characteristic zero field F . These multiplicities have been the subject of too many papers for us to attempt a list. Recent ones include [3], [7], [9] and [13]. An important tool in the study of the m_λ is the Poincaré series

$$P_n(x_1, \dots, x_n) = \sum m_\lambda S_\lambda(x_1, \dots, x_n),$$

where S_λ is the Schur function. The Poincaré series can be computed using complex integrals. It equals $(2\pi i)^{-k} (k!)^{-1}$ times

$$\oint_T \prod_{i,j=1}^k \prod_{\alpha=1}^n (1 - z_i z_j^{-1} x_\alpha)^{-1} \prod_{1 \leq i \neq j \leq k} (1 - z_i z_j^{-1})^{-1} \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_k}{z_k},$$

where T is the torus $|z_i| = 1$, $i = 1, \dots, k$, and the x_α are taken to be less than 1 in absolute value. If instead of Schur functions we work with hook Schur functions we get the double Poincaré series

$$P_{n,m}(t_1, \dots, t_n; u_1, \dots, u_m) = \sum m_\lambda H S_\lambda(t_1, \dots, t_n; u_1, \dots, u_m).$$

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Our main result is that this function can also be computed as a complex integral, namely, $(2\pi i)^{-k}(k!)^{-1}$ times

$$\oint_T \prod_{i,j=1}^k \prod_{\alpha=1}^n \prod_{\beta=1}^m (1 - z_i z_j^{-1} t_\alpha)^{-1} (1 + z_i z_j^{-1} u_\beta) \prod_{1 \leq i \neq j \leq k} (1 - z_i z_j^{-1})^{-1} \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_k}{z_k}.$$

We will also prove an analogous formula for mixed trace multiplicities, and to use them to compute some of the m_λ for $k = 3$. In the first section we prove the formula twice, the first proof being combinatorial and the second algebraic. Then, in the second section we apply the formula to determine the pure and mixed trace cocharacters m_λ and \bar{m}_λ for three by three matrices where λ has $\lambda_2 \leq 2$. These results are recorded in Tables 1 and 2.

1 Proof of the Formula

Definition 1.1. Let $f(z_1, \dots, z_k)$ be symmetric and let $\langle f, 1 \rangle$ denote $(2\pi i)^{-k}(k!)^{-1}$ times the integral

$$\oint_T f \prod_{1 \leq i \neq j \leq k} (1 - z_i z_j^{-1}) \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_k}{z_k},$$

where T is as in the introduction. As a special case of Weyl's integration theory, see [12] remark 26.20, if f is a character for the general linear group $GL_k(F)$, then $\langle f, 1 \rangle$ is the coefficient of the trivial character in f . Or, in terms of modules, if M is a $GL_k(F)$ -module with character f , then $\langle f, 1 \rangle$ is the dimension of the fixed points of M .

If λ is a partition we denoted by $S_\lambda(z_i z_j^{-1})$ the evaluation of the Schur function S_λ on the variables $z_i z_j^{-1}$, $i, j = 1, \dots, k$. The following theorems are from [10].

Theorem 1.2. Let m_λ and \bar{m}_λ be the multiplicities of χ^λ in the pure trace cocharacter and mixed trace cocharacter, respectively, of $M_k(F)$. Then

$$m_\lambda = \langle S_\lambda(z_i z_j^{-1}), 1 \rangle \text{ and } \bar{m}_\lambda = \left\langle \sum_{i,j=1}^k z_i z_j^{-1} S_\lambda(z_i z_j^{-1}), 1 \right\rangle.$$

In the introduction we defined the Poincaré series P_n and $P_{n,m}$ using the multiplicities m_λ . Analogously, we may define \bar{P}_n and $\bar{P}_{n,m}$ using \bar{m}_λ . In

order to make the transition from multiplicities to Poincaré series we need Berele and Remmel's generalization of the Cauchy identities from [5]:

$$\sum HS_\lambda(x_1, \dots, x_a; u_1, \dots, u_b) HS_\lambda(y_1, \dots, y_c; v_1, \dots, v_d) = \prod (1 + x_i v_j) \prod (1 + u_i y_j) \prod (1 - x_i y_j)^{-1} \prod (1 - u_i v_j)^{-1}. \quad (1)$$

Combining this formula with Theorem 1.2 we get our main theorem.

Theorem 1.3. $P_{n,m}$ equals

$$\left\langle \prod_{i,j=1}^k \prod_{\alpha=1}^n \prod_{\beta=1}^m (1 - z_i z_j^{-1} t_\alpha)^{-1} (1 + z_i z_j^{-1} u_\beta), 1 \right\rangle$$

and $\bar{P}_{n,m}$ equals

$$\left\langle \sum z_i z_j^{-1} \prod_{i,j=1}^k \prod_{\alpha=1}^n \prod_{\beta=1}^m (1 - z_i z_j^{-1} t_\alpha)^{-1} (1 + z_i z_j^{-1} u_\beta), 1 \right\rangle.$$

Proof. The two cases are similar. Here is the pure trace case:

$$\begin{aligned} P_{n,m} &= \sum_{\lambda} m_{\lambda} HS_{\lambda}(t; u) \\ &= \sum_{\lambda} \langle S_{\lambda}(z_i z_j^{-1}), 1 \rangle HS_{\lambda}(t; u) \\ &= \left\langle \sum_{\lambda} S_{\lambda}(z_i z_j^{-1}) HS_{\lambda}(t; u), 1 \right\rangle \\ &= \left\langle \prod (1 + z_i z_j^{-1} u) \prod (1 - z_i z_j^{-1} t)^{-1}, 1 \right\rangle \end{aligned}$$

□

We conclude this section by sketching an alternate proof using generic matrices instead of combinatorics. Let K be the free supercommutative algebra generated by the degree 0 (commuting) elements x_{ij}^{α} and the degree 1 (anticommuting) elements e_{ij}^{α} , $i, j = 1, 2, \dots, k$, $\alpha = 1, 2, \dots$. We construct the $k \times k$ matrices $X_{\alpha} = (x_{ij}^{\alpha})$ and $E_{\alpha} = (e_{ij}^{\alpha})$ and let $U^{n,m}$ be the algebra generated by X_1, \dots, X_n and E_1, \dots, E_m . The algebra $U^{n,m}$ is called the magnum of $M_k(F)$. It inherits a \mathbb{Z}_2 -grading from K and it has a $(n+m)$ -fold grading by degree in the generators. The map $str : U^{n,m} \rightarrow K$ given by

$str(a_{ij}) = \sum a_{ii}$ is a supertrace in the sense that if A and B are homogeneous with respect to the \mathbb{Z}_2 grading, then

$$str(AB) = (-1)^{\deg(A)\deg(B)}BA.$$

Denote by $STR^{n,m}$ the subalgebra of K generated by the image of $U^{n,m}$ under str . This algebra has two important properties which we record as lemmas. The former is theorem 4.3 of [2]. The latter is yet another variation of a standard lemma first proven in [8] and in [1], which has been proven about many types of cocharacters: ordinary, proper, with trace, with involution, with grading, etc.

Lemma 1.4. *The general linear group $GL_k(F)$ acts on K in such a way that $g(t_{ij}^\alpha)$ equals the (i, j) -entry of $gX_\alpha g^{-1}$ and $g(e_{ij}^\alpha)$ equals the (i, j) -entry of $gE_\alpha g^{-1}$, for all $g \in GL_k(F)$ and $STR^{n,m}$ is the fixed ring $K^{GL_k(F)}$.*

Lemma 1.5. *Let $f(t_1, \dots, t_n, u_1, \dots, u_m)$ be the Poincaré series of $STR^{n,m}$ with respect to its $(n+m)$ -fold grading. Then this series can be written as a series in hook Schur functions*

$$\sum_{\lambda} m_{\lambda} HS_{\lambda}(t_1, \dots, t_n, u_1, \dots, u_m),$$

where m_{λ} is the coefficient of χ^{λ} in the pure trace cocharacter of $M_k(F)$.

The rest of the proof follows from Weyl's integration formula. Note that the action of $GL_k(F)$ on $STR^{n,m}$ preserves the grading by degree. Let $D \in GL_k(F)$ be a generic diagonal matrix, $D = \text{diag}(z_1, \dots, z_k)$. Each x_{ij}^α and each e_{ij}^α in K is an eigenvector for D with eigenvalue equal to $z_i z_j^{-1}$. Hence, for each $\vec{i} \in \mathbb{N}^{n+m}$, the trace of D acting on the degree \vec{i} part of $STR^{n,m}$ equals the coefficient of $t_1^{v_1} \cdots u_m^{v_{n+m}}$ in

$$\prod_{i,j=1}^k \prod_{\alpha=1}^n \prod_{\beta=1}^m (1 - z_i z_j^{-1} t_{\alpha})^{-1} (1 + z_i z_j^{-1} u_{\beta}).$$

It follows from the two previous lemmas that the Poincaré series of the fixed ring is $P_{n,m}$ and then Weyl's integration formula implies that this Poincaré series is gotten by taking the inner product with 1. A similar argument holds for the mixed trace cocharacter.

2 Coefficients m_λ for 3×3 Matrices

We first do the case of $n = m = 1$ for 3×3 matrices. The series $P_{1,1}(t, u)$ is given by $\frac{1}{6(2\pi i)^3}$ times the integral over $|z_i| = 1$, $i = 1, 2, 3$ of

$$\prod_{i,j=1}^3 (1 + z_i z_j^{-1} u)(1 - z_i z_j^{-1} t)^{-1} \prod_{i \neq j} (1 - z_i z_j^{-1}) \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \frac{dz_3}{z_3}.$$

This can be computed using Cauchy's residue theorem, which we did using Maple. Although we are not including the program in this paper, the Appendix records the program we used to compute $P_{1,2}$ which can easily be adapted to the $P_{1,1}$ case. The result of the computation is a fraction which can be written as

$$P_{1,1}(t, u) = 1 + \frac{(t + u)N(t, u)}{(1 - t)(1 - t^2)(1 - t^3)}$$

where

$$\begin{aligned} N(t, u) = & (1 + t - t^3 - t^4 + t^5) + u(t + t^2 + t^3 - t^4) + u^2(1 + t + t^2 + t^3) \\ & + u^3(1 + 2t + 4t^2 + t^3) + u^4(1 + 2t + 4t^2 + 3t^3) + u^5(1 + 2t + 2t^2 + 3t^3) \\ & + u^6(t + 2t^2 + t^3) + u^7(1 + t^2) + u^8 \end{aligned} \quad (2)$$

Remark 2.1. The Poincaré series $P_{n,0}(t_1, \dots, t_n)$ satisfies the functional equation

$$P_{n,0}(t_1^{-1}, \dots, t_n^{-1}) = (-n)^k (t_1 \cdots t_n)^9 P_{n,0}(t_1, \dots, t_n).$$

Unfortunately, neither $P_{1,2}(t^{-1}, u)$ nor $P_{1,2}(t^{-1}, u^{-1})$ can be written in the form $\pm t^a u^b P_{1,2}(t, u)$.

In order to use the Poincaré series to compute the m_λ we need this fact about hook Schur functions.

Lemma 2.2. *Let $0 \neq \lambda \in H(1, 1)$, say $\lambda = (a + 1, 1^b)$. Then $HS_\lambda(t, u) = (t + u)t^a u^b$.*

Proof. Follows from theorem 6.20 of [4] with $k = \ell = 1$. □

We now do a special case that will prove important in subsequent computations.

Corollary 2.3. *$m_{(1^a)} = 1$ if $a = 0, 1, 3, 4, 5, 6, 8, 9$ and it equals zero otherwise.*

Proof. Setting $t = 0$ in $P_{1,1}(t, u)$ gives $P_{0,1}(u)$ and it equals

$$1 + u + u^3 + u^4 + u^5 + u^6 + u^8 + u^9.$$

On the other hand, $P_{0,1}(u) = \sum m_{(1^a)} u^a$. \square

It follows from Lemma 2.2 that the coefficient m_λ of $HS_\lambda(t, u)$ in $P_{1,1}$ equals the coefficient of $t^a u^b$ in $N(t, u)(1-t)^{-1}(1-t^2)^{-1}(1-t^3)^{-1}$. One could compute this by hand using partial fractions, but we used the `rgf_expand` command in Maple to perform the computation.

Theorem 2.4. *For each $b = 0, 1, \dots, 8$, the coefficient $m_{(a+1, 1^b)}$ equals $\alpha(b)a^2 + \beta(b)a + O(1)$, where $\alpha(b)$ and $\beta(b)$ are given by the table*

b	0	1	2	3	4	5	6	7	8
$\alpha(b)$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{12}$
$\beta(b)$	$\frac{2}{3}$	$\frac{2}{3}$	1	$\frac{11}{6}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{1}{2}$

To get the series $\bar{P}_{1,1}$ we simply add an additional factor of $\sum z_i z_j^{-1}$ to the integrand. This yields

$$\bar{P}_{1,1}(t, u) = 1 + \frac{(t+u)\bar{N}(t, u)}{(1-t)^2(1-t^2)}$$

where

$$\begin{aligned} \bar{N}(t, u) = & (2 - 2t^2 + t^3) + u(2 + 3t - t^2) + u^2(3 + 5t + 2t^2) \\ & + u^3(4 + 7t + 6t^2 + t^3) + u^4(4 + 8t + 7t^2 + 3t^3) + u^5(3 + 7t + 5t^2 + 3t^3) \\ & + u^6(2 + 4t + 3t^2 + t^3) + u^7(2 + t + t^2) + u^8 \end{aligned} \quad (3)$$

Here are the analogues of corollary 2.3 and theorem 2.4.

Corollary 2.5. *If $b = 0, 1, \dots, 9$ then the respective values of $\bar{m}_{(1^b)}$ are 1, 2, 2, 3, 4, 4, 3, 2, 2, 1*

Theorem 2.6. *For each $b = 0, 1, \dots, 9$, the coefficient $\bar{m}_{(a+1, 1^b)}$ equals $\alpha(b)a^2 + \beta(b)a + O(1)$, where $\alpha(b)$ and $\beta(b)$ are given by the table*

b	0	1	2	3	4	5	6	7	8
$\alpha(b)$	$\frac{1}{4}$	1	$\frac{5}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{9}{2}$	$\frac{5}{2}$	1	$\frac{1}{4}$
$\beta(b)$	$\frac{1}{2}$	$\frac{7}{2}$	$\frac{11}{2}$	7	$\frac{13}{2}$	5	$\frac{7}{2}$	$\frac{5}{2}$	1

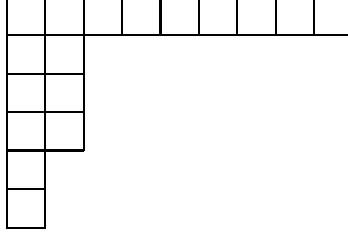


Figure 1: $\lambda = (9, 2^3, 1^2)$, $n = 7$ and $\mu = (5, 3)$

We now wish to compute m_λ and \bar{m}_λ for λ in $H(1, 2)$ and we need this analogue of lemma 2.2 for the one-by-two hook.

Lemma 2.7. *If $\lambda = (\lambda_1, 2^a, 1^b) \in H(1, 2)$ is a partition with $\lambda_1 \geq 2$ then*

$$HS_\lambda(t; u_1, u_2) = (t + u_1)(t + u_2)t^n S_\mu(u_1, u_2)$$

where $n = \lambda_1 - 2$ and $\mu = (a + b, b) = (\lambda'_1 - 1, \lambda'_2 - 1)$ and so $\lambda = (n + 2, 2^{\mu_1 - \mu_2}, 1^{\mu_2})$.

Based on Lemma 2.7 we will be making use of the correspondence $\lambda \leftrightarrow (n, \mu)$ between $H(1, 2)$ and $\mathbb{N} \times H(2, 0)$. See Figure 1 for an example.

Note that $\lambda_1 \leq 1$ if and only if $\lambda \in H(0, 1)$ and so in the case of $P_{1,2}$

$$\begin{aligned} P_{1,2}(t, u_1, u_2) &= \sum_{\lambda \in H(1,2)} m_\lambda HS_\lambda(t; u_1, u_2) \\ &= \sum_{\lambda \in H(0,1)} m_\lambda HS_\lambda(t; u_1, u_2) + \sum_{\lambda \in H(1,2)/H(0,1)} m_\lambda HS_\lambda(t; u_1, u_2). \end{aligned} \quad (4)$$

We already know the multiplicities m_λ for $\lambda \in H(0, 1)$ by corollary 2.3: $m_{(1^a)} = 1$ for $a = 0, 1, 3, 4, 5, 6, 8$ and 9 and it is zero for all other a . Hence

$$\begin{aligned} P_{1,2}^* &\stackrel{\text{DEF}}{=} \sum_{\lambda \in H(1,2)/H(0,1)} m_\lambda HS_\lambda(t; u_1, u_2) = \\ &P_{1,2}(t, u_1, u_2) - \sum \{HS_{(1^a)}(t, u_1, u_2) | a = 0, 1, 3, 4, 5, 6, 8, 9\}. \end{aligned} \quad (5)$$

Our strategy is to compute $P_{1,2}$ and $P_{1,2}^*$ using theorem 1.3 and a Maple computation which we included in the Appendix for the interested reader. The results are rational functions with denominator

$$(1 - t)(1 - t^2)(1 - t^3).$$

By lemma 2.7 $P_{1,2}^*$ can be re-written as

$$\begin{aligned} P_{1,2}^* &= (t + u_1)(t + u_2) \sum_{\mu} \sum_n m_{\lambda} t^n S_{\mu}(u_1, u_2) \\ &= (t + u_1)(t + u_2) \sum_{\mu} f_{\mu}(t) S_{\mu}(u_1, u_2) \\ &= (t + u_1)(t + u_2) \sum_{\mu} \frac{g_{\mu}(t) S_{\mu}(u_1, u_2)}{(1-t)(1-t^2)(1-t^3)} \end{aligned}$$

where $n = \lambda - 2$, $\mu = (\lambda'_1 - 1, \lambda'_2 - 1)$ and where $g_{\mu}(t)$ is a polynomial. We summarize in this lemma.

Lemma 2.8. *Let $\lambda \in H(1, 2)$ have $\lambda_1 \geq 2$ and let $\mu = (\lambda'_1 - 1, \lambda'_2 - 1)$. Then, for $g_{\mu}(t)$ as above m_{λ} equals the coefficient of $t^{\lambda_1 - 2} = t^n$ in $g_{\mu}(t)(1-t)^{-1}(1-t^2)^{-1}(1-t^3)^{-1}$.*

In order to read off the $g_{\mu}(t)$ we divide the numerator of $P_{1,2}^*$ by $(t + u_1)(t + u_2)$ and multiply by $u_1 - u_2$: g_{μ} will be the coefficient of $u_1^{\mu_1 + 1} u_2^{\mu_2}$. The result of the computation is in table 1. In the interest of saving space, we have omitted the cases of $\mu = (1^a)$ since this case was done in theorem 2.4, and the case of $\mu_1 = 8$ because of Formanek's theorem which says that if $\lambda_9 \geq 1$ then

$$m_{(\lambda_1, \dots, \lambda_9)} = m_{(\lambda_1 - 1, \dots, \lambda_9 - 1)},$$

and so in our case the partitions $\lambda = (\lambda_1, 2^a, 1^{8-a})$ and $\tilde{\lambda} = (\lambda_1 - 1, 1^a)$ have the same multiplicities, $m_{\lambda} = m_{\tilde{\lambda}}$.

The case of the mixed trace multiplicities \bar{m}_{λ} is similar. The integrand has an extra factor of $\sum z_i z_j^{-1}$ and the resulting Poincaré series $\bar{P}_{1,2}$ is a rational function with denominator $(1-t)^2(1-t^2)$. As in equation (5) we define

$$\begin{aligned} \bar{P}_{1,2}^* &= \sum_{\lambda \in H(1,2)/H(0,1)} m_{\lambda} HS_{\lambda}(t; u_1, u_2) \\ &= \bar{P}_{1,2} - \sum_{\lambda \in H(0,1)} m_{\lambda} HS_{\lambda}(t; u_1, u_2), \end{aligned}$$

where, by Corollary 2.5 the last sum is

$$\begin{aligned} HS_0 + 2HS_{(1)} + 2HS_{(1^2)} + 3HS_{(1^3)} + 4HS_{(1^4)} + 4HS_{(1^5)} + 3HS_{(1^6)} \\ + 2HS_{(1^7)} + 2HS_{(1^8)} + HS_{(1^9)}. \end{aligned}$$

The result is a function of the form

$$\bar{P}_{1,2} = (t + u_1)(t + u_2) \sum_{\mu} \frac{\bar{g}_{\mu} S_{\mu}(u_1, u_2)}{(1 - t)^2(1 - t^2)}.$$

Just as in the case of pure trace identities, we can compute the \bar{g}_{μ} and use them to compute the m_{λ} . The results are recorded in Table 2.

μ	$g_\mu(t)$	m_λ
(7,7)	$t^4 + 2t^2 + t + 2$	$\frac{n^2+3n}{2} + O(1)$
(7,6)	$t^5 + 3t^4 + 3t^3 + 6t^2 + 4t + 1$	$\frac{3n^2}{2} + 2n + O(1)$
(7,5)	$3t^5 + 5t^4 + 10t^3 + 9t^2 + 7t + 2$	$3n^2 + 3n$
(7,4)	$3t^5 + 7t^4 + 13t^3 + 13t^2 + 9t + 3$	$4n^2 + \frac{9n}{2} + O(1)$
(7,3)	$t^5 + 6t^4 + 9t^3 + 14t^2 + 9t + 3$	$\frac{7n^2+11n}{2} + O(1)$
(7,2)	$2t^4 + 5t^3 + 8t^2 + 6t + 3$	$2n^2 + \frac{9n}{2} + O(1)$
(7,1)	$-t^4 + 2t^3 + 3t^2 + 4t + 1$	$\frac{3n^2+10n}{4} + O(1)$
(6,6)	$t^6 + 2t^5 + 7t^4 + 5t^3 + 7t^2 + 5t + 3$	$\frac{5n^2}{2} + 2n + O(1)$
(6,5)	$3t^6 + 8t^5 + 16t^4 + 19t^3 + 18t^2 + 10t + 4$	$\frac{13n^2+3n}{2} + O(1)$
(6,4)	$3t^6 + 12t^5 + 22t^4 + 31t^3 + 26t^2 + 16t + 4$	$\frac{19n^2+5n}{2} + O(1)$
(6,3)	$t^6 + 8t^5 + 19t^4 + 29t^3 + 27t^2 + 19t + 5$	$9n^2 + 7n$
(6,2)	$2t^5 + 8t^4 + 16t^3 + 20t^2 + 14t + 6$	$\frac{11n^2}{2} + 9n + O(1)$
(6,1)	$t^4 + 4t^3 + 8t^2 + 7t + 4$	$2n^2 + \frac{11n}{2} + O(1)$
(5,5)	$6t^6 + 8t^5 + 18t^4 + 20t^3 + 18t^2 + 8t + 6$	$7n^2 + O(1)$
(5,4)	$8t^6 + 19t^5 + 32t^4 + 42t^3 + 37t^2 + 18t + 6$	$\frac{27n^2-n}{2} + O(1)$
(5,3)	$3t^6 + 18t^5 + 30t^4 + 46t^3 + 40t^2 + 25t + 6$	$14n^2 + \frac{11n}{2} + O(1)$
(5,2)	$6t^5 + 15t^4 + 30t^3 + 29t^2 + 20t + 8$	$9n^2 + 11n + O(1)$
(5,1)	$2t^4 + 9t^3 + 15t^2 + 10t + 6$	$\frac{7n^2+17n}{2} + O(1)$
(4,4)	$6t^6 + 13t^5 + 26t^4 + 27t^3 + 27t^2 + 14t + 7$	$10n^2 + n + O(1)$
(4,3)	$3t^6 + 16t^5 + 33t^4 + 40t^3 + 40t^2 + 23t + 7$	$\frac{27n^2+11n}{2} + O(1)$
(4,2)	$6t^5 + 17t^4 + 32t^3 + 32t^2 + 21t + 6$	$\frac{19n^2+21n}{2} + O(1)$
(4,1)	$2t^4 + 11t^3 + 16t^2 + 14t + 5$	$4n^2 + \frac{19n}{2}$
(3,3)	$t^6 + 5t^5 + 19t^4 + 18t^3 + 21t^2 + 13t + 7$	$7n^2 + 6n + O(1)$
(3,2)	$2t^5 + 11t^4 + 20t^3 + 24t^2 + 15t + 6$	$\frac{13n^2+19n}{2} + O(1)$
(3,1)	$8t^3 + 12t^2 + 12t + 4$	$3n^2 + 8n + O(1)$
(2,2)	$3t^4 + 6t^3 + 9t^2 + 6t + 6$	$\frac{5n^2}{2} + 6n + O(1)$
(2,1)	$-t^4 + 3t^3 + 7t^2 + 5t + 4$	$\frac{3n^2}{2} + 5n + O(1)$
(1,1)	$-t^3 + 2t^2 + 2t + 3$	$\frac{n^2+5n}{2} + O(1)$

Table 1: Multiplicities in the Pure Trace Cocharacter, λ, μ as in Lemma 2.7

μ	$g_\mu(t)$	m_λ
(7,7)	$t^4 + t^3 + 3t^2 + 3t + 4$	$3n^2 + 4n + O(1)$
(7,6)	$t^5 + 4t^4 + 7t^3 + 12t^2 + 13t + 7$	$11n^2 + \frac{9n}{2} + O(1)$
(7,5)	$3t^5 + 8t^4 + 18t^3 + 24t^2 + 26t + 11$	$\frac{45n^2+5n}{2}$
(7,4)	$3t^5 + 10t^4 + 23t^3 + 33t^2 + 34t + 15$	$\frac{59n^2}{2} + 6n + O(1)$
(7,3)	$t^5 + 7t^4 + 16t^3 + 30t^2 + 32t + 16$	$\frac{51n^2+31n}{2} + O(1)$
(7,2)	$2t^4 + 6t^3 + 15t^2 + 22t + 13$	$\frac{29n^2}{2} + 19n + O(1)$
(7,1)	$t^3 + t^2 + 11t + 8$	$\frac{21n^2}{4} + 13n + O(1)$
(6,6)	$t^6 + 3t^5 + 10t^4 + 14t^3 + 19t^2 + 16t + 9$	$18n^2 - \frac{13n}{2} + O(1)$
(6,5)	$3t^6 + 11t^5 + 27t^4 + 43t^3 + 53t^2 + 44t + 19$	$50n^2 - 30n + O(1)$
(6,4)	$3t^6 + 15t^5 + 37t^4 + 65t^3 + 79t^2 + 70t + 27$	$74n^2 - 36n + O(1)$
(6,3)	$t^6 + 9t^5 + 28t^4 + 56t^3 + 75t^2 + 74t + 31$	$\frac{137n^2-7n}{2} + O(1)$
(6,2)	$2t^5 + 10t^4 + 26t^3 + 44t^2 + 52t + 28$	$\frac{81n^2}{2} + 28n + O(1)$
(6,1)	$t^4 + 5t^3 + 11t^2 + 23t + 18$	$\frac{29n^2}{2} + 26n + O(1)$
(5,5)	$6t^6 + 14t^5 + 32t^4 + 46t^3 + 56t^2 + 40t + 18$	$53n^2 - 50n + O(1)$
(5,4)	$8t^6 + 27t^5 + 59t^4 + 93t^3 + 111t^2 + 88t + 34$	$105n^2 - 84n + O(1)$
(5,3)	$3t^6 + 21t^5 + 51t^4 + 94t^3 + 117t^2 + 108t + 42$	$109n^2 - \frac{79n}{2}$
(5,2)	$6t^5 + 21t^4 + 50t^3 + 74t^2 + 83t + 40$	$\frac{137n^2+53n}{2}$
(5,1)	$3t^4 + 11t^3 + 22t^2 + 38t + 28$	$\frac{51n^2+77n}{2} + O(1)$
(4,4)	$6t^6 + 19t^5 + 45t^4 + 66t^3 + 80t^2 + 61t + 27$	$76n^2 - 61n + O(1)$
(4,3)	$3t^6 + 19t^5 + 52t^4 + 89t^3 + 114t^2 + 100t + 43$	$105n^2 - 38n + O(1)$
(4,2)	$6t^5 + 23t^4 + 54t^3 + 81t^2 + 90t + 42$	$74n^2 + 28n + O(1)$
(4,1)	$3t^4 + 13t^3 + 24t^2 + 47t + 31$	$\frac{59n^2}{2} + 45n + O(1)$
(3,3)	$t^6 + 6t^5 + 25t^4 + 42t^3 + 59t^2 + 52t + 27$	$53n^2 - 4n + O(1)$
(3,2)	$2t^5 + 13t^4 + 32t^3 + 55t^2 + 64t + 34$	$50n^2 + 34n + O(1)$
(3,1)	$t^4 + 8t^3 + 15t^2 + 39t + 27$	$\frac{45n^2+83n}{2} + O(1)$
(2,2)	$3t^4 + 8t^3 + 18t^2 + 25t + 18$	$18n^2 + \frac{47n}{2} + O(1)$
(2,1)	$2t^3 + 4t^2 + 19t + 19$	$11n^2 + \frac{55n}{2} + O(1)$
(1,1)	$-t^2 + 4t + 9$	$3n^2 + 11n + O(1)$

Table 2: Multiplicities in the Mixed Trace Cocharacter, λ, μ as in Lemma 2.7

Appendix

```

> P:=(n- > product((1+z1/z2*u(i))*(1+z2/z1*u(i))*(1+z1/z3*u(i))
  *(1+z3/z1*u(i))*(1+z2/z3*u(i))*(1+z3/z2*u(i))*(1+u(i))^3,i=1..n)):
Q:=(n- > product((1-z1/z2*t(i))*(1-z2/z1*t(i))*(1-z1/z3*t(i))
  *(1-z3/z1*t(i))*(1-z2/z3*t(i))*(1-z3/z2*t(i))*(1-t(i))^3,i=1..n)):
> dis:=(1-z1/z2)*(1-z2/z1)*(1-z1/z3)*(1-z3/z1)*(1-z2/z3)*(1-z3/z2)
  /z1/z2/z3/6:
> A:=P(2)/Q(1)*dis: {A is the function we will integrate to compute
  P1,2. We omitted the factor of (2πi)-3 because it seems easiest for
  Maple to compute the integral using the residue command. To compute
  P̄1,2 we would need to add a factor of (z1+z2+z3)*(1/z1+1/z2+1/z3)}.
> A1:=residue(A,z1=0)+residue(A,z1=t(1)*z2)+residue(A,z1=t(1)*z3):
> A1:=simplify(A1):
> factor(denom(A1)); {We need the denominator to compute the next
  residues.}
  6z25z35(t(1)+1)(t(1)-1)3(t(1)z2-z3)(z2-z3t(1)2)(z2-t(1)z3)(z2-t(1)2z3)
> A2:=residue(A1,z2=0)+residue(A1,z2=z3*t(1))+residue(A1,z2=z3*t(1)^2):
> A2:=simplify(A2):
> factor(denom(A2));
  t(1)12(t(1)+1)6(t(1)-1)21z313(t(1)2+t(1)+1)2(t(1)2+1)
> A3:=residue(A2,z3=0); {A is P1,2}
> h := a-> sum(u(1)^i*u(2)^(a-i), i = 0 .. a); {h(n) is hn(u(1), u(2))}
> hs := n-> h(n)+t(1)*h(n-1); {hs(n) is HSn(t(1), t(2); u(1))}
> B := simplify(A3-hs(0)-hs(1)-hs(3)-hs(4)-hs(5)-hs(6)-hs(8)-hs(9)); {B
  is P̄1,2}
> B1 := numer(B)*(u(1)-u(2))/((t(1)+u(1))*(t(1)+u(2)));
> g:=-((a,b)->coeff(coeff(B1, u(1)^(a+1)),u(2)^b));
> for a from 1 to 9 do for b from 1 to a do print(g(a,b),a,b); end do end
  do

```

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