# LECTURES ON PRINCIPAL BUNDLES OVER PROJECTIVE VARIETIES

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In these notes we will always work with schemes over the field of complex numbers $\mathbb{C}$. Let $X$ be a scheme. A vector bundle of rank $r$ on $X$ is a scheme with a surjective morphism $p : \mathbb{V} \to X$ and an equivalence class of linear atlases. A linear atlas is an open cover $\{U_i\}$ of $X$ (in the Zariski topology) and isomorphisms $\psi_i : p^{-1}(U_i) \to U_i \times \mathbb{C}^r$, such that $p = p_X \circ \psi_i$, and $\psi_j \circ \psi_i^{-1}$ is linear on the fibers. Two atlases are equivalent if their union is an atlas. These two properties are usually expressed by saying that a vector bundle is locally trivial (in the Zariski topology), and the fibers have a linear structure.

An isomorphism of vector bundles on $X$ is an isomorphism $\varphi : \mathbb{V} \to \mathbb{V}'$ of schemes which is compatible with the linear structure. That is, $p = p' \circ \varphi$ and the covering $\{U_i\} \bigcup \{U_i'\}$ together with the isomorphisms $\psi_i$, $\psi_i' \circ \varphi$ is a linear structure on $\mathbb{V}$ as before.

The set of isomorphism classes of vector bundles of rank $r$ on $X$ is canonically bijective to the Cech cohomology set $\check{H}^1(X, \underline{\mathrm{GL}_r})$. Indeed, since the transition functions $\psi_j \circ \psi_i^{-1}$ are linear on the fibers, they are given by morphisms $\alpha_{ij} : U_i \cap U_j \to \mathrm{GL}_r$ which satisfy the cocycle condition.

Given a vector bundle $\mathbb{V} \to X$ we define the locally free sheaf $E$ of its sections, which to each open subset $U \subset X$, assigns $E(U) = \Gamma(U, p^{-1}(U))$. This provides an equivalence of categories between the categories of vector bundles and that of locally free sheaves ([Ha, Ex. II.5.18]). Therefore, if no confusion seems likely to arise, we will use the words "vector bundle" and "locally free sheaf" interchangeably. Note that a vector bundle of rank 1 is the same thing as a line bundle. We will be interested in constructing moduli spaces of vector bundles, which can then be considered as generalizations of the Jacobian. Sometimes it will be necessary to consider also torsion free sheaves. For instance, in order to compactify the moduli space of vector bundles.

Let $X$ be a smooth projective variety of dimension $n$ with an ample line bundle $\mathcal{O}_X(1)$ corresponding to a divisor $H$. Let $E$ be a torsion free sheaf on $X$. Its Chern classes are denoted $c_i(E) \in H^{2i}(X; \mathbb{C})$. We define the *degree* of $E$

$$\deg E = c_1(E) H^{n-1}$$

and its Hilbert polynomial

$$P_E(m) = \chi(E(m)),$$

where $E(m) = E \otimes \mathcal{O}_X(m)$ and $\mathcal{O}_X(m) = \mathcal{O}_X(1)^{\otimes m}$.

If $E$ is locally free, we define the determinant line bundle as $\det E = \bigwedge^r E$. If $E$ is torsion free, since $X$ is smooth, we can still define its determinant as follows. The maximal open subset $U \subset X$ where $E$ is locally free is *big* (with this we will mean that its complement has codimension at least two), because it is torsion free. Therefore, there is a line bundle $\det E|_U$ on $U$, and since $U$ is big and $X$ is smooth, this extends to a unique line bundle on $X$, which we call the determinant $\det E$ of $E$. It can be proved that $\deg E = \deg \det E$.

We will use the following notation. Whenever "(semi)stable" and "($\leq$)" appears in a sentence, two sentences should be read. One with "semistable" and "$\leq$", and another with "stable" and "$<$". Given two polynomials $p$ and $q$, we write $p < q$ if $p(m) < q(m)$ when $m \gg 0$.

A torsion free sheaf $E$ is *(semi)stable* if for all proper subsheaves $F \subset E$,

$$\frac{P_F}{\operatorname{rk} F} (\leq) \frac{P_E}{\operatorname{rk} E} \ .$$

A sheaf is called *unstable* if it is not semistable. Sometimes this is referred to as Gieseker (or Maruyama) stability.

A torsion free sheaf $E$ is *slope-(semi)stable* if for all proper subsheaves $F \subset E$ with $\operatorname{rk} F < \operatorname{rk} E$,

$$\frac{\deg F}{\operatorname{rk} F} (\leq) \frac{\deg E}{\operatorname{rk} E} \ .$$

The number $\deg E / \operatorname{rk} E$ is called the *slope* of $E$. A sheaf is called *slope-unstable* if it is not slope-semistable. Sometimes this is referred to as Mumford (or Takemoto) stability.

Using Riemann-Roch theorem, we find

$$P_E(m) = \operatorname{rk} E \frac{m^n}{n!} + (\deg E - \operatorname{rk} E \frac{\deg K}{2}) \frac{m^{n-1}}{(n-1)!} + \cdots$$

where $K$ is the canonical divisor. From this it follows that

$$\text{slope-stable} \Longrightarrow \text{stable} \Longrightarrow \text{semistable} \Longrightarrow \text{slope-semistable}$$

Note that, if $n = 1$, Gieseker and Mumford (semi)stability coincide, because the Hilbert polynomial has degree 1.

Let $E$ be a torsion free sheaf on $X$. There is a unique filtration, called the Harder-Narasimhan filtration,

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_l = E$$

such that $E^i = E_i / E_{i-1}$ is semistable and

$$\frac{P_{E^i}}{\operatorname{rk} E^i} > \frac{P_{E^{i+1}}}{\operatorname{rk} E^{i+1}}$$

for all $i$. In particular, any torsion free sheaf can be described as successive extensions of semistable sheaves.

There is also a Harder-Narasimhan filtration for slope stability: this is the unique filtration such that $E^i = E_i / E_{i-1}$ is slope-semistable and

$$\frac{\deg E^i}{\operatorname{rk} E^i} > \frac{\deg E^{i+1}}{\operatorname{rk} E^{i+1}}$$

for all $i$. We denote

$$\mu_{\max}(E) = \mu(E^1) \ , \quad \mu_{\min}(E) = \mu(E^l).$$

Of course, in general these two filtrations will be different.

Now let $E$ be a semistable sheaf. There is a filtration, called the Jordan-Hölder filtration,

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_l = E$$

such that $E^i = E_i/E_{i-1}$ is stable and

$$\frac{P_{E^i}}{\operatorname{rk} E^i} = \frac{P_{E^{i+1}}}{\operatorname{rk} E^{i+1}}$$

for all $i$. This filtration is not unique, but the associated graded sheaf

$$\operatorname{gr}_{\mathrm{JH}}(E) = \bigoplus_{i=1}^{l} E^i$$

is unique up to isomorphism. It is easy to check that $\operatorname{gr}_{\mathrm{JH}}(E)$ is semistable. Two semistable torsion free sheaves are called *$S$-equivalent* if $\operatorname{gr}_{\mathrm{JH}}(E)$ and $\operatorname{gr}_{\mathrm{JH}}(E')$ are isomorphic.

There is also a Jordan-Hölder filtration for slope stability, just replacing Hilbert polynomials with degrees.

A *family of coherent sheaves* parameterized by a scheme $T$ (also called $T$-family) is a coherent sheaf $E_T$ on $X \times T$, flat over $T$. For each closed point $t \in T$, we get a sheaf $E_t := f^* E_T$ on $X \times t \cong X$, where $f : X \times t \to X \times T$ is the natural inclusion. We say that $E_T$ is a family of torsion-free sheaves if $E_t$ is torsion-free sheaf for all closed points $t \in T$. We have analogous definitions for any open condition, and hence we can talk of families of (semi)stable sheaves, of families of sheaves with fixed Chern classes $c_i(E)$, etc... Two families are *isomorphic* if $E_T$ and $E'_T$ are isomorphic as sheaves.

To define the notion of moduli space, we will first look at the Jacobian $J$ of a projective scheme $X$. There is a bijection between isomorphism classes of line bundles $L$ with $0 = c_1(L) \in H^2(X; \mathbb{C})$ and closed points of $J$.

Furthermore, if we are given a family of line bundles $L_T$, with vanishing first Chern class, parameterized by a scheme $T$, we obtain a morphism $f : T \to J$ such that for all $t \in T$, the point $f(t) \in J$ is the point corresponding to the isomorphism class of $L_t$. And, conversely, if we are given a morphism $f : T \to J$, we obtain a family of line bundles parameterized by $T$ by pulling-back a Poincare line bundle: $L_T = (\mathrm{id}_X \times f)^* \mathcal{P}$.

Note that both constructions are not quite inverse to each other. On the one hand, if $M$ is a line bundle on $T$, the families $L_T$ and $L_T \otimes p_T^* M$ give the same morphism from $T$ to the Jacobian, and on the other hand, there is no unique Poincare line bundle: given a linen bundle $M$ on $J$, $\mathcal{P} \otimes p_J^* M$ is also a Poincare line bundle, and the family induced by $f$ will change to $L_T \otimes p_T^* f^* M$. This is why we declare two families of line bundles *equivalent* if they differ by the pullback of a line bundle on the parameter space $T$.

Using this equivalence, both constructions become inverse of each other. That is, there is a bijection between equivalence classes of families and morphisms to the Jacobian.

One could ask: is there a "better" version of the Jacobian?, i.e. some object $\mathcal{J}$ which provides a bijection between morphisms to it and families of line bundles up to isomorphism (not up to equivalence). The answer is yes, but this object $\mathcal{J}$ is not a scheme!. It is an algebraic stack (in the sense of Artin): the Jacobian stack. A stack is a generalization of the notion of scheme, but we will not consider it here.

We would like to have a scheme with the same properties as the Jacobian, but for torsion-free sheaves instead of line bundles. To be able to do this, we have to consider only the semistable ones. Then there will be a moduli scheme $\mathfrak{M}(r, c_i)$ such that a family of semistable torsion-free sheaves parameterized by $T$, with rank $r$ and

Chern classes $c_i$, will induce a morphism from the parameter space $T$ to $\mathfrak{M}(r,c_i)$. In particular, to each semistable torsion free sheaf we associate a closed point. If two stable torsion free sheaves on $X$ are not isomorphic, they will correspond to different points of $\mathfrak{M}(r,c_i)$, but it can happen that two strictly semistable torsion free sheaves on $X$ which are not isomorphic correspond to the same point of $\mathfrak{M}(r,c_i)$. In fact, $E$ and $E'$ correspond to the same point if and only if they are S-equivalent.

Another difference with the properties of the Jacobian is that in general there will be no "universal torsion free sheaf" on $X \times \mathfrak{M}(r,c_i)$, i.e. there will be no analogue of the Poincare bundle. In other words, given a morphism $f: T \to \mathfrak{M}(r,c_i)$, there might be no family parameterized by $T$ which induces $f$. If there is a universal torsion-free sheaf, we say that $\mathfrak{M}(r,c_i)$ is a *fine moduli space*, and if it does not exist, we say that it is a *coarse moduli space*.

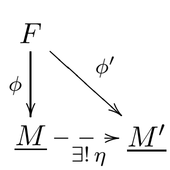
To explain this more precisely, it is useful to use the language of representable functors. Given a scheme $M$ over $\mathbb{C}$, we define a (contravariant) functor $\underline{M} := \mathrm{Mor}(-,M)$ from the category of $\mathbb{C}$-schemes $(\mathrm{Sch}/\mathbb{C})$ to the category of sets (Sets) by sending an $\mathbb{C}$-scheme $B$ to the set of morphisms $\mathrm{Mor}(B,M)$. On morphisms it is defined with composition, i.e., to a morphism $f: B \to B'$ we associate the map $\mathrm{Mor}(B',M) \to \mathrm{Mor}(B,M)$ which sends $\varphi'$ to $\varphi \circ f$.

**Definition 0.1** (Represents)**.** *A functor $F: (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{Sets})$ is represented by a scheme $M$ if there is an isomorphism of functors $F \cong \underline{M}$.*

Of course, not all functors from $(\mathrm{Sch}/\mathbb{C})$ to (Sets) are representable, but if a functor $F$ is, then the scheme $M$ is unique up to canonical isomorphism. Given a morphism $f: M \to M'$, we obtain an natural transformation $\underline{M} \to \underline{M'}$, and, by Yoneda's lemma, every natural transformation between representable functors is induced by a morphism of schemes. In other words, the category of schemes is a full subcategory of the category of functors $(\mathrm{Sch}/\mathbb{C})'$, whose objects are contravariant functors from $(\mathrm{Sch}/\mathbb{C})$ to (Sets) and whose morphisms are natural transformation. Therefore, we will denote by the same letter a morphism of schemes and the associated natural transformation.

For instance, let $F_J: (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{Sets})$ be the functor which sends a scheme $T$ to the set of equivalence classes of $T$-families of line bundles on $X$, with $c_1 = 0$. This functor is represented by the Jacobian, i.e., there is an isomorphism of functors $F_J \cong \underline{J}$. This is the translation, to the language of representable functors, of the fact that there is a natural bijection between the set of equivalence classes of these families and the set of morphisms from $T$ to $J$.

**Definition 0.2** (Corepresents)**.** *A functor $F: (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{Sets})$ is corepresented by a scheme $M$ if there is a natural transformation of functors $\phi: F \to \underline{M}$ such that given another scheme $M'$ and natural transformation $\phi': F \to \underline{M'}$, there is a unique morphism $\eta: M \to M'$ with $\phi' = \eta \circ \phi$.*



If $M$ corepresents $F$, then $M$ is unique up to canonical isomorphism. To explain why this is called "corepresentation", let $(\mathrm{Sch}/\mathbb{C})'$ be the above defined functor category. Then it can be seen that $M$ represents $F$ if and only if there is a natural bijection $\mathrm{Mor}(Y,M) = \mathrm{Mor}_{(\mathrm{Sch}/\mathbb{C})'}(\underline{Y},F)$ for all schemes $Y$. On the other hand, $M$ corepresents $F$ if and only if there is a natural bijection $\mathrm{Mor}(M,Y) = \mathrm{Mor}_{(\mathrm{Sch}/\mathbb{C})'}(F,\underline{Y})$ for all schemes $Y$. If $M$ represents $F$, then it corepresents it, but the converse is not true.

Let $X$ be a fixed $\mathbb{C}$-scheme. Define a functor $F^{ss}_{r,c_i}$ from the category of schemes over $\mathbb{C}$ to the category of sets, sending a scheme $T$ to the set $F^{ss}_{r,c_i}(T)$ of isomorphism classes of families of torsion-free sheaves on $X$ parameterized by $T$, with rank $r$ and Chern classes $c_i$. On morphisms it is defined by pullback, i.e., to a morphism $f: T \to T'$ we associate the map $F(T') \to F(T)$ which sends the family $E'_T$ to $(\mathrm{id}_X \times f)^* E'_T$. Analogously, we define the functor $F^{s}_{r,c_i}$ of families of stable torsion free sheaves.

It can be shown that, for any polarized smooth projective variety $X$, there is a scheme $\mathfrak{M}(r,c_i)$ corepresenting the above defined functor $F^{ss}_{r,c_i}$ ([Gi, Ma, Sesh, Si]). In section 1 we will sketch a proof of this result.

Note that the transformation of functors $\phi$ gives, for any $T$-family of semistable torsion free sheaves, a morphism $f: T \to \mathfrak{M}(r,c_i)$. As we mentioned before, there is a canonical bijection between closed points of $\mathfrak{M}(r,c_i)$ and S-equivalence classes of semistable torsion free sheaves.

Let $\hat{F}^{ss}_{r,c_i}$ be the functor of equivalence classes of families of semistable sheaves, where, as before, we declare two families equivalent if they differ by the pullback of a line bundle on the parameter space. There are some cases in which this functor is representable (for instance, if the rank $r$ and degree $c_1$ are coprime). In these cases, there is a universal family parameterized by the moduli space, and this universal family is unique up to the pullback of a line bundle on the moduli space.

**Definition 0.3** (Moduli space). *We say that $M$ is a moduli space for a set of objects, if it corepresents the functor of families of those objects.*

**Definition 0.4** (Coarse moduli). *A scheme $M$ is called a coarse moduli scheme for $F$ if it corepresents $F$ and furthermore the map*

$$\phi(\mathrm{Spec}\,\mathbb{C}) : F(\mathrm{Spec}\,\mathbb{C}) \to \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}, M)$$

*is bijective.*

Note that if a functor $F$ is corepresented by a scheme $M$, then it is a coarse moduli scheme for the functor $\widetilde{F}$ of S-equivalence classes of $F$, i.e., the functor defined as

$$\widetilde{F}(T) = \begin{cases} F(T) & \text{, if } T \neq \mathrm{Spec}\,\mathbb{C} \\ \text{S-equivalence classes of objects of } F(\mathrm{Spec}\,\mathbb{C}) & \text{, if } T = \mathrm{Spec}\,\mathbb{C} \end{cases}$$

## 1. Moduli space of torsion free sheaves

In this section we will sketch the proof of the existence of the moduli space of semistable torsion free sheaves. We will start by giving a brief idea of the construction. It can be shown that there is a scheme $Y$ classifying *semistable based sheaves*, that is, pairs $(f,E)$, where $E$ is a semistable sheaf and $f: V \to H^0(E(m))$ is an isomorphism between a fixed vector space $V$ and $H^0(E(m))$. The group $\mathrm{SL}(V)$ acts on $Y$ by "base change": an element $g \in \mathrm{SL}(V)$ sends the pair $(f,E)$ to $(f \circ g, E)$. Two pairs $(f,E)$ and $(f',E')$ are in the same orbit if and only if $E$ is isomorphic to

$E'$, therefore, the quotient of $Y$ by the action of $\mathrm{SL}(V)$ will be a moduli space of semistable sheaves. But, does this quotient exist in the category of schemes?, i.e., is there a scheme whose points are in bijection with the $\mathrm{SL}(V)$-orbits in $Y$?. In general the answer is no, but Geometric Invariant Theory (GIT) gives us something which is quite close to this, and is called the GIT quotient, and this will be the moduli space.

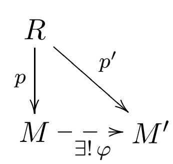
Note that we are using the group $\mathrm{SL}(V)$, and not $\mathrm{GL}(V)$. This is because if two isomorphisms $f$ and $f'$ only differ by multiplication with a scalar, then they correspond to the same point in $Y$. In other words, $Y$ classifies pairs $(f, E)$ up to scalar.

Let $G$ be an algebraic group. Recall that a right action on a scheme $R$ is a morphism $\sigma : R \times G \to R$, which we will usually denote $\sigma(z, g) = z \cdot g$, such that $z \cdot (gh) = (z \cdot g) \cdot h$ and $z \cdot e = z$, where $e$ is the identity element of $G$. A left action is analogously defined, with the associative condition $(hg) \cdot z = h \cdot (g \cdot z)$.

The orbit of a point $z \in R$ is the image $z \cdot G$. A morphism $p : R \to M$ between two schemes endowed with $G$-actions is called *$G$-equivariant* if it commutes with the actions, that is $f(z) \cdot g = f(z \cdot g)$. If the action on $M$ is trivial (i.e. $y \cdot g = y$ for all $g \in G$ and $y \in M$), then we also say that $f$ is *$G$-invariant*.

If $G$ acts on a projective variety $R$, a *linearization* of the action on a line bundle $\mathcal{O}_R(1)$ consists of giving, for each $g \in G$, an isomorphism of line bundles $\widetilde{g} : \mathcal{O}_R(1) \to \varphi_g^* \mathcal{O}_R(1)$, ($\varphi_g = \sigma(\cdot, g)$) which also satisfies the previous associative property. Giving a linearization is thus the same thing as giving an action on the total space $\mathbb{V}$ of the line bundle, which is linear along the fibers, and such that the projection $\mathbb{V} \to R$ is equivariant. If $\mathcal{O}_R(1)$ is very ample, then a linearization is the same thing as a representation of $G$ on the vector space $H^0(\mathcal{O}_R(1))$ such that the natural embedding $R \to \mathbb{P}(H^0(\mathcal{O}_R(1))^\vee)$ is equivariant.

**Definition 1.1** (Categorical quotient)**.** *Let $R$ be a scheme endowed with a $G$-action. A categorical quotient is a scheme $M$ with a $G$-invariant morphism $p : R \to M$ such that for every other scheme $M'$, and $G$-invariant morphism $p'$, there is a unique morphism $\varphi$ with $p' = \varphi \circ p$*




**Definition 1.2** (Good quotient)**.** *Let $R$ be a scheme endowed with a $G$-action. A good quotient is a scheme $M$ with a $G$-invariant morphism $p : R \to M$ such that*

(1) *$p$ is surjective and affine*
(2) *$p_*(\mathcal{O}_R^G) = \mathcal{O}_M$, where $\mathcal{O}_R^G$ is the sheaf of $G$-invariant functions on $R$.*
(3) *If $Z$ is a closed $G$-invariant subset of $R$, then $p(Z)$ is closed in $M$. Furthermore, if $Z_1$ and $Z_2$ are two closed $G$-invariant subsets of $R$ with $Z_1 \cap Z_2 = \emptyset$, then $f(Z_1) \cap f(Z_2) = \emptyset$.*

**Definition 1.3** (Geometric quotient)**.** *A geometric quotient $p : R \to M$ is a good quotient such that $p(x_1) = p(x_2)$ if and only if the orbit of $x_1$ is equal to the orbit of $x_2$.*

Clearly, a geometric quotient is a good quotient, and a good quotient is a categorical quotient.

Geometric Invariant Theory (GIT) is a technique to construct good quotients (cf. [Mu1]). Assume $R$ is projective, and the action of $G$ on $R$ has a linearization on an ample line bundle $\mathcal{O}_R(1)$. A closed point $z \in R$ is called *GIT-semistable* if, for some $m > 0$, there is a $G$-invariant section $s$ of $\mathcal{O}_R(m)$ such that $s(z) \neq 0$. If, moreover, the orbit of $z$ is closed in the open set of all GIT-semistable points, it is called *GIT-polystable*, and, if furthermore, this closed orbit has the same dimension as $G$ ( i.e., if $z$ has finite stabilizer), then $z$ is called a *GIT-stable* point. We say that a closed point of $R$ is *GIT-unstable* if it is not GIT-semistable.

We will use the following characterization in [Mu1] of GIT-(semi)stability. Let $\lambda : \mathbb{C}^* \to G$ be a one-parameter subgroup (by this we mean a nontrivial group homomorphism, even if $\lambda$ is not injective), and let $z \in R$. Then $\lim_{t\to 0} z \cdot \lambda(t) = z_0$ exists, and $z_0$ is fixed by $\lambda$. Let $t \mapsto t^a$ be the character by which $\lambda$ acts on the fiber of $\mathcal{O}_R(1)$. Defining $\mu(z, \lambda) = a$, Mumford proves that $z$ is GIT-(semi)stable if and only if, for all one-parameter subgroups, it is $\mu(z, \lambda)(\leq)0$.

**Proposition 1.4.** *Let $R^{ss}$ (respectively, $R^s$) be the subset of GIT-semistable points (respectively, GIT-stable). Both $R^{ss}$ and $R^s$ are open subsets. There is a good quotient $R^{ss} \to R/\!\!/G$, the image $R^s/\!\!/G$ of $R^s$ is open, $R/\!\!/G$ is projective, and the restriction $R^s \to R^s/\!\!/G$ is a geometric quotient.*

There is one important case in which a scheme is only quasi-projective but GIT can be applied to get a projective quotient: Assume that $R'$ is a $G$-acted scheme with a linearization on a line bundle $\mathcal{O}_{R'}(1)$, which is the restriction of a linearization on an ample line bundle $\mathcal{O}_R(1)$ on a projective variety $R$, and $R' = R^{ss}$, the open subset of GIT-semistable points of $R$. Then we define $R'/\!\!/G = R/\!\!/G$.

Now we are going to describe Grothendieck's Quot-scheme. This scheme parameterizes quotients of a fixed coherent sheaf $\mathcal{V}$ on $X$. That is, pairs $(q, E)$, where $q : \mathcal{V} \twoheadrightarrow E$ is a surjective homomorphism and $E$ is a coherent sheaf on $X$. An isomorphism of quotients is an isomorphism $\alpha : E \to E'$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\;q\;} & E \\ \| & & \cong\big\downarrow \alpha \\ \mathcal{V} & \xrightarrow{\;q'\;} & E' \end{array}$$

A family of quotients parameterized by $T$ is a pair $(q : p_X^*\mathcal{V} \twoheadrightarrow E_T, E_T)$ where $E_T$ is a coherent sheaf on $X \times T$, flat over $T$. An isomorphism of families is an isomorphism $\alpha : E_T \to E'_T$ such that $\alpha \circ q = q'$. Recall that $X$ is a projective scheme endowed with an ample line bundle $\mathcal{O}_X(1)$. Therefore, if $E_T$ is flat over $T$ then the Hilbert polynomial $P_{E_t}$ is locally constant as a function of $t \in T$. If $T$ is reduced, the converse is also true.

Fix a polynomial $P$ and a coherent sheaf $\mathcal{V}$ on $X$. Consider the contravariant functor which sends a scheme $T$ to the set of isomorphism classes of $T$-families of sheaves with Hilbert polynomial $P$ (and it is defined as pullback on morphisms). Grothendieck proved that there is a projective scheme $\mathrm{Quot}_X(\mathcal{V}, P)$, called the Quot scheme, which represents this functor. In particular, there is a universal quotient, i.e., a tautological family of quotients parameterized by $\mathrm{Quot}_X(\mathcal{V}, P)$. We will be interested in the case $\mathcal{V} = V \otimes_{\mathbb{C}} \mathcal{O}_X(-m)$, where $V$ is a vector space and $m$ is sufficiently large.

Given a coherent sheaf $E$, there is an integer $m(E)$ such that, if $m \geq m(E)$, then $E(m)$ is generated by global sections, $h^0(E(m)) = P_E(m)$, and $h^i(E(m)) = 0$

for $i > 0$ ([H-L, Def 1.7.1]). Assume that $m \geq m(E)$ and $\dim V = P_E(m)$. An isomorphism $f : V \to H^0(E(m))$ induces a quotient

$$q : V \otimes_{\mathbb{C}} \mathcal{O}_X(-m) \xrightarrow{\cong} H^0(E(m)) \otimes_{\mathbb{C}} \mathcal{O}_X(-m) \longrightarrow E$$

as above, and this is how a scheme parameterizing based sheaves $(f, E)$ appears as a subscheme of Grothendieck's Quot scheme.

Note that if we have a set $\mathcal{A}$ of isomorphism classes of sheaves, there might not be an integer $m$ large enough for all sheaves. A set $\mathcal{A}$ of isomorphism classes of sheaves on $X$ is called *bounded* if there is a family $E_S$ of torsion free sheaves parameterized by a scheme $S$ of finite type, such that for all $E \in \mathcal{A}$, there is at least one point $s \in S$ such that the corresponding sheaf $E_s$ is isomorphic to $E$. If a set $\mathcal{A}$ is bounded, then we can find an integer $m$ such that $m \geq m(E)$ for all $E \in \mathcal{A}$, thanks to the fact that $S$ is of finite type.

Maruyama proved that the set $\mathcal{A}$ of semistable sheaves with fixed Hilbert polynomial is bounded, and it follows that there is an integer $m_0$, depending only on the polynomial $P$ and $(X, \mathcal{O}_X(1))$, such that $m_0 \geq m(E)$ for all semistable sheaves $E$. This technical result is crucial in order to construct the moduli space. In fact, if $\dim X > 1$, he was able to prove it only if the base field has characteristic 0, and therefore he could only prove the existence of the moduli space in this case. Recently Langer was able to prove boundedness for characteristic $p > 0$, and therefore he was able to construct the corresponding moduli space [La].

Fix a Hilbert polynomial $P$, and let $m \geq m_0$. Let $Y \subset \mathrm{Quot}_X(V \otimes_{\mathbb{C}} \mathcal{O}_X(-m), P)$ be the open subset of quotients such that $E$ is torsion free and $q$ induces an isomorphism $V \cong H^0(E(m))$. Let $\overline{Y}$ be the closure of the open set $Y$ in $\mathrm{Quot}_X(V \otimes_{\mathbb{C}} \mathcal{O}_X(-m), P)$. Note that there is a natural action of $\mathrm{SL}(V)$ on $\mathrm{Quot}_X(V \otimes_{\mathbb{C}} \mathcal{O}_X(-m))$, which sends the quotient $q : V \otimes_{\mathbb{C}} \mathcal{O}_X(-m) \to E$ to the composition $q \circ (g \times \mathrm{id})$. It leaves $Y$ and $\overline{Y}$ invariant, and coincides with the previously defined action for based sheaves $(f, E)$.

To apply GIT, we also need an ample line bundle on $\overline{Y}$ and a linearization of the $\mathrm{SL}(V)$-action on it. This is done by giving an embedding of $\overline{Y}$ in $\mathbb{P}(V_1)$, where $V_1$ will be a vector space with a representation of $\mathrm{SL}(V)$.

There are different ways of doing this, corresponding to different representations $V_1$. One of them corresponds to Grothendieck's embedding of the Quot scheme. This is the method used by Simpson [Si]. Let $q : V \otimes_{\mathbb{C}} \mathcal{O}_X(-m) \twoheadrightarrow E$ be a quotient. Let $l > m$ be an integer and $W = H^0(\mathcal{O}_X(l - m))$. The quotient $q$ induces homomorphisms

$$\begin{array}{rcccl} q & : & V \otimes_{\mathbb{C}} \mathcal{O}_X(l-m) & \twoheadrightarrow & E(l) \\ q' & : & V \otimes W & \to & H^0(E(l)) \\ q'' & : & \bigwedge^{P(l)}(V \otimes W) & \to & \bigwedge^{P(l)} H^0(E(l)) \;\cong\; \mathbb{C} \end{array}$$

If $l$ is large enough, these homomorphisms are surjective, and give Grothendieck's embedding of the Quot scheme.

$$\mathrm{Quot}_X(V \otimes_{\mathbb{C}} \mathcal{O}_X(-m), P) \;\longrightarrow\; \mathbb{P}\Big(\bigwedge^{P(l)}(V^\vee \otimes W^\vee)\Big),$$

The natural representation of $\mathrm{SL}(V)$ in $\bigwedge^{P(l)}(V^\vee \otimes W^\vee)$ gives a linearization of the $\mathrm{SL}(V)$ action on the very ample line bundle $\mathcal{O}_{\overline{Y}}(1)$ induced by this embedding on $\overline{Y}$.

A theorem of Simpson says that a point $(q, E) \in \overline{Y}$ is GIT-(semi)stable if and only if the sheaf $E$ is (semi)stable and the induced linear map $f : V \to H^0(E(m))$ is an isomorphism. In other words, $Y = \overline{Y}^{ss}$. Therefore, the GIT quotient $\overline{Y}/\!\!/\mathrm{SL}(V)$ is the moduli space $\mathfrak{M}(P)$ of semistable sheaves with Hilbert polynomial $P$. The Chern classes $c_i \in H^{2i}(X,\mathbb{C})$ in a family of sheaves are locally constant, therefore the moduli space $\mathfrak{M}(r, c_i)$ of semistable sheaves with fixed rank and Chern classes is a union of connected components of the scheme $\mathfrak{M}(P)$.

Another choice of representation $V_1$ (and therefore, of line bundle on $Y$ and linearization of the action) is the one used by Gieseker and Maruyama. It is explained in the lectures of Schmitt.

## 2. Moduli space of tensors

A *tensor* of type $a$ is a pair $(E,\varphi)$ where $E$ is a torsion free sheaf and

$$\varphi : E \overbrace{\otimes \cdots \otimes}^{a} E \longrightarrow \mathcal{O}_X$$

is a homomorphism. An isomorphism between the tensors $(E,\varphi)$ and $(E',\varphi')$ is a pair $(f,\alpha)$ where $f$ is an isomorphism between $E$ and $E'$, $\alpha \in \mathbb{C}^*$, and the following diagram commutes

$$\begin{array}{ccc} E^{\otimes a} & \xrightarrow{\varphi} & \mathcal{O}_X \\ {\scriptstyle f^{\otimes a}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E'^{\otimes a} & \xrightarrow{\varphi'} & \mathcal{O}_X \end{array}$$

The definition of families of tensors and their isomorphisms are left to the reader ([G-S1, GLSS]).

To define the notion of stability for tensors, it is not enough to look at subsheaves. We have to consider filtrations $E_\bullet \subset E$. By this we always understand a $\mathbb{Z}$-indexed filtration

$$\ldots \subset E_{i-1} \subset E_i \subset E_{i+1} \subset \ldots$$

starting with 0 and ending with $E$ (i.e., $E_k = 0$ and $E_l = E$ for some $k$ and $l$). We say that the filtration is *saturated* if $E^i = E_i/E_{i-1}$ is torsion free for all $i$. If we delete, from 0 onward, all the non-strict inclusions, we obtain a filtration

$$0 \subsetneq E_{\lambda_1} \subsetneq E_{\lambda_2} \subsetneq \cdots \subsetneq E_{\lambda_t} \subsetneq E_{\lambda_{t+1}} = E \qquad \lambda_1 < \cdots < \lambda_{t+1}$$

Reciprocally, from a filtration $E_{\lambda_\bullet}$ we recover the $\mathbb{Z}$-indexed filtration $E_\bullet$ by defining $E_m = E_{\lambda_{i(m)}}$, where $i(m)$ is the maximum index with $\lambda_{i(m)} \le m$.

Let $\mathcal{I}_a = \{1,\ldots,t+1\}^{\times a}$ be the set of all multi-indexes $I = (i_1,\ldots,i_a)$ of cardinality $a$. Define

$$\mu_{\mathrm{tens}}(\varphi, E_{\lambda_\bullet}) = \min_{I\in\mathcal{I}_a} \left\{ \lambda_{i_1} + \cdots + \lambda_{i_a} : \phi|_{E_{\lambda_{i_1}}\otimes\cdots\otimes E_{\lambda_{i_a}}} \neq 0 \right\}, \tag{2.1}$$

or, in terms of the $\mathbb{Z}$-indexed filtration,

$$\mu_{\mathrm{tens}}(\varphi, E_\bullet) = \min_{I\in\mathcal{I}_a} \left\{ i_1 + \cdots + i_a : \phi|_{E_{i_1}\otimes\cdots\otimes E_{i_a}} \neq 0 \right\} \tag{2.2}$$

**Definition 2.1** (Balanced filtration). *A saturated filtration $E_\bullet \subset E$ of a torsion free sheaf $E$ is called a balanced filtration if $\sum i \,\mathrm{rk}\, E^i = 0$ for $E^i = E_i/E_{i-1}$. In terms of $E_{\lambda_\bullet}$, this is $\sum_{i=1}^{t+1} \lambda_i \,\mathrm{rk}(E^{\lambda_i}) = 0$ for $E^{\lambda_i} = E_{\lambda_i}/E_{\lambda_{i-1}}$.*

**Definition 2.2** (Stability of tensors)**.** *Let $\delta$ be a polynomial of degree at most $n-1$ (recall $n = \dim X$) with positive leading coefficient. We say that a tensor $(E,\varphi)$ is $\delta$-(semi)stable if $\varphi$ is not identically zero and for all balanced filtrations $E_{\lambda_\bullet}$ of $E$, it is*

$$\Big(\sum_{i=1}^{t}(\lambda_{i+1}-\lambda_i)\big(rP_{E_{\lambda_i}} - r_{\lambda_i}P\big)\Big) + \mu_{\mathrm{tens}}(\phi, E_{\lambda_\bullet})\,\delta\;(\leq)\;0 \tag{2.3}$$

We will always denote $r = \operatorname{rk} E$ and $r_i = \operatorname{rk} E_i$. The notion of stability for tensors looks complicated, but one finds that, in the applications, when the tensor has some geometric meaning, it can be simplified. We will see some examples.

A *framed bundle* is a tensor of the form $(E, \varphi : E \to \mathcal{O}_X)$. If $E$ is a vector bundle, then taking the dual we have a section of $E^\vee$, so this is equivalent to the pairs $(E, \varphi : \mathcal{O}_X \to E)$ considered by Bradlow, García-Prada and others. In this case, it is enough to look at filtrations with one step, i.e. subsheaves $E' \subsetneq E$.

An *orthogonal sheaf* is a tensor of the form $(E, \varphi : E \otimes E \to \mathcal{O}_X)$, where $E$ is torsion free and $\varphi$ is symmetric and non-degenerate (in the sense that the induced homomorphism $\det E \to \det E^\vee$ is an isomorphism). A *symplectic sheaf* is analogously defined, requiring the tensor $\varphi$ to be skew-symmetric instead of symmetric.

Given a subsheaf $E' \subset E$, its orthogonal $E'^\perp$ is defined as the kernel of the composition

$$E \xrightarrow{\widetilde{\varphi}} E^\vee \longrightarrow E'^\perp\;,$$

where $\widetilde{\varphi}$ is induced by $\varphi$.

**Definition 2.3.** *An orthogonal (or symplectic) sheaf is (semi)stable if for all orthogonal filtrations, that is, filtrations with*

$$E_i^\perp = E_{-i-1}$$

*for all $i$, the following holds*

$$\sum (rP_{E_i} - r_i P_E)(\leq) 0\,.$$

It it shown in [G-S1] that an orthogonal (or symplectic) sheaf is (semi)stable if and only if it is $\delta$-(semi)stable as a tensor, when $\delta$ has degree $n-1$.

A $T$-family of orthogonal sheaves is a $T$-family of tensors $(E_T, \varphi_T : E_T \otimes E_T \longrightarrow \mathcal{O}_{X\times T})$ such that $\varphi_T$ is symmetric and non-degenerate. Note that, since being symmetric is a closed condition, it is not enough to check that $\varphi_t$ is symmetric for every point $t \in T$. On the other hand, being non-degenerate is an open condition, so it is enough to check it for $\varphi_t$, for all points $t \in T$.

A *Lie algebra sheaf* is a pair $(E, \varphi)$ where $E$ is a torsion free sheaf and

$$\varphi : E \otimes E \longrightarrow E^{\vee\vee}$$

is a homomorphism such that for each point $x \in X$, where $E$ is locally free, the induced homomorphism on the fiber $\varphi(x) : E(x) \otimes E(x) \to E(x)$ is a Lie algebra structure. An isomorphism to another Lie algebra sheaf $(E', \varphi')$ is an isomorphism of sheaves $f : E \to E'$ with $\varphi' \circ (f \otimes f) = f \circ \varphi$.

At first sight, this does not seem to be included in the formalism of tensors, but, using the canonical isomorphism

$$(\bigwedge^{r-1} E)^\vee \otimes \det E \xrightarrow{\cong} E^{\vee\vee}, \tag{2.4}$$

a Lie sheaf becomes a tensor of the form

$$(F,\,\psi : F^{\otimes r+1} \longrightarrow \mathcal{O}_X), \tag{2.5}$$

with $E = F \otimes \det F$.

**Definition 2.4.** *A Lie tensor is a tensors of type $a = r+1$ which satisfies the following properties*

(1) *$\psi$ factors through $F \otimes F \otimes \bigwedge^{r-1} F$,*
(2) *the homomorphism $\widetilde{\psi} : F \otimes F \to F^{\vee\vee} \otimes \det F^{\vee}$ associated by (2.4) is skew-symmetric.*
(3) *the homomorphism $\widetilde{\psi}$ satisfies the Jacobi identity.*

There is a canonical bijection between the set of isomorphism classes of Lie sheaves $(E, \varphi : E \otimes E \to E^{\vee\vee})$ and Lie tensors $(F, \psi : F^{\otimes r+1} \longrightarrow \mathcal{O}_X)$ (with $E = F \otimes \det F$).

If the Lie algebra on the fiber $E(x)$ for all $x$ where $E$ is locally free is always isomorphic to a fixed semisimple Lie algebra $\mathfrak{g}$, then we say that it is a $\mathfrak{g}$-sheaf. Then, the Killing form gives an orthogonal structure $\kappa : E \otimes E \to \mathcal{O}_X$ to $E$.

**Definition 2.5.** *A $\mathfrak{g}$-sheaf is (semi)stable if for all orthogonal algebra filtrations, that is, filtrations with*

$$(1) \quad E_i^{\perp} = E_{-i-1} \quad \textit{and} \quad (2) \quad [E_i, E_i] \subset {E_{i+j}}^{\vee\vee}$$

*for all $i$, $j$, the following holds*

$$\sum (rP_{E_i} - r_i P_E)(\leq) 0\ .$$

It is shown in [G-S2] that a $\mathfrak{g}$-sheaf is (semi)stable if and only if the associated tensor is $\delta$-(semi)stable, when $\delta$ has degree $n-1$.

We will sketch how the moduli space of tensors is constructed. The idea is similar to the construction of the moduli space of torsion free sheaves. First we construct a scheme which classifies *$\delta$-semistable based tensors*, that is, triples $(f, E, \varphi)$ where $f : V \to H^0(E(m))$ is an isomorphism, up to a constant, and $(E, \varphi)$ is a $\delta$-semistable tensor. There is a natural embedding of this scheme in a product $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$, where $V_1$ and $V_2$ are representations of $\mathrm{SL}(V)$. An ample line bundle with a linearization of the $\mathrm{SL}(V)$ action is given by $\mathcal{O}_X(b_1, b_2)$. The choice of the integers $b_1$ and $b_2$ will depend on the polynomial $\delta$, and the moduli space of $\delta$-semistable tensors will be the GIT quotient.

To find $V_2$, note that the isomorphism $f : V \to H^0(E(m))$ and $\varphi$ induces a linear map

$$\Phi : V^{\otimes a} \ \longrightarrow\ H^0(E(m)^{\otimes a}) \ \longrightarrow\ H^0(\mathcal{O}_X(am)) \ =:\ B\ .$$

Therefore, the semistable based tensor $(f, E, \varphi)$ gives a point $(q, [\Phi])$

$$\mathcal{H} \times \mathbb{P}(V_2) \ :=\ \mathrm{Quot}_X(V \otimes_{\mathbb{C}} \mathcal{O}_X(-m), P) \times \mathbb{P}\big((V^{\otimes a})^{\vee} \otimes B\big)$$

The points obtained in this way have the property that the homomorphism $\Phi$ composed with evaluation factors as

$$\begin{array}{ccc} V^{\otimes a}\otimes \mathcal{O}_X(-am) & \xrightarrow{q^{\otimes a}} & E^{\otimes a} \\ \downarrow{\scriptstyle \Phi} & & \\ H^0(\mathcal{O}_X(am))\otimes \mathcal{O}_X(-am) & & \Big\downarrow{\scriptstyle \varphi} \\ \downarrow{\scriptstyle \mathrm{ev}} & & \\ \mathcal{O}_X & \longleftarrow & \end{array}$$

Let $Z'$ be the closed subscheme of $\mathcal{H}\times\mathbb{P}(V_2)$ where there is a factorization as above, and let $Z\subset Z'$ be the closure of the open subset $U\subset Z'$ of points $(q: V\otimes\mathcal{O}_X(-m)\to E,[\Phi])$ such that the tensor is $\delta$-semistable. Using Grothendieck's embedding $\mathcal{H}\to\mathbb{P}(V_1)$, explained in section 1, we obtain a closed embedding

$$Z\longrightarrow \mathbb{P}(V_1)\times\mathbb{P}(V_2)$$

We endow $Z$ with the polarization $\mathcal{O}_Z(b_1,b_1)$, where

$$\frac{b_2}{b_1}=\frac{P(l)\delta(m)-\delta(l)P(m)}{P(m)-a\delta(m)}$$

In other words, we use the Segre embedding

$$\mathbb{P}(V_1)\times\mathbb{P}(V_2)\longrightarrow \mathbb{P}(V_1^{\otimes b_1}\otimes V_2^{\otimes b_2})$$

and take the pullback of the ample line bundle $\mathcal{O}_{\mathbb{P}}(1)$.

It is proved in [G-S1] that a point in $Z$ is GIT-(semi)stable if and only if the induced linear map $f:V\to H^0(E(m))$ is an isomorphism and it corresponds to a $\delta$-(semi)stable based tensor. Therefore, the GIT quotient $Z/\!\!/\,\mathrm{SL}(V)$ is the moduli space of $\delta$-semistable tensors.

To show how this is used to obtain moduli spaces of related objects, we will sketch the construction of the moduli space of orthogonal sheaves. First we construct the projective scheme $Z$ as before, for tensors of type $a=2$, i.e., of the form $(E,\varphi: E\otimes E\to\mathcal{O}_X)$. The condition of being symmetric is closed, so it defines a closed subscheme $R\subset Z^{ss}$, and the GIT quotient $R/\!\!/\,\mathrm{SL}(V)$ is projective. On the other hand, the condition of being nondegenerate is open, so it defines an open subscheme $R_1\subset R$. How can we prove that, after we remove the points corresponding to degenerate bilinear forms, the quotient is still projective?. The idea is to show that, if $(E,\varphi)$ is degenerate, then it is $\delta$-unstable (we remark that, to prove this, we need the degree of $\delta$ to be $n-1$). Therefore, $R_1=R$, because all tensors corresponding to points in $R$ are semistable.

In other words, the moduli space of orthogonal sheaves $R_1/\!\!/\,\mathrm{SL}(V)$ is projective because the inclusion $R_1\hookrightarrow R$ is proper (in fact, it is the identity). Every time we impose a condition which is not closed, we have to prove a properness result of this sort, in order to show that the moduli space is projective.

The tensors defined in this section can easily be generalized to tensors of type $(a,b,c)$, that is, pairs $(E,\varphi)$ consisting of a torsion free sheaf and a homomorphism

$$\varphi:(E^{\otimes a})^{\otimes b}\longrightarrow(\det E)^{\otimes c}\,. \tag{2.6}$$

This more general notion will be needed in section 5.

## 3. Principal bundles

Recall that, in the étale topology, an open covering of a scheme $Y$ is a finite collection of morphisms $\{f_i : U_i \to Y\}_{i \in I}$ such that each $f_i$ is étale, and $Y$ is the union of the images of the $f_i$.

Note that an "open étale subset" of a scheme $Y$ is not really a subset of $Y$, but an étale morphism $U \to Y$. If $f : X \to Y$ is a morphism, by a slight abuse of language we will denote by $f^{-1}(U)$ the pull-back

$$\begin{array}{ccc} f^{-1}(U) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ U & \longrightarrow & Y \end{array}$$

Let $G$ be an algebraic group. A principal $G$-bundle on $X$ is a scheme $P$ with a right $G$-action and an invariant morphism $P \to X$ with a $G$-torsor structure. A $G$-torsor structure is given by an atlas consisting on an étale open covering $\{U_i\}$ of $X$ and $G$-equivariant isomorphisms $\psi_i : p^{-1}(U_i) \to U_i \times G$, with $p = p_{U_i} \circ \psi_i$ (the $G$-action on $U_i \times G$ is given by multiplication on the right). Two atlases give the same $G$-torsor structure if their union is an atlas. An isomorphism of principal bundles is a $G$-equivariant isomorphism $\varphi : P \to P'$.

In short, a principal bundle is locally trivial in the étale topology, and the fibers are $G$-torsors. We remark that, if we were working in arbitrary characteristic, an algebraic group could be non-reduced, and we should have used the flat topology.

Given a principal $G$-bundle as above, we obtain an element of the étale cohomology set $\check{H}^1_{\text{et}}(X, \underline{G})$, and this gives a bijection between isomorphism classes of principal $G$-bundles and elements of this set. Indeed, since the isomorphisms $\psi_i$ of an atlas are required to be $G$-invariants, the composition $\psi_j \circ \psi_i^{-1}$ is of the form $(x, g) \mapsto (x, \alpha_{ij}(x)g)$, where $\alpha_{ij} : U_i \cap U_j \to G$ is a morphism, which satisfies the cocycle condition and defines a class in $\check{H}^1_{\text{et}}(X, G)$.

Given a principal $G$-bundle $P \to X$ and a left action $\sigma$ of $G$ in a scheme $F$, we denote

$$P(\sigma, F) \; := \; P \times_G F \; = \; (P \times F)/G,$$

the associated fiber bundle. Sometimes this notation is shortened to $P(F)$ or $P(\sigma)$. In particular, for a representation $\rho$ of $G$ in a vector space $V$, $P(V)$ is a vector bundle on $X$, and if $\chi$ is a character of $G$, $P(\chi)$ is a line bundle.

If $\rho : G \to H$ is a group homomorphism, let $\sigma$ be the action of $G$ on $H$ defined by left multiplication $h \mapsto \rho(g)h$. Then, the associated fiber bundle is a principal $H$-bundle, and it is denoted $\rho_* P$. We say that this principal $H$-bundle is obtained by *extension of structure group.*

Let $\rho : H \to G$ be a homomorphism of groups, and let $P$ be a principal $G$-bundle on a scheme $Y$. A *reduction of structure group* of $P$ to $H$ is a pair $(P^H, \zeta)$, where $P^H$ is a principal $H$-bundle on $Y$ and $\zeta$ is an isomorphism between $\rho_* P^H$ and $P$. Two reductions $(P^H, \zeta)$ and $(Q^H, \theta)$ are isomorphic if there is an isomorphism $\alpha$ giving a commutative diagram

$$\begin{array}{ccccc} P^H & \qquad & \rho_* P^H & \xrightarrow{\zeta} & P \\ \cong \downarrow \alpha & & \downarrow \rho_* \alpha & & \| \\ Q^H & & \rho_* Q^H & \xrightarrow{\theta} & P \end{array} \tag{3.1}$$

The names "extension" and "reduction" come from the case in which $\rho$ is injective, but note that these notions are still defined if the homomorphism is not injective.

If $\rho$ is injective, giving a reduction is equivalent to giving a section $\sigma$ of the associated fibration $P(G/H)$, where $G/H$ is the quotient of $G$ by the right action of $H$. Indeed, such a section gives a reduction $P^H$ by pull-back

$$\begin{array}{ccc} P^H & \xrightarrow{\ i\ } & P \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ \sigma\ } & P(G/H) \end{array}$$

and the isomorphism $\zeta$ is induced by $i$. Conversely, given a reduction $(P^H,\zeta)$, the isomorphism $\zeta$ induces an embedding $i: P^H \to P$, and the quotient by $H$ of this morphism gives a section $\sigma$ as above.

For example, if $G = \mathrm{O}(r)$ and $H = \mathrm{GL}_r$, the quotient $H/G$ is the set of non-degenerate bilinear symmetric forms on the vector space $\mathbb{C}^r$, hence a section of $P(H/G)$ is just a non-degenerate bilinear symmetric morphism $E\otimes E \to \mathcal{O}_X$, where $E$ is the vector bundle associated to the principal $\mathrm{GL}_r$-bundle.

To construct the moduli space, we will assume that $G$ is a connected reductive algebraic group. Let $G' = [G,G]$ be the commutator subgroup, and let $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ be the Lie algebra of $G$, where $\mathfrak{g}'$ is the semisimple part and $\mathfrak{z}$ is the center.

Recall that, in the case of vector bundles, to obtain a projective moduli space when $\dim X > 1$, we had to consider also torsion free sheaves. Analogously, principal $G$-bundles are not enough if we want a projective moduli space, and this is why we also consider principal $G$-sheaves, which we will now define.

**Definition 3.1.** *A principal $G$-sheaf $\mathcal{P}$ over $X$ is a triple $\mathcal{P} = (P, E, \psi)$ consisting of a torsion free sheaf $E$ on $X$, a principal $G$-bundle $P$ on the maximal open set $U_E$ where $E$ is locally free, and an isomorphism of vector bundles*

$$\psi : P(\mathfrak{g}') \xrightarrow{\cong} E|_{U_E}\,.$$

This definition can be understood from two points of view. From the first point of view, we have a torsion free sheaf $E$ on $X$, together with a reduction to $G$, on the open set $U_E$, of the principal $\mathrm{GL}_r$-bundle corresponding to the vector bundle $E|_{U_E}$. Indeed, the pair $(P,\psi)$ is the same thing as a reduction to $G$ of the principal $\mathrm{GL}_r$-bundle on $U_E$ associated to the vector bundle $E|_{U_E}$. It can be shown that, if we are given a reduction to a principal $G$-bundle on a big open set $U' \subsetneq U_E$, this reduction can uniquely be extended to $U_E$.

From the other point of view, we have a principal $G$-bundle on a big open set $U$, hence a vector bundle $P(\mathfrak{g}')$, together with a given extension of this vector bundle on $U$ to a torsion free sheaf on the whole of $X$.

The Lie algebra structure of $\mathfrak{g}'$ is semisimple, hence the Killing form is non-degenerate. Correspondingly, the adjoint vector bundle $P(\mathfrak{g}')$ on $U$ has a Lie algebra structure $P(\mathfrak{g}')\otimes P(\mathfrak{g}') \to P(\mathfrak{g}')$ and an orthogonal structure, $\kappa: P(\mathfrak{g}')\otimes P(\mathfrak{g}') \to \mathcal{O}_U$. These uniquely extend to give orthogonal and $\mathfrak{g}'$-sheaf structure to $E$:

$$\kappa : E\otimes E \longrightarrow \mathcal{O}_X \qquad [\,,]: E\otimes E \longrightarrow E^{\vee\vee}$$

where we have to take $E^{\vee\vee}$ in the target because an extension $E\otimes E \to E$ does not always exist. The orthogonal structure assigns an orthogonal $F^{\perp} = \ker(E \hookrightarrow E^{\vee} \to F^{\vee})$ to each subsheaf $F \subset E$.

**Definition 3.2.** *A principal $G$-sheaf $\mathcal{P} = (P, E, \psi)$ is said to be (semi)stable if for all orthogonal algebra filtrations $E_\bullet \subset E$, that is, filtrations with*

$$(1) \quad E_i^\perp = E_{-i-1} \quad \textit{and} \quad (2) \quad [E_i, E_i] \subset E_{i+j}^{\vee\vee}$$

*for all $i$, $j$, the following holds*

$$\sum (rP_{E_i} - r_i P_E)(\leq) 0$$

Replacing the Hilbert polynomials $P_E$ and $P_{E_i}$ by degrees, we obtain the notion of *slope (semi)-stability.*

Clearly

$$\text{slope-stable} \Longrightarrow \text{stable} \Longrightarrow \text{semistable} \Longrightarrow \text{slope-semistable}$$

Since $G/G' \cong \mathbb{C}^{*q}$, given a principal $G$-sheaf, the principal bundle $P(G/G')$ obtained by extension of structure group provides $q$ line bundles on $U$, and since $\operatorname{codim} X \setminus U \geq 2$, these line bundles extend uniquely to line bundles on $X$. Let $d_1, \dots, d_q \in H^2(X, \mathbb{C})$ be their Chern classes. The rank $r$ of $E$ is clearly the dimension of $\mathfrak{g}'$. Let $c_i$ be the Chern classes of $E$.

**Definition 3.3** (Numerical invariants)**.** *We call the data $\tau = (d_1, \dots, d_q, c_i)$ the numerical invariants of the principal $G$-sheaf $(P, E, \psi)$.*

**Definition 3.4** (Family of semistable principal $G$-sheaves)**.** *A family of (semi)stable principal $G$-sheaves parameterized by a scheme $S$ is a triple $(P_S, E_S, \psi_S)$, with $E_S$ a family of torsion free sheaves, $P_S$ a principal $G$-bundle on the open set $U_{E_S}$ where $E_S$ is locally free, and $\psi : P_S(\mathfrak{g}') \to E_S|_{U_{E_S}}$ an isomorphism of vector bundles, such that for all closed points $s \in S$ the corresponding principal $G$-sheaf is (semi)stable with numerical invariants $\tau$.*

An isomorphism between two such families $(P_S, E_S, \psi_S)$ and $(P'_S, E'_S, \psi'_S)$ is a pair

$$(\beta : P_S \xrightarrow{\cong} P'_S, \gamma : E_S \xrightarrow{\cong} E'_S)$$

such that the following diagram is commutative

$$\begin{array}{ccc} P_S(\mathfrak{g}') & \xrightarrow{\psi} & E_S|_{U_{E_S}} \\ {\scriptstyle \beta(\mathfrak{g}')}\downarrow & & \downarrow{\scriptstyle \gamma|_{U_{E_S}}} \\ P'_S(\mathfrak{g}') & \xrightarrow{\psi'} & E'_S|_{U_{E_S}} \end{array}$$

where $\beta(\mathfrak{g}')$ is the isomorphism of vector bundles induced by $\beta$. Given an $S$-family $\mathcal{P}_S = (P_S, E_S, \psi_S)$ and a morphism $f : S' \to S$, the pullback is defined as $\widetilde{f}^* \mathcal{P}_S = (\widetilde{f}^* P_S, \overline{f}^* E_S, \widetilde{f}^* \psi_S)$, where $\overline{f} = \mathrm{id}_X \times f : X \times S \to X \times S'$ and $\widetilde{f} = i^*(\overline{f}) : U_{\overline{f}^* E_S} \to U_{E_S}$, denoting $i : U_{E_S} \to X \times S$ the inclusion of the open set where $E_S$ is locally free.

We can then define the functor of families of semistable principal $G$-sheaves

$$F_G^\tau : (\mathrm{Sch}/\mathbb{C}) \longrightarrow (\mathrm{Sets})$$

sending a scheme $S$, locally of finite type, to the set of isomorphism classes of families of semistable principal $G$-sheaves with numerical invariants $\tau$. As usual, it is defined on morphisms as pullback.

**Theorem 3.5.** *There is a projective moduli space of semistable $G$-sheaves on $X$ with fixed numerical invariants.*

This theorem is a generalization of the theorem of Ramanathan, asserting the existence of a moduli space of semistable principal bundles on a curve.

Note that in the definition of principal $G$-sheaf we have used the adjoint representation on the semisimple part $\mathfrak{g}'$ of the Lie algebra of $G$, to obtain a vector bundle $P(\mathfrak{g}')$ on a big open set of $X$, which we extend to the whole of $X$ by torsion free sheaf. If we use a different representation $\rho : G \to \mathrm{GL}_r$, we have the notion of principal $\rho$-sheaf:

**Definition 3.6.** *A principal $\rho$-sheaf $\mathcal{P}$ over $X$ is a triple $\mathcal{P} = (P, E, \psi)$ consisting of a torsion free sheaf $E$ on $X$, a principal $G$-bundle $P$ on the maximal open set $U_E$ where $E$ is locally free, and an isomorphism of vector bundles*

$$\psi : P(\rho) \xrightarrow{\cong} E|_{U_E} .$$

Now we will give some examples of principal $\rho$-sheaves which have already appeared:

- If $G = \mathrm{GL}_r$ and $\rho$ is the canonical representation, then a principal $\rho$-sheaf is a torsion free sheaf.
- If $G = \mathrm{O}(r)$ and $\rho$ is the canonical representation, then a principal $\rho$-sheaf is an orthogonal sheaf.
- If $G = \mathrm{SO}(r)$ and $\rho$ is the canonical representation, then a principal $\rho$-sheaf is a special orthogonal sheaf (cf. [G-S1]), that is, a triple $(E, \varphi, \psi)$ where $\varphi : E \otimes E \to \mathcal{O}_X$ symmetric and nondegenerate, and $\psi : \det E \to \mathcal{O}_X$ is an isomorphism such that $\det \varphi = \psi^{\otimes 2}$.
- If $G = \mathrm{Sp}(r)$ and $\rho$ is the canonical representation, then a principal $\rho$-sheaf is a symplectic sheaf.
- If $G$ is semisimple and $\rho$ is injective, then giving a principal $\rho$-sheaf is equivalent to giving a honest singular principal bundle [Sch1, Sch2] with respect to the dual representation $\rho^\vee$ (see section 5).

In all these cases (and also for principal $G$-sheaves, i.e., when $\rho : G \to \mathrm{GL}(\mathfrak{g}')$ is the adjoint representation), the stability condition is equivalent to the following:

**Definition 3.7** (Stability for principal $\rho$-sheaves)**.** *A principal $\rho$-sheaf $\mathcal{P} = (P, E, \psi)$ is said to be (semi)stable if for all reductions on any big open set $U \subset U_E$ of $P$ to a parabolic subgroup $Q \subsetneq G$, and all dominant characters of $Q$, which are trivial on the center of $Q$, the induced filtration of saturated torsion free sheaves*

$$\ldots \subset E_{i-1} \subset E_i \subset E_{i+1} \subset \ldots$$

*satisfies the following*

$$\sum (r P_{E_i} - r_i P_E) (\leq) 0$$

## 4. Construction of the moduli space of principal sheaves

In this section we will give a sketch of the construction of the moduli space in [G-S2]. The strategy is close to that of Ramanathan.

Let $r = \dim \mathfrak{g}'$, and consider the adjoint representation $\rho : G \to \mathrm{GL}_r$ of $G$ in $\mathfrak{g}'$. The idea of Ramanathan is to start by constructing a scheme $R_0$ which classifies based vector bundles of rank $r$, and then to construct another scheme $Q \to R_0$ such that the fiber over each based vector bundle $(f, E)$ parameterizes all reductions to $G$ of the principal $\mathrm{GL}_r$-bundle $E$. In other words, $Q$ classifies tuples $(f, P, E, \psi)$, where $f$ is an isomorphism of a fixed vector space $V$ with $H^0(E(m))$, $P$ is a principal $G$-bundle and $\psi$ is an isomorphism between the vector bundle $P(\rho, \mathfrak{g}')$ and $E$.

The problem is that $\rho$ is not injective in general, so it is not easy to construct a reduction of structure group from $\mathrm{GL}_r$ to $G$ in one step. Therefore, Ramanathan factors the representation $\rho$ into several group homomorphisms, and then constructs reductions step by step.

Recall that $G' = [G, G]$ is the commutator subgroup. Let $Z$ (respectively, $Z'$) be the center of $G$ (respectively, $G'$). Note that $Z' = G' \cap Z$. The adjoint representation factors as follows

$$G \xrightarrow{\rho_3} G/Z' \xrightarrow{\rho_2'} G/Z \xhookrightarrow{\rho_2} \mathrm{Aut}(\mathfrak{g}') \xhookrightarrow{\rho_1} \mathrm{GL}_r$$

and the schemes parameterizing these reductions are

$$R_3 \xrightarrow{f_3} R_2' \xrightarrow{f_2'} R_2 \xrightarrow{f_2} R_1 \longrightarrow R_0$$

In the case $\dim X = 1$ this works well because a principal $G$-bundle is semistable if and only if the associated vector bundle is semistable. This is no longer true if $X$ is not a curve, and this is why, for arbitrary dimension, we do not construct the scheme $R_0$, but instead start directly with a scheme $R_1$, classifying semistable based principal $\mathrm{Aut}(\mathfrak{g}')$-sheaves.

Here $\mathrm{Aut}(\mathfrak{g}')$ denotes the subgroup of $\mathrm{GL}_r$ of linear automorphisms which respect the Lie algebra structure. Therefore, a based principal $\mathrm{Aut}(\mathfrak{g}')$-sheaf is the same thing as a based $\mathfrak{g}'$-sheaf.

Using the isomorphism (2.4), we can describe a $\mathfrak{g}'$-sheaf as a Lie tensor (definition 2.4). such that the Lie algebra structure induced on the fibers of $E$, over points $x \in X$ where $E$ is locally free, is isomorphic to $\mathfrak{g}'$.

Choose a polynomial $\delta$ of degree $\dim X - 1$, with positive leading coefficient. We fix the first Chern class to be zero. This is because we are interested in $\mathfrak{g}'$-sheaves, and since $\mathfrak{g}'$ is semisimple, its Killing form is nondegenerate, hence induces an orthogonal structure on the sheaf, and this forces the first Chern class to be zero.

We start with the scheme $Z$, defined in section 2, classifying based tensors of type $a = r + 1$. This scheme has an open subset $Z^{ss}$ corresponding to $\delta$-semistable tensors. Conditions (1) to (3) in the definition of Lie tensor are closed, hence they define a closed subscheme $R \subset Z^{ss}$. Using the isomorphism (2.4), we see that the scheme $R$ parameterizes Lie sheaves. Recall that a Lie sheaf structure induces a Killing form $\kappa : E \otimes E \to \mathcal{O}_X$.

**Lemma 4.1.** *There is a subscheme $R_1 \subset R$ corresponding to those Lie tensors which are $\mathfrak{g}'$-tensors.*

The family of Lie sheaves parameterized by $R$ gives a family of Killing forms $E_R \otimes E_R \to \mathcal{O}_{X \times R}$, and hence a homomorphism $f : \det E_R \to \det E_R^\vee$. We have fixed the determinant of the tensors to be trivial, hence $\det E_R$ is the pullback of a line bundle on $R$, and therefore the homomorphism $f$ is nonzero on an open set of the form $X \times W$, where $W$ is an open set of $R$. The open set $W$ is in fact the whole of $R$. This is because if $z$ is a point in the complement, it corresponds to a Lie sheaf whose Killing form is non-degenerate, and hence has a nontrivial kernel. Using this, it is possible to construct a filtration which shows that this Lie sheaf is $\delta$-unstable when $\deg \delta = \dim X - 1$, but this contradicts the fact that $R \subset Y^{ss}$.

The Killing form of a Lie algebra is semisimple if and only if it is non-degenerate. Therefore, for all points $(x,t)$ in the open subset $\mathcal{U}_{E_R} \subset X \times R$ where $E_R$ is locally free, the Lie algebra is semisimple.

Semisimple Lie algebras are rigid, that is, if there is a family of Lie algebras, the subset of the parameter space corresponding to Lie algebras isomorphic to a given semisimple Lie algebra is open. Therefore, since $U_E$ is connected for all torsion free sheaves $E$, all points $(x,t) \in \mathcal{U}_{E_R} \subset X \times R$ where $t$ is in a fixed connected component of $R$, give isomorphic Lie algebras. Let $R_1$ be the union of those components whose Lie algebra is isomorphic to $\mathfrak{g}'$. The inclusion

$$i : R_1 \hookrightarrow R$$

is proper, and hence, since the GIT quotient $R/\!\!/ \mathrm{SL}(V)$ is proper, also the GIT quotient $R_1/\!\!/ \mathrm{SL}(V)$ is proper. Note that, to prove properness of $i$, two facts about semisimple Lie algebras were used: rigidity, and nondegeneracy of their Killing forms.

For simplicity of the exposition, to explain the successive reductions, first we will assume that for all $\mathfrak{g}'$-sheaves $(E,\varphi)$, the torsion free sheaf $E$ is locally free. In other words, $U_E = X$ (this holds, for instance, if $\dim X = 1$). At the end we will mention what has to be modified in order to consider the general case.

The group $G/Z$ is the connected component of identity of $\mathrm{Aut}(\mathfrak{g}')$. Therefore, giving a reduction of structure group of a principal $\mathrm{Aut}(\mathfrak{g}')$-bundle $P$ by $\rho_2$ is the same thing as giving a section of the finite étale morphism $P(F) \to X$, where $F$ is the finite group $\mathrm{Aut}(\mathfrak{g}')/(G/Z)$. This implies that $R_2 \to R_1$ is a finite étale morphism, whose image is a union of connected components of $R_1$.

There is an isomorphism of groups $G/Z' \cong G/G' \times G/Z$, and $\rho_2'$ is just the projection to the second factor. Therefore, a reduction to $G/Z'$ of a principal $G/Z$-bundle $P^{G/Z}$ is just a pair $(P^{G/G'}, P^{G/Z})$, where $P^{G/Z}$ is the original $G/Z$-bundle and $P^{G/G'}$ is a $G/G'$-bundle. But

$$G/G' \cong \mathbb{C}^* \overbrace{\times \cdots \times}^{q} \mathbb{C}^* \,,$$

hence this is just a collection of $q$ line bundles, whose Chern classes are given by the numerical invariants which have been fixed. This implies that there is an isomorphism

$$R_2' \cong J \overbrace{\times \cdots \times}^{q} J \times R_2 \,,$$

where $J$ is the Jacobian of $X$.

Finally, we have to consider reductions of a principal $G/Z'$-bundle to $G$, where $Z'$ is a finite subgroup of the center of $G$. There is an exact sequence of pointed sets (the distinguished point being the trivial bundle)

$$\check{H}^1_{\mathrm{et}}(X, \underline{Z'}) \longrightarrow \check{H}^1_{\mathrm{et}}(X, \underline{G}) \longrightarrow \check{H}^1_{\mathrm{et}}(X, \underline{G/Z'}) \stackrel{\delta}{\longrightarrow} \check{H}^2_{\mathrm{et}}(X, \underline{Z'}) \,.$$

Note that $Z'$ is abelian, therefore $H^i_{\mathrm{et}}(X, \underline{Z'})$ is an abelian group, and it is isomorphic to the singular cohomology group $H^i(X; Z')$, hence finite. A principal $G/Z'$-bundle admits a reduction to $G$ if and only if the image by $\delta$ of the corresponding point is 0. This is an open and closed condition, therefore there is a subscheme $\hat{R}_2'$ of $R_2'$, consisting of a union of connected components, corresponding to those principal $G/Z$-bundles admitting a reduction to $G$.

Let $(P^G,\zeta)$ be a reduction to $G$ of a principal $G/Z'$-bundle. It can be shown that the set of isomorphism classes of all reductions to $G$ is in bijection with the cohomology set $\check{H}^1_{\mathrm{et}}(X,\underline{Z'})$, with the unit element of this set corresponding to the chosen reduction $(P^G,\zeta)$. This cohomology set is an abelian group, because $Z'$ is abelian. Therefore, the set of reductions of a principal $G/Z'$-bundle to $G$ is an $\check{H}^1_{\mathrm{et}}(X,\underline{Z'})$-torsor, and this implies that $R_3\to\hat{R}'_2$ is a principal $\check{H}^1_{\mathrm{et}}(X,\underline{Z'})$-bundle. Using that this cohomology set is a finite set (in fact isomorphic to the singular cohomology group $H^1(X;Z')$), and that $\hat{R}'_2$ is a union of connected components of $R'_2$, it follows that $R_3\to R'_2$ is finite étale.

Ramanathan [Ra, Lemma 5.1] proves that, if $H$ is a reductive algebraic group, $f:Y\to S$ is an $H$-equivariant affine morphism, and $p:S\to\overline{S}$ is a good quotient, then $Y$ has a good quotient $q:\overline{Y}\to Y$ and the induced morphism $\overline{f}$ is affine. Moreover, if $f$ is finite, $\overline{f}$ is also finite. When $f$ is finite and $p$ is a geometric quotient, also $q$ is a geometric quotient.

The group $\mathrm{SL}(V)$ acts on all the schemes $R_i$, and the morphisms $f_2$ and $f_3$ are equivariant and finite. Therefore, we can apply Ramanathan's lemma to those morphisms.

The morphism $f'_2:J^{\times q}\times R_2\to R_2$ is just projection to a factor, and the group acts trivially on the fiber, therefore if $p_2:R_2\to\mathfrak{M}_2$ is a good quotient of $R_2$, $J^{\times q}\times\mathfrak{M}_2$ will give a good quotient of $R'_2$. Furthermore, if $p_2$ becomes a geometric quotient when restricting to an open set, the same will be true after taking the product with $J^{\times q}$.

Using GIT, we know that $R_1$ has a good quotient $\mathfrak{M}_1$, which is a geometric quotient when restricting to the open set of stable points. Therefore, the same holds for all these schemes, and the good quotient of $R_3$ is the moduli space of principal $G$-bundles.

The successive reductions in higher dimension are very similar to the reductions in the case $X$ is a curve, except for the technical difficulty that the principal bundles in general are not defined in the whole of $X$, but only in a big open set. To overcome this difficulty, we need "purity" results for open sets $U\subset X$ when $U$ is big. We will discuss them one by one.

First we consider reductions of a principal $\mathrm{Aut}(\mathfrak{g}')$-bundle $P$ to $G/Z$. These are parameterized by sections of the associated fibration $P(F)$, where $F=\mathrm{Aut}(\mathfrak{g}')/(G/Z)$ is a finite group. If $P$ is a principal bundle on a big open set $U$, $P(F)$ is a Galois cover of $U$, given by a representation of the algebraic fundamental group of $\pi(U)$ in $F$. Since $U$ is a big open set, $\pi(X)=\pi(U)$ (purity of fundamental group), and hence the Galois cover $P(F)$ of $U$ extends uniquely to a Galois cover of $X$. This implies that, even if $\dim X>1$, the morphism $R_2\to R_1$ is still finite étale, as in the curve case.

Giving a reduction of a principal $G/Z$-bundle on $U$ to a principal $G/Z'$-bundle is equivalent to giving $q$ line bundles on $U$. Since $U$ is a big open set, the Jacobians of $U$ and $X$ are isomorphic (purity of Jacobian), and hence we still have $R'_2=J(X)^{\times q}\times R_2$.

Finally, we have to consider reductions of principal $G/Z'$-bundles to $G$. Using the fact that $U$ is a big open set, there are isomorphisms $\check{H}^i_{\mathrm{et}}(X,\underline{Z'})\cong\check{H}^i_{\mathrm{et}}(U,\underline{Z'})$ for $i=1,2$. Therefore, the arguments used for the case $U=X$ still hold in general, and it follows that $R_3\to R'_2$ is étale finite.

## 5. Construction of the moduli space of principal $\rho$-sheaves

In [Sch1, Sch2], A. Schmitt fixes a semisimple group $G$ and a faithful representation $\rho$, defines semisimple honest singular principal bundle with respect to this data (see definition below), and constructs the corresponding projective moduli space. Giving such an object is equivalent to giving a principal $\rho^\vee$-bundle, where $\rho^\vee$ is the dual representation in $V^\vee$. In this section we will give a sketch of Schmitt's construction.

Let $G$ be a semisimple group, and $\rho : G \to \mathrm{GL}(V)$ a faithful representation. A honest singular principal $G$-bundle is a pair $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a torsion free sheaf on $X$ and

$$\tau : \mathrm{Sym}^*(\mathcal{A} \otimes V)^G \longrightarrow \mathcal{O}_X$$

is a homomorphism of $\mathcal{O}_X$-algebras such that, if $\sigma : X \to Hom(V \otimes \mathcal{O}_U, \mathcal{A}|_U^\vee)/\!\!/G$ is the induced morphism, then

$$\sigma(U) \subset Isom(V \otimes \mathcal{O}_U, \mathcal{A}|_U^\vee)/G \subset Hom(V \otimes \mathcal{O}_U, \mathcal{A}|_U^\vee)/\!\!/G\,.$$

It can be shown that the points in the affine $U$-scheme $Isom(V \otimes_{\mathbb{C}} \mathcal{O}_U, \mathcal{A}|_U^\vee)$ are in the open set of GIT-polystable points of $Hom(V \otimes_{\mathbb{C}} \mathcal{O}_U, \mathcal{A}|_U^\vee)$, under the natural action of $G$, therefore the previous inclusion makes sense.

Note that the homomorphism $\tau$ is uniquely defined by its restriction to $U \subset X$, therefore, giving a honest singular principal $G$-bundle is equivalent to giving a principal $\rho^\vee$-sheaf $(P, E, \psi)$, where $\rho^\vee : G \to \mathrm{GL}(V^\vee)$ is the dual representation, $P$ is a principal $\mathrm{GL}_n$-bundle, $E = \mathcal{A}$, and $\psi$ is induced by $\sigma|_U$.

In other words, in a principal $\rho$-sheaf, we extend to the whole of $X$, as a torsion free sheaf $E$, the vector bundle associated to $\rho$, whereas, is a honest singular principal $G$-bundle associated to $\rho$, we extend the dual of the vector bundle associated to $\rho$.

The idea of Schmitt's construction is to transform $\tau$ into a tensor. Note that $\tau$ is an infinite collection of $\mathcal{O}_X$-module homomorphisms

$$\tau_i : \mathrm{Sym}^i(\mathcal{A} \otimes V)^G \longrightarrow \mathcal{O}_X, \tag{5.1}$$

but, since $\mathrm{Sym}^*(\mathcal{A} \otimes V)^G$ is finitely generated as a $\mathcal{O}_X$-algebra, there is an integer $s$ such that

(1) the sheaf

$$\bigoplus_{i=1}^{s} \mathrm{Sym}^i(\mathcal{A} \otimes V)^G$$

contains a set of generators of the algebra, and

(2) the subalgebra

$$\mathrm{Sym}^{(s!)}(\mathcal{A} \otimes V)^G \;:=\; \bigoplus_{m=0}^{\infty} \mathrm{Sym}^{s!m}(\mathcal{A} \otimes V)^G$$

is generated by elements in $\mathrm{Sym}^{s!}(\mathcal{A} \otimes V)^G$.

Using the homomorphisms $\tau_s$, we construct a homomorphism of $\mathcal{O}_X$-modules

$$\bigoplus_{\sum i d_i = s!} \Big( \bigotimes_{i=1}^{s} \mathrm{Sym}^{d_i} \left( \mathrm{Sym}^i(\mathcal{A} \otimes V)^G \right) \Big) \twoheadrightarrow \mathrm{Sym}^{s!}(\mathcal{A} \otimes V)^G \xrightarrow{\tau_s} \mathcal{O}_X \tag{5.2}$$

Note that the vector space

$$\bigoplus_{\sum id_i = s!} \Big( \bigotimes_{i=1}^{s} \mathrm{Sym}^{d_i} \big( \mathrm{Sym}^i(\mathbb{C}^r \otimes V)^G \big) \Big)$$

has a canonical representation of $\mathrm{GL}_n$, homogeneous of degree $s!$, and hence it is a quotient of the representation

$$(\mathbb{C}^{\otimes a})^{\oplus b} \otimes (\bigwedge^{r} \mathbb{C}^r)^{-\otimes c}$$

for appropriate values of $a$, $b$ and $c$. Therefore, there is a surjection

$$(\mathcal{A}^{\otimes a})^{\oplus b} \otimes (\det \mathcal{A})^{-\otimes c} \twoheadrightarrow \bigoplus_{\sum id_i = s!} \Big( \bigotimes_{i=1}^{s} \mathrm{Sym}^{d_i} \big( \mathrm{Sym}^i(\mathcal{A} \otimes V)^G \big) \Big) \tag{5.3}$$

and composing (5.2) with (5.3) we obtain a tensor of type $(a, b, c)$, as in (2.6).

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