On the dimension of spline spaces on planar T-subdivisions

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Abstract

We analyze the space $\mathcal{S}^{r,r'}_{m,m'}(\mathcal{T})$ of bivariate functions that are piecewise polynomial of bidegree (m, m') and class $C^{r,r'}$ over the planar T-subdivision $\mathcal{T}$. We give a new formula for the dimension of this space by exploiting homological techniques. We relate this dimension to the number of nodes on the maximal interior segments of the subdivision, give combinatorial lower and upper bounds on the dimension of these spline spaces for general hierarchical T-subdivisions. We show that these bounds are exact, for high enough degrees or if the subdivision is enough regular. Finally, we analyse cases of small degrees and regularities.

Introduction

Standard parametrisations of surfaces in Computer Aided Geometric Design are based on tensor product bspline functions, defined from a grid of nodes over a rectangular domain. These representations are easy to control but their refinement has some drawback. Inserting a node in one direction of the parameter domain implies the insertion of several control points in the other directions. If for instance, regions along the diagonal of the parameter domain should be refined, this create a fine grid at some region where it is not needed. To avoid this problem, while extending the standard tensor product representation of CAGD, representations associated to subdivisions with T-junctions instead of a grid, have been analyzed. Such a T-subdivision is a partition of an axis-aligned box Ω (e.g. the unit square) into smaller axis-aligned boxes, called the cells of the subdivision.

The first type of T-splines introduced in [17, 16], are defined by blending functions which are product of univariate bspline basis function associated to some nodes of the subdivision. They are piecewise polynomial functions, but the pieces were these functions are polynomial do not match with the cells of the T-subdivision. There is no proof that these piecewise polynomial functions are linearly independent. Indeed, [2] shows that in some cases, these blending T-spline functions are not linear independent. Moreover, there is no characterisation of the vector space span by these functions. For this reason, the partition of unity properties useful in CAGD are not available directly in this space. The spline functions have to be combined into piecewise rational functions, so that these piecewise rational functions sum to 1. However, this construction tends to complexify the practical use of such T-splines.

Being able to describe a basis of the vector space of piecewise polynomials on a T-subdivision is an important but non-trivial issue. It yields a construction of piecewise polynomial functions on the T-subdivision which form a partition of unity so that the use of piecewise rational functions is not needed. It has also a direct impact in approximation problems such as surface reconstruction or isogeometric analysis, where controlling the space of functions used to approximate a solution is critical. In CAGD, it also provides more degrees of freedom to control a shape. That is why further works have been developed to understand better the space of piecewise polynomial functions on a T-subdivision.

In [4], [6], a second family of splines on hierarchical T-subdivisions have been studied to tackle these issues. These splines are piecewise polynomial and form a basis of the space of piecewise polynomial functions on a hierarchical T-subdivision. They are called PHT-splines (Polynomial Hierarchical T-splines). Dimension formulae of the spline space on such a subdivision have been proposed when the degree is high enough compared to the regularity [4], [10], [12] and in some cases for biquadratic C^1 piecewise polynomial functions [5]. The construction of a basis is described for bicubic C^1 spline spaces in terms of the coefficients of the polynomials in the Bernstein basis attached to a cell. When a cell is subdivided into 4 subcells, the Bernstein coefficients of the basis functions of the old level are modified and independent functions are introduced, using Bernstein bases on the cells at the new level.

In this paper, we analyse the space $\mathcal{S}^{r,r'}_{m,m'}(\mathcal{T})$ of bivariate functions that are piecewise polynomial of bidegree (m, m') and class $C^{r,r'}$ over a planar T-subdivision $\mathcal{T}$. We give a new formula for its dimension. By extending homological techniques developed in [1] and [15], we relate this dimension to the number of nodes on the maximal interior segments of the subdivision. We show that for $m \geqslant 2r+1$ and $m' \geqslant 2r'+1$, the dimension depends directly on the number of faces, interior edges and interior points, providing a new proof of the dimension formula proved in [4], [10], [12] for a hierarchical T-subdivision. The algebraic approach provides an homological interpretation of the so-called Smoothing Cofactor-Conformality method [19]. It allows us to generalize the dimension formulae obtained by this technique [12], [10]. In particular, it yields combinatorial lower and upper bounds on the dimension of these spline spaces for general T-subdivisions. We also show that these bounds are exact for T-subdivisions which are enough regular. As a consequence, we give the dimension of the space of Locally Refined splines described in [7]. We do not consider the problem of constructing explicit bases for these spline spaces, which will be analyzed in a forthcoming paper [13].

In the first section, we recall the notations, the polynomial properties and the homological constructions which are needed in the following. Section 2 deals with the properties of T-subdivisions that we will use. In section 3, we detail the construction of the topological complex, describe its homology and proof the dimension formulae. In the last section, we analyse some examples for small degree and regularity.

1 Planar T-splines

1.1 Notations

We denote by $R = \mathbb{K}[s,t]$ the space of polynomials in the variables s, t with coefficients in the field $\mathbb{K}$ of characteristic 0. Let $R_{m,m'}$ denote the space of polynomials of degree $\leqslant m$ in s and degree $\leqslant m'$ in t.

For any line L of $\mathbb{K}^2$ with equation $\Delta_L(s,t) = 0$ and for $r \in \mathbb{N}$, let $I_{L,r} = (\Delta_L(s,t)^{r+1})$. For two such lines L, L' which intersect at a point $\gamma \in \mathbb{K}^2$ and for $m, m' \in \mathbb{N}$, the ideal $I_{L,r} + I_{L',r'} = (\Delta_L(s,t)^{r+1}) + (\Delta_{L'}(s,t)^{r'+1})$ defines the point γ with multiplicity $(r+1) \times (r'+1)$.

Let $\mathcal{T}$ be a subdivision of a rectangular domain $D_0 \subset \mathbb{R}^2$ into a rectangular complex.

- The faces of dimension 2 of $\mathcal{T}$ are rectangles. The set 2-faces of $\mathcal{T}$ is denoted by $\mathcal{T}_2$ and the number of elements in $\mathcal{T}_2$ is f_2.

- The faces of dimension 1 of $\mathcal{T}$ are edges which are either vertical or horizontal. The set 1-faces of $\mathcal{T}$ is denoted by $\mathcal{T}_1$. The set of interior faces, that is those which intersect the interior of Ω is denoted by $\mathcal{T}_1^o$. The number of edges in $\mathcal{T}_1$ is f_1. Let f_1^h (resp. f_1^v) be the number of interior horizontal (resp. vertical) edges.The number of interior edges is $f_1^o = f_1^h + f_1^v$.

  Let $\Delta_\tau(s,t) = 0$ be the equation associated to line supporting τ. If an edge is vertical, we define $\delta(\tau) = (r+1, 0)$ and $\Delta_\tau^{(r,r')} = \Delta_\tau^{r+1}$. Otherwise $\delta(\tau) = (0, r'+1)$ and $\Delta_\tau^{(r,r')} = \Delta_\tau^{r'+1}$. We denote $\mathfrak{I}^{r,r'}(\tau) = (\Delta_\tau^{(r,r')})$ and $\mathfrak{I}^{r,r'}_{m,m'}(\tau) = \mathfrak{I}^{r,r'}(\tau) \cap R_{m,m'}$.

Notice that two horizontal (resp. vertical) edges τ_1, τ_2 which intersect define the same ideal $\mathfrak{I}^{r,r'}(\tau_1) = \mathfrak{I}^{r,r'}(\tau_2)$.

- The faces of dimension 0 of $\mathcal{T}$ are the vertices of the subdivision. The set of vertices of $\mathcal{T}$ is denoted $\mathcal{T}_0$. The set of interior vertices is denoted $\mathcal{T}_0^o$. Let f_0 be the number of vertices of $\mathcal{T}_0$, f_0^o be the number of interior vertices, f_0^+ be the number of interior crossing vertices, f_0^T the number of interior T-vertices, f_0^b the number of boundary vertices, counting the 4 corner points. There are $f_0^b - 4$ boundary points which are T-vertices.

  A vertex $\gamma \in \mathcal{T}_0$ is the intersection of a vertical edge τ_v with an horizontal edge τ_h. Let $\mathfrak{I}^{r,r'}(\gamma) = \mathfrak{I}^{r,r'}(\tau_v) + \mathfrak{I}^{r,r'}(\tau_h) = (\Delta_{\tau_v}^{(r,r')}, \Delta_{\tau_h}^{(r,r')})$. We denote by $\mathfrak{I}^{r,r'}_{m,m'}(\gamma) = \mathfrak{I}^{r,r'}_{m,m'}(\tau_v) + \mathfrak{I}^{r,r'}_{m,m'}(\tau_h)$. Notice that these definitions are independent of the choice (if any) of the vertical edge τ_v and horizontal edge τ_h which contain γ.

In the following, we are going to analyse the following space.

Definition 1.1 *Let $\mathcal{S}^{r,r'}_{m,m'}(\mathcal{T})$ be the vector space of functions which are polynomials in $R_{m,m'}$ on each face $\sigma \in \mathcal{T}_2$ and of class C^r in s and $C^{r'}$ in t on Ω.*

We will say that $f \in \mathcal{S}^{r,r'}_{m,m'}(\mathcal{T})$ is $C^{r,r'}$ regular. Our aim is to describe its dimension in terms of combinatorial quantities attached to $\mathcal{T}$ and an explicit basis.

Given an element $f \in \mathcal{S}^{r,r'}_{m,m'}(\mathcal{T})$, we denote by f^ε the function on $\mathbb{K}^2$, which coincide with f on Ω and which is 0 outside. We are interested in the analysis of the following space.

1.2 Polynomial properties

We recall here basic results on the dimension of the vector spaces involved in the analysis of $\mathcal{S}^{r,r'}_{m,m'}(\mathcal{T})$:

Lemma 1.2

- $\dim R_{m,m'} = (m+1) \times (m'+1)$.
- $\dim R_{m,m'}/\mathfrak{I}^{r,r'}_{m,m'}(\tau) = \begin{cases} (m+1)\times(r'+1) & \text{if } \tau \text{ is } a \text{ horizontal edge} \\ (r+1)\times(m'+1) & \text{if } \tau \text{ is } a \text{ vertical edge} \end{cases}$
- $\dim R_{m,m'}/\mathfrak{I}^{r,r'}_{m,m'}(\gamma)) = (r+1) \times (r'+1)$.

An algebraic way to characterise the $C^{r,r'}$-regularity is given by the next lemma [1]:

Lemma 1.3 *Let $\tau \in \mathcal{T}_1$ and let p_1, p_2 be two polynomials. Their derivatives coincide on τ up to order (r,r') iff $p_1 - p_2 \in \mathfrak{I}^{r,r'}(\tau)$.*

In the following, we will use the apolar product defined on polynomials of degree $\leqslant n$ of $\mathbb{K}[x]$ by:

$$\langle f, g\rangle_n = \sum_{i=0}^{n} \binom{n}{i} f_i g_i$$

where $f = \sum_{i=1}^n f_i x^i$, $g = \sum_{i=1}^n g_i x^i \in \mathbb{K}[x]_n$. One of the property that we will need is the following [14], [11], [8]:

Lemma 1.4 *Let $g \in \mathbb{K}[x]_n$, $d < n$ and $a \in \mathbb{K}$. Then g is orthogonal to $(x-a)^d R_{n-d}$ for the apolar product iff*

$$\partial^k g(a) = 0, k = 0 \dots n-d.$$

Proposition 1.5 *Let $a_1, \ldots, a_l \in \mathbb{K}$ be l distinct points. Then*

$$\dim\left(\sum_{i=1}^{l}(x-a_i)^d\mathbb{K}[x]_{n-d}\right) = \min(n+1, (n-d+1)l).$$

Proof. In order to compute the dimension of $V := \sum_{i=1}^{l}(x-a_iy)^d\mathbb{K}[x]_{n-d} \subset \mathbb{K}[x]_n$, we compute the dimension of the orthogonal $V^\perp$ in $\mathbb{K}[x]_n$ of V for the apolar product. Let $g \in \mathbb{K}[x]_n$ be an element of the orthogonal $V^\perp$ of V. By lemma 1.4, $\partial^k g(a_i) = 0, k = 0\ldots n-d, i = 1\ldots l$. In other words, g is divisible by $(x-a_i)^{n-d+1}$ for $i = 1, \ldots, l$. As the points a_i are distinct, g is divisible by

$$\Delta := \prod_{i=1}^{l}(x-a_i)^{n-d+1}.$$

Conversely, any multiple of Δ of degree $\leqslant n$ is in $V^\perp$. Thus $V^\perp = (\Delta) \cap \mathbb{K}[x]_n$.

This vector space $V^\perp$ of multiples of Δ in degree n is of dimension $\max(0, n+1-\deg(\Delta))$, so V is of dimension

$$n+1-\max(0, n+1-\deg(\Delta)) = \min(n+1, \deg(\Delta)) = \min(n+1, (n-d+1)l).$$

□

We are going to use an equivalent formulation of this result:

$$\dim\left(R_n / \sum_{i=1}^{l}(x-a_i)^d\mathbb{R}[x]_{n-d}\right) = (n+1-(n-d+1)l)_+. \tag{1}$$

1.3 Complexes and homology

Let us recall here the basic properties that we will need on complexes of vector spaces. Given a sequence of $\mathbb{K}$-vectors spaces A_i, $i = 0, \ldots, l$ and linear maps $\partial_i : A_i \to A_{i+1}$, we say that we have a complex

$$\mathcal{A} : A_l \to A_{l-1} \to \cdots A_i \to A_{i-1} \to \cdots A_1 \to A_0$$

if $\operatorname{im}\partial_i \subset \ker\partial_{i-1}$.

Definition 1.6 *The i^{th} homology $H_i(\mathcal{A})$ of $\mathcal{A}$ is $\ker\partial_{i-1}/\operatorname{im}\partial_i$ for $i = 1, \ldots, l$.*

The complex $\mathcal{A}$ is called *exact* (or an exact sequence) if $H_i(\mathcal{A}) = 0$ (ie. $\operatorname{im}\partial_i = \ker\partial_{i-1}$) for $i = 1, \ldots, l$. If the complex is exact and $A_l = A_0 = 0$, we have

$$\sum_{i=1}^{l-1}(-1)^i \dim A_i = 0.$$

Given complexes $\mathcal{A} = (A_i)_{i=0,\ldots,l}$ $\mathcal{B} = (\mathcal{B}_i)_{i=0,\ldots,l}$, $\mathcal{C} = (C_i)_{i=0,\ldots,l}$ and exact sequences

$$0 \to A_i \to B_i \to C_i \to 0$$

for $i = 0, \ldots, l$, we have a long exact sequence [3], [18][p. 182]:

$$\cdots \to H_{i+1}(\mathcal{C}) \to H_i(\mathcal{A}) \to H_i(\mathcal{B}) \to H_i(\mathcal{C}) \to H_{i-1}(\mathrm{A}) \to \cdots$$

2 T-subdivisions

In this section, we describe the properties that we need on T-subdivisions, starting with some well-known enumeration results.

2.1 Combinatorial properties

Lemma 2.1

- $f_2 = f_0^+ + \frac{1}{2} f_0^T + \frac{1}{2} f_0^b - 1$
- $f_1^o = 2 f_0^+ + \frac{3}{2} f_0^T + \frac{1}{2} f_0^b - 2$
- $f_0^o = f_0^+ + f_0^T$

Proof. Each face $\sigma \in \mathcal{T}_2$ is a rectangle with 4 corners. If we count these corners for all cells in $\mathcal{T}_2$, we enumerate 4 times the crossing vertices, 2 times the T-vertices which are interior or on the boundary and one time the corner vertices of Ω. This yields the relation

$$4 f_2 = 4 f_0^+ + 2(f_0^T + (f_0^b - 4)) + 4.$$

As each interior edge $\tau \in \mathcal{T}_1^o$ has two end points. Counting these end points for all interior edges, we count 4 times the crossing vertices, 3 times the T-vertices with are interior and one time the T-vertices on the boundary:

$$2 f_1^o = 4 f_0^+ + 3 f_0^T + (f_0^b - 4).$$

Finally, as an interior vertex is a crossing vertex or a T-vertex, we have

$$f_0^o = f_0^+ + f_0^T.$$

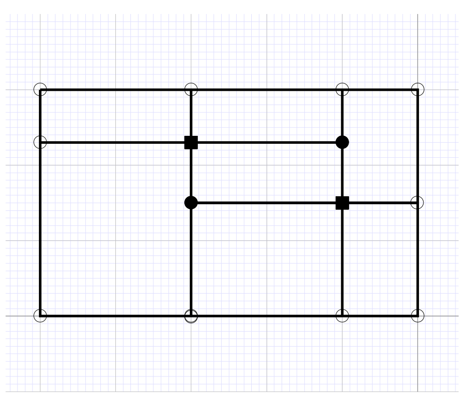
□

2.2 Hierarchical subdivision

Definition 2.2 *A hierarchical T-subdivision is either the initial square or obtained from a hierarchical T-subdivision by splitting a cell along a vertical or horizontal line.*

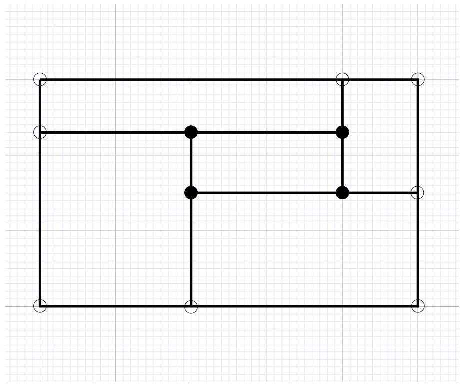
A hierarchical T-subdivision can be represented by a subdivision tree where the nodes are the cells obtained during the subdivision and the children of a cell σ are the cells obtained by subdividing σ.

Example 2.3 *Here is a hierarchical T-subdivision*



The crossing vertices are marked with a square, the interior T-vertices with a black disk and the boundary vertices with a circle. For this subdivision, we have $f_2 = 7$, $f_1^o = 10$, $f_0^o = 4$, $f_0^+ = 2$, $f_0^T = 2$, $f_0^b = 10$.

Here is a non-hierarchical T-subdivision:



For this subdivision, we have $f_2 = 5$, $f_1^o = 8$, $f_0^o = 4$, $f_0^+ = 0$, $f_0^T = 4$, $f_0^b = 8$.

In a hierarchical T-subdivision $\mathcal{T}$, we can define the level of a cell as the distance of this cell to the root Ω of the subdivision tree.

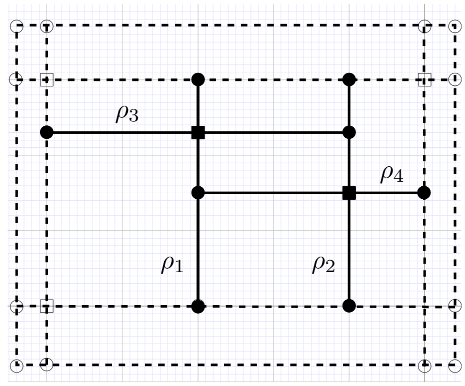
2.3 Maximal interior segments

In order to simplify the dimension analysis of $\mathcal{S}_{m,m'}^{r,r'}(\mathcal{T})$, we introduce the following definitions:

- For any interior edge $\tau \in \mathcal{T}_1^o$, we define ρ as the *maximal segment* made of edges $\in \mathcal{T}_1^o$ of the same direction as τ, which contains τ and such that their union is connected. We say the maximal segment ρ is interior if it does not intersect the boundary of Ω.
- As all the edges belonging to a maximal segment s have the same supporting line, we can define $\Delta^{r,r'}(\rho) = \Delta^{r,r'}(\tau)$ for any edge τ belonging to ρ.
- The set of all maximal interior segments is denoted $\mathrm{MIS}(\mathcal{T})$. The set of horizontal (resp. vertical) maximal interior segments of $\mathcal{T}$ is denoted by $\mathrm{MIS}_h(\mathcal{T})$ (resp. $\mathrm{MIS}_v(\mathcal{T})$).
- The degree of $\rho \in \mathrm{MIS}(\mathcal{T})$ is $\delta(\rho) = \delta(\tau)$ for any $\tau \subset \rho$.
- A vertex $\gamma \in \rho$ iff γ belongs to an edge τ that composes ρ.
- For each interior vertex $\gamma \in \mathcal{T}_0^o$, which is the intersection of an horizontal edge $\tau_h \in \mathcal{T}_1^o$ with a vertical edge $\tau_v \in \mathcal{T}_1^o$, let $\rho_h(\gamma)$ (resp. $\rho_v(\gamma)$) is the corresponding horizontal (resp. vertical) maximal segment. We denote by $\Delta_h^{r,r'}(\gamma)$ (resp. $\Delta_v^{r,r'}(\gamma)$) the equations of the corresponding supporting lines with multiplicity $r' + 1$ (resp. $r + 1$).
- We say that $\rho \in \mathrm{MIS}(\mathcal{T})$ is *blocking* $\rho' \in \mathrm{MIS}(\mathcal{T})$ if one of the end points of ρ' is in the interior of ρ.

Notice that the connections between maximal segments $\in \mathrm{MIS}(\mathcal{T})$ are also T-junctions or crossing junctions.

Example 2.4 *In the figure below, the maximal interior edges are indicated by plain segments.*



In this example, ρ_1 is blocking ρ_4 and ρ_2 is blocking ρ_3.

Notice that in a hierarchical subdivision, if a segment ρ is blocking ρ', then ρ' appeared after ρ in the subdivision.

2.4 Regular subdivision

We introduce the notion of regular subdivision. This notion is related to the notion of node multiplicity distribution, which we describe now.

For each edge of the subdivision, we consider the line supporting this edge and we associate to it a positive integer $\in \mathbb{N}$, called its node multiplicity. This defines a node multiplicity distribution on the subdivision $\mathcal{T}$. For a given segment s and a vertex γ on ρ, the node multiplicity of γ on ρ is the node multiplicity of the line l containing γ and of direction different from ρ (so that γ is the intersection of ρ with l).

Example 2.5 *For $m, m', r \leqslant m, r' \leqslant m' \in \mathbb{N}$, we define the $^{r,r'}_{m,m'}$ -node multiplicity distribution on the subdivision $\mathcal{T}$ in the following way:*

- *Each boundary vertical edge has multiplicity $m+1$.*
- *Each boundary horizontal edge has multiplicity $m'+1$.*
- *Each interior vertical edge has multiplicity $m-r$.*
- *Each interior horizontal edge has multiplicity $m'-r'$.*

A subdivision $\mathcal{T}$ with this node multiplicity distribution will be called hereafter a $^{r,r'}_{m,m'}$ -subdivision.

Definition 2.6 *Given an order $\rho_1, \rho_2, \dots$ on the maximal interior segments of $\mathcal{T}$ and a node multiplicity distribution, we say that*

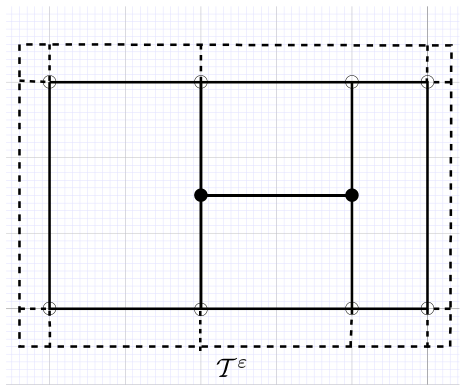
- *a segment ρ_j is k-regular if it contains k vertices (counted with multiplicity) which are not on maximal interior segments ρ_i with $i > j$;*
- *the subdivision $\mathcal{T}$ is k-regular if every maximal interior segment is k-regular.*

For a hierarchical subdivision $\mathcal{T}$, the natural order associated to the maximal interior segments is the order in which they appear during the subdivision.

Notice that a T-subdivision with no interior segment is k-regular for any k.

2.5 Extended T-subdivision

Definition 2.7 *An extended T-subdivision associated to a subdivision $\mathcal{T}$ of a rectangle Ω is obtained by enclosing Ω into a cocentric rectangle Ω' and by extending the edges of $\mathcal{T}$ touching the boundary of Ω to the boundary of the outer rectangle. It is denoted by $\mathcal{T}^{\varepsilon}$.*



Notice that the interior crossing vertices of $\mathcal{T}^{\varepsilon}$ are the boundary and interior crossing vertices of $\mathcal{T}$.

2.6 Dual topological complex

The dual complex $\mathcal{T}^{\star}$ of the subdivision $\mathcal{T}$, is such that we have the following properties.

- a face $\sigma \in \mathcal{T}_2$ is a vertex of the dual complex $\mathcal{T}^{\star}$.
- an edge of $\mathcal{T}^{\star}$ is connecting two elements $\sigma, \sigma' \in \mathcal{T}_2$ if they share a common (interior) edge $\tau \in \mathcal{T}_1^o$. Thus it is identified with the edge τ of $\mathcal{T}$ between σ, σ';
- a face of $\mathcal{T}^{\star}$ corresponds to an elements $\gamma \in \mathcal{T}_0^o$. It is either a triangle if γ is a T-junction or a quadrangle if γ is a crossing vertex.

Notice that the boundary cells of $\mathcal{T}$ correspond to boundary vertices of $\mathcal{T}^{\star}$. They are connected by boundary edges which belong to a single face of $\mathcal{T}^{\star}$.

3 Topological chain complexes

In this section, we describe the tools from algebraic topology, that we will use. For more details, see eg. [18], [9].

3.1 Definitions

We consider the following complexes:

$$
\begin{array}{lcccccccc}
 & & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \\
\mathfrak{I}^{r,r'}_{m,m'}(\mathcal{T}^o): & 0 & \to & \bigoplus_{\tau\in\mathcal{T}_1^o}[\tau]\mathfrak{I}^{r,r'}_{m,m'}(\tau) & \to & \bigoplus_{\gamma\in\mathcal{T}_0^o}[\gamma]\mathfrak{I}^{r,r'}_{m,m'}(\gamma) & \to & 0\\
 & \downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{R}_{m,m'}(\mathcal{T}^o): & \bigoplus_{\sigma\in\mathcal{T}_2}[\sigma]R_{m,m'} & \to & \bigoplus_{\tau\in\mathcal{T}_1^o}[\tau]R_{m,m'} & \to & \bigoplus_{\gamma\in\mathcal{T}_0^o}[\gamma]R_{m,m'} & \to & 0\\
 & \downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}^o): & \bigoplus_{\sigma\in\mathcal{T}_2}[\sigma]R_{m,m'} & \to & \bigoplus_{\tau\in\mathcal{T}_1^o}[\tau]R_{m,m'}/\mathfrak{I}^{r,r'}_{m,m'}(\tau) & \to & \bigoplus_{\gamma\in\mathcal{T}_0^o}[\gamma]R_{m,m'}/\mathfrak{I}^{r,r'}_{m,m'}(\gamma) & \to & 0\\
 & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & & 0 & & 0 & &
\end{array}
$$

The different vector spaces of these complexes are obtained as the components in bidegree $\leqslant (m, m')$ of R-modules generated by (formal) independent elements $[\sigma], [\tau], [\gamma]$ indexed respectively by the faces, the interior edges and interior points of $\mathcal{T}$. An oriented edge $\tau \in \mathcal{T}_1$ is be represented as: $[\tau] = [ab]$ where $a, b \in \mathcal{T}_0$ are the end points. The opposite edge is represented by $[ba]$. By convention $[ba] = -[ab]$.

The maps of the complex $\mathfrak{R}_{m,m'}(\mathcal{T}^o)$ are defined as follows:

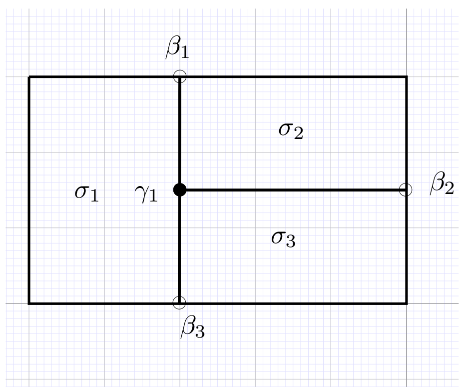
- for each cell $\sigma \in \mathcal{T}_2$ with its counter-clockwise boundary formed by edges $\tau_1 = a_1a_2, \dots, \tau_l = a_la_1$, $\partial_2(\sigma) = [\tau_1] \oplus \cdots \oplus [\tau_l] = [a_1a_2] \oplus \cdots \oplus [a_la_1]$;
- for each interior edge $\tau \in \mathcal{T}_1^o$ from γ_1 to $\gamma_2 \in \mathcal{T}_0$, $\partial_1([\tau]) = [\gamma_1] - [\gamma_2]$ where $[\gamma] = 0$ if $\gamma \notin \mathcal{T}_0^o$;
- for each interior point $\gamma \in \mathcal{T}_o^o$, $\partial_0([\gamma]) = 0$.

By construction, we have $\partial_i \circ \partial_{i+1} = 0$ for $i = 0, 1$. The maps of the complex $\mathfrak{I}_{m,m'}^{r,r'}(\mathcal{T}^o)$ are obtained from those of $\mathfrak{R}_{m,m'}(\mathcal{T}^o)$ by restriction, those of the complex $\mathfrak{S}_{m,m'}^{r,r'}(\mathcal{T}^o)$ are obtained by taking the quotient by the corresponding vector spaces of $\mathfrak{I}_{m,m'}^{r,r'}(\mathcal{T}^o)$. They are denoted $\bar{\partial}_i$.

For each column of the diagram, the vertical maps are respectively the inclusion map and the quotient map.

The complex $\mathfrak{R}_{m,m'}(\mathcal{T}^o)$ is also known as the chain complex of $\mathcal{T}$ relative to its boundary $\partial\mathcal{T}$.

Example 3.1 *We consider the following subdivision $\mathcal{T}$ of a rectangle Ω:*



We have

- $\partial_2([\sigma_1]) = [\gamma_1\beta_1] + [\beta_3\gamma_1], \partial_2([\sigma_2]) = [\beta_1\gamma_1] + [\gamma_1\beta_2], \partial_2([\sigma_3]) = [\gamma_1\beta_3] + [\beta_2\gamma_1]$,
- $\partial_1([\beta_1\gamma_1]) = [\gamma_1], \partial_1([\beta_2\gamma_1]) = [\gamma_1], \partial_1([\beta_3\gamma_1] = [\gamma_1]$,
- $\partial_0([\gamma_1]) = 0$.

This defines the following complex:

$$\mathfrak{R}_{m,m'}(\mathcal{T}): \quad \bigoplus_{i=1}^3 [\sigma_i] R_{m,m'} \quad \to \quad \bigoplus_{i=1}^3 [\beta_i\gamma_1] R_{m,m'} \quad \to \quad [\gamma_1] R_{m,m'} \quad \to \quad 0$$

The matrices of these maps in the canonical (monomial) bases are

$$[\partial_2] = \begin{pmatrix} -I & I & 0 \\ 0 & -I & I \\ I & 0 & -I \end{pmatrix}, [\partial_1] = \begin{pmatrix} I & I & I \end{pmatrix}$$

where I is the identity matrix of size $(m+1) \times (m'+1)$ (ie. the dimension of $R_{m,m'}$).

The matrices of the complex $\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T})$ are

$$[\bar\partial_2] = \begin{pmatrix} -\Pi_1 & \Pi_1 & 0 \\ 0 & -\Pi_2 & \Pi_2 \\ \Pi_3 & 0 & -\Pi_3 \end{pmatrix}, [\bar\partial_1] = \begin{pmatrix} P_1 & P_2 & P_3 \end{pmatrix}$$

where Π_i (resp. P_i) is the matrix of the projection of $R_{m,m'}$ (resp. $R_{m,m'}/\mathfrak{I}^{r,r'}_{m,m'}(\beta_i\gamma_1)$) onto $R_{m,m'}/\mathfrak{I}^{r,r'}_{m,m'}(\beta_i\gamma_1)$ (resp. $R_{m,m'}/\mathfrak{I}^{r,r'}_{m,m'}(\gamma_1)$).

3.2 Their homology

In this section, we analyse the homology of the different complexes. The homology of $\mathfrak{R}_{m,m'}(\mathcal{T}^o)$ is well-known [18][Chap. 4], [9][Chap. 2], but we give simple proofs for self-content purposes.

3.2.1 The 0-homology. We start by analysing the homology on the vertices.

Lemma 3.2 $H_0(\mathfrak{R}_{m,m'}(\mathcal{T}^o)) = H_0(\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}^o)) = 0.$

Proof. Let $\gamma \in \mathcal{T}^o_0$. There is a sequence of edges $\tau_0 = \gamma_0\gamma_1,\ \tau_1 = \gamma_1\gamma_2, \ldots,\ \tau_l = \gamma_l\gamma_{l+1}$, such that $\tau_i \in \mathcal{T}^o_1$, $\gamma_0 \notin \mathcal{T}^o_0$ and $\gamma_{l+1} = \gamma$. Then

$$\partial_1([\tau_0] + \cdots + [\tau_l]) = [\gamma_1] - [\gamma_0] + \cdots + [\gamma_{l+1}] - [\gamma_l] = [\gamma]$$

since $[\gamma_0] = 0$ and $[\gamma_{l+1}] = [\gamma]$. Multiplying by any element in $R_{m,m'}$ and taking the quotient by $\mathfrak{I}^{r,r'}_{m,m'}(\tau_i)$, we get that $[\gamma]R_{m,m'} \subset \operatorname{im}\partial_1$ (resp. $[\gamma]R_{m,m'}/\mathfrak{I}^{r,r'}_{m,m'}(\gamma) \subset \operatorname{im}\bar\partial_1$) and thus that

$$H_0(\mathfrak{R}_{m,m'}(\mathcal{T}^o)) = H_0(\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}^o)) = 0.$$

□

Let us describe in more details $H_0(\mathfrak{I}^{r,r'}_{m,m'}(\mathcal{T}^o)) = \bigoplus_{\gamma\in\mathcal{T}^o_0}[\gamma]\mathfrak{I}^{r,r'}(\gamma)/\partial_1(\bigoplus_{\tau\in\mathcal{T}^o_1}[\tau]\mathfrak{I}^{r,r'}(\tau))$. We consider the free R module generated by the elements $e_{\gamma,\tau}$, for all interior edges $\tau \in \mathcal{T}^o_1$ and all vertices $\gamma \in \tau$, with the convention that $e_{\gamma,\tau} = 0$ if $\gamma \notin \mathcal{T}^o_0$. Let $\gamma \in \mathcal{T}^o_0$ and $E(\gamma)$ be the set of interior edges that contain γ. We consider first the map

$$\begin{aligned} \varphi_\gamma : \bigoplus_{\tau\in E(\gamma)} e_{\gamma,\tau}R_{(m,m')-\delta(\tau)} &\to [\gamma]\mathfrak{I}^{r,r'}_{m,m'}(\gamma) \\ e_{\gamma,\tau} &\mapsto [\gamma]\Delta^{r,r'}_{\tau} \end{aligned}$$

Its kernel is denoted $\mathfrak{K}^{r,r'}_{m,m'}(\gamma)$. Let $P_h(\gamma)$ (resp. $P_v(\gamma)$) be the set of pairs of horizontal (resp. vertical) interior edges which contain γ. If γ is a T-junction, one of the two sets is empty and the other is a singleton containing one pair. If γ is a crossing vertex, each set is a singleton.

Proposition 3.3

$$\begin{aligned} \mathfrak{K}^{r,r'}_{m,m'}(\gamma) = & \sum_{(\tau,\tau')\in P_h(\gamma)} (e_{\gamma,\tau} - e_{\gamma,\tau'})R_{(m,m')-\delta(\tau)} + \sum_{(\tau,\tau')\in P_v(\gamma)} (e_{\gamma,\tau} - e_{\gamma,\tau'})R_{(m,m')-\delta(\tau)} \\ & + \sum_{\tau\in E_h(\gamma),\tau'\in E_v(\gamma)} (\Delta^{r,r'}(\tau')e_{\gamma,\tau} - \Delta^{r,r'}(\tau)e_{\gamma,\tau'})R_{(m-r-1,m'-r'-1)} \end{aligned}$$

Proof. Let us suppose first that γ is a crossing vertex and let us denote by τ_1, τ_2 the horizontal edges, τ_3, τ_4 the vertical edges at γ. The matrix of the map φ_γ in the basis e_{γ,τ_i} is

$$[\varphi_\gamma] = \begin{pmatrix} \Delta & \Delta & \Delta' & \Delta' \end{pmatrix}$$

where $\Delta = \Delta_{\tau_1}^{r,r'} = \Delta_{\tau_2}^{r,r'}$, $\Delta' = \Delta_{\tau_3}^{r,r'} = \Delta_{\tau_4}^{r,r'}$. Since Δ and Δ' have no common factor, the kernel of the matrix $[\varphi_\gamma]$ is generated by the elements $e_{\gamma,\tau_1} - e_{\gamma,\tau_2}$, $\Delta' e_{\gamma,\tau_2} - \Delta e_{\gamma,\tau_3}$, $e_{\gamma,\tau_3} - e_{\gamma,\tau_4}$, which give the description of $\mathfrak{K}_{m,m'}^{r,r'}(\gamma)$ in degree $\le (m,m')$. A similar proof applies when there is only one horizontal or vertical edge at γ. This proves the result. □

We use the maps $(\varphi_\gamma)_{\gamma\in\mathcal{T}_0^o}$ to define the map

$$\varphi : \bigoplus_{\gamma\in\mathcal{T}_0^o} \bigoplus_{\tau\in E_\gamma} e_{\gamma,\tau} R_{(m,m')-\delta(\tau)} \quad\to\quad \bigoplus_{\gamma\in\mathcal{T}_0^o} [\gamma]\mathfrak{I}_{m,m'}^{r,r'}(\gamma),$$

so that we have the following exact sequence:

$$0 \to \bigoplus_{\gamma\in\mathcal{T}_0^o} \mathfrak{K}_{m,m'}^{r,r'}(\gamma) \to \bigoplus_{\gamma\in\mathcal{T}_0^o} \bigoplus_{\tau\in E_\gamma} e_{\gamma,\tau} R_{(m,m')-\delta(\tau)} \to \bigoplus_{\gamma\in\mathcal{T}_0^o} [\gamma]\mathfrak{I}_{m,m'}^{r,r'}(\gamma) \to 0$$

Using this exact sequence, we can now identify $\bigoplus_{\gamma\in\mathcal{T}_1^o} [\gamma]\mathfrak{I}_{m,m'}^{r,r'}(\gamma)$ with the quotient

$$\bigoplus_{\gamma\in\mathcal{T}_1^o} \bigoplus_{\tau\in E_\gamma} e_{\gamma,\tau} R_{(m,m')-\delta(\tau)} / \sum_{\gamma\in\mathcal{T}_1^o} \mathfrak{K}_{m,m'}^{r,r'}(\gamma).$$

We are going to show that we can extend this identification to $\operatorname{im}\partial_1$ and $H_0(\mathfrak{I}_{m,m'}^{r,r'}(\mathcal{T}^o))$:

Proposition 3.4 *We have*

$$\begin{aligned} H_0(\mathfrak{I}_{m,m'}^{r,r'}(\mathcal{T}^o)) \quad = & \quad \bigoplus_{\gamma\in\mathcal{T}_0^o} \bigoplus_{\tau\in E_\gamma} e_{\gamma,\tau} R_{(m,m')-\delta(\tau)} \\ / & \quad \Bigg(\sum_{(\tau,\tau')\in E_\gamma^v} (e_{\gamma,\tau} - e_{\gamma,\tau'}) R_{(m,m')-\delta(\tau)} + \sum_{(\tau,\tau')\in E_\gamma^h} (e_{\gamma,\tau} - e_{\gamma,\tau'}) R_{(m,m')-\delta(\tau)} \\ & \quad + \sum_{\tau=(\gamma,\gamma')\in\mathcal{T}_1^o} (e_{\gamma,\tau} - e_{\gamma',\tau}) R_{(m,m')-\delta(\tau)} \\ & \quad + \sum_{\tau\in E_\gamma^h, \tau'\in E_\gamma^v} (\Delta_{\tau'}^{r,r'} e_{\gamma,\tau} - \Delta_{\tau}^{r,r'} e_{\gamma,\tau'}) R_{(m-r-1,m'-r'-1)} \Bigg). \end{aligned}$$

Proof. The application

$$\partial_1 : \bigoplus_{\tau\in\mathcal{T}_1^o} [\tau]\mathfrak{I}_{m,m'}^{r,r'}(\tau) \quad\to\quad \bigoplus_{\gamma\in\mathcal{T}_0^o} [\gamma]\mathfrak{I}_{m,m'}^{r,r'}(\gamma)$$

lift to an application:

$$\begin{aligned} \tilde\partial_1 : \bigoplus_{\tau\in\mathcal{T}_1^o} [\tau] R_{(m,m')-\delta(\tau)} \quad &\to \quad \bigoplus_{\gamma\in\mathcal{T}_0^o} \bigoplus_{\tau\in E_\gamma} e_{\gamma,\tau} R_{(m,m')-\delta(\tau)} \\ \tau \quad &\mapsto \quad e_{\gamma,\tau} - e_{\gamma',\tau} \end{aligned}$$

so that the image of ∂_1 lift in $\bigoplus_{\gamma\in\mathcal{T}_0^o} \bigoplus_{\tau\in E_\gamma} e_{\gamma,\tau} R_{(m,m')-\delta(\tau)}$ to

$$\operatorname{im}\tilde\partial_1 \quad = \quad \sum_{\tau\in\mathcal{T}_1^o} (e_{\gamma,\tau} - e_{\gamma',\tau}) R_{(m,m')-\delta(\tau)}.$$

Consequently,

$$H_0(\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}^o)) \quad = \quad \bigoplus_{\gamma\in\mathcal{T}^o_0}\bigoplus_{\tau\in E_\gamma} e_{\gamma,\tau}R_{(m,m')-\delta(\tau)}/\left(\operatorname{im}\tilde{\partial}_1+\sum_{\gamma\in\mathcal{T}^o_0}\mathfrak{K}^{r,r'}_{m,m'}(\gamma)\right),$$

which yields the desired description of $H_0(\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}))$. □

Proposition 3.5 *We have*

$$\begin{array}{rcl} H_0(\mathfrak{J}^{r,r'}_{m,m'}(\mathcal{T}^o)) & = & \oplus_{\rho\in \mathrm{MIS}(\mathcal{T})}[\rho]R_{(m,m')-\delta(\rho)} \\ & / & \left(\sum_{\gamma\in\mathcal{T}^o_0}(\Delta^{r,r'}_h(\gamma)[\rho_v(\gamma)]-\Delta^{r,r'}_v(\gamma)[\rho_h(\gamma)])R_{(m-r-1,m'-r'-1)}\right). \end{array}$$

Proof. Let $B=\bigoplus_{\gamma\in\mathcal{T}^o_0}\bigoplus_{\tau\in E_\gamma} e_{\gamma,\tau}R_{(m,m')-\delta(\tau)}$, $K=\operatorname{im}\tilde{\partial}_1+\sum_{\gamma\in\mathcal{T}^o_0}\mathfrak{K}^{r,r'}_{m,m'}(\gamma)$ and

$$\begin{array}{rcl} K' & = & \left(\sum_{(\tau,\tau')\in E^v_\gamma}(e_{\gamma,\tau}-e_{\gamma,\tau'})R_{(m,m')-\delta(\tau)}+\sum_{(\tau,\tau')\in E^h_\gamma}(e_{\gamma,\tau}-e_{\gamma,\tau'})R_{(m,m')-\delta(\tau)}\right. \\ & & \left.+\sum_{\tau=(\gamma,\gamma')\in\mathcal{T}^o_1}(e_{\gamma,\tau}-e_{\gamma',\tau})R_{(m,m')-\delta(\tau)}\right). \end{array}$$

As $K'\subset K\subset B$, we have $B/K\equiv(B/K')/(K/K')$. Taking the quotient by K' means, that we identify the horizontal (resp. vertical) edges which share a vertex. Thus all horizontal (resp. vertical) edges which are contained in a maximal segment ρ of $\mathcal{T}$ are identify to a single element, that we denote $[\rho]$. As $e_{\gamma,\tau}=0$ if $\gamma\notin\mathcal{T}^o_0$, $[\rho]=0$ if the maximal segment ρ intersects the boundary of Ω and we obtain the description of $H_0(\mathfrak{J}^{r,r'}_{m,m'}(\mathcal{T}))$ of the proposition. □

Definition 3.6 *Let* $h^{r,r'}_{m,m'}(\mathcal{T})=\dim H_0(\mathfrak{J}^{r,r'}_{m,m'}(\mathcal{T}^o))$.

3.2.2 The 1-homology. We consider now the homology on the edges.

Proposition 3.7 $H_1(\mathfrak{R}_{m,m'}(\mathcal{T}^o))=0$.

Proof. Let $p=\sum_{\tau\in\mathcal{T}^o_1}[\tau]p_\tau\in\ker\partial_1$ with $p_\tau\in R_{m,m'}$. Let us prove that p is in the image of ∂_2. For each $\gamma\in\mathcal{T}^o_0$ and each edge τ which contains γ, we have $\sum\varepsilon_\tau p_\tau=0$ with $\varepsilon_\tau=1$ if τ ends at γ, $\varepsilon_\tau=-1$ if τ starts at γ and $\varepsilon_\tau=0$ otherwise.

For any $\sigma\in\mathcal{T}^o_2$ and $\tau\in\mathcal{T}^o_1$, we define $\varepsilon_{\sigma,\tau}=1$ if τ is oriented counter-clockwise on the boundary of σ, $\varepsilon_{\sigma,\tau}=-1$ if τ is oriented counter-clockwise on the boundary of σ and $\varepsilon_{\sigma,\tau}=0$ otherwise.

For any oriented edge of the dual graph $\mathcal{T}^\star$ from σ' to σ, let us define $\partial^\star_1([\sigma',\sigma])=\varepsilon_{\sigma,\tau}p_\tau$. Notice that $\partial^\star_1([\sigma,\sigma'])=\varepsilon_{\sigma',\tau}p_\tau=-\varepsilon_{\sigma,\tau}p_\tau=-\partial^\star_1([\sigma',\sigma])$, since the orientation of τ on the boundary of σ and σ' are opposite.

Let σ_0 be the cell of $\mathcal{T}$ in the low left corner. We order the cells $\sigma\in\mathcal{T}_2$ according to their distance to σ_0 in this dual graph $\mathcal{T}^\star$. We define an element $q=\sum_{\sigma\in\mathcal{T}_2}q_\sigma[\sigma]$ where $q_\sigma\in R_{m,m'}$ by induction using this order, as follows:

- $q_{\sigma_0}=0$;
- For any $\sigma,\sigma'\in\mathcal{T}_2$, if $\sigma>\sigma'$ and σ and σ' share a common edge τ, then $q_\sigma=q_{\sigma'}+\partial^\star_1([\sigma',\sigma])$.

Thus, if $[\sigma_0,\sigma_1],[\sigma_1,\sigma_2],\ldots,[\sigma_{k-1},\sigma_k]$ is a path of $\mathcal{T}^\star$ connecting σ_0 to $\sigma_k=\sigma$ with $\sigma_{i+1}>\sigma_i$ then $q_\sigma=\sum_{i=0}^{k-1}\partial^\star_1([\sigma_i,\sigma_{i+1}])$. Let us prove that this definition does not depend on the chosen path between σ_0 and σ.

We first show that for any face $\gamma^\star$ of $\mathcal{T}^\star$ attached to a vertex γ, if its counter-clockwise boundary is formed by the edges $[\sigma,\sigma'],[\sigma',\sigma''],\ldots,[\sigma''',\sigma]$ corresponding to the edges $\tau,\tau',\tau'',\ldots$ of $\mathcal{T}$, then

$$\partial_1^\star([\sigma,\sigma'])+\partial_1^\star([\sigma',\sigma'']+\cdots+\partial_1^\star([\sigma''',\sigma])=\varepsilon_{\sigma,\tau}p_\tau+\varepsilon_{\sigma',\tau'}p_{\tau'}+\varepsilon_{\sigma'',\tau''}p_{\tau''}+\cdots=0.$$

By changing the orientation of an edge τ, we replace p_τ by $-p_\tau$ and $\varepsilon_{\sigma,\tau}$ by $-\varepsilon_{\sigma,\tau}$ so that the quantity $\varepsilon_{\sigma,\tau}p_\tau$ is not changed. Thus we can assume that all the edges $\tau,\tau',\tau'',\ldots$ are pointing to γ. As $p\in\ker\partial_1$, we have $p_\tau+p_{\tau'}+p_{\tau''}+\cdots=0$.

Now as the cells $\sigma,\sigma',\sigma'',\ldots,\sigma$ are ordered counter-clockwise around γ and as the edges are pointing to γ, we have $\varepsilon_{\sigma,\tau}=\varepsilon_{\sigma',\tau'}=\varepsilon_{\sigma'',\tau''}=\cdots=1$, so that if the sum $\partial_1^\star([\sigma,\sigma'])+\partial_1^\star([\sigma',\sigma'']+\cdots+\partial_1^\star([\sigma''',\sigma])$ over a the boundary of a face $\gamma^\star$ of $\mathcal{T}^\star$ is 0.

By composition, for any loop of $\mathcal{T}^\star$, the sum on the corresponding oriented edges is 0. This shows that the definition of q_σ does not depend on the oriented path from σ_0 to σ.

By construction, we have

$$\partial_2(q)=\sum_{\sigma\in\mathcal{T}_2}(\sum_{\tau\in\mathcal{T}_1^o}\varepsilon_{\sigma,\tau}q_\sigma[\tau])=\sum_{\tau\in\mathcal{T}_1^o}(\sum_{\sigma\in\mathcal{T}_2}\varepsilon_{\sigma,\tau}q_\sigma)[\tau].$$

For each interior edge $\tau\in\mathcal{T}_1^o$, there are two faces $\sigma_1>\sigma_2$, which are adjacent to τ. Thus, we have $\varepsilon_{\sigma_1,\tau}=-\varepsilon_{\sigma_2,\tau}$ and $q_{\sigma_1}=q_{\sigma_2}+\varepsilon_{\sigma_1,\tau}p_\tau$. We deduce that

$$(\sum_\sigma\varepsilon_{\sigma,\tau}q_\sigma)=\varepsilon_{\sigma_1,\tau}q_{\sigma_1}+\varepsilon_{\sigma_2,\tau}q_{\sigma_2}=\varepsilon_{\sigma_1,\tau}(q_{\sigma_2}+\varepsilon_{\sigma_1,\tau}p_\tau)+\varepsilon_{\sigma_2,\tau}q_{\sigma_2}=p_\tau.$$

This shows that $\partial_2(q)=p$. In other words, $\operatorname{im}\partial_2=\ker\partial_1$ and $H_1(\mathfrak{R}_{m,m'}(\mathcal{T}^o))=0$. □

Proposition 3.8 $H_1(\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}^o))=H_0(\mathfrak{J}^{r,r'}_{m,m'}(\mathcal{T}^o))$.

Proof. As $H_0(\mathfrak{R}_{m,m'}(\mathcal{T}^o))=0$ and $H_1(\mathfrak{R}_{m,m'}(\mathcal{T}^o))=0$, we deduce from the long exact sequence (see Section 1.3)

$$\cdots\to H_1(\mathfrak{R}_{m,m'}(\mathcal{T}^o))\to H_1(\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}^o))\to H_0(\mathfrak{J}^{r,r'}_{m,m'}(\mathcal{T}^o))\to H_0(\mathfrak{R}_{m,m'}(\mathcal{T}^o))\to\cdots$$

that $H_1(\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}^o))\sim H_0(\mathfrak{J}^{r,r'}_{m,m'}(\mathcal{T}^o))$, since $H_1(\mathfrak{R}_{m,m'}(\mathcal{T}^o))=H_0(\mathfrak{R}_{m,m'}(\mathcal{T}^o))=0$. □

3.2.3 The 2-homology. Finally, the homology on the 2-cells will give us information on the spline space $\mathcal{S}^{r,r'}_{m,m'}(\mathcal{T})$.

Proposition 3.9 $H_2(\mathfrak{R}_{m,m'}(\mathcal{T}^o))=R_{m,m'}$.

Proof. An element of $H_2(\mathfrak{R}_{m,m'}(\mathcal{T}^o))=\ker\partial_2$ is a collection of polynomials $(p_\sigma)_{\sigma\in\mathcal{T}_2}$ such that $p_\sigma\in R_{m,m'}$ and $p_\sigma=p_{\sigma'}$ if σ and σ' share an (internal) edges. As $\mathcal{T}$ is a subdivision of a rectangle D_0, all faces $\sigma\in\mathcal{T}_2$ share pairwise an edge. Thus $p_\sigma=p_{\sigma'}$ for all $\sigma,\sigma'\in\mathcal{T}_2$ and $H_2(\mathfrak{R}_{m,m'}(\mathcal{T}))=R_{m,m'}$. □

Notice that by counting the dimensions in the exact sequence $\mathfrak{R}_{m,m'}(\mathcal{T})$, we recover the well-known Euler formula: $f_2-f_1+f_0=1$ (the domain Ω has one connected component).

Proposition 3.10 $H_2(\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}^o))=\ker\partial_2=\mathcal{S}^{r,r'}_{m,m'}(\mathcal{T})$.

Proof. An element of $H_2(\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}^o))=\ker\overline{\partial}_2$ is collection of polynomials $(p_\sigma)_{\sigma\in\mathcal{T}_2}$ such that $p_\sigma\in R_{m,m'}$ and $p_\sigma\equiv p_{\sigma'}\bmod\mathfrak{J}^{r,r'}_\tau(\tau)$ if σ and σ' share the (internal) edge τ. By Lemma 1.3, this implies that the piecewise polynomial function which is p_σ on σ and $p_{\sigma'}$ on σ' is of class $C^{r,r'}$ across the edge τ. As this is true for all interior edges, $(p_\sigma)_{\sigma\in\mathcal{T}_2}\in\ker\partial_2$ is a piecewise polynomial function of $R_{m,m'}$ which is of class $C^{r,r'}$, that is an element of $\mathcal{S}^{r,r'}_{m,m'}(\mathcal{T})$. □

3.3 Dimension formula

In this section, are the main results on the dimension of the spline spaces.

Theorem 3.11 *We have*

$$\begin{aligned}
&\dim \mathcal{S}^{r,r'}_{m,m'}(\mathcal{T}) \\
&= \quad (m+1)(m'+1)f_2 - (m+1)(r'+1)f_1^h - (m'+1)(r+1)f_1^v + (r+1)(r'+1)f_0 \qquad (2)\\
&+ \quad h^{r,r'}_{m,m'}(\mathcal{T}).
\end{aligned}$$

where

- f_2 *is the number of 2-faces* $\in \mathcal{T}_2$,
- f_1^h *(resp.* f_1^v*) is the number of horizontal (resp. vertical) interior edges* $\in \mathcal{T}_1^o$,
- f_0 *is the number of interior vertices* $\in \mathcal{T}_0^o$,
- $h^{r,r'}_{m,m'}(\mathcal{T}) = \dim H_0(\mathfrak{I}^{r,r'}_{m,m'}(\mathcal{T}^o))$.

Proof. The complex

$$\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}^o) : \oplus_{\sigma\in\mathcal{T}_2}[\sigma]R_{m,m'} \longrightarrow \oplus_{\tau\in\mathcal{T}_1^o}[\tau]R_{m,m'}/\mathfrak{I}^{r,r'}_{m,m'}(\tau) \longrightarrow \oplus_{\gamma\in\mathcal{T}_0^o}[\gamma]R_{m,m'}/\mathfrak{I}^{r,r'}_{m,m'}(\gamma) \longrightarrow 0$$

induces the following relations

$$\dim(\oplus_{\sigma\in\mathcal{T}_2}[\sigma]R_{m,m'}) - \dim(\oplus_{\tau\in\mathcal{T}_1^o}[\tau]R_{m,m'}/\mathfrak{I}^{r,r'}_{m,m'}(\tau)) + \dim(\oplus_{\gamma\in\mathcal{T}_0^o}[\gamma]R_{m,m'}/\mathfrak{I}^{r,r'}_{m,m'}(\gamma))$$

$$= \dim(H_2(\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}^o)) - \dim(H_1(\mathfrak{S}^{r,r}_{m,m'}(\mathcal{T}^o)) + \dim(H_0(\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}^o))$$

As $H_2(\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}^o)) = \mathcal{S}^{r,r'}_{m,m'}(\mathcal{T})$, $H_0(\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}^o)) = 0$ and $H_1(\mathfrak{S}^{r,r'}_{m,m'}(\mathcal{T}^o)) = H_0(\mathfrak{I}^{r,r'}_{m,m'}(\mathcal{T}^o))$, we deduce that

$$\begin{aligned}
\dim \mathcal{S}^{r,r'}_{m,m'}(\mathcal{T}) \quad &= \quad \dim(\oplus_{\sigma\in\mathcal{T}_2}[\sigma]R_{m,m'}) - \dim(\oplus_{\tau\in\mathcal{T}_1^o}[\tau]R_{m,m'}/\mathfrak{I}^{r,r'}_{m,m'}(\tau)) \\
&+ \quad \dim(\oplus_{\gamma\in\mathcal{T}_0^o}[\gamma]R_{m,m'}/\mathfrak{I}^{r,r'}_{m,m'}(\gamma)) + \dim(H_0(\mathfrak{I}^{r,r'}_{m,m'}(\mathcal{T}))
\end{aligned}$$

which yields the dimension formula, using Lemma 1.2. □

As an immediate corollary of this theorem and Proposition 3.5, we deduce the following result:

Corollary 3.12 *If the T-subdivision* $\mathcal{T}$ *has no maximal interior segments then* $h^{r,r'}_{m,m'}(\mathcal{T})$*=0.*

For a hierarchical T-subdivision, the maximal interior segments are ordered as $\rho_1, \rho_2, \ldots, \rho_l$ in the way they appear during the subdivision process. In this case, if ρ_i is blocking ρ_j then $j > i$.

Definition 3.13 *Let* $\lambda(\rho_i)$ *be the number of vertices of* ρ_i*, which are not on a maximal interior segment of index* $> i$.

Then we have the following theorem:

Theorem 3.14 *Let* $\mathcal{T}$ *be a hierarchical T-subdivision. Then*

$$\begin{aligned}
h^{r,r'}_{m,m'}(\mathcal{T}) \quad &\leqslant \quad \sum_{\rho\in \mathrm{MIS}_h(\mathcal{T})} (m+1-(m-r)\lambda(\rho))_+ \times (m'-r') \\
&+ \quad \sum_{\rho\in \mathrm{MIS}_v(\mathcal{T})} (m-r) \times (m'+1-(m'-r')\lambda(\rho))_+ \,.
\end{aligned}$$

Moreover, equality holds in

- $\forall \rho \in \mathrm{MIS}_h(\mathcal{T})$, $(m-r)\lambda(\rho) \geqslant m+1$ *and* $\forall \rho \in \mathrm{MIS}_v(\mathcal{T})$, $(m'-r')\lambda(\rho) \geqslant m'+1$*; or*
- $\forall \rho \in \mathrm{MIS}_h(\mathcal{T}), (m-r)\lambda(\rho) \leqslant m+1 \,\text{and}\, \forall \rho \in \mathrm{MIS}_v(\mathcal{T}), (m'-r')\lambda(\rho) \leqslant m'+1.$

Proof. By Proposition 3.5, the dimension $h^{r,r'}_{m,m'}(\mathcal{T})$ is the dimension of the quotient of $M := \sum_{i=1}^{l} R_{m,m'-\delta(\rho_i)}[\rho_i]$ by the module $K_{m,m'}$ generated in degree $\leq (m,m')$ by the following relations: For each vertex $\gamma \in \mathcal{T}_0^o$ which is on a maximal interior segment, we have the relation

- $\Delta_{\rho_j}[\rho_i] - \Delta_{\rho_i}[\rho_j]$ if γ is the intersection of two maximal interior segments ρ_i and ρ_j or
- $\Delta_\tau[\rho_i]$ if γ is the intersection of the maximal interior segment ρ_i with a non-interior segment τ.

To compute the dimension of this quotient, we use a graduation on M given by the indices of the segments: The initial of an element $\sum_i p_i[\rho_i] \in M$ (with $p_i \in R_{m,m'-\delta(\rho_i)}$) is $p_{i_0}[\rho_{i_0}]$ where i_0 is the minimal index such that $p_i \neq 0$. The dimension $h^{r,r'}_{m,m'}(\mathcal{T})$ is

$$h^{r,r'}_{m,m'}(\mathcal{T}) = \dim(M/K) = \dim(M/\,\mathrm{In}(K))$$

where $\mathrm{In}(K)$ is the initial of the module K for the chosen graduation.

Notice that $\mathrm{In}(K)$ contains the multiples in degree $\leqslant (m,m')$ of

- $\Delta_{\rho_j}[\rho_i]$ if γ is the intersection of two maximal interior segments ρ_i and ρ_j with $i > j$,
- $\Delta_\tau[\rho_i]$ if γ is the intersection of the maximal interior segment ρ_i with a non-interior segment τ.

The number of such relations associated to the segment ρ_i is by definition $\lambda(\rho_i)$. Let L_i be the vector space span by these initials in degree $\leqslant (m,m')$. As $[\rho_i]$ is of degree $\delta(\rho_i)$, by proposition 1.5, L_i is of dimension

- $\min(m+1,(r+1)\lambda(\rho_i)) \times (m'-r')$ if ρ_i is horizontal,
- $\min(m'+1,(r'+1)\lambda(\rho_i,)) \times (m-r)$ if ρ_i is vertical.

Thus the dimension of $R_{m,m'-\delta(\rho_i)}[\rho_i]/L_i$ is

- $(m+1-(r+1)l(\rho_i))_+ \times (m'-r')$ if $\rho_i \in \mathrm{MIS}_h(\mathcal{T})$,
- $(m'+1-(r'+1)l(\rho_i)) \times (m-r)$ if $\rho_i \in \mathrm{MIS}_v(\mathcal{T})$.

As $\mathrm{In}(K) \supset \sum_i L_i$, we have

$$h^{r,r'}_{m,m'}(\mathcal{T}) \leqslant \dim(\sum_i R_{m,m'-\delta(\rho_i)}[\rho_i]/L_i) = \sum_i \dim(R_{m,m'-\delta(\rho_i)}[\rho_i]/L_i).$$

This gives the announced bound on $h^{r,r'}_{m,m'}(\mathcal{T})$.

In the case where $\forall s \in \mathrm{MIS}_h(\mathcal{T}), (m-r)\lambda(s) \geqslant m+1$ and $\forall s \in \mathrm{MIS}_v(\mathcal{T}), (m'-r')\lambda(s) \geqslant m'+1$, we directly deduce that $h^{r,r'}_{m,m'}(\mathcal{T}) = 0$, so that the equality holds.

In the case where $\forall \rho \in \mathrm{MIS}_h(\mathcal{T}), (m-r)\lambda(\rho) \leqslant m+1$ and $\forall \rho \in \mathrm{MIS}_v(\mathcal{T}), (m'-r')\lambda(\rho) \leqslant m'+1$, Proposition 1.5 implies that there is no relations in degree $\leqslant (m,m')$ of the monomial multiples of $\Delta(\rho_j)[\rho_i]$, $\Delta(\tau)[\rho_i]$ for $i = 1,\ldots,l$ and we have $\mathrm{In}(K) = \sum_i L_i$, which shows that equality also holds in this case. ☐

As a corollary, we deduce the following result, also proved in [4], [12], [10]:

Proposition 3.15 *Let $\mathcal{T}$ be a hierarchical subdivision. For $m \geqslant 2r+1$ and $m' \geqslant 2r'+1$, we have* $h^{r,r'}_{m,m'}(\mathcal{T}) = 0$.

Proof. Let us order the maximal interior segments in the way they appear during the subdivision. Then if a segment $\rho_i \in \mathrm{MIS}(\mathcal{T})$ blocks $\rho_j \in \mathrm{MIS}(\mathcal{T})$, we must have $i < j$. This shows that the end points of ρ_i contribute to $\lambda(\rho_i)$. Thus, $\lambda(\rho_i) \geqslant 2$.

As $m \geqslant 2r+1$, we have

$$(m-r)\lambda(\rho_i) \geqslant 2(m-r) \geqslant m + (m-2r) \geqslant m+1.$$

Thus, $(m+1-(m-r)\lambda(\rho_i))_+ = 0$. Similarly $(m'+1-(m'-r')\lambda(\rho_i))_+ = 0$ holds since $m' \geqslant 2r'+1$. By Theorem 3.14, we deduce that $h_{m,m'}^{r,r'}(\mathcal{T}) = 0$. □

Theorem 3.16 *Let $k \geqslant \max\{m+1, m'+1\}$ and let $\mathcal{T}$ be a hierarchical T-subdivision, which is k-regular for the ${}_{m,m'}^{r,r'}$ node multiplicity distribution. Then $h_{m,m'}^{r,r'}(\mathcal{T}) = 0$.*

Proof. A hierarchical k-regular T-subdivision $\mathcal{T}$ is such that each maximal interior segment ρ_i contains at least k vertices, counted with multiplicity, which are not on segments ρ_j with $j > i$.

Let us consider first the case where ρ_i is a horizontal segment. As the multiplicity of an interior vertex is $m-r$, the number of this interior vertices not on segments of bigger index is bigger than $\frac{m+1}{m-r}$. Thus, we have $\lambda(\rho_i) \geqslant \frac{m+1}{m-r}$, which implies that $(m+1-(m-r)\lambda(s))_+ = 0$.

A similar argument applies if ρ_i is a vertical segment, which yields $(m'+1-(m'-r')\lambda(\rho_i))_+ = 0$. By Theorem 3.14, we deduce that $h_{m,m'}^{r,r'}(\mathcal{T}) = 0$. □

This leads to the following rules to construct a hierarchical T-subdivision $\mathcal{T}$ for which $h_{m,m'}^{r,r'}(\mathcal{T}) = 0$.

Algorithm 3.17 (Hierarchical k-regular ${}_{m,m'}^{r,r'}$-subdivision)
$k := \max\{m+1, m'+1\}$;
repeat

1. Choose a cell to subdivide and split it with the new edge τ;
2. If the edge τ does not extend an existing segment, extend τ (on one side and/or the other) so that the maximal segment containing τ is k-regular.

until no more subdivision is needed.

In such a construction,

- either a new interior segment is constructed so that it contains $k' \leq k$ vertices (counted with multiplicity) which are on interior segments of smaller indices or on segments intersecting the boundary;
- or an existing segment ρ_i (which is k-regular) is extended.

In both cases, each constructed maximal interior segment ρ_i is k-regular. By theorem 3.16, $h_{m,m'}^{r,r'}(\mathcal{T}) = 0$ and the dimension of $\dim \mathcal{S}_{m,m'}^{r,r'}(\mathcal{T})$, given by formula (2), depends only on the number of cells, interior segments and interior vertices of $\mathcal{T}$. From this analysis, we deduce the dimension formula of the space of Locally Refined splines described in [7].

4 Examples

In this section, we analyse the dimension formula for small bidegrees and regularities. The construction of explicit basis for these spline spaces will be studied in [13].

4.1 Bilinear C^0 T-splines

We consider first piecewise bilinear polynomials on $\mathcal{T}$, which are continuous, that is $m = m' = 1$ and $r = r' = 0$. By Proposition 3.15, we have $h_{1,1}^{0,0}(\mathcal{T}) = 0$. Using Theorem 3.11 and Lemma 2.1, we obtain:

$$\dim \mathcal{S}_{1,1}^{0,0}(\mathcal{T}) = 4f_2 - 2f_1^o + f_0^o = f_0^+ + f_0^b = f_0^{+,\varepsilon} \tag{3}$$

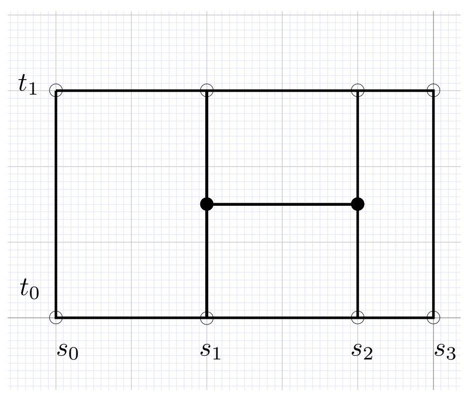
4.2 Biquadratic C^1 T-splines

Let us consider now the set of piecewise biquadratic functions on a T-subdivision $\mathcal{T}$, which are C^1. For $m = m' = 2$ and $r = r' = 1$, Theorem 3.11 and Lemma 2.1 yield

$$\dim \mathcal{S}_{2,2}^{1,1}(\mathcal{T}) = 9f_2 - 6f_1^o + 4f_0^o + h_{2,2}^{1,1}(\mathcal{T}) = f_0^+ - \frac{1}{2}f_0^T + \frac{3}{2}f_0^b + 3 + h_{2,2}^{1,1}(\mathcal{T}). \tag{4}$$

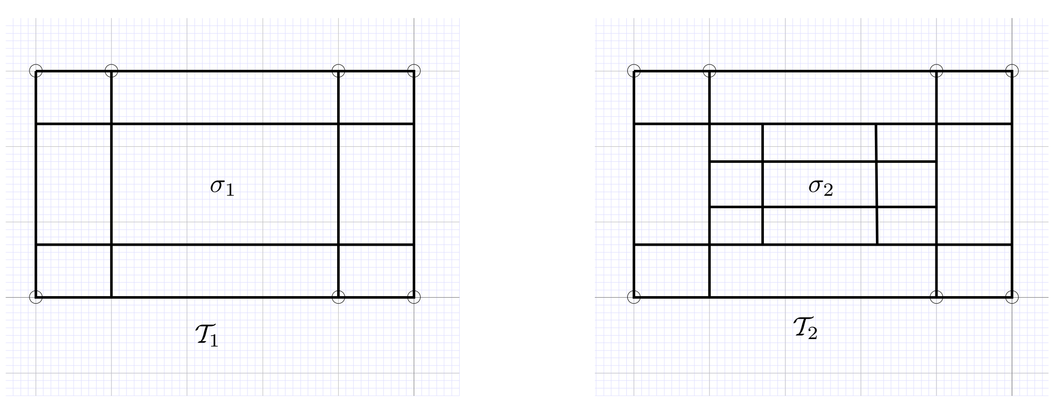
If the subdivision $\mathcal{T}$ is 3-regular for the $^{1,1}_{2,2}$-node multiplicity distribution, then by Theorem 3.16, we have a $h_{2,2}^{1,1}(\mathcal{T}) = 0$, but this is not always the case. Here are examples where $h_{2,2}^{1,1}(\mathcal{T}) \neq 0$.

Example 4.1 *Here is an example where $h_{2,2}^{1,1}(\mathcal{T}) = 1$ by theorem 3.14, since there is one maximal interior segment s, with $\lambda(s) = 2$:*



The dimension of $\mathcal{S}_{2,2}^{1,1}(\mathcal{T})$ is $9\times4-6\times5+4\times2+h_{2,2}^{1,1}(\mathcal{T}) = 14+1 = 15$. Notice that the dimension is the same without the (horizontal) interior segment. Thus a basis of $\mathcal{S}_{2,2}^{1,1}(\mathcal{T})$ is the tensor product bspline basis corresponding to the nodes $s_0, s_0, s_0, s_1, s_2, s_3, s_3, s_3$ in the horizontal direction and the nodes $t_0, t_0, t_0, t_1, t_1, t_1$ in the vertical direction.

Example 4.2 *Here is another example. We subdivide the T-subdivision $\mathcal{T}_1$ to obtain the second T-subdivision $\mathcal{T}_2$:*



Doing this, we increase the number of cells by $9-1=8$, *the number of interior edges by* $24-4=20$, *the number of interior points by* $16-4=12$. *The dimension of the spline space increases by* $9\times 8-6\times 20+4\times 12+h_{2,2}^{1,1}(\mathcal{T}_2)-h_{2,2}^{1,1}(\mathcal{T}_1)=h_{2,2}^{1,1}(\mathcal{T}_2)$. *Since there is no maximal interior segment in* $\mathcal{T}_1$, *by corollary 3.12 we have* $h_{2,2}^{1,1}(\mathcal{T}_1)=0$. *Choosing a proper ordering of the interior segments, we deduce by theorem 3.11 that* $h_{2,2}^{1,1}(\mathcal{T}_2)\leqslant 1$. *Suppose that* $\sigma_1=[a_0,a_3]\times[b_0,b_3]$ *and* $\sigma_2=[a_1,a_2]\times[b_1,b_2]$. *Then the piecewise polynomial function*

$$N(s;a_0,a_1,a_2,a_3)\times N(t;b_0,b_1,b_2,b_3)$$

is an element of $\mathcal{S}_{2,2}^{1,1}(\mathcal{T}_2)$, *with support in* σ_1. *It is not in* $\mathcal{S}_{2,2}^{1,1}(\mathcal{T}_1)$, *since the function is not polynomial on* σ_1. *Thus we have* $\dim\mathcal{S}_{2,2}^{1,1}(\mathcal{T}_2)=\dim\mathcal{S}_{2,2}^{1,1}(\mathcal{T}_1)+1$.

Notice that $\mathcal{T}_2$ *is* 2*-regular but not* 3*-regular, since any new maximal segment intersects two of other new maximal segments.*

4.3 Bicubic C^1 T-splines

For $m=m'=3$ and $r=r'=1$, that is for piecewise bicubic polynomial functions which are C^1, Proposition 3.15 yields $h_{3,3}^{1,1}(\mathcal{T})=0$. Using Theorem 3.11 and Lemma 2.1, we obtain:

$$\dim\mathcal{S}_{3,3}^{2,2}(\mathcal{T})=16f_2-8f_1^o+4f_0^o=4(f_0^++f_0^b)=4f_0^{+,\varepsilon}. \tag{5}$$

4.4 Bicubic C^2 T-splines

For $m=m'=3$ and $r=r'=2$, by Theorem 3.11 and Lemma 2.1, we have:

$$\dim\mathcal{S}_{3,3}^{2,2}(\mathcal{T})=16f_2-12f_1^o+9f_0^o+h_{3,3}^{2,2}(\mathcal{T})=f_0^+-f_0^T+2f_0^b+8+h_{3,3}^{2,2}(\mathcal{T}). \tag{6}$$

If $\mathcal{T}$ is a hierarchical 5-regular subdivision for the ${}^{2,2}_{3,3}$ node-multiplicity distribution then by Theorem 3.16, we have a $h_{3,3}^{2,2}(\mathcal{T})=0$.

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