Lefschetz trace formula for open adic spaces

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Abstract. In this article, we discuss the Lefschetz trace formula for an adic space which is separated smooth of finite type but not necessarily proper over an algebraically closed non-archimedean field. Under a certain condition on the absence of set-theoretical fixed points on the boundary, we obtain a fixed point formula. As an application, we can establish a trace formula for some formal schemes, which is applicable to the Rapoport-Zink tower for GSp(4). A partial generalization of Fujiwara's trace formula for contracting morphisms is also given.

1 Introduction

In this paper, we consider the Lefschetz trace formula for open adic spaces over an algebraically closed non-archimedean field. First recall the Lefschetz-Verdier trace formula for schemes. Let X be a scheme which is separated smooth of finite type over an algebraically closed field k, $X \hookrightarrow \overline{X}$ a dense compactification and $\overline{f}\colon \overline{X} \longrightarrow \overline{X}$ a k-morphism which induces a proper k-morphism $f\colon X \longrightarrow X$. Assume for simplicity that the fixed scheme $\operatorname{Fix} f$ defined by the cartesian diagram

$$\begin{tikzcd} \operatorname{Fix} f \arrow[r] \arrow[d] & X \arrow[d, "\text{diagonal}"] \\ X \arrow[r, "f\times\mathrm{id}"] & X \times X \end{tikzcd}$$

is discrete. Then, for a prime ℓ which is invertible in k, the alternating sum of the traces $\sum_i (-1)^i \operatorname{Tr}(f^*; H^i_c(X, \mathbb{Q}_\ell))$ is equal to $\#\operatorname{Fix} f + \sum_{D\in\pi_0(\operatorname{Fix}\overline{f}\cap(\overline{X}\setminus X))} \operatorname{loc}_D(f)$, where $\#\operatorname{Fix} f$ is the number of fixed points by f counted with multiplicity, $\pi_0(-)$ denotes the set of connected components and $\operatorname{loc}_D(f)$ denotes the "local term at D" (*cf.* [Fuj97, §1.2]). In particular, if $\operatorname{Fix}\overline{f} \subset X$ then $\sum_i (-1)^i \operatorname{Tr}(f^*; H^i_c(X, \mathbb{Q}_\ell))$ coincides with $\#\operatorname{Fix} f$, which gives a nice fixed point formula.

It seems natural to expect the similar formula for adic spaces. However, there is an obvious counterexample. Let k be an algebraically closed non-archimedean field

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and $\mathbb{D}^1 = \mathrm{Spa}(k\langle T\rangle, k\langle T\rangle^\circ)$ the unit disk. We can compactify it by taking its closure $\overline{\mathbb{D}^1}$ in $(\mathbb{A}^1)^{\mathrm{ad}}$. Consider the isomorphism $\mathbb{A}^1 \longrightarrow \mathbb{A}^1$ given by $T \longmapsto T+1$, which induces the isomorphisms $f\colon \mathbb{D}^1 \longrightarrow \mathbb{D}^1$ and $\overline{f}\colon \overline{\mathbb{D}^1} \longrightarrow \overline{\mathbb{D}^1}$. Since $\mathrm{Fix}\, f = \mathrm{Fix}\, \overline{f} = \varnothing$, we expect to have $\sum_i (-1)^i \mathrm{Tr}(f^*; H^i_c(\mathbb{D}^1, \mathbb{Q}_\ell)) = 0$. Nevertheless the left hand side is equal to 1, and thus the analogue of the Lefschetz-Verdier trace formula do not hold.

Actually, this phenomenon has already been observed by Fujiwara [Fuj97] and Huber [Hub01]. Fujiwara proved his topological Lefschetz trace formula under the condition that there exists no topological fixed point on the boundary. Huber established a trace formula for open curves, which says that the alternating sum of the traces on the cohomology of an open adic curve X is the sum of the number of fixed points and the contribution at each set-theoretical fixed point on the boundary $X^c \setminus X$, where X^c denotes the universal compactification of X. Our main theorem is also in this line. Here we will give a slightly simplified statement. As above, let k be an algebraically closed non-archimedean field and ℓ a prime which is invertible in the residue field of k. Let X be a purely d-dimensional adic space which is separated smooth of finite type over k, $X \hookrightarrow \overline{X}$ a dense compactification and $\overline{f}\colon \overline{X} \longrightarrow \overline{X}$ a k-morphism which induces a proper k-morphism $f\colon X \longrightarrow X$.

Theorem 1.1 *Assume that for every $x \in \overline{X} \setminus X$, the points x and $f(x)$ can be separated by closed constructible subsets; namely, there exists closed constructible subsets W_1 and W_2 of $\overline{X}$ such that $x \in W_1$, $f(x) \in W_2$ and $W_1 \cap W_2 = \varnothing$. Then we have*

$$\mathrm{Tr}\bigl(f^*; R\Gamma_c(X, \mathbb{Z}/\ell^n\mathbb{Z})\bigr) = \#\,\mathrm{Fix}\, f.$$

If moreover the characteristic of k is 0, then

$$\sum_{i=0}^{2d} (-1)^i \mathrm{Tr}\bigl(f^*; H^i_c(X, \mathbb{Q}_\ell)\bigr) = \#\,\mathrm{Fix}\, f.$$

Since we have no intersection theory for adic spaces yet (at least the author do not know), we need to clarify the meaning of "the number of fixed points" $\#\,\mathrm{Fix}\, f$. The definition is given by using cohomology theory (see Definition 2.6).

The statement above is similar to [Fuj97, Theorem 2.2.8], but our theorem is valid for a non-algebraizable case. Our proof is very different from that in [Fuj97]; we use neither formal geometry nor the Lefschetz-Verdier trace formula for schemes. Our proof is purely rigid-geometric. The first step of our proof is to observe that why the proof of the Lefschetz-Verdier trace formula in [SGA5, Exposé III] cannot be applied to the case of adic spaces; actually, the only obstruction is the failure of the Künneth formula for push-forward $(Rf_*F) \overset{\mathbb{L}}{\boxtimes} (Rg_*G) \cong R(f \times g)_*(F \overset{\mathbb{L}}{\boxtimes} G)$ (*cf.* Remark 3.8). Therefore, our main strategy is to find a suitable isomorphism induced by the Künneth homomorphism $(Rf_*F) \overset{\mathbb{L}}{\boxtimes} (Rg_*G) \longrightarrow R(f \times g)_*(F \overset{\mathbb{L}}{\boxtimes} G)$ by using the assumption in Theorem 1.1, so that the analogous proof as in [SGA5, Exposé III] works. This idea is also useful for finding other trace formulas than Theorem

1.1. For example, we will give another formula for open curves which is very similar to the formula by Huber, and a partial generalization of Fujiwara's trace formula for contracting correspondences to the non-algebraizable case.

The author's main motivation for this work is to establish the Lefschetz trace formula which is applicable to the Rapoport-Zink towers. For a classical case, there are such works by Faltings [Fal94] and Strauch [Str08]; the former is on the Drinfeld tower and the latter is on the Lubin-Tate tower. As an application of Theorem 1.1, we will establish the Lefschetz trace formula for formal schemes (Theorem 4.5), which is applicable to the Rapoport-Zink tower for GSp(4) considered in [IM10]. The author has a joint project with Matthias Strauch to investigate the cohomology of this Rapoport-Zink tower by means of the trace formula in this paper. He also hopes that there are a few more Rapoport-Zink towers to which our trace formula can apply.

We sketch the outline of the paper. In Section 2, we will consider the action γ^* of a correspondence γ on the étale cohomology of adic spaces, and give the definition of "the number of fixed points" $\#\operatorname{Fix}\gamma$. We also show that it is étale local and compatible with the comparison functor. These properties justify our definition, though it is cohomological and far from geometric. In Section 3, we will prove our main theorem. First we discuss the Künneth formula for $R\Gamma$, which leads us to a weaker form of the Lefschetz trace formula (Proposition 3.9). Next we refine this weaker version by using our assumption on points of the boundary to get our main theorem. We also remark on the trace formula for open curves. In Section 4, we prove the Lefschetz trace formula for formal schemes and give some interesting examples. In Section 5, we give a simple trace formula for a morphism which is contracting near fixed points. It is a partial generalization of a result of Fujiwara [Fuj97, Theorem 3.2.4].

Notation Let k be an algebraically closed non-archimedean field (*cf.* [Hub96, Definition 1.1.3]) and denote its valuation ring by k^+. Put $S = \operatorname{Spa}(k, k^+)$. Fix a prime ℓ which is invertible in k^+ and put $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ for an integer $n \geq 1$.

Every sheaf and cohomology are considered in the étale topology. We simply write $f^!$ for the functor $R^+f^!$ introduced in [Hub96, Theorem 7.1.1].

2 Correspondences on adic spaces

2.1 Trace maps and Gysin maps

Let X be a purely d-dimensional adic space which is separated, locally of finite type and taut over S. Assume that X is generically smooth over S, namely, the dimension of the singular locus Z of X is strictly less than d. We will construct the trace map $\operatorname{Tr}_X\colon H_c^{2d}(X, \Lambda(d)) \longrightarrow \Lambda$ for such X.

Note that the complement U of Z is taut [Hub96, Lemma 5.1.4 i)], thus we may define the compactly supported cohomology of X, Z and U. By the exact sequence

$$H_c^{2d-1}\bigl(Z,\Lambda(d)\bigr) \longrightarrow H_c^{2d}\bigl(U,\Lambda(d)\bigr) \longrightarrow H_c^{2d}\bigl(X,\Lambda(d)\bigr) \longrightarrow H_c^{2d}\bigl(Z,\Lambda(d)\bigr)$$

and the vanishing $H_c^{2d-1}(Z,\Lambda(d)) = H_c^{2d}(Z,\Lambda(d)) = 0$ ([Hub96, Proposition 5.5.8, Corollary 1.8.8]), the canonical homomorphism $H_c^{2d}(U,\Lambda(d)) \longrightarrow H_c^{2d}(X,\Lambda(d))$ is an isomorphism.

On the other hand, since U is smooth over S, the trace map $\mathrm{Tr}_U\colon H_c^{2d}(U,\Lambda(d)) \longrightarrow \Lambda$ has already been constructed in [Hub96, Theorem 7.3.4]. We will define Tr_X as the composite

$$H_c^{2d}\bigl(X,\Lambda(d)\bigr) \xleftarrow{\cong} H_c^{2d}\bigl(U,\Lambda(d)\bigr) \xrightarrow{\mathrm{Tr}_U} \Lambda.$$

Since $H_c^i(X,\Lambda(d)) = 0$ for $i > 2d$, Tr_X induces the map $R\Gamma_c(X,\Lambda(d)[2d]) \longrightarrow \Lambda$, which is also denoted by Tr_X.

Proposition 2.1 *Let X, X' be purely d-dimensional adic spaces which are separated, locally of finite type, generically smooth and taut over S. Let $\pi\colon X' \longrightarrow X$ be an étale S-morphism between them. Then the composite $H_c^{2d}(X',\Lambda(d)) \xrightarrow{\pi_*} H_c^{2d}(X,\Lambda(d)) \xrightarrow{\mathrm{Tr}_X} \Lambda$ is equal to $\mathrm{Tr}_{X'}$.*

Proof. First assume that X is smooth over S. Then the claim follows from the characterizing properties of the trace morphisms ((Var 3) and (Var 4) in [Hub96, Theorem 7.3.4]). In the general case, let Z (resp. Z') be the singular locus of X (resp. X') and put $U = X \setminus Z$ (resp. $U' = X' \setminus Z'$). Note that we have $\pi^{-1}(U) \subset U'$, for π is étale. Thus we have the following commutative diagram:

$$\begin{tikzcd}
H_c^{2d}\bigl(X',\Lambda(d)\bigr) \arrow[d, equal] & H_c^{2d}\bigl(U',\Lambda(d)\bigr) \arrow[l, "\cong"'] \arrow[r, "\mathrm{Tr}_{U'}"] & \Lambda \arrow[d, equal] \\
H_c^{2d}\bigl(X',\Lambda(d)\bigr) \arrow[d, "\pi_*"] & H_c^{2d}\bigl(\pi^{-1}(U),\Lambda(d)\bigr) \arrow[l, "(*)"'] \arrow[u] \arrow[d, "\pi_*"] \arrow[r, "\mathrm{Tr}_{\pi^{-1}(U)}"] & \Lambda \arrow[d, equal] \\
H_c^{2d}\bigl(X,\Lambda(d)\bigr) & H_c^{2d}\bigl(U,\Lambda(d)\bigr) \arrow[l, "\cong"'] \arrow[r, "\mathrm{Tr}_U"'] & \Lambda.
\end{tikzcd}$$

Here the homomorphism $(*)$ is an isomorphism, since the closed subscheme $\pi^{-1}(Z)$ of X', which is étale over Z, has dimension less than d. Therefore the claim immediately follows from the diagram above. ∎

Proposition 2.2 *Let X be a purely d-dimensional scheme which is separated of finite type over k and $X^{\mathrm{ad}} = X \times_{\operatorname{Spec} k} S$ the associated adic space. Assume that X is generically smooth over k. Then the composite of the canonical comparison map $H_c^{2d}(X,\Lambda(d)) \longrightarrow H_c^{2d}(X^{\mathrm{ad}},\Lambda(d))$ and $\mathrm{Tr}_{X^{\mathrm{ad}}}\colon H_c^{2d}(X^{\mathrm{ad}},\Lambda(d)) \longrightarrow \Lambda$ is equal to the trace map $\mathrm{Tr}_X\colon H_c^{2d}(X,\Lambda(d)) \longrightarrow \Lambda$ for X, whose definition is similar to that of $\mathrm{Tr}_{X^{\mathrm{ad}}}$.*

First we consider the smooth case.

Proposition 2.3 *Let $f\colon X \longrightarrow Y$ be a separated smooth morphism of finite type with relative dimension d between k-schemes of finite type. Let $f^{\mathrm{ad}}\colon X^{\mathrm{ad}} \longrightarrow Y^{\mathrm{ad}}$ be the induced morphism of adic spaces. Denote the natural morphisms of sites $(X^{\mathrm{ad}})_{\text{ét}} \longrightarrow X_{\text{ét}}$ and $(Y^{\mathrm{ad}})_{\text{ét}} \longrightarrow Y_{\text{ét}}$ by ε. Then the following diagram is commutative:*

$$\begin{tikzcd}
\varepsilon^* Rf_!\Lambda(d)[2d] \arrow[r, "\varepsilon^* \mathrm{Tr}_f"] \arrow[d] & \Lambda \arrow[d, equal] \\
Rf^{\mathrm{ad}}_!\Lambda(d)[2d] \arrow[r, "\varepsilon^* \mathrm{Tr}_{f^{\mathrm{ad}}}"] & \Lambda.
\end{tikzcd}$$

The left vertical arrow is defined in [Hub96, Theorem 5.7.2].

Proof. By using [SGA4, Exposé XVIII, Lemme 2.2] and [Hub96, Lemma 7.3.5], we have only to consider the case where $X = \mathbb{A}^m_Y$ or the case where f is étale. The former case immediately follows from the construction of $\mathrm{Tr}_{f^{\mathrm{ad}}}$ ([Hub96, proof of Theorem 7.3.4]). For the latter case, every morphism is defined by the adjointness and the commutativity is formal. ∎

Proof of Proposition 2.2. Let Z be the singular locus of X and put $U = X \setminus Z$. Then, by Proposition 2.3, the following diagram is commutative:

$$\begin{tikzcd}
H^{2d}_c\big(X, \Lambda(d)\big) \arrow[d] & H^{2d}_c\big(U, \Lambda(d)\big) \arrow[l, "\cong"'] \arrow[r, "\mathrm{Tr}_U"] \arrow[d] & \Lambda \arrow[d, equal] \\
H^{2d}_c\big(X^{\mathrm{ad}}, \Lambda(d)\big) & H^{2d}_c\big(U^{\mathrm{ad}}, \Lambda(d)\big) \arrow[l, "(*)"'] \arrow[r, "\mathrm{Tr}_{U^{\mathrm{ad}}}"] & \Lambda.
\end{tikzcd}$$

Since the dimension of Z^{ad} is less than d, the map $(*)$ is an isomorphism. Moreover, in the same way as Proposition 2.1, we can prove that the composite of $H^{2d}_c(X^{\mathrm{ad}}, \Lambda(d)) \xleftarrow{\cong} H^{2d}_c(U^{\mathrm{ad}}, \Lambda(d)) \xrightarrow{\mathrm{Tr}_{U^{\mathrm{ad}}}} \Lambda$ coincides with $\mathrm{Tr}_{X^{\mathrm{ad}}}$. Thus we have the desired compatibility. ∎

Let X (resp. Y) be a purely d-dimensional (resp. d'-dimensional) adic space which is separated, locally of finite type and taut over S. Put $c = d - d'$. Assume X (resp. Y) is smooth (resp. generically smooth) over S. Let $f\colon Y \longrightarrow X$ be an S-morphism between them. Let us denote the structure map of X (resp. Y) by $a\colon X \longrightarrow S$ (resp. $b\colon Y \longrightarrow S$). By the construction above, we have $\mathrm{Tr}_X\colon Ra_!\Lambda(d)[2d] \longrightarrow \Lambda$ and $\mathrm{Tr}_Y\colon Rb_!\Lambda(d')[2d'] \longrightarrow \Lambda$. By the adjointness, these correspond to the maps $\mathrm{Gys}_a\colon \Lambda \longrightarrow a^!\Lambda(-d)[-2d]$ and $\mathrm{Gys}_b\colon \Lambda \longrightarrow b^!\Lambda(-d')[-2d']$. Since a is smooth, Gys_a is an isomorphism ([Hub96, Theorem 7.5.3]). Therefore we have an isomorphism $b^!\Lambda(-d')[-2d'] = f^!a^!\Lambda(-d')[-2d'] \cong f^!\Lambda(c)[2c]$, and finally we obtain a map $\mathrm{Gys}_f\colon \Lambda \longrightarrow f^!\Lambda(c)[2c]$, which is called the Gysin map associated with f. Since $\mathrm{Hom}(\Lambda, f^!\Lambda(c)[2c]) = H^{2c}(Y, f^!\Lambda(c))$, it gives an element of $H^{2c}(Y, f^!\Lambda(c))$.

If moreover f is proper, then f is naturally decomposed as $Y \xrightarrow{f'} f(Y) \xrightarrow{i} X$, where f' is proper and i is a closed immersion (here $f(Y) := (X, f(Y))$ is a pseudo-adic space; *cf.* [Hub96, §1.10]). Thus we have the map

$$\begin{aligned} H^{2c}\bigl(Y, f^!\Lambda(c)\bigr) &= H^{2c}\bigl(f(Y), Rf'_! f'^! i^! \Lambda(c)\bigr) \\ &\xrightarrow{\text{adj}} H^{2c}\bigl(f(Y), i^!\Lambda(c)\bigr) = H^{2c}_{f(Y)}\bigl(X, \Lambda(c)\bigr). \end{aligned}$$

We denote the image of Gys_f under this map by $\mathrm{cl}(f)$, and call it the cohomology class associated with f.

Proposition 2.4 *In the setting above (we do not assume that f is proper), consider the following commutative diagram:*

$$\begin{tikzcd} Y' \arrow[r, "g"] \arrow[d, "\pi'"] & X' \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X, \end{tikzcd}$$

where π and π' are étale. Then the image of Gys_f under the map

$$\pi^*\colon H^{2c}\bigl(Y, f^!\Lambda(c)\bigr) \xrightarrow{\pi'^*} H^{2c}\bigl(Y, \pi'^* f^!\Lambda(c)\bigr) = H^{2c}\bigl(Y, g^!\Lambda(c)\bigr)$$

coincides with Gys_g. If moreover the diagram above is cartesian and f is proper, then the image of $\mathrm{cl}(f)$ under the map $\pi^*\colon H^{2c}_{f(Y)}(X, \Lambda(c)) \longrightarrow H^{2c}_{g(Y')}(X', \Lambda(c))$ coincides with $\mathrm{cl}(g)$.

Proof. First we will prove $\pi^* \mathrm{Gys}_f = \mathrm{Gys}_g$. Denote the structure map of X' (resp. Y') by a' (resp. b'). By Proposition 2.1 and the adjointness, we have the following commutative diagrams:

$$\begin{tikzcd} \Lambda \arrow[r, "\mathrm{Gys}_{a'}", "\cong"'] \arrow[d, equal] & a'^!\Lambda(-d)[-2d] \arrow[d, equal] \\ \pi^*\Lambda \arrow[r, "\pi^*\mathrm{Gys}_a", "\cong"'] & \pi^* a^!\Lambda(-d)[-2d], \end{tikzcd} \qquad \begin{tikzcd} \Lambda \arrow[r, "\mathrm{Gys}_{b'}"] \arrow[d, equal] & b'^!\Lambda(-d')[-2d'] \arrow[d, equal] \\ \pi'^*\Lambda \arrow[r, "\pi'^*\mathrm{Gys}_b"] & \pi'^* b^!\Lambda(-d')[-2d']. \end{tikzcd}$$

Namely, we have $\mathrm{Gys}_{a'} = \pi^*\mathrm{Gys}_a$ and $\mathrm{Gys}_{b'} = \pi'^*\mathrm{Gys}_b$. On the other hand, by the definition, $\mathrm{Gys}_f = f^!\,\mathrm{Gys}_a^{-1}(c)[2c] \circ \mathrm{Gys}_b$ and $\mathrm{Gys}_g = g^!\,\mathrm{Gys}_{a'}^{-1}(c)[2c] \circ \mathrm{Gys}_{b'}$. Therefore we have

$$\begin{aligned} \pi^*\mathrm{Gys}_f &= \pi^* f^!\,\mathrm{Gys}_a^{-1}(c)[2c] \circ \pi^*\mathrm{Gys}_b = g^! \pi'^*\,\mathrm{Gys}_a^{-1}(c)[2c] \circ \pi^*\mathrm{Gys}_b \\ &= g^!\,\mathrm{Gys}_{a'}^{-1}(c)[2c] \circ \mathrm{Gys}_{b'} = \mathrm{Gys}_g, \end{aligned}$$

as desired.

Assume that the diagram in the proposition is cartesian and f is proper. To prove $\pi^* \operatorname{cl}(f) = \operatorname{cl}(g)$, it suffices to observe the commutativity of the diagram below:

$$\begin{tikzcd}
H^{2c}\big(Y, f^!\Lambda(c)\big) \arrow[r] \arrow[d, "\pi'^*"] & H^{2c}_{f(Y)}\big(X, \Lambda(c)\big) \arrow[d, "\pi^*"] \\
H^{2c}\big(Y', g^!\Lambda(c)\big) \arrow[r] & H^{2c}_{g(Y')}\big(X', \Lambda(c)\big).
\end{tikzcd}$$

Let $Y' \xrightarrow{g'} g(Y') \xrightarrow{i'} X$ be the factorization of g. Note that the following diagram is cartesian due to [Hub94, Lemma 3.9 (i)]:

$$\begin{tikzcd}
Y' \arrow[r, "g'"] \arrow[d, "\pi'"] & g(Y') \arrow[r, "i'"] \arrow[d, "\pi"] & X' \arrow[d, "\pi"] \\
Y \arrow[r, "f'"] & f(Y) \arrow[r, "i"] & X.
\end{tikzcd}$$

For every object L of $D^+(f(Y), \Lambda)$, it is straightforward to check the commutativity of the diagram below:

$$\begin{tikzcd}
Rf'_! f'^! L \arrow[r, "\mathrm{adj}"] \arrow[d, "\mathrm{adj}"] & L \arrow[d, "\mathrm{adj}"] \\
R\pi_*\pi^* Rf'_! f'^! L \arrow[r, "\mathrm{adj}"] & R\pi_*\pi^* L \arrow[dd, equal] \\
R\pi_* Rg'_! \pi'^* f'^! L \arrow[u, leftrightarrow, "\cong"', "\text{base change}"] \arrow[d, equal] & \\
R\pi_* Rg'_! g'^! \pi^* L \arrow[r, "\mathrm{adj}"] & R\pi_*\pi^* L.
\end{tikzcd}$$

Setting $L = i^!\Lambda(c)$ and taking $H^{2c}(f(Y), -)$, we obtain the desired commutativity. ■

Next we will prove a comparison result for the Gysin maps and the cohomology classes associated with proper morphisms. Let X (resp. Y) be a purely d-dimensional (resp. d'-dimensional) scheme which is separated of finite type over k. Assume X (resp. Y) is smooth (resp. generically smooth) over k and put $c = d - d'$. Let $f\colon Y \longrightarrow X$ be a morphism of finite type over k. Then, by the same way as above, we may define the Gysin map $\mathrm{Gys}_f\colon \Lambda \longrightarrow f^!\Lambda(c)[2c]$. If moreover f is proper, we can also define the cohomology class $\operatorname{cl}(f) \in H^{2c}_{f(Y)}(X, \Lambda(c))$ associated with f. Let $f^{\mathrm{ad}}\colon Y^{\mathrm{ad}} \longrightarrow X^{\mathrm{ad}}$ be the morphism of adic spaces associated with f and denote the natural morphisms of locally ringed spaces $X^{\mathrm{ad}} \longrightarrow X$ and $Y^{\mathrm{ad}} \longrightarrow Y$ by ε. Note that X^{ad}, Y^{ad} and f^{ad} satisfy all the assumptions above, thus we may define the Gysin map $\mathrm{Gys}_{f^{\mathrm{ad}}}$, and the class $\operatorname{cl}(f^{\mathrm{ad}})$ if f is proper.

Proposition 2.5 *The image of* Gys_f *under the map*

$$H^{2c}\bigl(Y, f^!\Lambda(c)\bigr) \xrightarrow{\varepsilon^*} H^{2c}\bigl(Y, \varepsilon^* f^!\Lambda(c)\bigr) \xrightarrow[(*)]{\cong} H^{2c}\bigl(Y, f^{\mathrm{ad}!}\varepsilon^*\Lambda(c)\bigr) = H^{2c}\bigl(Y, f^{\mathrm{ad}!}\Lambda(c)\bigr)$$

coincides with $\mathrm{Gys}_{f^{\mathrm{ad}}}$ *(for the construction of the isomorphism* $(*)$*, see [Mie10b, Proposition 4.37]). If moreover* f *is proper, the image of* $\mathrm{cl}(f)$ *under the canonical map* $\varepsilon^*\colon H^{2d}_{f(Y)}(X, \Lambda(d)) \longrightarrow H^{2d}_{f^{\mathrm{ad}}(Y^{\mathrm{ad}})}(X^{\mathrm{ad}}, \Lambda(d))$ *coincides with* $\mathrm{cl}(f^{\mathrm{ad}})$ *(note that* $f^{\mathrm{ad}}(Y^{\mathrm{ad}}) = \varepsilon^{-1}(f(Y))$ *due to [Hub94, Lemma 3.9 (ii)]).*

Proof. First we will observe $\varepsilon^* \mathrm{Gys}_f = \mathrm{Gys}_{f^{\mathrm{ad}}}$. Let us denote the structure morphism of X (resp. Y) by a (resp. b). By Proposition 2.2 and the adjointness, we have the commutativity of the left diagram below. On the other hand, by the definition of Gys_b, the right diagram below is also commutative.

$$\begin{tikzcd}
\varepsilon^*\Lambda \arrow[r, "\mathrm{Gys}_{b^{\mathrm{ad}}}"] \arrow[d, "\mathrm{adj}"] & b^{\mathrm{ad}!}\varepsilon^*\Lambda(-d')[-2d'] \arrow[dd, equal] \\
b^{\mathrm{ad}!}Rb^{\mathrm{ad}}_!\varepsilon^*\Lambda \arrow[d, leftrightarrow, "\mathrm{b.c.}", "\cong"'] & \\
b^{\mathrm{ad}!}\varepsilon^* Rb_!\Lambda \arrow[r, "b^{\mathrm{ad}!}\varepsilon^*\mathrm{Tr}_b"] & b^{\mathrm{ad}!}\varepsilon^*\Lambda(-d')[-2d'],
\end{tikzcd}
\quad
\begin{tikzcd}
\varepsilon^*\Lambda \arrow[r, "\varepsilon^*\mathrm{Gys}_b"] \arrow[d, "\mathrm{adj}"] & \varepsilon^* b^!\Lambda(-d')[-2d'] \arrow[d, equal] \\
\varepsilon^* b^! Rb_!\Lambda \arrow[r, "\varepsilon^* b^!\mathrm{Tr}_b"] \arrow[d, "\mathrm{b.c.}", "\cong"'] & \varepsilon^* b^!\Lambda(-d')[-2d'] \arrow[d, "\mathrm{b.c.}", "\cong"'] \\
b^{\mathrm{ad}!}\varepsilon^* Rb_!\Lambda \arrow[r, "b^{\mathrm{ad}!}\varepsilon^*\mathrm{Tr}_b"] & b^{\mathrm{ad}!}\varepsilon^*\Lambda(-d')[-2d'].
\end{tikzcd}$$

Moreover, it is easy to show that the following diagram is commutative:

$$\begin{tikzcd}
\varepsilon^*\Lambda \arrow[r, "\mathrm{adj}"] \arrow[d, "\mathrm{adj}"] & b^{\mathrm{ad}!}Rb^{\mathrm{ad}}_!\varepsilon^*\Lambda \\
\varepsilon^* b^! Rb_!\Lambda \arrow[r, "\mathrm{b.c.}"] & b^{\mathrm{ad}!}\varepsilon^* Rb_!\Lambda. \arrow[u, "\mathrm{b.c.}"']
\end{tikzcd}$$

By these three commutative diagrams, we have the following commutative diagram:

$$\begin{tikzcd}
\varepsilon^*\Lambda \arrow[r, "\mathrm{Gys}_{b^{\mathrm{ad}}}"] \arrow[d, equal] & \varepsilon^* b^!\Lambda \arrow[d, "\mathrm{b.c.}"] \\
\varepsilon^*\Lambda \arrow[r, "\varepsilon^*\mathrm{Gys}_b"] & b^{\mathrm{ad}!}\varepsilon^*\Lambda.
\end{tikzcd}$$

Namely, we have $\mathrm{Gys}_{b^{\mathrm{ad}}} = \varepsilon^* \mathrm{Gys}_b$. Similarly we have $\mathrm{Gys}_{a^{\mathrm{ad}}} = \varepsilon^* \mathrm{Gys}_a$. Therefore, in the same way as in the proof of Proposition 2.4, we can obtain $\varepsilon^* \mathrm{Gys}_f = \mathrm{Gys}_{f^{\mathrm{ad}}}$.

Assume that f is proper. To prove $\varepsilon^* \mathrm{cl}(f) = \mathrm{cl}(f^{\mathrm{ad}})$, it suffices to show the commutativity of the diagram below:

$$\begin{tikzcd}
H^{2c}\bigl(Y, f^!\Lambda(c)\bigr) \arrow[r] \arrow[d, "\varepsilon^*"] & H^{2c}_{f(Y)}\bigl(X, \Lambda(c)\bigr) \arrow[d, "\varepsilon^*"] \\
H^{2c}\bigl(Y^{\mathrm{ad}}, f^{\mathrm{ad}!}\Lambda(c)\bigr) \arrow[r] & H^{2c}_{f^{\mathrm{ad}}(Y^{\mathrm{ad}})}\bigl(X^{\mathrm{ad}}, \Lambda(c)\bigr).
\end{tikzcd}$$

Put $Z = f(Y)$ and endow it with the structure of a reduced closed subscheme of X. Then f factors as $Y \xrightarrow{f'} Z \xhookrightarrow{i} X$. As we mentioned above, a closed subset $f^{\mathrm{ad}}(Y^{\mathrm{ad}})$ of X^{ad} coincides with Z^{ad}. Therefore, by [Hub96, Corollary 2.3.8], $i^{\mathrm{ad}}\colon Z^{\mathrm{ad}} \hookrightarrow X^{\mathrm{ad}}$ induces an equivalence between the étale topos of Z^{ad} and that of $f^{\mathrm{ad}}(Y^{\mathrm{ad}})$. Hence the diagram above can be identified with the following diagram:

$$\begin{tikzcd} H^{2c}\big(Y, f^!\Lambda(c)\big) \arrow[r] \arrow[d, "\varepsilon^*"] & H^{2c}\big(Z, i^!\Lambda(c)\big) \arrow[d, "\varepsilon^*"] \\ H^{2c}\big(Y^{\mathrm{ad}}, f^{\mathrm{ad}!}\Lambda(c)\big) \arrow[r] & H^{2c}\big(Z^{\mathrm{ad}}, i^{\mathrm{ad}!}\Lambda(c)\big). \end{tikzcd}$$

Now we can show the commutativity in the same way as in the proof of Proposition 2.4. This completes the proof. ∎

2.2 Correspondences

Let X and Γ be purely d-dimensional adic spaces which are separated, locally of finite type and taut over S. Assume that X (resp. Γ) is smooth (resp. generically smooth) over S. Let $\gamma\colon \Gamma \longrightarrow X \times_S X$ be a morphism over S and put $\gamma_i = \mathrm{pr}_i \circ \gamma$.

Then we may apply the construction in the previous subsection and have the cohomology class $\mathrm{Gys}_\gamma \in H^{2d}(\Gamma, \gamma^!\Lambda(d))$. Using it, we will define "the number of points fixed by γ".

Definition 2.6 Consider the following cartesian diagram in which $\delta\colon X \longrightarrow X \times_S X$ denotes the diagonal morphism:

$$\begin{tikzcd} \mathrm{Fix}\,\gamma \arrow[r, "\gamma_0"] \arrow[d] & X \arrow[d, "\delta"] \\ \Gamma \arrow[r, "\gamma"] & X \times_S X. \end{tikzcd}$$

Let D be an open and closed subset of $\mathrm{Fix}\,\gamma$ which is proper over S, and denote the open and closed immersion $D \hookrightarrow \mathrm{Fix}\,\gamma$ by j. Put $j_0 = \gamma_0 \circ j$. Then we have the canonical maps

$$\begin{aligned} H^{2d}\big(\Gamma, \gamma^!\Lambda(d)\big) &\xrightarrow{\delta^*} H^{2d}\big(\mathrm{Fix}\,\gamma, \gamma_0^!\Lambda(d)\big) \longrightarrow H^{2d}\big(D, j^*\gamma_0^!\Lambda(d)\big) = H^{2d}\big(D, j_0^!\Lambda(d)\big) \\ &= H_c^{2d}\big(X, Rj_{0!}j_0^!\Lambda(d)\big) \xrightarrow{\mathrm{adj}} H_c^{2d}\big(X, \Lambda(d)\big) \xrightarrow{\mathrm{Tr}} \Lambda. \end{aligned}$$

We denote the image of Gys_γ under these maps by $\#\,\mathrm{Fix}_D\,\gamma$. If $D = \mathrm{Fix}\,\gamma$, we write $\#\,\mathrm{Fix}\,\gamma$ for $\#\,\mathrm{Fix}_D\,\gamma$.

Remark 2.7 Assume that γ is proper. Then $\#\,\mathrm{Fix}\,\gamma$ can be calculated from $\mathrm{cl}(\gamma)$ as follows. Denote the inverse image of $\gamma(\Gamma)$ under δ by $\Delta_X \cap \gamma(\Gamma)$. Since we are implicitly assuming that $\mathrm{Fix}\,\gamma$ is proper over S, the pseudo-adic space $(X, \Delta_X \cap \gamma(\Gamma))$ is proper over S. Thus we have the following natural maps:

$$H^{2d}_{\gamma(\Gamma)}\bigl(X \times_S X, \Lambda(d)\bigr) \xrightarrow{\delta^*} H^{2d}_{\Delta_X \cap \gamma(\Gamma)}\bigl(X, \Lambda(d)\bigr) \longrightarrow H^{2d}_c\bigl(X, \Lambda(d)\bigr) \xrightarrow{\mathrm{Tr}_X} \Lambda.$$

The image of $\mathrm{cl}(\gamma)$ under these maps coincides with $\#\,\mathrm{Fix}\,\gamma$.

Remark 2.8 Obviously the construction of $\#\,\mathrm{Fix}\,\gamma$ is compatible with a change of Λ. Namely, if we denote $\#\,\mathrm{Fix}\,\gamma$ for $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ by $\#_{\ell^n}\mathrm{Fix}\,\gamma$, then $(\#_{\ell^n}\mathrm{Fix}\,\gamma)_{n\ge 1}$ gives an element of $\mathbb{Z}_\ell = \varprojlim_n \mathbb{Z}/\ell^n\mathbb{Z}$. We also denote it by $\#\,\mathrm{Fix}\,\gamma$.

Example 2.9 Let $f\colon X \longrightarrow X$ be a morphism over S. Then $\gamma_f = f \times \mathrm{id}\colon X \longrightarrow X \times_S X$ lies in the situation above. In this case, $\mathrm{Fix}\,\gamma_f$ is a closed adic subspace of X. We simply write $\mathrm{Fix}\,f$, $\#\,\mathrm{Fix}_D\,f$ and $\#\,\mathrm{Fix}\,f$ for $\mathrm{Fix}\,\gamma_f$, $\#\,\mathrm{Fix}_D\,\gamma_f$ and $\#\,\mathrm{Fix}\,\gamma_f$, respectively. Denote the image of γ_f by Γ_f; note that it has the natural structure of a closed adic subspace of $X \times_S X$ since γ_f is a closed immersion.

By the results in the previous subsection, we can prove that the number $\#\,\mathrm{Fix}_D\,\gamma$ is étale local and compatible with the comparison functor:

Proposition 2.10 *Let $\gamma\colon \Gamma \longrightarrow X \times_S X$ be as above and $\gamma'\colon \Gamma' \longrightarrow X' \times_S X'$ be another morphism satisfying the conditions above. Let $\pi\colon \Gamma' \longrightarrow \Gamma$ and $\pi'\colon X' \longrightarrow X$ be étale morphisms over S such that $\gamma \circ \pi = (\pi' \times \pi') \circ \gamma'$. Let D (resp. D') be an open and closed subset of $\mathrm{Fix}\,\gamma$ (resp. $\mathrm{Fix}\,\gamma'$) which is proper over S. Assume that π induces an isomorphism from D' to D. Then we have $\#\,\mathrm{Fix}_D\,\gamma = \#\,\mathrm{Fix}_{D'}\,\gamma'$.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
H^{2d}\bigl(\Gamma, \gamma^!\Lambda(d)\bigr) \arrow[r, "\delta^*"] \arrow[d, "\pi^*"] & H^{2d}\bigl(\mathrm{Fix}\,\gamma, \gamma_0^!\Lambda(d)\bigr) \arrow[r] \arrow[d, "\pi^*"] & H^{2d}\bigl(D, j_0^!\Lambda(d)\bigr) \arrow[d, "\pi^*"] \\
H^{2d}\bigl(\Gamma', \gamma'^!\Lambda(d)\bigr) \arrow[r, "\delta'^*"] & H^{2d}\bigl(\mathrm{Fix}\,\gamma', \gamma_0'^!\Lambda(d)\bigr) \arrow[r] & H^{2d}\bigl(D', j_0'^!\Lambda(d)\bigr).
\end{tikzcd}$$

Thus, by Proposition 2.1 and Proposition 2.4, it suffices to show the commutativity of the diagram below:

$$\begin{tikzcd}
H^{2d}\bigl(D, j_0^!\Lambda(d)\bigr) \arrow[r, equal] \arrow[d, "\pi^*"] & H^{2d}_c\bigl(X, Rj_{0!}j_0^!\Lambda(d)\bigr) \arrow[r] & H^{2d}_c\bigl(X, \Lambda(d)\bigr) \\
H^{2d}\bigl(D', j_0'^!\Lambda(d)\bigr) \arrow[r, equal] & H^{2d}_c\bigl(X', Rj'_{0!}j_0'^!\Lambda(d)\bigr) \arrow[r] & H^{2d}_c\bigl(X', \Lambda(d)\bigr). \arrow[u, "\pi'_*"]
\end{tikzcd}$$

By the commutative diagram

$$\begin{tikzcd}
D' \arrow[r, "j_0'"] \arrow[d, "\pi", "\cong"'] & X' \arrow[d, "\pi'"] \\
D \arrow[r, "j_0"] & X,
\end{tikzcd}$$

we can construct the map $\pi_*\colon H^{2d}(D', j_0'^!\Lambda(d)) \longrightarrow H^{2d}(D, j_0^!\Lambda(d))$ as the composite

$$\begin{aligned} H^{2d}\bigl(D', j_0'^!\Lambda(d)\bigr) &= H^{2d}\bigl(D, \pi_! j_0'^!\Lambda(d)\bigr) = H^{2d}\bigl(D, \pi_! j_0'^!\pi'^!\Lambda(d)\bigr) \\ &= H^{2d}\bigl(D, \pi_!\pi^! j_0^!\Lambda(d)\bigr) \xrightarrow{\mathrm{adj}} H^{2d}\bigl(D, j_0^!\Lambda(d)\bigr). \end{aligned}$$

The commutativity of the following diagram is immediate:

$$\begin{tikzcd} H^{2d}\bigl(D, j_0^!\Lambda(d)\bigr) \arrow[r, equal] & H^{2d}_c\bigl(X, Rj_{0!}j_0^!\Lambda(d)\bigr) \arrow[r] & H^{2d}_c\bigl(X, \Lambda(d)\bigr) \\ H^{2d}\bigl(D', j_0'^!\Lambda(d)\bigr) \arrow[r, equal] \arrow[u, "\pi_*"] & H^{2d}_c\bigl(X', Rj'_{0!}j_0'^!\Lambda(d)\bigr) \arrow[r] & H^{2d}_c\bigl(X', \Lambda(d)\bigr). \arrow[u, "\pi'_*"] \end{tikzcd}$$

Thus it suffices to show that $\pi_* \circ \pi^*\colon H^{2d}(D, j_0^!\Lambda(d)) \longrightarrow H^{2d}(D, j_0^!\Lambda(d))$ is the identity map. This map is induced from the composite of $j_0^!\Lambda \xrightarrow{\mathrm{adj}} R\pi_*\pi^* j_0^!\Lambda = \pi_!\pi^! j_0^!\Lambda \xrightarrow{\mathrm{adj}} j_0^!\Lambda$, which is the identity map since $R\pi_* = \pi_!$ is the quasi-inverse of $\pi^* = \pi^!$ by the assumption that $\pi\colon D' \longrightarrow D$ is an isomorphism. Now the proof is complete. ∎

Proposition 2.11 *Let X and Γ be purely d-dimensional schemes which are separated of finite type over k, Assume that X (resp. Γ) is smooth (resp. generically smooth) over k. Let $\gamma\colon \Gamma \longrightarrow X \times_k X$ be a morphism over k.*

For an open and closed subset D of $\mathrm{Fix}\,\gamma := \Gamma \times_{X\times_k X} X$ which is proper over k, we can define $\#\,\mathrm{Fix}_D(\gamma)$ in the same way as in Definition 2.6. Then we have $\#\,\mathrm{Fix}_{D^{\mathrm{ad}}}(\gamma^{\mathrm{ad}}) = \#\,\mathrm{Fix}_D(\gamma)$. In particular, if $\mathrm{Fix}\,\gamma$ is proper over k, then $\#\,\mathrm{Fix}(\gamma^{\mathrm{ad}})$ coincides with the number of points fixed by γ in the usual (intersection-theoretic) sense.

Proof. We denote the natural morphism $\mathrm{Fix}\,\gamma \longrightarrow X$ by γ_0, the open and closed immersion $D \hookrightarrow \mathrm{Fix}\,\gamma$ by j, and put $j_0 = \gamma_0 \circ j$. Then the proposition is clear from Proposition 2.5 and the following commutative diagrams:

$$\begin{tikzcd} H^{2d}\bigl(\Gamma, \gamma^!\Lambda(d)\bigr) \arrow[r, "\delta^*"] \arrow[d, "\varepsilon^*"] & H^{2d}\bigl(\mathrm{Fix}\,\gamma, \gamma^!\Lambda(d)\bigr) \arrow[r, "j^*"] \arrow[d, "\varepsilon^*"] & H^{2d}\bigl(D, j^*\gamma^!\Lambda(d)\bigr) \arrow[d, "\varepsilon^*"] \\ H^{2d}\bigl(\Gamma^{\mathrm{ad}}, \gamma^{\mathrm{ad}!}\Lambda(d)\bigr) \arrow[r, "\delta^{\mathrm{ad}*}"] & H^{2d}\bigl(\mathrm{Fix}\,\gamma^{\mathrm{ad}}, \gamma^{\mathrm{ad}!}\Lambda(d)\bigr) \arrow[r, "j^{\mathrm{ad}*}"] & H^{2d}\bigl(D^{\mathrm{ad}}, j^{\mathrm{ad}*}\gamma^{\mathrm{ad}!}\Lambda(d)\bigr), \end{tikzcd}$$

$$\begin{tikzcd} H^{2d}\bigl(D, j^*\gamma^!\Lambda(d)\bigr) \arrow[r, equal] \arrow[d, "\varepsilon"] & H^{2d}_c\bigl(X, Rj_{0!}j_0^!\Lambda(d)\bigr) \arrow[r, "\mathrm{adj}"] \arrow[d, "\varepsilon^*"] & H^{2d}_c\bigl(X, \Lambda(d)\bigr) \arrow[r, "\mathrm{Tr}_X"] \arrow[d, "\varepsilon^*"] & \Lambda \arrow[d, equal] \\ H^{2d}\bigl(D^{\mathrm{ad}}, j^{\mathrm{ad}*}\gamma^{\mathrm{ad}!}\Lambda(d)\bigr) \arrow[r, equal] & H^{2d}_c\bigl(X^{\mathrm{ad}}, Rj^{\mathrm{ad}}_{0!}j_0^{\mathrm{ad}!}\Lambda(d)\bigr) \arrow[r, "\mathrm{adj}"] & H^{2d}_c\bigl(X^{\mathrm{ad}}, \Lambda(d)\bigr) \arrow[r, "\mathrm{Tr}_{X^{\mathrm{ad}}}"] & \Lambda. \end{tikzcd}$$

∎

Remark 2.12 By Proposition 2.10 and Proposition 2.11, we may often calculate $\#\operatorname{Fix}\gamma$. For example, we can apply the method in [Str08, §2.6] to calculate the number of fixed points on some Rapoport-Zink spaces. Since the period space for a Rapoport-Zink space $\mathscr{M}$ is an open adic subspace of an algebraic variety, we can use Proposition 2.11 for counting fixed points on the period space. As the period map from $\mathscr{M}$ to the period space is étale, Proposition 2.10 enables us to count fixed points on $\mathscr{M}$.

Definition 2.13 In the setting introduced at the beginning of this subsection, assume moreover that γ_1 is proper. We define the action γ^* of γ on $R\Gamma_c(X,\Lambda)$ as follows:

$$\gamma^*\colon R\Gamma_c(X,\Lambda) \xrightarrow{\gamma_1^*} R\Gamma_c(\Gamma,\Lambda) \xrightarrow{\mathrm{Gys}_{\gamma_2}} R\Gamma_c(\Gamma,\gamma_2^!\Lambda) = R\Gamma_c(X, R\gamma_{2!}\gamma_2^!\Lambda) \xrightarrow{\mathrm{adj}} R\Gamma_c(X,\Lambda).$$

Example 2.14 Let $f\colon X \longrightarrow X$ be a proper morphism over S. Then γ_f^* (*cf.* Example 2.9) obviously coincides with f^*.

In the sequel we assume that X and Γ are quasi-compact and γ_1 (and hence γ) is proper. We will describe γ^* by means of a compactification of $\gamma\colon \Gamma \longrightarrow X\times_S X$.

Definition 2.15 A compactification of $\gamma\colon \Gamma \longrightarrow X\times_S X$ is a triple $(X\hookrightarrow \overline{X}, \Gamma\hookrightarrow \overline{\Gamma}, \overline{\gamma})$, where $X \hookrightarrow \overline{X}$ and $\Gamma \hookrightarrow \overline{\Gamma}$ are dense open immersions into pseudo-adic spaces which are proper over S and $\overline{\gamma}$ is a proper S-morphism which makes the following diagram commutative:

$$\begin{tikzcd} \Gamma \arrow[r, "\gamma"] \arrow[d] & X\times_S X \arrow[d] \\ \overline{\Gamma} \arrow[r, "\overline{\gamma}"] & \overline{X}\times_S \overline{X}. \end{tikzcd}$$

For simplicity, we often write $\overline{\gamma}\colon \overline{\Gamma} \longrightarrow \overline{X}\times_S \overline{X}$ for $(X\hookrightarrow \overline{X}, \Gamma\hookrightarrow \overline{\Gamma}, \overline{\gamma})$.

Remark 2.16 For a compactification $\overline{\gamma}\colon \overline{\Gamma} \longrightarrow \overline{X}\times_S \overline{X}$, we have $\Gamma = \overline{\gamma}_1^{-1}(X)$. Indeed, since γ_1 is proper, the open immersion $\Gamma \hookrightarrow \overline{\gamma}_1^{-1}(X)$ is proper. On the other hand, since Γ is assumed to be dense in $\overline{\Gamma}$, it is also dense in $\overline{\gamma}_1^{-1}(X)$. Thus we have $\Gamma = \overline{\gamma}_1^{-1}(X)$. In other words, $\overline{\gamma}_1^{-1}(X)$ is contained in $\overline{\gamma}_2^{-1}(X)$.

Example 2.17 Let $X \hookrightarrow X^c$ and $\Gamma \hookrightarrow \Gamma^c$ be the universal compactifications of X and Γ over S, respectively (*cf.* [Hub96, Definition 5.1.1, Theorem 5.1.5, Corollary 5.1.6]). Then the morphism $\gamma^c\colon \Gamma^c \longrightarrow X^c\times_S X^c$ over S is naturally induced from γ, and gives a compactification of $\gamma\colon \Gamma \longrightarrow X\times_S X$.

If $\gamma\colon \Gamma \longrightarrow X\times_S X$ can be extended to a morphism $\gamma'\colon \Gamma' \longrightarrow X'\times_S X'$ where X' (resp. Γ') is an adic space which is partially proper taut over S and contains X (resp. Γ) as an open adic subspace, then we can construct another compactification of γ. Let $\overline{X}$ (resp. $\overline{\Gamma}$) be the closure of X (resp. Γ) in X' (resp. Γ') and regard it as a pseudo-adic space. Then $\overline{X}$ and $\overline{\Gamma}$ are proper over S and γ' induces a morphism $\overline{\gamma}\colon \overline{\Gamma} \longrightarrow \overline{X}\times_S \overline{X}$. It gives a compactification of γ. This construction should be more convenient for practical use.

Take a compactification $\overline{\gamma}\colon \overline{\Gamma} \longrightarrow \overline{X} \times_S \overline{X}$ of γ and denote the open immersion $X \hookrightarrow \overline{X}$ by j. We denote the open immersions $X \times_S X \hookrightarrow \overline{X} \times_S X$, $X \times_S \overline{X} \hookrightarrow \overline{X} \times_S \overline{X}$ by $j \times 1$, and $X \times_S X \hookrightarrow X \times_S \overline{X}$, $\overline{X} \times_S X \hookrightarrow \overline{X} \times_S \overline{X}$ by $1 \times j$. Consider the natural isomorphisms

$$\begin{aligned} H^{2d}_{\overline{\gamma}(\overline{\Gamma})}\big(\overline{X} \times_S \overline{X}, (j \times 1)_*(1 \times j)_!\Lambda(d)\big) &\xrightarrow{\cong} H^{2d}_{\gamma(\Gamma)}\big(X \times_S \overline{X}, (1 \times j)_!\Lambda(d)\big) \\ &\xrightarrow{\cong} H^{2d}_{\gamma(\Gamma)}\big(X \times_S X, \Lambda(d)\big). \end{aligned}$$

Note that the first isomorphy is a consequence of $\Gamma = \overline{\gamma}_1^{-1}(X) = \overline{\gamma}^{-1}(X \times_S \overline{X})$. We also denote by $\mathrm{cl}(\gamma)$ the element of $H^{2d}_{\overline{\gamma}(\overline{\Gamma})}(\overline{X} \times_S \overline{X}, (j \times 1)_*(1 \times j)_!\Lambda(d))$ that is mapped to $\mathrm{cl}(\gamma)$ by the homomorphism above. Since the projection formula gives

$$\begin{aligned} (j \times 1)_!\Lambda \overset{\mathbb{L}}{\otimes} (j \times 1)_*(1 \times j)_!\Lambda &= (j \times 1)_!\big(\Lambda \overset{\mathbb{L}}{\otimes} (j \times 1)^*(j \times 1)_*(1 \times j)_!\Lambda\big) \\ &= (j \times j)_!\Lambda, \end{aligned}$$

the cup product with $\mathrm{cl}(\gamma)$ induces the map

$$R\Gamma\big(\overline{X} \times_S \overline{X}, (j \times 1)_!\Lambda\big) \longrightarrow R\Gamma\big(\overline{X} \times_S \overline{X}, (j \times j)_!\Lambda(d)[2d]\big).$$

Proposition 2.18 *The map γ^* coincides with the composite below:*

$$\begin{aligned} R\Gamma_c(X, \Lambda) = R\Gamma(\overline{X}, j_!\Lambda) &\xrightarrow{\mathrm{pr}_1^*} R\Gamma(\overline{X} \times_S \overline{X}, \mathrm{pr}_1^* j_!\Lambda) = R\Gamma\big(\overline{X} \times_S \overline{X}, (j \times 1)_!\Lambda\big) \\ &\xrightarrow{\cup\,\mathrm{cl}(\gamma)} R\Gamma\big(\overline{X} \times_S \overline{X}, (j \times j)_!\Lambda(d)[2d]\big) = R\Gamma_c\big(X \times_S X, \Lambda(d)[2d]\big) \\ &\xrightarrow{\mathrm{Gys}_{\mathrm{pr}_2}} R\Gamma_c(X \times_S X, \mathrm{pr}_2^! \Lambda) = R\Gamma_c(X, R\,\mathrm{pr}_{2!}\,\mathrm{pr}_2^! \Lambda) \xrightarrow{\mathrm{adj}} R\Gamma_c(X, \Lambda). \end{aligned}$$

Proof. We also denote the open immersion $\Gamma \hookrightarrow \overline{\Gamma}$ by j. First let us prove the commutativity of the following diagram:

$$\begin{tikzcd} R\Gamma\big(\overline{X} \times_S \overline{X}, (j \times 1)_!\Lambda\big) \arrow[r, "\cup\,\mathrm{cl}(\gamma)"] \arrow[d, "\overline{\gamma}^*"] & R\Gamma\big(\overline{X} \times_S \overline{X}, (j \times j)_!\Lambda(d)[2d]\big) \\ R\Gamma\big(\overline{\Gamma}, \overline{\gamma}^*(j \times 1)_!\Lambda\big) \arrow[d, equal] & R\Gamma\big(\overline{X} \times_S \overline{X}, (j \times j)_! R\gamma_!\gamma^!\Lambda(d)[2d]\big) \arrow[u, "\mathrm{adj}"'] \arrow[d, equal] \\ R\Gamma(\overline{\Gamma}, j_!\Lambda) \arrow[r, "j_!\,\mathrm{Gys}_\gamma"] & R\Gamma\big(\overline{\Gamma}, j_!\gamma^!\Lambda(d)[2d]\big). \end{tikzcd}$$

By the adjointness of $(j \times 1)_!$ and $(j \times 1)^*$, it suffices to show the commutativity of the following diagram, where $\gamma' := (1\times j)\circ\gamma$:

$$\begin{tikzcd}
\Lambda \arrow[r, "\mathrm{cl}(\gamma)"] \arrow[d, "\mathrm{adj}"] & (1\times j)_!\Lambda(d)[2d] \\
R\gamma'_*\gamma'^*\Lambda \arrow[d, equal] & (1\times j)_!R\gamma_!\gamma^!\Lambda(d)[2d] \arrow[u, "\mathrm{adj}"] \arrow[d, equal] \\
R\gamma'_*\Lambda \arrow[r, "R\gamma'_*\,\mathrm{Gys}_\gamma"] & R\gamma'_*\gamma^!\Lambda(d)[2d].
\end{tikzcd}$$

Here $\mathrm{cl}(\gamma)$ is regarded as an element of $\mathrm{Hom}(\Lambda,(1\times j)_!\Lambda(d)[2d])$ by the maps

$$\begin{aligned}
&H^{2d}_{\gamma(\Gamma)}(X\times_S X,\Lambda(d)) \xleftarrow{\cong} H^{2d}_{\gamma(\Gamma)}(X\times_S \overline{X},(1\times j)_!\Lambda(d))\\
&\qquad\longrightarrow H^{2d}(X\times_S\overline{X},(1\times j)_!\Lambda(d)) = \mathrm{Hom}\bigl(\Lambda,(1\times j)_!\Lambda(d)[2d]\bigr).
\end{aligned}$$

By the construction, it is obtained by the composite

$$\begin{aligned}
&\Lambda \xrightarrow{\mathrm{adj}} R\gamma'_*\gamma'^*\Lambda = R\gamma'_*\Lambda \xrightarrow{R\gamma'_*\,\mathrm{Gys}_\gamma} R\gamma'_*\gamma^!\Lambda(d)[2d] = R\gamma'_!\gamma'^!(1\times j)_!\Lambda(d)[2d]\\
&\quad\xrightarrow{\mathrm{adj}} (1\times j)_!\Lambda(d)[2d].
\end{aligned}$$

Since it is easy to see that two maps

$$R\gamma'_*\gamma^!\Lambda = R\gamma'_!\gamma'^!(1\times j)_!\Lambda \xrightarrow{\mathrm{adj}} (1\times j)_!\Lambda,$$
$$R\gamma'_*\gamma^!\Lambda = (1\times j)_!R\gamma_!\gamma^!\Lambda \xrightarrow{\mathrm{adj}} (1\times j)_!\Lambda$$

coincide, we have the desired commutativity.

Therefore, the composite of

$$\begin{aligned}
R\Gamma_c(X,\Lambda) &= R\Gamma(\overline{X},j_!\Lambda) \xrightarrow{\mathrm{pr}_1^*} R\Gamma(\overline{X}\times_S\overline{X},\mathrm{pr}_1^*j_!\Lambda) = R\Gamma\bigl(\overline{X}\times_S\overline{X},(j\times 1)_!\Lambda\bigr)\\
&\xrightarrow{\cup\mathrm{cl}(\gamma)} R\Gamma\bigl(\overline{X}\times_S\overline{X},(j\times j)_!\Lambda(d)[2d]\bigr) = R\Gamma_c\bigl(X\times_S X,\Lambda(d)[2d]\bigr)
\end{aligned}$$

coincides with the composite of

$$\begin{aligned}
R\Gamma_c(X,\Lambda) &\xrightarrow{\gamma_1^*} R\Gamma_c(\Gamma,\Lambda) \xrightarrow{\mathrm{Gys}_\gamma} R\Gamma_c\bigl(\Gamma,\gamma^!\Lambda(d)[2d]\bigr) = R\Gamma_c\bigl(X\times_S X,R\gamma_!\gamma^!\Lambda(d)[2d]\bigr)\\
&\xrightarrow{\mathrm{adj}} R\Gamma_c\bigl(X\times_S X,\Lambda(d)[2d]\bigr).
\end{aligned}$$

Therefore it suffices to show that the composite of

$$\begin{aligned}
R\Gamma_c(\Gamma,\Lambda) &\xrightarrow{\mathrm{Gys}_\gamma} R\Gamma_c\bigl(\Gamma,\gamma^!\Lambda(d)[2d]\bigr) = R\Gamma_c\bigl(X\times_S X,R\gamma_!\gamma^!\Lambda(d)[2d]\bigr)\\
&\xrightarrow{\mathrm{adj}} R\Gamma_c\bigl(X\times_S X,\Lambda(d)[2d]\bigr) \xrightarrow{\mathrm{Gys}_{\mathrm{pr}_2}} R\Gamma_c\bigl(X\times_S X,\mathrm{pr}_2^!\Lambda\bigr)\\
&= R\Gamma_c\bigl(X,R\,\mathrm{pr}_{2!}\,\mathrm{pr}_2^!\Lambda\bigr) \xrightarrow{\mathrm{adj}} R\Gamma_c(X,\Lambda)
\end{aligned}$$

is equal to the composite of

$$R\Gamma_c(\Gamma,\Lambda) \xrightarrow{\mathrm{Gys}_{\gamma_2}} R\Gamma_c(\Gamma,\gamma_2^!\Lambda) = R\Gamma_c(X, R\gamma_{2!}\gamma_2^!\Lambda) \xrightarrow{\mathrm{adj}} R\Gamma_c(X,\Lambda).$$

It is an easy consequence of $\mathrm{Gys}_{\gamma_2} = \gamma^! \mathrm{Gys}_{\mathrm{pr}_2}(d)[2d] \circ \mathrm{Gys}_\gamma$, which can be proved directly from the construction of the Gysin maps. ∎

3 Lefschetz trace formula for open adic spaces

3.1 Künneth formula

Lemma 3.1 *Let X be a finite-dimensional pseudo-adic space which is quasi-separated of weakly finite type over S. Let $\mathcal{F}$ be a Λ-sheaf on X and L a bounded complex of Λ-modules. Then we have an isomorphism $R\Gamma(X, \mathcal{F} \overset{\mathbb{L}}{\otimes} L_X) \cong R\Gamma(X,\mathcal{F}) \overset{\mathbb{L}}{\otimes} L$.*

Proof. We may assume L is a Λ-module. By [Hub96, Corollary 2.8.3, Corollary 1.8.8], the cohomological dimension of $R\Gamma$ is finite. Therefore, by taking a free resolution of L, we may reduce to the case where L is a free Λ-module. Then the claim holds, since $R\Gamma$ commutes with any direct sum if X is quasi-compact and quasi-separated ([Hub96, Lemma 2.3.13 i)]). ∎

Similarly, we can prove the following:

Lemma 3.2 *Let X be a finite-dimensional pseudo-adic space which is separated, locally of $+$-weakly finite type and taut over S. Let $\mathcal{F}$ be a Λ-sheaf on X and L a bounded complex of Λ-modules. Then we have an isomorphism $R\Gamma_c(X, \mathcal{F} \overset{\mathbb{L}}{\otimes} L_X) \cong R\Gamma_c(X,\mathcal{F}) \overset{\mathbb{L}}{\otimes} L$.*

Proof. Since $R\Gamma_c$ has finite cohomological dimension ([Hub96, Proposition 5.5.8, Corollary 1.8.8]) and commutes with any direct sum ([Hub96, Proposition 5.4.5 i)]), the proof is exactly the same as the previous lemma. ∎

Corollary 3.3 *Let X be a finite-dimensional pseudo-adic space which is separated, locally of $+$-weakly finite type and taut over S. Assume that $H_c^i(X,\Lambda)$ is a finitely generated Λ-module for every i. Then $R\Gamma_c(X,\Lambda)$ is a perfect Λ-complex.*

In particular, if $\underline{X}$ is separated of finite type over S and $|X|$ is a locally closed constructible subset of $\underline{X}$, then $R\Gamma_c(X,\Lambda)$ is a perfect Λ-complex.

Proof. Since $R\Gamma_c(X,\Lambda)$ is bounded with finitely generated cohomology, by [SGA4$\frac{1}{2}$, [Rapport], Lemme 4.5.1] it suffices to show that $R\Gamma_c(X,\Lambda)$ has finite tor-dimension. Let M be a Λ-module. Then Lemma 3.2 says $R\Gamma_c(X,\Lambda) \overset{\mathbb{L}}{\otimes} M \cong R\Gamma_c(X, M_X)$. In particular, the ith cohomology of $R\Gamma_c(X,\Lambda) \overset{\mathbb{L}}{\otimes} M$ vanishes unless $0 \le i \le 2\dim X$. This means that $R\Gamma_c(X,\Lambda)$ has finite tor-dimension.

The latter part follows from the finiteness results due to Huber ([Hub98c, Corollary 2.3], [Hub07, Corollary 5.4]). ∎

Let X and Y be finite-dimensional pseudo-adic spaces which are quasi-separated of weakly finite type over S, and denote their structure maps by $a\colon X \longrightarrow S$ and $b\colon Y \longrightarrow S$. Denote the first (resp. second) projection by $\mathrm{pr}_1\colon X \times_S Y \longrightarrow X$ (resp. $\mathrm{pr}_2\colon X \times_S Y \longrightarrow Y$). Let $\mathcal{F}$ be a sheaf on X and $\mathcal{G}$ a sheaf on Y. Put $\mathcal{F} \overset{\mathbb{L}}{\boxtimes} \mathcal{G} = \mathrm{pr}_1^* \mathcal{F} \overset{\mathbb{L}}{\otimes} \mathrm{pr}_2^* \mathcal{G}$. Then we have the canonical homomorphism

$$R\Gamma(X, \mathcal{F}) \overset{\mathbb{L}}{\otimes} R\Gamma(Y, \mathcal{G}) \longrightarrow R\Gamma(X \times_S Y, \mathcal{F} \overset{\mathbb{L}}{\boxtimes} \mathcal{G}),$$

which is called the Künneth homomorphism.

Lemma 3.4 *If the canonical map*

$$\mathcal{F} \overset{\mathbb{L}}{\otimes} R\,\mathrm{pr}_{1*} \mathrm{pr}_2^* \mathcal{G} \longrightarrow R\,\mathrm{pr}_{1*}(\mathrm{pr}_1^* \mathcal{F} \overset{\mathbb{L}}{\otimes} \mathrm{pr}_2^* \mathcal{G})$$

is an isomorphism, the Künneth homomorphism is also an isomorphism.

Proof. By the quasi-compact/generalizing base change theorem ([Hub96, Theorem 4.3.1]), we have $R\,\mathrm{pr}_{1*} \mathrm{pr}_2^* \mathcal{G} \cong R\Gamma(Y, \mathcal{G})_X$. Then by Lemma 3.4, we have an isomorphism

$$\begin{aligned} R\Gamma(X, \mathcal{F}) \overset{\mathbb{L}}{\otimes} R\Gamma(Y, \mathcal{G}) &\cong R\Gamma\big(X, \mathcal{F} \overset{\mathbb{L}}{\otimes} R\Gamma(Y, \mathcal{G})_X\big) \cong R\Gamma\big(X, \mathcal{F} \overset{\mathbb{L}}{\otimes} R\,\mathrm{pr}_{1*} \mathrm{pr}_2^* \mathcal{G}\big) \\ &\xrightarrow{\cong} R\Gamma\big(X, R\,\mathrm{pr}_{1*}(\mathrm{pr}_1^* \mathcal{F} \overset{\mathbb{L}}{\otimes} \mathrm{pr}_2^* \mathcal{G})\big) \cong R\Gamma(X \times_S Y, \mathcal{F} \overset{\mathbb{L}}{\boxtimes} \mathcal{G}). \end{aligned}$$

It is easy to see that the isomorphism above is actually the Künneth homomorphism. ∎

Proposition 3.5 *In the case $\mathcal{F} = \Lambda$, the Künneth homomorphism*

$$R\Gamma(X, \Lambda) \overset{\mathbb{L}}{\otimes} R\Gamma(Y, \mathcal{G}) \longrightarrow R\Gamma(X \times_S Y, \Lambda \overset{\mathbb{L}}{\boxtimes} \mathcal{G})$$

is an isomorphism.

Proof. Clear by Lemma 3.4. ∎

Proposition 3.6 *Let $j\colon U \hookrightarrow Y$ be a quasi-compact open immersion. If $\mathcal{G} = j_!\Lambda$, then the Künneth homomorphism*

$$R\Gamma(X, \mathcal{F}) \overset{\mathbb{L}}{\otimes} R\Gamma(Y, j_!\Lambda) \longrightarrow R\Gamma(X \times_S Y, \mathcal{F} \overset{\mathbb{L}}{\boxtimes} j_!\Lambda)$$

is an isomorphism.

Proof. By Lemma 3.4, we have only to prove that the natural map

$$(R\,\mathrm{pr}_{2*}\,\mathrm{pr}_1^*\,\mathcal{F}) \overset{\mathbb{L}}{\otimes} j_!\Lambda \longrightarrow R\,\mathrm{pr}_{2*}(\mathrm{pr}_1^*\,\mathcal{F} \overset{\mathbb{L}}{\otimes} \mathrm{pr}_2^*\,j_!\Lambda)$$

is an isomorphism. Put $Z = (\underline{Y}, |Y| \setminus |U|)$ and denote the closed immersion $Z \hookrightarrow Y$ of pseudo-adic spaces by i. By the distinguished triangle $j_!\Lambda \longrightarrow \Lambda \longrightarrow i_*\Lambda \xrightarrow{+1} j_!\Lambda[1]$, it suffices to prove that the natural map

$$(R\,\mathrm{pr}_{2*}\,\mathrm{pr}_1^*\,\mathcal{F}) \overset{\mathbb{L}}{\otimes} i_*\Lambda \longrightarrow R\,\mathrm{pr}_{2*}(\mathrm{pr}_1^*\,\mathcal{F} \overset{\mathbb{L}}{\otimes} \mathrm{pr}_2^*\,i_*\Lambda)$$

is an isomorphism.

Consider the following commutative diagram whose rectangles are cartesian:

$$\begin{tikzcd}
X \times_S Z \arrow[r, "1\times i"] \arrow[d, "\mathrm{pr}_2'"] & X \times_S Y \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & X \arrow[d, "a"] \\
Z \arrow[r, "i"] & Y \arrow[r, "b"] & S.
\end{tikzcd}$$

By the quasi-compact/generalizing base change theorem, the base change map $i^*R\,\mathrm{pr}_{2*}\,\mathrm{pr}_1^*\,\mathcal{F} \longrightarrow i^*b^*Ra_*\mathcal{F}$ and $R\,\mathrm{pr}'_{2*}(1\times i)^*\,\mathrm{pr}_1^*\,\mathcal{F} \longrightarrow i^*b^*Ra_*\mathcal{F}$ are isomorphisms. Thus the base change map $i^*R\,\mathrm{pr}_{2*}\,\mathrm{pr}_1^*\,\mathcal{F} \longrightarrow R\,\mathrm{pr}'_{2*}(1\times i)^*\,\mathrm{pr}_1^*\,\mathcal{F}$ is also an isomorphism. By this, we have

$$\begin{aligned}
(R\,\mathrm{pr}_{2*}\,\mathrm{pr}_1^*\,\mathcal{F}) \overset{\mathbb{L}}{\otimes} i_*\Lambda \cong i_*i^*R\,\mathrm{pr}_{2*}\,\mathrm{pr}_1^*\,\mathcal{F} &\xrightarrow{\cong} i_*R\,\mathrm{pr}'_{2*}(1\times i)^*\,\mathrm{pr}_1^*\,\mathcal{F} \\
&= R\,\mathrm{pr}_{2*}(1\times i)_*(1\times i)^*\,\mathrm{pr}_1^*\,\mathcal{F} \cong R\,\mathrm{pr}_{2*}(\mathrm{pr}_1^*\,\mathcal{F} \overset{\mathbb{L}}{\otimes} (1\times i)_*\Lambda) \\
&\cong R\,\mathrm{pr}_{2*}(\mathrm{pr}_1^*\,\mathcal{F} \overset{\mathbb{L}}{\otimes} \mathrm{pr}_2^*\,i_*\Lambda),
\end{aligned}$$

which completes the proof. ∎

Corollary 3.7 *Let $U \subset X$ and $V \subset Y$ be quasi-compact open adic subspaces. Denote the open immersions $U \hookrightarrow X$, $V \hookrightarrow Y$ by j, j' respectively. We write $j\times 1$ for $U\times_S V \hookrightarrow X\times_S V$ and $U\times_S Y \hookrightarrow X\times_S Y$, $1\times j'$ for $U\times_S V \hookrightarrow U\times_S Y$ and $X\times_S V \hookrightarrow X\times_S Y$. Since $(j\times 1)^*(j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j'_!\Lambda) \cong (j\times 1)^*\,\mathrm{pr}_2^*\,j'_!\Lambda \cong (1\times j')_!\Lambda$, we have the canonical morphism $j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j'_!\Lambda \longrightarrow (j\times 1)_*(1\times j')_!\Lambda$ which is denoted by τ. Then the map*

$$R\Gamma(X\times_S Y, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j'_!\Lambda) \longrightarrow R\Gamma\big(X\times_S Y, (j\times 1)_*(1\times j')_!\Lambda\big)$$

induced by τ is an isomorphism.

Proof. Clear from Proposition 3.6 and the commutative diagram below:

$$\begin{tikzcd}
R\Gamma\bigl(X\times_S Y, j_*\Lambda\overset{\mathbb{L}}{\boxtimes} j'_!\Lambda\bigr) \arrow[r] & R\Gamma\bigl(X\times_S Y, (j\times 1)_*(1\times j')_!\Lambda\bigr) \arrow[d, "\cong"] \\
& R\Gamma\bigl(U\times_S Y, (1\times j')_!\Lambda\bigr) \\
R\Gamma(X, j_*\Lambda)\overset{\mathbb{L}}{\otimes} R\Gamma(Y, j'_!\Lambda) \arrow[uu, "\cong"] \arrow[r, "\cong"] & R\Gamma(U,\Lambda)\overset{\mathbb{L}}{\otimes} R\Gamma(Y, j'_!\Lambda). \arrow[u, "\cong"]
\end{tikzcd}$$

∎

Remark 3.8 Unlike the case of schemes, the morphism τ itself is not an isomorphism in general. For example, put $U = V = \mathbb{D}^1$ and let X, Y be the closure $\overline{\mathbb{D}^1}$ of $\mathbb{D}^1$ in $(\mathbb{A}^1)^{\mathrm{ad}}$. Take a continuous valuation $|\ |\colon k \longrightarrow \mathbb{R}_{\geq 0}$ on k and consider the point x of $(\mathbb{A}^1)^{\mathrm{ad}}$ corresponding to the valuation

$$k[T] \longrightarrow \mathbb{R}_{\geq 0}\times\mathbb{Z};\quad \sum_i a_iT^i \longmapsto \max\bigl\{(|a_i|, i)\bigr\}$$

(here we endow $\mathbb{R}_{\geq 0}\times\mathbb{Z}$ with the lexicographic order). Then x lies in the closure of $\mathbb{D}^1$ in $(\mathbb{A}^1)^{\mathrm{ad}}$ but does not lie in $\mathbb{D}^1$ itself. In $(\mathbb{A}^1)^{\mathrm{ad}}$ it has a unique generalization y which is given by the following valuation:

$$k[T] \longrightarrow \mathbb{R}_{\geq 0};\quad \sum_i a_iT^i \longmapsto \max\{|a_i|\}.$$

It is easy to see that y belongs to $\mathbb{D}^1$.

By the diagonal map $(\mathbb{A}^1)^{\mathrm{ad}} \longrightarrow (\mathbb{A}^2)^{\mathrm{ad}}$, we regard x and y as points of $(\mathbb{A}^2)^{\mathrm{ad}}$. Then x lies in $\overline{\mathbb{D}^1}\times_S\overline{\mathbb{D}^1} = \mathrm{pr}_1^{-1}(\overline{\mathbb{D}^1})\cap\mathrm{pr}_2^{-1}(\overline{\mathbb{D}^1})$ and y is a unique generalization of x in $(\mathbb{A}^2)^{\mathrm{ad}}$. Let j be the open immersion $\mathbb{D}^1 \hookrightarrow \overline{\mathbb{D}^1}$, which is quasi-compact. It is clear that $(j_*\Lambda\overset{\mathbb{L}}{\boxtimes} j'_!\Lambda)_{\overline{x}} = 0$. On the other hand, by [Hub96, Proposition 2.6.4] and its proof, we have $((j\times 1)_*(1\times j)_!\Lambda)_{\overline{x}} = ((1\times j)_!\Lambda)_{\overline{y}} = \Lambda$. Thus τ is not an isomorphism.

3.2 Unlocalized Lefschetz trace formula

In the remaining part of this section, we use the same notation as in §2.2; let $\gamma\colon\Gamma \longrightarrow X\times_S X$ be an S-morphism between purely d-dimensional adic spaces which are separated of finite type over S, and assume that X (resp. Γ) is smooth (resp. generically smooth) over S and γ_1 is proper. Fix a compactification $\overline{\gamma}\colon\overline{\Gamma} \longrightarrow \overline{X}\times_S\overline{X}$ and denote the open immersion $X \hookrightarrow \overline{X}$ by j.

As in §2, γ defines the element $\mathrm{cl}(\gamma)$ of $H^{2d}(\overline{X}\times_S\overline{X}, (j\times 1)_*(1\times j)_!\Lambda(d))$. Moreover, by Corollary 3.7, the map $H^{2d}(\overline{X}\times_S\overline{X}, j_*\Lambda\overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d)) \longrightarrow H^{2d}(\overline{X}\times_S\overline{X}, (j\times$

$1)_*(1\times j)_!\Lambda(d))$ induced by the canonical morphism $\tau\colon j_*\Lambda\overset{\mathbb{L}}{\boxtimes}j_!\Lambda \longrightarrow (j\times 1)_*(1\times j)_!\Lambda$ is an isomorphism. Therefore γ defines an element of $H^{2d}(Y\times_S Y, (1\times j)_!(j\times 1)_*\Lambda(d))$ denoted by $[\gamma]$. The diagonal morphism $\overline{\delta}\colon \overline{X} \longrightarrow \overline{X}\times_S \overline{X}$ induces the pull-back map $\overline{\delta}^*\colon H^{2d}(\overline{X}\times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d)) \longrightarrow H^{2d}(\overline{X}, j_!\Lambda(d)) = H^{2d}_c(X, \Lambda(d))$.

Proposition 3.9 *In the situation above, we have the equality*

$$\mathrm{Tr}\bigl(\gamma^*; R\Gamma_c(X,\Lambda)\bigr) = \mathrm{Tr}_X\bigl(\overline{\delta}^*[\gamma]\bigr).$$

The proof of this proposition is similar to that in the scheme case. However, we will include it for the completeness. Let us consider the following diagram:

$$\begin{tikzcd}
R\Gamma\bigl(X,\Lambda(d)[2d]\bigr) \overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda) \arrow[r, "\cong"] \arrow[d, "\cong"] & R\,\mathrm{Hom}\bigl(R\Gamma_c(X,\Lambda),\Lambda\bigr) \overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda) \arrow[dd] \\
R\Gamma\bigl(\overline{X}\times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d)[2d]\bigr) \arrow[d] \arrow[dd, bend right=60, "\overline{\delta}^*"'] & \\
R\Gamma\bigl(\overline{X}\times_S \overline{X}, (j\times 1)_*(1\times j)_!\Lambda(d)[2d]\bigr) \arrow[r, "(*)"] & R\,\mathrm{Hom}\bigl(R\Gamma_c(X,\Lambda), R\Gamma_c(X,\Lambda)\bigr) \arrow[d, "\mathrm{Tr}"] \\
R\Gamma_c\bigl(X,\Lambda(d)[2d]\bigr) \arrow[r, "\mathrm{Tr}_X"] & \Lambda.
\end{tikzcd}$$

Here $(*)$ is the map induced by

$$\begin{aligned}
&R\Gamma_c(X,\Lambda) \overset{\mathbb{L}}{\otimes} R\Gamma\bigl(\overline{X}\times_S \overline{X}, (j\times 1)_*(1\times j)_!\Lambda(d)[2d]\bigr) \\
&\xrightarrow{\mathrm{pr}_1^*\otimes \mathrm{id}} R\Gamma\bigl(\overline{X}\times_S \overline{X}, (j\times 1)_!\Lambda\bigr) \overset{\mathbb{L}}{\otimes} R\Gamma\bigl(\overline{X}\times_S \overline{X}, (j\times 1)_*(1\times j)_!\Lambda(d)[2d]\bigr) \\
&\xrightarrow{\cup} R\Gamma\bigl(\overline{X}\times_S \overline{X}, (j\times 1)_!(1\times j)_!\Lambda(d)[2d]\bigr) = R\Gamma_c\bigl(X\times_S X, \Lambda(d)[2d]\bigr) \\
&\xrightarrow{\mathrm{pr}_{2*}} R\Gamma_c\bigl(X,\Lambda\bigr),
\end{aligned}$$

where pr_{2*} is the composite of

$$\begin{aligned}
R\Gamma_c\bigl(X\times_S X, \Lambda(d)[2d]\bigr) &\xrightarrow{\mathrm{Gys}_{\mathrm{pr}_2}} R\Gamma_c\bigl(X\times_S X, \mathrm{pr}_2^!\,\Lambda\bigr) = R\Gamma_c(X, R\,\mathrm{pr}_{2!}\,\mathrm{pr}_2^!\,\Lambda) \\
&\xrightarrow{\mathrm{adj}} R\Gamma_c(X,\Lambda),
\end{aligned}$$

or equivalently, the composite of

$$R\Gamma_c\bigl(X\times_S X, \Lambda(d)[2d]\bigr) = R\Gamma_c\bigl(X, R\,\mathrm{pr}_{2!}\,\Lambda(d)[2d]\bigr) \xrightarrow{\mathrm{Tr}_{\mathrm{pr}_2}} R\Gamma_c(X,\Lambda).$$

Since $R\Gamma_c(X,\Lambda)$ is a perfect Λ-complex (Corollary 3.3), we may define the map

$$\mathrm{Tr}\colon R\operatorname{Hom}(R\Gamma_c(X,\Lambda),R\Gamma_c(X,\Lambda))\longrightarrow\Lambda$$

([SGA6, Expose I]). It is a unique map such that the composite

$$R\operatorname{Hom}\bigl(R\Gamma_c(X,\Lambda),\Lambda\bigr)\overset{\mathbb{L}}{\otimes}R\Gamma_c(X,\Lambda)\xrightarrow{\cong}R\operatorname{Hom}\bigl(R\Gamma_c(X,\Lambda),R\Gamma_c(X,\Lambda)\bigr)\xrightarrow{\mathrm{Tr}}\Lambda$$

coincides with the evaluation map ev.

By Proposition 2.18, $(*)$ maps $\mathrm{cl}(\gamma)$ to γ^*. Thus the proposition follows from the commutativity of the lower part of the diagram above, i.e., the commutativity of the diagram below:

$$\begin{tikzcd}
R\Gamma\bigl(\overline{X}\times_S\overline{X},j_*\Lambda\overset{\mathbb{L}}{\boxtimes}j_!\Lambda(d)[2d]\bigr) \arrow[r, "\overline{\delta}^*"] \arrow[d] & R\Gamma_c\bigl(X,\Lambda(d)[2d]\bigr) \arrow[dd, "\mathrm{Tr}_X"] \\
R\Gamma\bigl(\overline{X}\times_S\overline{X},(j\times 1)_*(1\times j)_!\Lambda(d)[2d]\bigr) \arrow[d, "(*)"] & \\
R\operatorname{Hom}\bigl(R\Gamma_c(X,\Lambda),R\Gamma_c(X,\Lambda)\bigr) \arrow[r, "\mathrm{Tr}"] & \Lambda.
\end{tikzcd}$$

To show it, it is sufficient to show the commutativities of the upper part and the outer part of the diagram above. We will divide their proofs into two propositions:

Proposition 3.10 *The following diagram is commutative:*

$$\begin{tikzcd}
R\Gamma\bigl(X,\Lambda(d)[2d]\bigr)\overset{\mathbb{L}}{\otimes}R\Gamma_c(X,\Lambda) \arrow[r, "\cong"] \arrow[d, "\cong"] & R\operatorname{Hom}\bigl(R\Gamma_c(X,\Lambda),\Lambda\bigr)\overset{\mathbb{L}}{\otimes}R\Gamma_c(X,\Lambda) \arrow[dd, "\mathrm{ev}"] \\
R\Gamma\bigl(\overline{X}\times_S\overline{X},j_*\Lambda\overset{\mathbb{L}}{\boxtimes}j_!\Lambda(d)[2d]\bigr) \arrow[d, "\overline{\delta}^*"] & \\
R\Gamma_c\bigl(X,\Lambda(d)[2d]\bigr) \arrow[r, "\mathrm{Tr}_X"] & \Lambda.
\end{tikzcd}$$

Proof. Notice that the composite of the left vertical arrows is nothing but the cup product. Thus the lemma follows from the next lemma, which is easy to see. ∎

Lemma 3.11 *Let K (resp. L) be an object of $D^+(\Lambda)$ (resp. $D^-(\Lambda)$) and $\Phi\colon K\overset{\mathbb{L}}{\otimes}L\longrightarrow\Lambda$ a map. We have a natural map $\Phi'\colon K\longrightarrow R\operatorname{Hom}(L,\Lambda)$ induced from Φ. Then the following diagram is commutative:*

$$\begin{tikzcd}
K\overset{\mathbb{L}}{\otimes}L \arrow[rr, "\Phi'\otimes\mathrm{id}"] \arrow[dr, "\Phi"'] & & R\operatorname{Hom}(L,\Lambda)\overset{\mathbb{L}}{\otimes}L \arrow[dl, "\mathrm{ev}"] \\
& \Lambda. &
\end{tikzcd}$$

Proposition 3.12 *The following diagram is commutative:*

$$\begin{tikzcd}
R\Gamma\big(X, \Lambda(d)[2d]\big) \overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda) \arrow[r, "\cong"] \arrow[d, "\cong"] & R\operatorname{Hom}\big(R\Gamma_c(X,\Lambda), \Lambda\big) \overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda) \arrow[dd] \\
R\Gamma\big(\overline{X} \times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d)[2d]\big) \arrow[d] & \\
R\Gamma\big(\overline{X} \times_S \overline{X}, (j\times 1)_*(1\times j)_!\Lambda(d)[2d]\big) \arrow[r, "(*)"] & R\operatorname{Hom}\big(R\Gamma_c(X,\Lambda), R\Gamma_c(X,\Lambda)\big).
\end{tikzcd}$$

Proof. Recall that by the definition the morphism $(*)$ is decomposed as

$$\begin{aligned}
R\Gamma\big(\overline{X} \times_S \overline{X}, (j\times 1)_*(1\times j)_!\Lambda(d)[2d]\big) &\longrightarrow R\operatorname{Hom}\big(R\Gamma_c(X,\Lambda), R\Gamma_c(X\times_S X, \Lambda(d)[2d])\big) \\
&\longrightarrow R\operatorname{Hom}\big(R\Gamma_c(X,\Lambda), R\Gamma_c(X,\Lambda)\big),
\end{aligned}$$

where the second arrow is induced from $\mathrm{pr}_{2*}\colon R\Gamma_c(X\times_S X, \Lambda(d)[2d]) \longrightarrow R\Gamma_c(X,\Lambda)$. Hence we may divide the diagram above into two parts:

$$\begin{tikzcd}
R\Gamma\big(X, \Lambda(d)[2d]\big) \overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda) \arrow[r, "(\dagger)"] \arrow[d, "\cong"] & R\operatorname{Hom}\big(R\Gamma_c(X,\Lambda), R\Gamma_c(X,\Lambda(d)[2d]) \overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda)\big) \arrow[dd, "\cong"] \\
R\Gamma\big(\overline{X} \times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d)[2d]\big) \arrow[d] & \\
R\Gamma\big(\overline{X} \times_S \overline{X}, (j\times 1)_*(1\times j)_!\Lambda(d)[2d]\big) \arrow[r] & R\operatorname{Hom}\big(R\Gamma_c(X,\Lambda), R\Gamma_c(X\times_S X, \Lambda(d)[2d])\big)
\end{tikzcd}$$

and

$$\begin{tikzcd}
R\Gamma\big(X, \Lambda(d)[2d]\big) \overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda) \arrow[r] \arrow[d, "(\dagger)"] & R\operatorname{Hom}\big(R\Gamma_c(X,\Lambda), \Lambda\big) \overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda) \arrow[d] \\
R\operatorname{Hom}\big(R\Gamma_c(X,\Lambda), R\Gamma_c(X,\Lambda(d)[2d]) \overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda)\big) \arrow[r, "\mathrm{Tr}_X \otimes \mathrm{id}"] \arrow[d] & R\operatorname{Hom}\big(R\Gamma_c(X,\Lambda), R\Gamma_c(X,\Lambda)\big) \\
R\operatorname{Hom}\big(R\Gamma_c(X,\Lambda), R\Gamma_c(X\times_S X, \Lambda(d)[2d])\big). \arrow[ur] &
\end{tikzcd}$$

Here $(\dagger)$ is induced from

$$R\Gamma_c(X,\Lambda) \overset{\mathbb{L}}{\otimes} R\Gamma(X, \Lambda(d)[2d]) \overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda) \xrightarrow{\cup\otimes\mathrm{id}} R\Gamma_c(X, \Lambda(d)[2d]) \overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda).$$

Let us prove the commutativity of the former. By the adjointness, this is equivalent to the commutativity of the following:

$$\begin{tikzcd}
R\Gamma_c(X,\Lambda)\overset{\mathbb{L}}{\otimes} R\Gamma\bigl(X,\Lambda(d)[2d]\bigr)\overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda) \arrow[r, "\cup\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes(\cup\circ(\mathrm{pr}_1^*\otimes\mathrm{pr}_2^*))"] & R\Gamma_c(X,\Lambda(d)[2d])\overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda) \arrow[dd, "\cup\circ(\mathrm{pr}_1^*\otimes\mathrm{pr}_2^*)", "\cong"'] \\
R\Gamma_c(X,\Lambda)\overset{\mathbb{L}}{\otimes} R\Gamma\bigl(\overline{X}\times_S\overline{X}, j_*\Lambda\overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d)[2d]\bigr) \arrow[d] & \\
R\Gamma_c(X,\Lambda)\overset{\mathbb{L}}{\otimes} R\Gamma\bigl(\overline{X}\times_S\overline{X}, (j\times 1)_*(1\times j)_!\Lambda(d)[2d]\bigr) \arrow[r, "\cup\circ(\mathrm{pr}_1^*\otimes\mathrm{id})"] & R\Gamma_c(X\times_S X,\Lambda(d)[2d]).
\end{tikzcd}$$

It easily follows from the associativity of cup products.

Next we will prove the commutativity of the latter. The triangle is commutative, since the trace map for a smooth morphism is compatible with base change. By the adjointness, the commutativity of the rectangle is equivalent to that of the following:

$$\begin{tikzcd}
R\Gamma_c(X,\Lambda)\overset{\mathbb{L}}{\otimes} R\Gamma\bigl(X,\Lambda(d)[2d]\bigr)\overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda) \arrow[r] \arrow[d, "\cup\otimes\mathrm{id}"] & R\Gamma_c(X,\Lambda)\overset{\mathbb{L}}{\otimes} R\operatorname{Hom}\bigl(R\Gamma_c(X,\Lambda),\Lambda\bigr)\overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda) \arrow[d, "\mathrm{ev}\otimes\mathrm{id}"] \\
R\Gamma_c\bigl(X,\Lambda(d)[2d]\bigr)\overset{\mathbb{L}}{\otimes} R\Gamma_c(X,\Lambda) \arrow[r, "\mathrm{Tr}_X\otimes\mathrm{id}"] & R\Gamma_c(X,\Lambda).
\end{tikzcd}$$

Since it is obtained from the diagram in Proposition 3.10 by taking tensor products with $R\Gamma_c(X,\Lambda)$, it is commutative. ∎

Now the proof of Proposition 3.9 is complete.

3.3 Localization

Let the notation be the same as in the previous subsection. In this subsection, we will prove the main theorem in this paper, whose statement is the following:

Theorem 3.13 *Assume that for every $z\in\overline{\Gamma}\setminus\Gamma$, the points $\overline{\gamma}_1(z)$ and $\overline{\gamma}_2(z)$ can be separated by closed constructible subsets; namely, there exists closed constructible subsets W_1 and W_2 of $\overline{X}$ such that $\overline{\gamma}_1(z)\in W_1$, $\overline{\gamma}_2(z)\in W_2$ and $W_1\cap W_2=\varnothing$. Then we have*

$$\mathrm{Tr}\bigl(\gamma^*;R\Gamma_c(X,\Lambda)\bigr)=\#\operatorname{Fix}\gamma.$$

Remark 3.14 The condition in Theorem 3.13 implies that $\overline{\gamma}^{-1}(\Delta_{\overline{X}}\setminus\Delta_X)=\varnothing$, thus $\operatorname{Fix}\gamma=\operatorname{Fix}\overline{\gamma}$. In particular, $\operatorname{Fix}\gamma$ is proper over S and $\#\operatorname{Fix}\gamma$ makes sense. Note also that $\Delta_X\cap\gamma(\Gamma)=\Delta_{\overline{X}}\cap\overline{\gamma}(\overline{\Gamma})$, which will be used in the proof of Theorem 3.13.

Remark 3.15 The underlying topological space $|\overline{X}|$ of a pseudo-adic space $\overline{X}$ which is proper over S is spectral. Indeed, it is quasi-compact, quasi-separated and locally pro-constructible in the locally spectral space $(\overline{X})_-$ (*cf.* [Hub93, Remark 2.1 (iv)]). Therefore, a subset W of $\overline{X}$ is closed constructible if and only if $\overline{X} \setminus W$ is quasi-compact open (*cf.* [Hub93, Remark 2.1 (i)]).

We will begin the proof of Theorem 3.13. It suffices to compare the right hand side of Proposition 3.9 with $\#\operatorname{Fix}\gamma$. The idea is to localize (or refine) the isomorphism

$$H^{2d}\big(\overline{X} \times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d)\big) \xrightarrow{\cong} H^{2d}\big(\overline{X} \times_S \overline{X}, (j \times 1)_*(1 \times j)_!\Lambda(d)\big)$$

used in the previous subsection. Although the element $\mathrm{cl}(\gamma)$ lies in the local cohomology $H^{2d}_{\overline{\gamma}(\overline{\Gamma})}(\overline{X} \times_S \overline{X}, (j \times 1)_*(1 \times j)_!\Lambda(d))$, the map

$$H^{2d}_{\overline{\gamma}(\overline{\Gamma})}\big(\overline{X} \times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d)\big) \longrightarrow H^{2d}_{\overline{\gamma}(\overline{\Gamma})}\big(\overline{X} \times_S \overline{X}, (j \times 1)_*(1 \times j)_!\Lambda(d)\big)$$

is not necessary an isomorphism (*cf.* Remark 3.8). Thus we will slightly enlarge the closed subset $\overline{\gamma}(\overline{\Gamma})$ so that the morphism above becomes an isomorphism. For the precise statement, see Proposition 3.20.

Before doing it, we need some preparation.

Lemma 3.16 *Let U and V be quasi-compact open subsets of $\overline{X}$. Then the map $R\Gamma_{U\times_S V}(\overline{X} \times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda) \longrightarrow R\Gamma_{U\times_S V}(\overline{X} \times_S \overline{X}, (j \times 1)_*(1 \times j)_!\Lambda)$ induced by τ is an isomorphism.*

Proof. By easy observation, we have

$$\begin{aligned}(j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda)|_{U\times_S V} &\cong j'_*\Lambda \overset{\mathbb{L}}{\boxtimes} j''_!\Lambda,\\ \big((j \times 1)_*(1 \times j)_!\Lambda\big)\big|_{U\times_S V} &\cong (j' \times 1)_*(1 \times j'')_!\Lambda,\end{aligned}$$

where j' (resp. j'') is the open immersion $U \cap X \hookrightarrow U$ (resp. $V \cap X \hookrightarrow V$). Hence we have

$$\begin{aligned}R\Gamma_{U\times_S V}(\overline{X} \times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda) &= R\Gamma\big(U \times_S V, j'_*\Lambda \overset{\mathbb{L}}{\boxtimes} j''_!\Lambda\big),\\ R\Gamma_{U\times_S V}\big(\overline{X} \times_S \overline{X}, (j \times 1)_*(1 \times j)_!\Lambda\big) &= R\Gamma\big(U \times_S V, (j' \times 1)_*(1 \times j'')_!\Lambda\big).\end{aligned}$$

Note that $U \cap X$ and $V \cap X$ are also quasi-compact, since $\overline{X}$ is quasi-separated. Therefore by Corollary 3.7, the canonical map

$$R\Gamma(U \times_S V, j'_*\Lambda \overset{\mathbb{L}}{\boxtimes} j''_!\Lambda) \longrightarrow R\Gamma\big(U \times_S V, (j' \times 1)_*(1 \times j'')_!\Lambda\big)$$

is an isomorphism. Now the proof is complete. ∎

Corollary 3.17 *Let $U_1,\ldots,U_m,U'_1,\ldots,U'_m$ be quasi-compact open subsets of X and put $V=\bigcup_{i=1}^m U_i\times_S U'_i$. Then the map $R\Gamma_V(\overline{X}\times_S\overline{X}, j'_*\Lambda\overset{\mathbb{L}}{\boxtimes} j''_!\Lambda)\longrightarrow R\Gamma_V(\overline{X}\times_S\overline{X},(j\times 1)_*(1\times j)_!\Lambda)$ induced by τ is an isomorphism.*

Proof. Recall that the underlying topological space of $U_i\times_S U'_i$ is equal to $\mathrm{pr}_1^{-1}(U_i)\cap\mathrm{pr}_2^{-1}(U'_i)$. Therefore, for every subset $I\subset\{1,\ldots,m\}$, we have $\bigcap_{i\in I}U_i\times_S U'_i=(\bigcap_{i\in I}U_i)\times_S(\bigcap_{i\in I}U'_i)$. Then an easy Mayer-Vietoris argument we may reduce to the case where $m=1$, which is already proven in the previous lemma. ∎

Corollary 3.18 *Let $V\subset\overline{X}\times_S\overline{X}$ be an open subset of the type in the above corollary. Put $W=(\overline{X}\times_S\overline{X})\setminus V$. Then the map $R\Gamma_W(\overline{X}\times_S\overline{X}, j_*\Lambda\overset{\mathbb{L}}{\boxtimes} j_!\Lambda)\longrightarrow R\Gamma_W(\overline{X}\times_S\overline{X},(j\times 1)_*(1\times j)_!\Lambda)$ induced by τ is an isomorphism.*

Proof. Clear from the distinguished triangle

$$R\Gamma_V(\overline{X}\times_S\overline{X},-)\longrightarrow R\Gamma(\overline{X}\times_S\overline{X},-)\longrightarrow R\Gamma_W(\overline{X}\times_S\overline{X},-)\xrightarrow{+1}$$

and Corollary 3.17. ∎

Corollary 3.19 *Let $W_1,\ldots,W_m,W'_1,\ldots,W'_m$ be constructible closed subsets of $\overline{X}$ and put $Z=\bigcup_{i=1}^m W_i\times_S W'_i$. Then the map $R\Gamma_Z(\overline{X}\times_S\overline{X}, j_*\Lambda\overset{\mathbb{L}}{\boxtimes} j_!\Lambda)\longrightarrow R\Gamma_Z(\overline{X}\times_S\overline{X},(j\times 1)_*(1\times j)_!\Lambda)$ induced by τ is an isomorphism.*

Proof. In the same way as in the proof of Corollary 3.17, we can reduce to the case where $m=1$. This is the special case of Corollary 3.18, since $W_1\times_S W'_1$ is the complement of $(W_1^c\times_S\overline{X})\cup(\overline{X}\times_S W_1'^c)$ (*cf.* Remark 3.15). ∎

Now assume the condition in Theorem 3.13. Then, for every $y\in\overline{\gamma}(\overline{\Gamma})\setminus(X\times_S X)$, there exist closed constructible subsets W_y and W'_y of $\overline{X}$ such that $y\in W_y\times_S W'_y$ and $W_y\cap W'_y=\varnothing$. Since $\overline{\gamma}(\overline{\Gamma})\setminus(X\times_S X)$ is closed and $W_y\times_S W'_y$ is constructible in $\overline{X}\times_S\overline{X}$, we may choose finitely many $W_1\times_S W'_1,\ldots,W_m\times_S W'_m$ among $\{W_y\times_S W'_y\}$ so that they cover $\overline{\gamma}(\overline{\Gamma})\setminus(X\times_S X)$. Indeed, if we endow $\overline{X}\times_S\overline{X}$ with the patch topology, $\overline{\gamma}(\overline{\Gamma})\setminus(X\times_S X)$ becomes compact and $W_y\times_S W'_y$ becomes open (*cf.* Remark 3.15, [Hoc69, §2]).

Put $Z=\bigcup_{i=1}^m W_i\times_S W'_i$. The following proposition is crucial for the proof of Theorem 3.13:

Proposition 3.20 *The map*

$$R\Gamma_{\overline{\gamma}(\overline{\Gamma})\cup Z}(\overline{X}\times_S\overline{X}, j_*\Lambda\overset{\mathbb{L}}{\boxtimes} j_!\Lambda)\longrightarrow R\Gamma_{\overline{\gamma}(\overline{\Gamma})\cup Z}\big(\overline{X}\times_S\overline{X},(j\times 1)_*(1\times j)_!\Lambda\big)$$

induced by τ is an isomorphism.

Proof. Put $V = (\overline{X} \times_S \overline{X}) \setminus Z$. By the distinguished triangle

$$R\Gamma_Z(\overline{X} \times_S \overline{X}, -) \longrightarrow R\Gamma_{\overline{\gamma}(\overline{\Gamma}) \cup Z}(\overline{X} \times_S \overline{X}, -) \longrightarrow R\Gamma_{\overline{\gamma}(\overline{\Gamma}) \setminus Z}\big((V, (-)|_V\big) \xrightarrow{+1}$$

and Corollary 3.19, it suffices to show that the map

$$R\Gamma_{\overline{\gamma}(\overline{\Gamma}) \setminus Z}\big(V, (j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda)|_V\big) \longrightarrow R\Gamma_{\overline{\gamma}(\overline{\Gamma}) \setminus Z}\big(V, ((j \times 1)_*(1 \times j)_!\Lambda)|_V\big)$$

is an isomorphism. By the assumption on Z, $\overline{\gamma}(\overline{\Gamma}) \setminus Z$ has an open neighborhood $V \cap (X \times_S X)$ on which $\tau \colon j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda \longrightarrow (j \times 1)_*(1 \times j)_!\Lambda$ is an isomorphism. Thus the map above is also an isomorphism. ∎

We define the element $[\gamma]_Z$ of $H^{2d}_{\overline{\gamma}(\overline{\Gamma}) \cup Z}(\overline{X} \times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d))$ as the image of $\mathrm{cl}(\gamma)$ under the composite of following maps:

$$\begin{aligned} H^{2d}_{\gamma(\Gamma)}\big(X \times_S X, \Lambda(d)\big) &\cong H^{2d}_{\overline{\gamma}(\overline{\Gamma})}\big(\overline{X} \times_S \overline{X}, (j \times 1)_*(1 \times j)_!\Lambda(d)\big) \\ &\longrightarrow H^{2d}_{\overline{\gamma}(\overline{\Gamma}) \cup Z}\big(\overline{X} \times_S \overline{X}, (j \times 1)_*(1 \times j)_!\Lambda(d)\big) \\ &\xleftarrow{\cong} H^{2d}_{\overline{\gamma}(\overline{\Gamma}) \cup Z}\big(\overline{X} \times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d)\big). \end{aligned}$$

By the definition, the image of $[\gamma]_Z$ under the natural map

$$H^{2d}_{\overline{\gamma}(\overline{\Gamma}) \cup Z}\big(\overline{X} \times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d)\big) \longrightarrow H^{2d}\big(\overline{X} \times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d)\big)$$

coincides with $[\gamma]$.

Since $\Delta_{\overline{X}} \cap Z = \varnothing$, we have $\Delta_{\overline{X}} \cap (\overline{\gamma}(\overline{\Gamma}) \cup Z) = \Delta_X \cap \gamma(\Gamma)$. Therefore $\overline{\delta}$ induces the maps

$$H^{2d}\big(\overline{X} \times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d)\big) \xrightarrow{\overline{\delta}^*} H^{2d}_{\Delta_X \cap \gamma(\Gamma)}\big(\overline{X}, j_!\Lambda(d)\big) \xrightarrow{\cong} H^{2d}_{\Delta_X \cap \gamma(\Gamma)}\big(X, \Lambda(d)\big).$$

We denote the image of $[\gamma]_Z$ under the maps above by $\overline{\delta}^*([\gamma]_Z)$.

Lemma 3.21 *The image of $\overline{\delta}^*([\gamma]_Z)$ under the canonical map $H^{2d}_{\Delta_X \cap \gamma(\Gamma)}(X, \Lambda(d)) \longrightarrow H^{2d}_c(X, \Lambda(d))$ coincides with $\overline{\delta}^*[\gamma]$.*

Proof. Clear from the following commutative diagram:

$$\begin{tikzcd} H^{2d}_{\overline{\gamma}(\overline{\Gamma}) \cup Z}\big(\overline{X} \times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d)\big) \arrow[r] \arrow[d, "\overline{\delta}^*"] & H^{2d}\big(\overline{X} \times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d)\big) \arrow[d, "\overline{\delta}^*"] \\ H^{2d}_{\Delta_X \cap \gamma(\Gamma)}\big(\overline{X}, j_!\Lambda(d)\big) \arrow[r] \arrow[d, "\cong"] & H^{2d}\big(\overline{X}, j_!\Lambda(d)\big) \arrow[d, equal] \\ H^{2d}_{\Delta_X \cap \gamma(\Gamma)}\big(X, \Lambda(d)\big) \arrow[r] & H^{2d}_c\big(X, \Lambda(d)\big). \end{tikzcd}$$

∎

Lemma 3.22 *The image of* $\mathrm{cl}(\gamma)$ *under the map* $\delta^*\colon H^{2d}_{\gamma(\Gamma)}(X \times_S X, \Lambda(d)) \longrightarrow H^{2d}_{\Delta_X \cap \gamma(\Gamma)}(X, \Lambda(d))$ *coincides with* $\overline{\delta}^*([\gamma]_Z)$.

Proof. Clear from the following commutative diagram:

$$\begin{tikzcd}
H^{2d}_{\overline{\gamma}(\overline{\Gamma})}\bigl(\overline{X} \times_S \overline{X}, (j \times 1)_*(1 \times j)_!\Lambda(d)\bigr) \arrow[r, "|_{X \times_S X}", "\cong"'] \arrow[d] & H^{2d}_{\gamma(\Gamma)}\bigl(X \times_S X, \Lambda(d)\bigr) \arrow[r, "\delta^*"] \arrow[d] & H^{2d}_{\Delta_X \cap \gamma(\Gamma)}\bigl(X, \Lambda(d)\bigr) \arrow[d, equal] \\
H^{2d}_{\overline{\gamma}(\overline{\Gamma}) \cup Z}\bigl(\overline{X} \times_S \overline{X}, (j \times 1)_*(1 \times j)_!\Lambda(d)\bigr) \arrow[r, "|_{X \times_S X}"] & H^{2d}_{(X \times_S X) \cap (\overline{\gamma}(\overline{\Gamma}) \cup Z)}\bigl(X \times_S X, \Lambda(d)\bigr) \arrow[r, "\delta^*"] & H^{2d}_{\Delta_X \cap \gamma(\Gamma)}\bigl(X, \Lambda(d)\bigr) \arrow[d, equal] \\
H^{2d}_{\overline{\gamma}(\overline{\Gamma}) \cup Z}\bigl(\overline{X} \times_S \overline{X}, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(d)\bigr) \arrow[u, "\cong"'] \arrow[r, "\overline{\delta}^*"] & H^{2d}_{\Delta_X \cap \gamma(\Gamma)}\bigl(\overline{X}, j_!\Lambda(d)\bigr) \arrow[r, "|_X", "\cong"'] & H^{2d}_{\Delta_X \cap \gamma(\Gamma)}\bigl(X, \Lambda(d)\bigr).
\end{tikzcd}$$

∎

Proof of Theorem 3.13. By Remark 2.7, $\#\operatorname{Fix}\gamma$ is the image of $\mathrm{cl}(\gamma)$ under the maps

$$H^{2d}_{\gamma(\Gamma)}\bigl(X \times_S X, \Lambda(d)\bigr) \xrightarrow{\delta^*} H^{2d}_{\Delta_X \cap \gamma(\Gamma)}\bigl(X, \Lambda(d)\bigr) \longrightarrow H^{2d}_c\bigl(X, \Lambda(d)\bigr) \xrightarrow{\mathrm{Tr}_X} \Lambda.$$

By Lemma 3.21 and Lemma 3.22, it coincides with $\mathrm{Tr}_X(\overline{\delta}^*[\gamma])$. Therefore by Proposition 3.9, we conclude that $\#\operatorname{Fix}\gamma = \mathrm{Tr}(\gamma^*; R\Gamma_c(X, \Lambda))$. ∎

Remark 3.23 So far, we considered the case of torsion coefficient. However, at least when the characteristic of k is 0 (*cf.* [Hub98a, Theorem 3.1]), we may obtain the Lefschetz trace formula for ℓ-adic coefficient simply by taking projective limit. For the definition of $\#\operatorname{Fix}\gamma$ for the ℓ-adic case, see Remark 2.8.

3.4 Lefschetz trace formula for open adic curves

In this subsection, we will establish a Lefschetz trace formula for quasi-compact smooth adic curve by using the same idea as in the proof of Theorem 3.13. Let X be a 1-dimensional quasi-compact adic space which is separated and smooth over S, and $j\colon X \hookrightarrow X^c$ the universal compactification over S. It is known that $\partial X := X^c \setminus X$ is a finite discrete set ([Hub01, Lemma 5.12]). In particular, every $x \in \partial X$ is a constructible closed subset of X^c.

Let $f\colon X \longrightarrow X$ be a proper morphism over S and $f^c\colon X^c \longrightarrow X^c$ the induced morphism. Since f is proper and X is dense in X^c, we have $(f^c)^{-1}(\partial X) = \partial X$. We will use the notation in Example 2.9. Assume that $\operatorname{Fix} f$ is proper over S; thus $\#\operatorname{Fix} f$ can be defined. Put $\partial X^{\mathrm{fix}} := \{x \in \partial X \mid f^c(x) = x\}$. For $x \in \partial X^{\mathrm{fix}}$, we will define "the contribution from x" in the Lefschetz trace formula for X.

Definition 3.24 Put $\Gamma' = \Gamma_f \cup \bigcup_{x\in\partial X}(f^c(x)\times_S x)$. Since x is a closed constructible subset of X^c, the natural map

$$H^2_{\Gamma'}\big(X^c\times_S X^c, j_*\Lambda \overset{\mathbb{L}}{\otimes} j_!\Lambda(1)\big) \longrightarrow H^2_{\Gamma'}\big(X^c\times_S X^c, (j\times 1)_*(1\times j)_!\Lambda(1)\big)$$

is an isomorphism (*cf.* Proposition 3.20). We denote by $[f]_\partial$ the element of $H^2_{\Gamma'}(X^c\times_S X^c, j_*\Lambda \overset{\mathbb{L}}{\otimes} j_!\Lambda(1))$ that is mapped to

$$\mathrm{cl}(\gamma_f)\in H^2_{\Gamma_f}\big(X\times_S X, \Lambda(1)\big) = H^2_{\Gamma'}\big(X^c\times_S X^c, (j\times 1)_*(1\times j)_!\Lambda(1)\big).$$

On the other hand, since $\operatorname{Fix} f$ is proper over S, it is a closed subset of X^c. Therefore we have $\Delta_{X^c}\cap\Gamma' = \operatorname{Fix} f \amalg \coprod_{x\in\partial X^{\mathrm{fix}}} x$ as a topological space, and thus $H^2_{\Delta_{X^c}\cap\Gamma'}(X^c, j_!\Lambda(1)) = H^2_{\operatorname{Fix} f}(X^c, j_!\Lambda(1)) \oplus \bigoplus_{x\in\partial X^{\mathrm{fix}}} H^2_x(X^c, j_!\Lambda(1))$. For $x\in\partial X^{\mathrm{fix}}$, we define $\mathrm{loc}(x)$ as the image of $[f]_\partial$ under the composite of

$$\begin{aligned} H^2_{\Gamma'}\big(X^c\times_S X^c, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(1)\big) &\xrightarrow{(\delta^c)^*} H^2_{\Delta_{X^c}\cap\Gamma'}\big(X^c, j_!\Lambda(1)\big)\\ &= H^2_{\operatorname{Fix} f}\big(X^c, j_!\Lambda(1)\big) \oplus \bigoplus_{x\in\partial X^{\mathrm{fix}}} H^2_x\big(X^c, j_!\Lambda(1)\big) \longrightarrow H^2_x\big(X^c, j_!\Lambda(1)\big)\\ &\longrightarrow H^2\big(X^c, j_!\Lambda(1)\big) = H^2_c\big(X, \Lambda(1)\big) \xrightarrow{\mathrm{Tr}_X} \Lambda, \end{aligned}$$

where δ^c denotes the diagonal morphism for X^c.

The following is our Lefschetz trace formula for adic curves:

Theorem 3.25 *In the setting above, we have*

$$\mathrm{Tr}\big(f^*; R\Gamma_c(X,\Lambda)\big) = \#\operatorname{Fix} f + \sum_{x\in\partial X^{\mathrm{fix}}} \mathrm{loc}(x).$$

Proof. It is easy to see that the image of $[f]_\partial$ under the composite of

$$\begin{aligned} H^2_{\Gamma'}\big(X^c\times_S X^c, j_*\Lambda \overset{\mathbb{L}}{\boxtimes} j_!\Lambda(1)\big) &\xrightarrow{(\delta^c)^*} H^2_{\Delta_{X^c}\cap\Gamma'}\big(X^c, j_!\Lambda(1)\big)\\ &= H^2_{\operatorname{Fix} f}\big(X^c, j_!\Lambda(1)\big) \oplus \bigoplus_{x\in\partial X^{\mathrm{fix}}} H^2_x\big(X^c, j_!\Lambda(1)\big) \longrightarrow H^2_{\operatorname{Fix} f}\big(X^c, j_!\Lambda(1)\big)\\ &\longrightarrow H^2\big(X^c, j_!\Lambda(1)\big) = H^2_c\big(X, \Lambda(1)\big) \xrightarrow{\mathrm{Tr}_X} \Lambda \end{aligned}$$

is equal to $\#\operatorname{Fix} f$ (*cf.* the proof of Lemma 3.22). Therefore, the theorem immediately follows from Proposition 3.9. ∎

Remark 3.26 The formula in Theorem 3.4 is very similar to Huber's trace formula for open curves ([Hub01, Theorem 6.3]). However, the definition of his local term is different from ours; that is given by purely algebraic manner and depends only on the homomorphism induced on the valuation ring corresponding to $x\in\partial X^{\mathrm{fix}}$. The author expects that these two local terms coincide, and will consider this problem in his future work.

4 Lefschetz trace formula for formal schemes

In this section, we deduce a Lefschetz trace formula for formal schemes from Theorem 3.13. For formal schemes, we will use the same notation as in [Mie10b, §4]. Let us recall some of them.

Let R be a complete discrete valuation ring with separably closed residue field and k an algebraic closure of the fraction field F of R. Put $\mathcal{S} = \operatorname{Spf} R$. Let $\mathcal{X}$ be a quasi-compact special formal scheme which is separated over $\mathcal{S}$. Then we can associate $\mathcal{X}$ with the adic spaces $t(\mathcal{X})_a$, $t(\mathcal{X})_\eta$ and $t(\mathcal{X})_{\overline{\eta}}$. The adic space $t(\mathcal{X})_a$ is an open adic subspace of $t(\mathcal{X})$ consisting of analytic points of $t(\mathcal{X})$. It is quasi-compact. The adic space $t(\mathcal{X})_\eta$ is the "generic fiber" of $t(\mathcal{X})$, which is locally of finite type, separated and taut over $\operatorname{Spa}(F,R)$. The adic space $t(\mathcal{X})_{\overline{\eta}}$ is the base change of $t(\mathcal{X})_\eta$ from $\operatorname{Spa}(F,R)$ to $S = \operatorname{Spa}(k,k^+)$. Note that $t(\mathcal{X})_\eta$ and $t(\mathcal{X})_{\overline{\eta}}$ are not necessarily quasi-compact. In the sequel, we write X, X_η and $X_{\overline{\eta}}$ for $t(\mathcal{X})_a$, $t(\mathcal{X})_\eta$ and $t(\mathcal{X})_{\overline{\eta}}$, respectively. On the other hand, we denote the special fiber of $\mathcal{X}$ (resp. X) by $\mathcal{X}_s$ (resp. X_s).

Let $\mathcal{T}$ be a finite set equipped with a partial order and $\{\mathcal{Y}_\alpha\}_{\alpha\in\mathcal{T}}$ a family of closed formal subschemes of $\mathcal{X}_s$ indexed by $\mathcal{T}$. We put $Y_\alpha = t(\mathcal{Y}_\alpha)_a = t(\mathcal{Y}_\alpha)\times_{t(\mathcal{X})} X$. We assume the following:

Assumption 4.1 i) $X_s = \bigcup_{\alpha\in\mathcal{T}} Y_\alpha$.

ii) For $\alpha\in\mathcal{T}$, put $Y_{(\alpha)} = Y_\alpha\backslash\bigcup_{\beta>\alpha} Y_\beta$. Then, for $\alpha,\beta\in\mathcal{T}$ with $\alpha\neq\beta$, $Y_{(\alpha)}\cap Y_{(\beta)} = \varnothing$.

Example 4.2 Let X be a scheme which is separated of finite type over $\operatorname{Spec} R$ and $\{\mathsf{Y}_\alpha\}_{\alpha\in\mathcal{T}}$ a family of closed subschemes of the special fiber X_s of X. Assume the following conditions:

i) $\mathsf{X}_s = \bigcup_{\alpha\in\mathcal{T}} \mathsf{Y}_\alpha$.

ii) For $\alpha\in\mathcal{T}$, put $\mathsf{Y}_{(\alpha)} = \mathsf{Y}_\alpha\setminus\bigcup_{\beta>\alpha}\mathsf{Y}_\beta$. Then, for $\alpha,\beta\in\mathcal{T}$ with $\alpha\neq\beta$, $\mathsf{Y}_{(\alpha)}\cap\mathsf{Y}_{(\beta)} = \varnothing$.

iii) There exists the unique maximal element α_0 in $\mathcal{T}$.

Denote the completion of X along Y_{α_0} by $\mathcal{X}$ and put $\mathcal{Y}_\alpha = \mathsf{Y}_\alpha\times_{\mathsf{X}}\mathcal{X}$ for each $\alpha\in\mathcal{T}$. Then $\mathcal{X}$ and $\{\mathcal{Y}_\alpha\}_{\alpha\in\mathcal{T}\setminus\{\alpha_0\}}$ satisfy the assumption above. Indeed, we have the natural morphism of locally ringed spaces $(X,\mathcal{O}_X)\longrightarrow \mathsf{X}\backslash\mathsf{Y}_{\alpha_0}$ (*cf.* [Hub94, Remark 4.6 (iv)]) such that the inverse image of $\mathsf{Y}_\alpha\setminus\mathsf{Y}_{\alpha_0}$ is equal to Y_α.

Let us consider an isomorphism $f\colon \mathcal{X}\xrightarrow{\cong}\mathcal{X}$ over $\mathcal{S}$. We also denote the induced isomorphism $X\xrightarrow{\cong}X$ by the same symbol f. The induced isomorphisms $X_\eta\xrightarrow{\cong}X_\eta$ and $X_{\overline{\eta}}\xrightarrow{\cong}X_{\overline{\eta}}$ are denoted by f_η and $f_{\overline{\eta}}$, respectively. We will make the following assumption on f:

Assumption 4.3 There exists an order-preserving bijection $f\colon \mathcal{T} \xrightarrow{\cong} \mathcal{T}$ and a system of constructible open (resp. closed) subsets $\{Y_\alpha(n)\}_{n\ge 1}$ (resp. $\{Y^\circ_\alpha(n)\}_{n\ge 1}$) of X for each $\alpha \in \mathcal{T}$ satisfying the following:

i) $Y_\alpha(n+1) \subset Y^\circ_\alpha(n) \subset Y_\alpha(n)$ for every $n \ge 1$.

ii) $\bigcap_{n\ge 1} Y_\alpha(n) = Y_\alpha$.

iii) $f(Y_\alpha(n)) = Y_{f(\alpha)}(n)$ and $f(Y^\circ_\alpha(n)) = Y^\circ_{f(\alpha)}(n)$ for every $\alpha \in \mathcal{T}$ and $n \ge 1$.

iv) $f(\alpha) \ne \alpha$ for every $\alpha \in \mathcal{T}$.

Remark 4.4 Later we will give some conditions for the existence of $\{Y_\alpha(n)\}_{n\ge 1}$ and $\{Y^\circ_\alpha(n)\}_{n\ge 1}$ in Assumption 4.3. In fact, one of the following is sufficient (Proposition 4.18, Proposition 4.19):

- For every $\alpha \in \mathcal{T}$, the isomorphism $f\colon \mathcal{X} \xrightarrow{\cong} \mathcal{X}$ induces an isomorphism of formal schemes $\mathcal{Y}_\alpha \xrightarrow{\cong} \mathcal{Y}_{f(\alpha)}$.
- For every $\alpha \in \mathcal{T}$, the isomorphism $f\colon \mathcal{X} \xrightarrow{\cong} \mathcal{X}$ induces a set-theoretic bijection $Y_\alpha \xrightarrow{\cong} Y_{f(\alpha)}$. Furthermore, for every ideal of definition $\mathcal{I}$ of $\mathcal{X}$, there exists an integer $N \ge 1$ such that $f^N \equiv \mathrm{id} \pmod{\mathcal{I}}$.

Now we can state our Lefschetz trace formula for $\mathcal{X}$:

Theorem 4.5 *In addition to Assumption 4.1 and Assumption 4.3, assume that $\mathcal{X}$ is locally algebraizable ([Mie10b, Definition 3.18]) and X_η is partially proper and smooth over* $\mathrm{Spa}(F,R)$. *Then $R\Gamma_c(X_{\overline{\eta}},\Lambda)$ is a perfect Λ-complex,* $\mathrm{Fix}\, f_{\overline{\eta}}$ *(cf. Example 2.9) is proper over S and we have*

$$\mathrm{Tr}\bigl(f^*_{\overline{\eta}}; R\Gamma_c(X_{\overline{\eta}},\Lambda)\bigr) = \#\,\mathrm{Fix}\, f_{\overline{\eta}}.$$

In order to prove this theorem, we need some preparations. First we observe the finiteness of the cohomology of $X_{\overline{\eta}}$.

Proposition 4.6 *Let $\mathcal{X}$ be a quasi-compact special formal scheme which is locally algebraizable and separated over $\mathcal{S}$. Then $H^i_c(X_{\overline{\eta}},\Lambda)$ is a finitely generated Λ-module for every i. Thus $R\Gamma_c(X_{\overline{\eta}},\Lambda)$ is a perfect Λ-complex by Corollary 3.3. More generally, let L be a locally closed constructible subset of X. Then $H^i_c(L_{\overline{\eta}},\Lambda)$ is a finitely generated Λ-module for every i, and $R\Gamma_c(L_{\overline{\eta}},\Lambda)$ is a perfect Λ-complex.*

Proof. We way assume that $\mathcal{X}$ is algebraizable. First we consider the case where L is a quasi-compact open subset of X. Then there exists an admissible blow-up $\mathcal{X}' \longrightarrow \mathcal{X}$ and an open formal subscheme $\mathcal{U}' \subset \mathcal{X}'$ such that $L = t(\mathcal{U}')_a$. Since $\mathcal{U}'$ is algebraizable (*cf.* [Mie07, Lemma 7.1.4]), replacing $\mathcal{X}$ by $\mathcal{U}'$, we may assume that $L = X$. Moreover we may assume that $\mathcal{X}$ is affine, and thus pseudo-compactifiable (*cf.* [Mie10b, Definition 4.21 i), Example 4.22 i)]). Therefore, by [Mie10b, Theorem

4.32], we have $H^i_c(X_{\overline{\eta}},\Lambda)\cong H^i_c(\mathcal{X}_{\mathrm{red}},R\Psi_{\mathcal{X},c}\Lambda)$. On the other hand, by [Mie10b, Proposition 3.20], $R\Psi_{\mathcal{X},c}\Lambda$ is constructible, and thus $H^i_c(\mathcal{X}_{\mathrm{red}},R\Psi_{\mathcal{X},c}\Lambda)$ is a finitely generated Λ-module. Hence $H^i_c(X_{\overline{\eta}},\Lambda)$ is also finitely generated, which concludes the proof for a quasi-compact open L.

A general L can be expressed as $L_2\setminus L_1$, where L_1 and L_2 are quasi-compact open subsets of X with $L_1\subset L_2$. Thus the proposition follows from the exact sequence

$$H^i_c(L_{2,\overline{\eta}},\Lambda)\longrightarrow H^i_c\bigl((L_2\setminus L_1)_{\overline{\eta}},\Lambda\bigr)\longrightarrow H^{i+1}_c(L_{1,\overline{\eta}},\Lambda).$$ ∎

Now we use the notation in Assumption 4.3.

Lemma 4.7 *For every $\alpha\in\mathcal{T}$, $\{Y_\alpha(n)\}_{n\geq 1}$ (or $\{Y^\circ_\alpha(n)\}_{n\geq 1}$) form a fundamental system of open neighborhoods of Y_α with respect to the patch topology of X.*

Proof. Let U be a subset of X containing Y_α, which is open in the patch topology. We will find $n\geq 1$ such that $Y_\alpha(n)\subset U$. By Assumption 4.3 ii), $X\setminus U$ is covered by $\{X\setminus Y_\alpha(n)\}_{n\geq 1}$. Since $X\setminus U$ is compact and $X\setminus Y_\alpha(n)$ is an open subset of X with respect to the patch topology, there exists an integer $n\geq 1$ such that $X\setminus U\subset X\setminus Y_\alpha(n)$. In other words, $Y_\alpha(n)$ is contained in U. ∎

The following construction is crucial for the proof of Theorem 4.5:

Lemma 4.8 *We can find an integer $n_\alpha\geq 1$ for each $\alpha\in\mathcal{T}$ satisfying the following conditions:*

- *For every $\alpha\in\mathcal{T}$, $n_\alpha=n_{f(\alpha)}$.*
- *For $\alpha\in\mathcal{T}$, put $U_\alpha=Y^\circ_\alpha(n_\alpha)\setminus\bigcup_{\beta>\alpha}Y^\circ_\beta(n_\beta)$ and $W_\alpha=Y^\circ_\alpha(n_\alpha)\setminus\bigcup_{\beta>\alpha}Y_\beta(n_\beta+1)$. Then we have $U_\alpha\cap U_\beta=\varnothing$ for every $\alpha,\beta\in\mathcal{T}$ with $\alpha\neq\beta$, and $W_\alpha\cap W_{f(\alpha)}=\varnothing$ for every $\alpha\in\mathcal{T}$.*

By the assumption, $Y^\circ(n)$ is an open subset of X with respect to the patch topology. Therefore, the lemma is reduced to the following:

Lemma 4.9 *Let X be a compact topological space. Let $\mathcal{T}$ be a finite set equipped with a partial order and $\{Y_\alpha\}_{\alpha\in\mathcal{T}}$ a family of closed subsets of X indexed by $\mathcal{T}$. Put $Y_{(\alpha)}=Y_\alpha\setminus\bigcup_{\beta>\alpha}Y_\beta$ and assume that $Y_{(\alpha)}\cap Y_{(\beta)}=\varnothing$ if $\alpha\neq\beta$. Let $f\colon\mathcal{T}\xrightarrow{\cong}\mathcal{T}$ be an order-preserving bijection such that $f(\alpha)\neq\alpha$ for every $\alpha\in\mathcal{T}$. Assume that we are given a fundamental system of open neighborhoods $\{Y_\alpha(n)\}_{n\geq 1}$ of Y_α for each $\alpha\in\mathcal{T}$ such that $Y_\alpha(n+1)\subset Y_\alpha(n)$ for every $n\geq 1$. Then, for every integer $N\geq 1$ we can find an integer $n_\alpha\geq N$ for each $\alpha\in\mathcal{T}$ satisfying the following conditions:*

- *For every $\alpha\in\mathcal{T}$, $n_\alpha=n_{f(\alpha)}$.*
- *For $\alpha\in\mathcal{T}$, put $U_\alpha=Y_\alpha(n_\alpha)\setminus\bigcup_{\beta>\alpha}Y_\beta(n_\beta)$ and $V_\alpha=Y_\alpha(n_\alpha)\setminus\bigcup_{\beta>\alpha}Y_\beta(n_\beta+1)$. Then we have $U_\alpha\cap U_\beta=\varnothing$ for every $\alpha,\beta\in\mathcal{T}$ with $\alpha\neq\beta$, and $V_\alpha\cap V_{f(\alpha)}=\varnothing$ for every $\alpha\in\mathcal{T}$.*

Proof. Use the induction on the cardinality of $\mathcal{T}$. If $\mathcal{T}$ is empty, then the lemma is clear. Assume that $\mathcal{T}$ is non-empty. Take a maximal element α_0 of $\mathcal{T}$ and put $\mathcal{T}_0 = \{f^m(\alpha_0) \mid m \in \mathbb{Z}\}$. Note that every element of $\mathcal{T}_0$ is maximal.

Consider an element (α, β) of $\mathcal{T}_0 \times \mathcal{T}$ such that $\beta \not\leq \alpha$. As Y_β is contained in $\coprod_{\gamma \geq \beta} Y_{(\gamma)}$, $Y_\alpha = Y_{(\alpha)}$ does not intersect Y_β by the assumption. Since X is compact, two closed subsets Y_α and Y_β can be separated by open neighborhoods. Thus we can find an integer $n \geq N$ such that $Y_\alpha(n) \cap Y_\beta(n) = \varnothing$. Since there are only finitely many such elements (α, β), we can take $n \geq N$ such that $Y_\alpha(n) \cap Y_\beta(n) = \varnothing$ for every $\alpha \in \mathcal{T}_0$ and $\beta \in \mathcal{T}$ with $\beta \not\leq \alpha$. Put $n_\alpha = n$ for every $\alpha \in \mathcal{T}_0$.

Put $X' = X \setminus \bigcup_{\alpha \in \mathcal{T}_0} Y_\alpha(n_\alpha + 1)$, $\mathcal{T}' = \mathcal{T} \setminus \mathcal{T}_0$, $Y'_\beta = X' \cap Y_\beta$ and $Y'_\beta(n) = X' \cap Y'_\beta(n)$ for $\beta \in \mathcal{T}'$. Note that f induces an order-preserving bijection $\mathcal{T}' \xrightarrow{\cong} \mathcal{T}'$, which is also denoted by f. Then, these satisfy the assumptions in the lemma. Therefore, by the induction hypothesis, we can find $n_\alpha \geq n_{\alpha_0}$ for each $\alpha \in \mathcal{T}'$. We will observe that $\{n_\alpha\}_{\alpha \in \mathcal{T}}$ satisfies the conditions in the lemma. The first condition $n_\alpha = n_{f(\alpha)}$ is clear from the construction. For the second condition, put $U'_\alpha = Y'_\alpha(n_\alpha) \setminus \bigcup_{\beta \in \mathcal{T}', \beta > \alpha} Y'_\beta(n_\beta)$ and $V'_\alpha = Y'_\alpha(n_\alpha) \setminus \bigcup_{\beta \in \mathcal{T}', \beta > \alpha} Y'_\beta(n_\beta + 1)$ for $\alpha \in \mathcal{T}'$. Then,

$$\begin{aligned} U'_\alpha \setminus \bigcup_{\gamma \in \mathcal{T}_0} Y_\gamma(n_\gamma) &= \Big(Y_\alpha(n_\alpha) \setminus \bigcup_{\beta \in \mathcal{T}', \beta > \alpha} Y_\beta(n_\beta)\Big) \setminus \bigcup_{\gamma \in \mathcal{T}_0} Y_\gamma(n_\gamma) \\ &= \Big(Y_\alpha(n_\alpha) \setminus \bigcup_{\beta \in \mathcal{T}', \beta > \alpha} Y_\beta(n_\beta)\Big) \setminus \bigcup_{\gamma \in \mathcal{T}_0, \gamma \geq \alpha} Y_\gamma(n_\gamma) \\ &= Y_\alpha(n_\alpha) \setminus \bigcup_{\beta \in \mathcal{T}, \beta > \alpha} Y_\beta(n_\beta) = U_\alpha. \end{aligned}$$

The second equality follows from $Y_\alpha(n_\alpha) \cap Y_\gamma(n_\gamma) \subset Y_\alpha(n_{\alpha_0}) \cap Y_\gamma(n_{\alpha_0}) = \varnothing$ for $\gamma \in \mathcal{T}_0$ with $\gamma \not\geq \alpha$. Similarly, we can check that $V'_\alpha = V_\alpha$ for $\alpha \in \mathcal{T}'$.

Let us take $\alpha, \beta \in \mathcal{T}$ with $\alpha \neq \beta$ and prove $U_\alpha \cap U_\beta = \varnothing$. If $\alpha, \beta \in \mathcal{T}_0$, $U_\alpha \cap U_\beta = Y_\alpha(n_{\alpha_0}) \cap Y_\beta(n_{\alpha_0}) = \varnothing$ since $\beta \not\geq \alpha$. If $\alpha \in \mathcal{T}_0$ and $\beta \in \mathcal{T}'$, then $U_\alpha \cap U_\beta = Y_\alpha(n_\alpha) \cap (U'_\beta \setminus \bigcup_{\gamma \in \mathcal{T}_0} Y_\gamma(n_\gamma)) = \varnothing$. The case where $\alpha \in \mathcal{T}'$ and $\beta \in \mathcal{T}_0$ is similar. Finally if $\alpha, \beta \in \mathcal{T}'$, then $U_\alpha \cap U_\beta \subset U'_\alpha \cap U'_\beta = \varnothing$ by the induction hypothesis.

Let us take $\alpha \in \mathcal{T}$ and prove $V_\alpha \cap V_{f(\alpha)} = \varnothing$. If $\alpha \in \mathcal{T}_0$ then $V_\alpha \cap V_{f(\alpha)} = Y_\alpha(n_{\alpha_0}) \cap Y_{f(\alpha)}(n_{\alpha_0}) = \varnothing$ since α and $f(\alpha)$ are disjoint elements in $\mathcal{T}_0$. If $\alpha \in \mathcal{T}'$ then $V_\alpha \cap V_{f(\alpha)} = V'_\alpha \cap V'_{f(\alpha)} = \varnothing$ by the induction hypothesis.

Now the proof is complete. ∎

Fix $\{n_\alpha\}_{\alpha \in \mathcal{T}}$ as in Lemma 4.8 and put $W = \bigcup_{\alpha \in \mathcal{T}} Y^\circ_\alpha(n_\alpha)$, $X_0 = X \setminus W$. By Assumption 4.3 iii), we have $f(W) = W$ and $f(X_0) = X_0$.

Proposition 4.10 *We have* $\mathrm{Tr}(f^*_{\overline{\eta}}; R\Gamma_c(X_{\overline{\eta}}, \Lambda)) = \mathrm{Tr}(f^*_{\overline{\eta}}; R\Gamma_c(X_{0,\overline{\eta}}, \Lambda))$.

Lemma 4.11 *Let L, L' be locally closed constructible subsets of X such that $f(L) = L$, $f(L') = L'$ and L' is an open subset of L. Put $L'' = L \setminus L'$. Then* *we have*

$$\mathrm{Tr}\big(f_{\overline{\eta}}^*; R\Gamma_c(L_{\overline{\eta}},\Lambda)\big)=\mathrm{Tr}\big(f_{\overline{\eta}}^*; R\Gamma_c(L'_{\overline{\eta}},\Lambda)\big)+\mathrm{Tr}\big(f_{\overline{\eta}}^*; R\Gamma_c(L''_{\overline{\eta}},\Lambda)\big).$$

Proof. First note that $R\Gamma_c(L_{\overline{\eta}},\Lambda)$, $R\Gamma_c(L'_{\overline{\eta}},\Lambda)$ and $R\Gamma_c(L''_{\overline{\eta}},\Lambda)$ are perfect Λ-complexes by Proposition 4.6, and the traces make sense.

Let $j\colon (X_{\overline{\eta}},L'_{\overline{\eta}})\longrightarrow (X_{\overline{\eta}},L_{\overline{\eta}})$ and $i\colon (X_{\overline{\eta}},L''_{\overline{\eta}})\longrightarrow (X_{\overline{\eta}},L_{\overline{\eta}})$ be the natural immersions of pseudo-adic spaces. Consider the filtered sheaf $\mathcal{F}=(F_1\mathcal{F}=j_!\Lambda\subset\Lambda=F_0\mathcal{F})$. We have a natural morphism $f_{\overline{\eta}}^*\mathcal{F}\longrightarrow\mathcal{F}$ of filtered sheaves, which induces a morphism $f_{\overline{\eta}}^*\colon R\Gamma_c(L_{\overline{\eta}},\mathcal{F})\longrightarrow R\Gamma_c(L_{\overline{\eta}},f_{\overline{\eta}}^*\mathcal{F})\longrightarrow R\Gamma_c(L_{\overline{\eta}},\mathcal{F})$ in the filtered derived category of Λ-modules (*cf.* [Ill71, Chapitre V]). It is easy to see that the morphism on gr^0 (resp. gr^1) induced by $f_{\overline{\eta}}^*$ coincides with the pull-back map $f_{\overline{\eta}}^*$ on $R\Gamma_c(L_{\overline{\eta}},i_*\Lambda)=R\Gamma_c(L''_{\overline{\eta}},\Lambda)$ (resp. $R\Gamma_c(L_{\overline{\eta}},j_!\Lambda)=R\Gamma_c(L'_{\overline{\eta}},\Lambda)$). Therefore the equality follows from [Ill71, Corollaire 3.7.7, Remarque 3.7.7.1]. ∎

Proof of Proposition 4.10. By Lemma 4.11, it suffices to show $\mathrm{Tr}(f_{\overline{\eta}}^*; R\Gamma_c(W_{\overline{\eta}},\Lambda))=0$. Take a maximal element α_0 of $\mathcal{T}$ and put $\mathcal{T}_0=\{f^m(\alpha_0)\mid m\in\mathbb{Z}\}$, $W_0=\bigcup_{\alpha\in\mathcal{T}_0}Y_\alpha^\circ(n_\alpha)=\coprod_{\alpha\in\mathcal{T}_0}U_\alpha$. Obviously $R\Gamma_c(W_{0,\overline{\eta}},\Lambda)=\bigoplus_{\alpha\in\mathcal{T}_0}R\Gamma_c(U_{\alpha,\overline{\eta}},\Lambda)$ and $f(W_0)=W_0$. Since $f(U_\alpha)=U_{f(\alpha)}$ and $f(\alpha)\neq\alpha$ by Assumption 4.3 iv), it is immediate to see $\mathrm{Tr}(f_{\overline{\eta}}^*; R\Gamma_c(W_{0,\overline{\eta}},\Lambda))=0$.

Put $W'=W\backslash W_0$ and $\mathcal{T}'=\mathcal{T}\backslash\mathcal{T}_0$. Then $\mathrm{Tr}(f_{\overline{\eta}}^*; R\Gamma_c(W_{\overline{\eta}},\Lambda))=\mathrm{Tr}(f_{\overline{\eta}}^*; R\Gamma_c(W'_{\overline{\eta}},\Lambda))$ by Lemma 4.11. If $\mathcal{T}'$ is non-empty, take a maximal element α_1 of $\mathcal{T}'$ and put $\mathcal{T}_1=\{f^m(\alpha_1)\mid m\in\mathbb{Z}\}$, $W_1=\bigcup_{\alpha\in\mathcal{T}_1}Y_\alpha^\circ(n_\alpha)\setminus W_0=\coprod_{\alpha\in\mathcal{T}_1}U_\alpha$. In the same way as above, we can prove that $\mathrm{Tr}(f_{\overline{\eta}}^*; R\Gamma_c(W_{1,\overline{\eta}},\Lambda))=0$. Put $W''=W'\setminus W_1$. Then $\mathrm{Tr}(f_{\overline{\eta}}^*; R\Gamma_c(W'_{\overline{\eta}},\Lambda))=\mathrm{Tr}(f_{\overline{\eta}}^*; R\Gamma_c(W''_{\overline{\eta}},\Lambda))$ by Lemma 4.11 (note that W_1 is closed in W'). We repeat this procedure to obtain $\mathrm{Tr}(f_{\overline{\eta}}^*; R\Gamma_c(W_{\overline{\eta}},\Lambda))=0$. ∎

Next lemma ensures that we may apply Theorem 3.13 to $X_{0,\overline{\eta}}$.

Lemma 4.12 i) *The adic space X_0 is a quasi-compact open adic subspace of X_η. In particular, $X_{0,\overline{\eta}}$ is smooth, separated of finite type over S.*

ii) *For $x\in X_{\overline{\eta}}\setminus X_{0,\overline{\eta}}$, there exists a closed constructible subset W_x of $X_{\overline{\eta}}$ such that $f_{\overline{\eta}}(W_x)\cap W_x=\varnothing$.*

iii) *The closed adic subspace $\mathrm{Fix}\, f_{\overline{\eta}}$ of $X_{\overline{\eta}}$ is contained in $X_{0,\overline{\eta}}$. In particular, $\mathrm{Fix}\, f_{\overline{\eta}}=\mathrm{Fix}(f_{\overline{\eta}}|_{X_{0,\overline{\eta}}})$ is proper over S and $\#\,\mathrm{Fix}\, f_{\overline{\eta}}=\#\,\mathrm{Fix}(f_{\overline{\eta}}|_{X_{0,\overline{\eta}}})$.*

Proof. i) Since X is a spectral space and W is a closed constructible subset of X, X_0 is a quasi-compact open subset of X. On the other hand, $X_0\subset X\setminus\bigcup_{\alpha\in\mathcal{T}}Y_\alpha=X_\eta$ by Assumption 4.1 i). Thus X_0 is a quasi-compact open subset of X_η.
ii) As $x\in W_{\overline{\eta}}=\coprod_{\alpha\in\mathcal{T}}U_{\alpha,\overline{\eta}}\subset\bigcup_{\alpha\in\mathcal{T}}W_{\alpha,\overline{\eta}}$, there exists $\alpha\in\mathcal{T}$ such that $x\in W_{\alpha,\overline{\eta}}$. By Assumption 4.3 iii), we have $W_{\alpha,\overline{\eta}}\cap f_{\overline{\eta}}(W_{\alpha,\overline{\eta}})=W_{\alpha,\overline{\eta}}\cap W_{f(\alpha),\overline{\eta}}=\varnothing$ (we use the second condition in Lemma 4.8). Since W_α is a closed constructible subset of X, $W_{\alpha,\overline{\eta}}$ is a closed constructible subset of $X_{\overline{\eta}}$.
iii) Clear from ii) and Proposition 2.10. ∎

Proof of Theorem 4.5. Since $X_{\overline{\eta}}$ is partially proper and taut over S by the assumption, the closure $\overline{X_{0,\overline{\eta}}}$ of $X_{0,\overline{\eta}}$ in $X_{\overline{\eta}}$ is proper over S. By Lemma 4.12 i), ii), we can apply Theorem 3.13 to $X_{0,\overline{\eta}} \hookrightarrow \overline{X_{0,\overline{\eta}}}$. Together with Proposition 4.10 and Lemma 4.12 iii), we can conclude

$$\mathrm{Tr}\bigl(f_{\overline{\eta}}^*; R\Gamma_c(X_{\overline{\eta}},\Lambda)\bigr) = \mathrm{Tr}\bigl(f_{\overline{\eta}}^*|_{X_{0,\overline{\eta}}}; R\Gamma_c(X_{0,\overline{\eta}},\Lambda)\bigr) = \#\,\mathrm{Fix}(f_{\overline{\eta}}|_{X_{0,\overline{\eta}}}) = \#\,\mathrm{Fix}\, f_{\overline{\eta}}. \qquad \blacksquare$$

Remark 4.13 At least when the characteristic of k is 0, we can deduce from Theorem 4.5 the analogous result for ℓ-adic coefficient simply by taking projective limit (*cf.* [Mie10b, proof of Corollary 4.40]).

Next we discuss the existence of systems of neighborhoods in Assumption 4.3. Let $f\colon \mathcal{X} \xrightarrow{\cong} \mathcal{X}$ and $\{\mathcal{Y}_\alpha\}_{\alpha\in\mathcal{T}}$ be as in the beginning of this section, and $f\colon \mathcal{T} \xrightarrow{\cong} \mathcal{T}$ a bijection (we do not need to take the order on $\mathcal{T}$ into account). We want to find a system of open constructible subsets $\{Y_\alpha(n)\}_{n\geq 1}$ and that of closed constructible subsets $\{Y_\alpha^\circ(n)\}_{n\geq 1}$ satisfying i), ii), iii) in Assumption 4.3. To construct them, we introduce a "tubular neighborhood" of a closed formal subscheme $\mathcal{Y}$ of $\mathcal{X}$.

Definition 4.14 Let $\mathcal{Y}$ be a closed formal subscheme of $\mathcal{X}$ and $\mathcal{I}$ be an ideal of definition of $\mathcal{X}$. We will define the subsets $Y(\mathcal{I})$ and $Y^\circ(\mathcal{I})$ of $X = t(\mathcal{X})_a$ as follows. First assume that $\mathcal{X} = \mathrm{Spf}\, A$ is affine. Then $\mathcal{Y}$ is defined by an ideal J of A. Put $I = \Gamma(\mathcal{X},\mathcal{I})$ and

$$Y(\mathcal{I}) = \bigl\{x \in X \bigm| \max_{f\in J}|f(x)| \leq \max_{g\in I}|g(x)|\bigr\},$$
$$Y^\circ(\mathcal{I}) = \bigl\{x \in X \bigm| \max_{f\in J}|f(x)| < \max_{g\in I}|g(x)|\bigr\}.$$

Note that $\max_{f\in J}|f(x)| = \max_{1\leq i\leq m}|f_i(x)|$ for every system of generators $f_1,\ldots,f_m$ of J; in particular $\max_{f\in J}|f(x)|$ exists. Similar for $\max_{g\in I}|g(x)|$.

Obviously we can globalize the construction by patching, and get $Y(\mathcal{I})$ and $Y^\circ(\mathcal{I})$ for the general case.

Proposition 4.15 *The subset $Y(\mathcal{I})$ is open in X and $Y^\circ(\mathcal{I})$ is closed in X. These are constructible subsets in X.*

Proof. We may assume that $\mathcal{X} = \mathrm{Spf}\, A$ is affine. Let $J \subset A$ be the defining ideal of $\mathcal{Y}$ and put $I = \Gamma(\mathcal{X},\mathcal{I})$. Take a system of generators $f_1,\ldots,f_m$ (resp. $g_1,\ldots,g_n$) of J (resp. I). Then, noting that $X = \{x \in t(\mathcal{X}) \mid \max_{1\leq j\leq n}|g_j(x)| \neq 0\}$, we have

$$Y(\mathcal{I}) = \bigcup_{1\leq j\leq n} R\Bigl(\frac{f_1,\ldots,f_m,g_1,\ldots,g_n}{g_j}\Bigr), \quad X \setminus Y^\circ(\mathcal{I}) = X \cap \bigcup_{1\leq i\leq m} R\Bigl(\frac{g_1,\ldots,g_n}{f_i}\Bigr),$$

where $R(-)$ denotes a rational subset of $t(\mathcal{X}) = \mathrm{Spa}(A,A)$. Since every rational subset is quasi-compact and open, $Y(\mathcal{I})$ and $X \setminus Y^\circ(\mathcal{I})$ are quasi-compact open subsets of X. This completes the proof. $\blacksquare$

The following two lemmas are clear from the definition:

Lemma 4.16 *For an ideal of definition $\mathcal{I}$ of $\mathcal{X}$, we have the following:*

i) $Y(\mathcal{I}^{n+1}) \subset Y^\circ(\mathcal{I}^n) \subset Y(\mathcal{I}^n)$ *for every* $n \geq 1$.

ii) $Y := t(\mathcal{Y})_a = \bigcap_{n\geq 1} Y(\mathcal{I}^n)$.

Lemma 4.17 *Let $f\colon \mathcal{X}' \longrightarrow \mathcal{X}$ be an adic morphism over $\mathcal{S}$ and put $\mathcal{Y}' = \mathcal{Y}\times_{\mathcal{X}}\mathcal{X}'$. For an ideal of definition $\mathcal{I}$ of $\mathcal{X}$, put $\mathcal{I}' = (f^{-1}\mathcal{I})\mathcal{O}_{\mathcal{X}'}$. Then $f\colon X' := t(\mathcal{X}')_a \longrightarrow X$ induces a map from $Y'(\mathcal{I}')$ to $Y(\mathcal{I})$.*

Now we can give fairly simple conditions for existence of systems of neighborhoods in Assumption 4.3.

Proposition 4.18 *Assume that the isomorphism $f\colon \mathcal{X} \xrightarrow{\cong} \mathcal{X}$ induces an isomorphism of formal schemes $\mathcal{Y}_\alpha \xrightarrow{\cong} \mathcal{Y}_{f(\alpha)}$ for every $\alpha \in \mathcal{T}$. Then there exists a system of constructible open (resp. closed) subsets $\{Y_\alpha(n)\}_{n\geq 1}$ (resp. $\{Y^\circ_\alpha(n)\}_{n\geq 1}$) of X satisfying i), ii), iii) in Assumption 4.3.*

Proof. Let $\mathcal{I}$ be the maximal ideal of definition of $\mathcal{X}$ (it exists since $\mathcal{X}$ is noetherian) and put $Y_\alpha(n) = Y_\alpha(\mathcal{I}^n)$, $Y^\circ_\alpha(n) = Y^\circ_\alpha(\mathcal{I}^n)$ for each $\alpha \in \mathcal{T}$ and $n \geq 1$. By Proposition 4.15, $Y_\alpha(n)$ (resp. $Y^\circ_\alpha(n)$) is a constructible open (resp. closed) subset of X. Moreover, by Lemma 4.16, $\{Y_\alpha(n)\}_{n\geq 1}$ and $\{Y^\circ_\alpha(n)\}_{n\geq 1}$ satisfy the conditions i), ii) in Assumption 4.3. Finally, the condition iii) follows from Lemma 4.17, since $\mathcal{I}$ is preserved by the isomorphism f. ∎

Proposition 4.19 *Assume that the isomorphism $f\colon \mathcal{X} \xrightarrow{\cong} \mathcal{X}$ induces a set-theoretic bijection $Y_\alpha \xrightarrow{\cong} Y_{f(\alpha)}$ for every $\alpha \in \mathcal{T}$. Assume moreover that for every ideal of definition $\mathcal{I}$ of $\mathcal{X}$, there exists an integer $N \geq 1$ such that $f^N \equiv \mathrm{id} \pmod{\mathcal{I}}$. Then there exists a system of constructible open (resp. closed) subsets $\{Y_\alpha(n)\}_{n\geq 1}$ (resp. $\{Y^\circ_\alpha(n)\}_{n\geq 1}$) of X satisfying i), ii), iii) in Assumption 4.3.*

First we will show:

Lemma 4.20 *Assume that the isomorphism $f\colon \mathcal{X} \xrightarrow{\cong} \mathcal{X}$ satisfies the latter condition in Proposition 4.19. Then, for every constructible subset V of X, there exists an integer $N \geq 1$ such that $f^N(V) = V$.*

Proof. Replacing f by its power if necessary, we may assume that f induces the identity on the underlying space of $\mathcal{X}$. Therefore we may assume that $\mathcal{X}$ is affine. Moreover, we can reduce to the case where V is a rational subset of $t(\mathcal{X})$. Now the lemma is clear from [Hub93, Lemma 3.10]. ∎

Proof Proposition 4.19. Take an ideal of definition $\mathcal{I}$ of $\mathcal{X}$.

Let us decompose $\mathcal{T}$ into f-orbits $\mathcal{T}_1 \amalg \cdots \amalg \mathcal{T}_m$. Fix an element $\alpha_i \in \mathcal{T}_i$ for each $1 \le i \le m$; then $\mathcal{T}_i = \{f^j(\alpha_i) \mid j \in \mathbb{Z}\}$. For $\alpha \in \mathcal{T}_i$ and an integer $n \ge 1$, put $Y_\alpha(n) = \bigcap_{j\in\mathbb{Z}, f^j(\alpha_i)=\alpha} f^j(Y_{\alpha_i}(\mathcal{I}^n))$ and $Y_\alpha^\circ(n) = \bigcap_{j\in\mathbb{Z}, f^j(\alpha_i)=\alpha} f^j(Y^\circ_{\alpha_i}(\mathcal{I}^n))$. By the previous lemma, these intersections are essentially finite. Therefore, $Y_\alpha(n)$ (resp. $Y^\circ_\alpha(n)$) is a constructible open (resp. closed) subset of X. By the assumption $f(Y_\alpha) = Y_{f(\alpha)}$ and Lemma 4.16, $\{Y_\alpha(n)\}_{n\ge1}$ and $\{Y^\circ_\alpha(n)\}_{n\ge1}$ satisfy the condition i), ii) in Assumption 4.3. On the other hand, $f(Y_\alpha(n)) = Y_{f(\alpha)}(n)$ and $f(Y^\circ_\alpha(n)) = Y^\circ_{f(\alpha)}(n)$ are clear from the construction. Now the proof is complete. ∎

We finish this section by two examples of Rapoport-Zink spaces.

Example 4.21 Let $\mathcal{O}$ be a complete discrete valuation ring with finite residue field $\mathbb{F}_q$. Denote the completion of the strict henselization of $\mathcal{O}$ by $\breve{\mathcal{O}}$ and the fraction field of $\breve{\mathcal{O}}$ by $\breve{F}$. Fix an integer $d \ge 1$ and denote by $\mathbb{X}$ a formal $\mathcal{O}$-module over $\overline{\mathbb{F}}_q$ with $\mathcal{O}$-height d (such $\mathbb{X}$ is unique up to isomorphism). For an integer $m \ge 0$, $\mathcal{X}_m$ denotes the universal deformation space over $\breve{\mathcal{O}}$ of $\mathbb{X}$ with Drinfeld m-level structures. For the precise definition, see [Str08, §2.1] for example. Recall that $\mathcal{X}_0$ is isomorphic to $\operatorname{Spf} \breve{\mathcal{O}}[[T_1, \ldots, T_{d-1}]]$ and the natural morphism $\mathcal{X}_m \longrightarrow \mathcal{X}_0$ is finite. In particular, $\mathcal{X}_m$ is special over $\operatorname{Spf} \breve{\mathcal{O}}$ and its generic fiber $X_m = t(\mathcal{X}_m)_\eta$ is partially proper over $\operatorname{Spa}(\breve{F}, \breve{\mathcal{O}})$. Moreover, it is known that the morphism $X_m \longrightarrow X_0$ induced on the generic fibers is étale. Therefore X_m is smooth over $\operatorname{Spa}(\breve{F}, \breve{\mathcal{O}})$.

More generally, we can associate to a compact open subgroup K of $K_0 = \mathrm{GL}_d(\mathcal{O})$ the formal scheme $\mathcal{X}_K$ (*cf.* [Str08, §2.2]). Put $K_m = \operatorname{Ker}(\mathrm{GL}_d(\mathcal{O}) \to \mathrm{GL}_d(\mathcal{O}/\mathfrak{m}^m))$ for an integer $m \ge 1$, where $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}$. Take $m \ge 0$ such that $K_m \subset K$; then $\mathcal{X}_K$ is defined as the quotient in the sense of invariant theory of the action of the finite group K/K_m on $\mathcal{X}_m$ (the action of $K \subset K_0$ on $\mathcal{X}_m$ is given via the Drinfeld level structures). It is easy to see that $\mathcal{X}_K$ is special over $\operatorname{Spf} \breve{\mathcal{O}}$ and its generic fiber $X_K = t(\mathcal{X}_K)_\eta$ is partially proper and smooth over $\operatorname{Spa}(\breve{F}, \breve{\mathcal{O}})$.

Let D be the central division algebra over F with invariant $1/d$. The formal scheme $\mathcal{X}_K$ is endowed with a right action of the subgroup of $\mathrm{GL}_d(F) \times D^\times$ consisting of elements (g,h) such that $v_F(\det g) + v_F(\operatorname{Nrd} h) = 0$ and $gKg^{-1} = K$, where v_F denotes the normalized valuation of F. We would like to explain that we can use Theorem 4.5 to calculate the trace $\operatorname{Tr}((g,h)^*; R\Gamma_c(X_{K,\overline{\eta}}, \Lambda))$, under the assumption that gK consists of regular elliptic elements of $\mathrm{GL}_d(F)$ and that h is a regular elliptic element in $D^\times$.

For an integer $m \ge 1$, let $\mathcal{S}_m$ be the ordered set of $\mathcal{O}/\mathfrak{m}^m$-submodules of $(\mathfrak{m}^{-m}/\mathcal{O})^d$ which are direct summands. Put $\mathcal{S}'_m = \mathcal{S}_m \setminus \{0, (\mathfrak{m}^{-m}/\mathcal{O})^d\}$. For each $I \in \mathcal{S}_m$, we can construct the closed formal subscheme $\mathcal{Y}_I$ of $\mathcal{X}_{m,s}$ (*cf.* [Mie10a, Definition 4.1]); roughly speaking, it is the locus where the universal Drinfeld m-level structure vanishes on I. More generally, for an compact open subgroup K of K_0, we put $\mathcal{S}_K = (K/K_m)\backslash\mathcal{S}_m$ and $\mathcal{S}'_K = (K/K_m)\backslash\mathcal{S}'_m$, where $m \ge 1$ is an integer with $K_m \subset K$. We endow them with the induced partial orders. For $I \in \mathcal{S}_K$, we can also define the closed formal subscheme $\mathcal{Y}_I$ of $\mathcal{X}_{K,s}$ so that $Y_{(I)} = Y_I \setminus \bigcup_{I' \in \mathcal{S}_K, I' > I} Y_{I'}$, where Y_I denotes $t(\mathcal{Y}_I)_a$, coincides with the boundary subset $\partial_I M_K$ in [Str08, Paragraph 3.1.1]; in fact, we can take $\mathcal{Y}_I$ as the closed formal subscheme defined by $\mathfrak{p}_K$ in [Str08, Proposition 3.1.3 (i)] (note that the partial order on $\mathcal{S}_K$ in [Str08] is the inverse of ours). However, the author could not find any natural moduli interpretation of $\mathcal{Y}_I$.

Now we can easily see that the family of closed formal subschemes $\{\mathcal{Y}_I\}_{I\in\mathcal{S}'_m}$ satisfies Assumption 4.1. On the other hand, we can define the action of g on $\mathcal{S}'_K$ as in [Str08, p. 914]. Since we are assuming that gK consists of regular elliptic elements, we have $I \neq g^{-1}I$ for every $I \in \mathcal{S}'_K$. Moreover, it is easy to see that the action of (g,h) maps Y_I onto $Y_{g^{-1}I}$ at least set-theoretically (*cf.* [Str08, Lemma 3.2.2 (ii)]). Since (g,h) is regular elliptic, the action of (g,h) on $\mathcal{X}_K$ satisfies the second condition in Proposition 4.19 by the first part of the proof of [Str08, Proposition 3.2.4]. Therefore, by Proposition 4.19, the action of (g,h) on $\mathcal{X}_K$ satisfies Assumption 4.3. Finally, it is known that $\mathcal{X}_K$ is algebraizable ([Str08, Theorem 2.3.1]). Hence all the assumptions in Theorem 4.5 are satisfied and we obtain

$$\operatorname{Tr}\bigl((g,h)^*; R\Gamma_c(X_{K,\overline{\eta}},\Lambda)\bigr) = \#\operatorname{Fix}(g,h).$$

This recovers a result [Str08, Theorem 3.3.1] of Strauch. Recall that the right hand side has been calculated in [Str08, Theorem 2.6.8], and as a consequence of the trace formula above, we can get a purely local proof of the fact that the ℓ-adic cohomology of the Lubin-Tate tower $(X_m)_{m\geq 0}$ realizes the local Jacquet-Langlands correspondence ([Str08, Theorem 4.1.3]).

The advantage of our proof is that it does not require algebraization of the action. The proof of [Str08, Theorem 3.3.1] uses careful approximation of the action of (g,h) by an algebraizable morphism (*cf.* [Str08, Proposition 3.2.4 (ii), §5.2]), and it seems difficult to extend that method to the non-affine case.

Example 4.22 Let p be a prime. For a compact open subgroup K^p of $\mathrm{GSp}_4(\mathbb{A}^{\infty,p})$ and an integer $m\geq 0$, let Sh_{m,K^p} be the Shimura variety over $\mathbb{Z}_{p^\infty} = W(\overline{\mathbb{F}}_p)$ introduced in [IM10, §4]; namely, it is the moduli space parametrizing polarized abelian surfaces with K^p-level structures outside p and Drinfeld m-level structures at p. Let $\mathcal{S}_m$ be the ordered set of direct summands of $(\mathbb{Z}/p^m\mathbb{Z})^4$ whose ranks are greater than 1 and put $\mathcal{S}'_m = \mathcal{S}_m \setminus \{(\mathbb{Z}/p^m\mathbb{Z})^4\}$. Fix a perfect alternating bilinear form on $\mathbb{Z}_p^4$ and denote the subset of $\mathcal{S}_m$ (resp. $\mathcal{S}'_m$) consisting of coisotropic direct summands by $\mathcal{S}_m^{\mathrm{coi}}$ (resp. $\mathcal{S}_m^{\prime\mathrm{coi}}$). Then, as in Example 4.21, we can define the closed subscheme $\overline{\mathrm{Sh}}_{m,K^p,[I]}$ of $\overline{\mathrm{Sh}}_{m,K^p} = \mathrm{Sh}_{m,K^p}\otimes_{\mathbb{Z}_{p^\infty}}\overline{\mathbb{F}}_p$ for $I\in\mathcal{S}_m$ (*cf.* [IM10, Definition 5.1]). It is known that every $\overline{\mathbb{F}}_p$-rational point of $Y_{m,K^p} := \overline{\mathrm{Sh}}_{m,K^p,[(\mathbb{Z}/p^m\mathbb{Z})^4]}$ corresponds to a supersingular abelian variety (note that the definition of Y_{m,K^p} here is different from that in [IM10], but they coincide up to nilpotent elements; see [IM10, Lemma 5.3 iii)]).

Let Sh'_{m,K^p} be the closed subscheme of Sh_{m,K^p} defined by the quasi-coherent ideal of $\mathcal{O}_{\mathrm{Sh}_{m,K^p}}$ consisting of elements killed by p^l for some integer $l\geq 0$. Put $\overline{\mathrm{Sh}}'_{m,K^p} = \overline{\mathrm{Sh}}_{m,K^p}\times_{\mathrm{Sh}_{m,K^p}}\mathrm{Sh}'_{m,K^p}$, $\overline{\mathrm{Sh}}'_{m,K^p,[I]} = \overline{\mathrm{Sh}}_{m,K^p,[I]}\times_{\mathrm{Sh}_{m,K^p}}\mathrm{Sh}'_{m,K^p}$ and $Y'_{m,K^p} =$

$Y_{m,K^p} \times_{\mathrm{Sh}_{m,K^p}} \mathrm{Sh}'_{m,K^p}$. Then, by [IM10, Lemma 5.3, proof of Proposition 5.7] the family of closed subschemes $\{\overline{\mathrm{Sh}}'_{m,K^p,[I]}\}_{I\in\mathcal{S}_m^{\mathrm{coi}}}$ satisfies the conditions in Example 4.2. Let us denote the completion of Sh_{m,K^p} (resp. Sh'_{m,K^p}) along Y_{m,K^p} (resp. Y'_{m,K^p}) by $(\mathrm{Sh}_{m,K^p})^\wedge_{/Y_{m,K^p}}$ (resp. $(\mathrm{Sh}'_{m,K^p})^\wedge_{/Y'_{m,K^p}}$).

Let $\breve{\mathscr{M}}$ be the Rapoport-Zink space for GSp(4) considered in [IM10] and $\breve{\mathscr{M}}_m$ be as in [IM10, §3.2]. The formal scheme $\breve{\mathscr{M}}_m$ is endowed with the action of $\mathrm{GSp}_4(\mathbb{Z}_p)\times J(\mathbb{Q}_p)$, where J is an inner form of GSp_4. By the p-adic uniformization theorem of Rapoport-Zink, there is a $\mathrm{GSp}_4(\mathbb{Z}_p)$-equivariant morphism $\theta_{m,K^p}\colon \breve{\mathscr{M}}_m \longrightarrow (\mathrm{Sh}_{m,K^p})^\wedge_{/Y_{m,K^p}}$, which induces an open and closed immersion $\breve{\mathscr{M}}_m/\Gamma \hookrightarrow (\mathrm{Sh}_{m,K^p})^\wedge_{/Y_{m,K^p}}$ for some discrete subgroup Γ of $J(\mathbb{Q}_p)$ ([IM10, Theorem 4.2]). Put $\breve{\mathscr{M}}_{m,[I]} = \breve{\mathscr{M}}_m \times_{\mathrm{Sh}_{m,K^p}} \overline{\mathrm{Sh}}_{m,K^p,[I]}$, $\breve{\mathscr{M}}'_m = \breve{\mathscr{M}}_m \times_{\mathrm{Sh}_{m,K^p}} \mathrm{Sh}'_{m,K^p}$ and $\breve{\mathscr{M}}'_{m,[I]} = \breve{\mathscr{M}}_m \times_{\mathrm{Sh}_{m,K^p}} \overline{\mathrm{Sh}}'_{m,K^p,[I]}$. As mentioned in [IM10, proof of Proposition 5.10], $\breve{\mathscr{M}}_{m,[I]}$ is preserved by the action of J on $\breve{\mathscr{M}}_m$. On the other hand, it is easy to see that the defining ideal of $\breve{\mathscr{M}}'_m$ in $\breve{\mathscr{M}}_m$ consists of elements of $\mathcal{O}_{\breve{\mathscr{M}}_m}$ which is killed by p^l for some integer $l>0$. Therefore the actions of J on $\breve{\mathscr{M}}'_m$ and $\breve{\mathscr{M}}'_{m,[I]}$ are naturally induced. Moreover, we have an open and closed immersion $\breve{\mathscr{M}}'_m/\Gamma \hookrightarrow (\mathrm{Sh}'_{m,K^p})^\wedge_{/Y'_{m,K^p}}$. It is also easy to observe that $t(\breve{\mathscr{M}}'_m)_\eta$ coincides with $t(\breve{\mathscr{M}}_m)_\eta$.

Now it is easy to see that the formal scheme $\breve{\mathscr{M}}'_m/\Gamma$, a family of closed formal subscheme $\{\breve{\mathscr{M}}'_{m,[I]}/\Gamma\}_{I\in\mathcal{S}_m'^{\mathrm{coi}}}$ and the action of $(g,j)\in\mathrm{GSp}_4(\mathbb{Z}_p)\times J(\mathbb{Q}_p)$ where gK_m consists of regular elliptic elements of $\mathrm{GSp}_4(\mathbb{Q}_p)$ and j normalizes Γ satisfy all the assumptions in Theorem 4.5. Indeed, Assumption 4.1 has already been observed. Assumption 4.3 follows from [IM10, Proposition 5.15], Proposition 4.18 and the fact that if gK_m consists of regular elliptic then $g^{-1}I\neq I$ for every $I\in\mathcal{S}_m'^{\mathrm{coi}}$ (otherwise gK_m intersects a proper parabolic subgroup). Since every irreducible component of $(\breve{\mathscr{M}}_m)_{\mathrm{red}}$ is projective over $\overline{\mathbb{F}}_p$ ([RZ96, Proposition 2.32]), Y_{m,K^p} and Y'_{m,K^p} are proper over $\overline{\mathbb{F}}_p$ and thus $\breve{\mathscr{M}}'_m/\Gamma$ is partially proper over $\mathrm{Spa}(\mathbb{Q}_{p^\infty},\mathbb{Z}_{p^\infty})$. Hence we get the formula

$$\mathrm{Tr}\bigl((g,j)^*; R\Gamma_c(t(\breve{\mathscr{M}}_m)_{\overline{\eta}}/\Gamma,\Lambda)\bigr) = \#\,\mathrm{Fix}(g,j).$$

The author expects that the right hand side can be calculated in the similar way as in [Str08, §2.6], and plans to work this out in a forthcoming paper.

5 Lefschetz trace formula for contracting morphisms

5.1 Statement

In this section, we generalize Fujiwara's trace formula for contracting morphisms ([Fuj97, Theorem 3.2.4]) to rigid spaces which are not necessarily algebraizable.

Let X be a purely d-dimensional adic space which is proper and smooth over S. For a closed adic subspace Y of X and $\varepsilon \in |k^\times| \subset \mathbb{R}$, we can construct the open tubular neighborhood $Y(\varepsilon)$ and the closed tubular neighborhood $Y^\circ(\varepsilon)$ (*cf.* [Hub98c, §2.6], in which $Y(\varepsilon)$ is denoted by $T(\varepsilon)$ and $Y^\circ(\varepsilon)$ by $S(\varepsilon)$). If Y is defined by $f_1, \ldots, f_m \in \Gamma(X, \mathcal{O}_X)$, then $Y(\varepsilon)$ (resp. $Y^\circ(\varepsilon)$) is given by $\{x \in X \mid |f_i(x)| \leq \varepsilon\}$ (resp. $\{x \in X \mid |f_i(x)| < \varepsilon\}$). Note that $Y(\varepsilon)$ and $Y^\circ(\varepsilon)$ are constructible and $Y = \bigcap_{\varepsilon \in |k^\times|} Y(\varepsilon)$.

Let $f\colon X \longrightarrow X$ be an S-morphism. We will use the notation in Example 2.9. We denote the set of connected components of $\operatorname{Fix} f$ by $\pi_0(\operatorname{Fix} f)$. It is a finite set since $H^0(\operatorname{Fix} f, \Lambda)$ is a finitely generated Λ-module. Therefore every element of $\pi_0(\operatorname{Fix} f)$ is open and closed in $\operatorname{Fix} f$.

Definition 5.1 Let D be a connected component of $\operatorname{Fix} f$. We say that f is contracting near D if there exists a strictly decreasing sequence $(\varepsilon_n)_{n \geq 0}$ in $|k^\times|$ converging to 0 such that $f(D(\varepsilon_n)) \subset D(\varepsilon_{n+1})$ for every $n \geq 0$.

If f is contracting near every connected component of $\operatorname{Fix} f$, we say that f is contracting near its fixed points.

The goal of this section is the following theorem:

Theorem 5.2 *Assume that the characteristic of k is equal to 0 and f is contracting near $D \in \pi_0(\operatorname{Fix} f)$. Then we have*

$$\# \operatorname{Fix}_D f = \chi(D, \Lambda),$$

where we put $\chi(D, \Lambda) = \operatorname{Tr}(\mathrm{id}; R\Gamma(D, \Lambda))$.

This theorem will be proved in §5.3. The following corollary is immediate from Theorem 5.2 and Proposition 3.9:

Corollary 5.3 *Assume that the characteristic of k is equal to 0 and f is contracting near its fixed points. Then we have*

$$\operatorname{Tr}\bigl(f^*; R\Gamma(X, \Lambda)\bigr) = \sum_{D \in \pi_0(\operatorname{Fix} f)} \chi(D, \Lambda).$$

In particular, if every fixed point of f is isolated, the right hand side is equal to the number of the fixed points.

5.2 Lefschetz trace formula for proper pseudo-adic spaces

In order to prove Theorem 5.2, we need a variant of Proposition 3.9 for a pseudo-adic space which is proper but not necessarily smooth over S. In this subsection, let X be a finite-dimensional pseudo-adic space which is proper over S. We assume that $H^i(X, \Lambda)$ is a finitely generated Λ-module for every integer i; then $R\Gamma(X, \Lambda)$ is a perfect Λ-complex (Corollary 3.3).

Let $f\colon X \longrightarrow X$ be an S-morphism. Since $R\Gamma(X,\Lambda)$ is a perfect complex, the trace $\mathrm{Tr}(f; R\Gamma(X,\Lambda))$ makes sense. Our purpose is to express this trace by a cohomology class analogous to $[\gamma]$ in Proposition 3.9.

First we will construct an analogue of $\mathrm{cl}(\gamma)$.

Definition 5.4 Since $\mathrm{pr}_2 \circ \gamma_f = \mathrm{id}$, we have $H^0_{\Gamma_f}(X \times_S X, \mathrm{pr}_2^! \Lambda) \cong H^0(X,\Lambda)$. We denote by $\mathrm{cl}(f)$ the element of $H^0_{\Gamma_f}(X \times_S X, \mathrm{pr}_2^! \Lambda)$ that corresponds to $1 \in H^0(X,\Lambda)$ under the isomorphism above.

Remark 5.5 If X is purely d-dimensional and smooth over S, then $H^0_{\Gamma_f}(X \times_S X, \mathrm{pr}_2^! \Lambda) \cong H^{2d}_{\Gamma_f}(X \times_S X, \Lambda(d))$ and by this isomorphism $\mathrm{cl}(f)$ corresponds to $\mathrm{cl}(\gamma_f)$ in §2.

The following is an analogue of Proposition 2.18, whose proof is similar:

Proposition 5.6 *The map $f^*\colon R\Gamma(X,\Lambda) \longrightarrow R\Gamma(X,\Lambda)$ coincides with the composite of*

$$\begin{aligned} R\Gamma(X,\Lambda) &\xrightarrow{\mathrm{pr}_1^*} R\Gamma(X \times_S X, \Lambda) \xrightarrow{\cup\,\mathrm{cl}(f)} R\Gamma(X \times_S X, \mathrm{pr}_2^! \Lambda) = R\Gamma(X, R\,\mathrm{pr}_{2!}\,\mathrm{pr}_2^! \Lambda) \\ &\xrightarrow{\mathrm{adj}} R\Gamma(X,\Lambda). \end{aligned}$$

Next we will establish an analogue of Corollary 3.7. Denote the structure morphism of X by $a\colon X \longrightarrow S$ and put $K_X = a^!\Lambda$. Let $\tau\colon K_X \overset{\mathbb{L}}{\boxtimes} \Lambda \longrightarrow \mathrm{pr}_2^! \Lambda$ be the base change map $K_X \overset{\mathbb{L}}{\boxtimes} \Lambda = \mathrm{pr}_1^* a^! \Lambda \longrightarrow \mathrm{pr}_2^! a^* \Lambda = \mathrm{pr}_2^! \Lambda$.

Proposition 5.7 *The map τ above induces an isomorphism*

$$R\Gamma(X \times_S X, K_X \overset{\mathbb{L}}{\boxtimes} \Lambda) \xrightarrow{\cong} R\Gamma(X \times_S X, \mathrm{pr}_2^! \Lambda).$$

Proof. By Proposition 3.5, the Künneth morphism $R\Gamma(X, K_X) \overset{\mathbb{L}}{\otimes} R\Gamma(X,\Lambda) \longrightarrow R\Gamma(X \times_S X, K_X \overset{\mathbb{L}}{\boxtimes} \Lambda)$ is an isomorphism. On the other hand, we have isomorphisms

$$\begin{aligned} R\Gamma(X \times_S X, \mathrm{pr}_2^! \Lambda) &= R\Gamma(X, R\,\mathrm{pr}_{1*}\,\mathrm{pr}_2^! \Lambda) \cong R\Gamma(X, a^! Ra_*\Lambda) = R\Gamma\big(X, a^! R\Gamma(X,\Lambda)\big) \\ &\underset{(1)}{\cong} R\Gamma\big(X, a^!\Lambda \overset{\mathbb{L}}{\otimes} R\Gamma(X,\Lambda)_X\big) \underset{(2)}{\cong} R\Gamma(X, a^!\Lambda) \overset{\mathbb{L}}{\otimes} R\Gamma(X,\Lambda) \\ &= R\Gamma(X, K_X) \overset{\mathbb{L}}{\otimes} R\Gamma(X,\Lambda). \end{aligned}$$

For (1), note that the natural map $a^!\Lambda \overset{\mathbb{L}}{\otimes} R\Gamma(X,\Lambda)_X \longrightarrow a^! R\Gamma(X,\Lambda)$ is an isomorphism. This is an easy consequence of the fact that $R\Gamma(X,\Lambda)$ is a perfect Λ-complex. The isomorphy of (2) is due to Lemma 3.1.

It is easy to see that this isomorphism fits into the following diagram:

$$\begin{tikzcd} R\Gamma(X, K_X) \overset{\mathbb{L}}{\otimes} R\Gamma(X, \Lambda) \arrow[r, equal] \arrow[d, "\cong"] & R\Gamma(X, K_X) \overset{\mathbb{L}}{\otimes} R\Gamma(X, \Lambda) \arrow[d, "\cong"] \\ R\Gamma(X \times_S X, K_X \overset{\mathbb{L}}{\boxtimes} \Lambda) \arrow[r, "\tau"] & R\Gamma(X \times_S X, \mathrm{pr}_2^! \Lambda). \end{tikzcd}$$

This completes the proof. ∎

Definition 5.8 Let $[f]$ be the element of $H^0(X \times_S X, K_X \overset{\mathbb{L}}{\boxtimes} \Lambda)$ that corresponds to $\mathrm{cl}(f)$ by the isomorphism in Proposition 5.7.

Proposition 5.9 *We have* $\mathrm{Tr}(f^*; R\Gamma(X, \Lambda)) = \mathrm{Adj}_X(\delta^*[f])$, *where* Adj_X *denotes the natural adjunction map* $H^0(X, K_X) \longrightarrow \Lambda$.

Proof. By Proposition 5.6, it suffices to show the commutativity of the lower part of the following diagram:

$$\begin{tikzcd} R\Gamma(X, K_X) \overset{\mathbb{L}}{\otimes} R\Gamma(X, \Lambda) \arrow[r, "\cong"] \arrow[d, "\cong"] & R\,\mathrm{Hom}\big(R\Gamma(X, \Lambda), \Lambda\big) \overset{\mathbb{L}}{\otimes} R\Gamma(X, \Lambda) \arrow[dd] \\ R\Gamma(X \times_S X, K_X \overset{\mathbb{L}}{\boxtimes} \Lambda) \arrow[d] \arrow[dd, bend right=60, "\delta^*"'] & \\ R\Gamma(X \times_S X, \mathrm{pr}_2^! \Lambda) \arrow[r, "(*)"] & R\,\mathrm{Hom}\big(R\Gamma(X, \Lambda), R\Gamma(X, \Lambda)\big) \arrow[d, "\mathrm{Tr}"] \\ R\Gamma(X, K_X) \arrow[r, "\mathrm{Adj}_X"] & \Lambda, \end{tikzcd}$$

where $(*)$ is given by

$$\begin{aligned} R\Gamma(X, \Lambda) \overset{\mathbb{L}}{\otimes} R\Gamma(X \times_S X, \mathrm{pr}_2^! \Lambda) &\xrightarrow{\mathrm{pr}_1^* \cup \mathrm{id}} R\Gamma(X \times_S X, \mathrm{pr}_2^! \Lambda) \\ &= R\Gamma(X, R\,\mathrm{pr}_{2!} \mathrm{pr}_2^! \Lambda) \xrightarrow{\mathrm{adj}} R\Gamma(X, \Lambda). \end{aligned}$$

As in the proof of Proposition 3.9, we can prove the commutativities of the upper part and the outer part of the diagram above. ∎

We can apply the technique in §3.3 to calculate $\mathrm{Adj}_X(\delta^*[f])$:

Lemma 5.10 *Let U be an open adic subspace of X which is purely d-dimensional and smooth over S, and Y an closed constructible subset of X contained in U. Assume that $f(X) \subset Y$. Then we have* $\mathrm{Adj}_X(\delta^*[f]) = \#\,\mathrm{Fix}\, f|_U$.

Proof. First note that $\mathrm{Fix}\, f|_U = \mathrm{Fix}\, f$ since $\mathrm{Fix}\, f \subset Y$. Therefore the adic space $\mathrm{Fix}\, f|_U$ is proper over S and $\#\,\mathrm{Fix}\, f|_U$ makes sense.

In the same way as Corollary 3.19, we can deduce from Proposition 5.7 that the map $R\Gamma_{Y\times_S X}(X\times_S X, K_X \overset{\mathbb{L}}{\boxtimes} \Lambda) \longrightarrow R\Gamma_{Y\times_S X}(X\times_S X, \mathrm{pr}_2^! \Lambda)$ is an isomorphism.

Consider the following commutative diagram:

$$\begin{tikzcd}
H^0_{\Gamma_f}(X\times_S X, \mathrm{pr}_2^!\Lambda) \arrow[r] \arrow[d] & H^{2d}_{\Gamma_{f|_U}}\big(U\times_S U, \Lambda(d)\big) \arrow[d] \\
H^0_{Y\times_S X}(X\times_S X, \mathrm{pr}_2^!\Lambda) \arrow[r] & H^{2d}_{Y\times_S U}\big(U\times_S U, \Lambda(d)\big) \arrow[d, equal] \\
H^0_{Y\times_S X}(X\times_S X, K_X \overset{\mathbb{L}}{\boxtimes} \Lambda) \arrow[r] \arrow[u, "\cong"'] \arrow[d, "\delta^*"] & H^{2d}_{Y\times_S U}\big(U\times_S U, \Lambda(d)\big) \arrow[d, "\delta_U^*"] \\
H^0_Y(X, K_X) \arrow[r, "\cong"] \arrow[d] & H^{2d}_Y\big(U, \Lambda(d)\big) \arrow[d] \\
H^0(X, K_X) \arrow[d, "\mathrm{Adj}_X"] & H^{2d}_c\big(U, \Lambda(d)\big) \arrow[l] \arrow[d, "\mathrm{Tr}_U"] \\
\Lambda \arrow[r, equal] & \Lambda.
\end{tikzcd}$$

It is easy to see that the image of $\mathrm{cl}(f) \in H^0_{\Gamma_f}(X\times_S X, \mathrm{pr}_2^!\Lambda)$ under the composite of the arrows in the left column is equal to $\mathrm{Adj}_X(\delta^*[f])$. On the other hand, by Remark 5.5, the image of $\mathrm{cl}(f)$ under the top horizontal arrow is $\mathrm{cl}(f|_U)$. Since the image of $\mathrm{cl}(f|_U)$ under the composite of the arrows in the right column is $\#\,\mathrm{Fix}\, f|_U$, we get the lemma. ∎

5.3 Proof of Theorem 5.2

We go back to the notation introduced in §5.1. Let D be a connected component of $\mathrm{Fix}\, f$ and assume that f is contracting near D. Take a strictly decreasing sequence $(\varepsilon_n)_{n\geq 0}$ in $|k^\times|$ converging to 0 such that $f(D(\varepsilon_n)) \subset D(\varepsilon_{n+1})$ for every $n \geq 0$. Take a sequence $(\varepsilon'_n)_{n\geq 0}$ in $|k^\times|$ such that $\varepsilon_n > \varepsilon'_n > \varepsilon_{n+1}$ for every $n \geq 0$. Then we have $f(D^\circ(\varepsilon_n)) \subset f(D(\varepsilon_n)) \subset D(\varepsilon_{n+1}) \subset D^\circ(\varepsilon'_n) \subset D(\varepsilon'_n) \subset D^\circ(\varepsilon_n)$. Fix an integer $n \geq 0$. The pseudo-adic space $D^\circ(\varepsilon_n) := (X, D^\circ(\varepsilon_n))$ is proper over S and $H^i(D^\circ(\varepsilon_n), \Lambda)$ is a finitely generated Λ-module ([Hub98c, Corollary 2.3], [Hub07, Corollary 5.4]). Hence we may apply Proposition 5.9 to $D^\circ(\varepsilon_n)$. Moreover, we can also apply Lemma 5.10 to $D^\circ(\varepsilon'_n) \subset D(\varepsilon'_n) \subset D^\circ(\varepsilon_n)$. Summing up, we get the formula $\mathrm{Tr}(f^*; R\Gamma(D^\circ(\varepsilon_n), \Lambda)) = \#\,\mathrm{Fix}(f|_{D(\varepsilon'_n)})$.

On the other hand, for every $D' \in \pi_0(\mathrm{Fix}\, f)$ distinct from D, $D(\varepsilon'_n)$ does not intersect D'; otherwise D' also intersects $\bigcap_{m\geq 0} f^m(D(\varepsilon'_n)) \subset \bigcap_{m\geq n} D(\varepsilon_m) = D$.

Thus we have $\mathrm{Fix}(f|_{D(\varepsilon'_n)}) = D$ and $\#\,\mathrm{Fix}(f|_{D(\varepsilon'_n)}) = \#\,\mathrm{Fix}_D(f|_{D(\varepsilon'_n)}) = \#\,\mathrm{Fix}_D\, f$ (the final equality is due to Proposition 2.10).

Now assume that the characteristic of k is 0. Then, by [Hub98b, Theorem 3.6], the restriction map $H^i(D^\circ(\varepsilon_n), \Lambda) \longrightarrow H^i(D, \Lambda)$ is an isomorphism for sufficiently large n. Therefore, by the commutative diagram

$$\begin{CD} R\Gamma\bigl(D^\circ(\varepsilon_n), \Lambda\bigr) @>{f^*}>> R\Gamma\bigl(D^\circ(\varepsilon_n), \Lambda\bigr) \\ @V{\cong}VV @VV{\cong}V \\ R\Gamma(D, \Lambda) @>{\mathrm{id}}>> R\Gamma(D, \Lambda), \end{CD}$$

we have $\mathrm{Tr}(f^*; R\Gamma(D^\circ(\varepsilon_n), \Lambda)) = \chi(D, \Lambda)$ for such n. Hence we have $\#\,\mathrm{Fix}_D\, f = \chi(D, \Lambda)$, as desired.

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