A “missing“ family of classical orthogonal polynomials

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Abstract

We study a family of ”classical” orthogonal polynomials which satisfy (apart from a 3-term recurrence relation) an eigenvalue problem with a differential operator of Dunkl-type. These polynomials can be obtained from the little q-Jacobi polynomials in the limit $q = -1$. We also show that these polynomials provide a nontrivial realization of the Askey-Wilson algebra for $q = -1$.

Keywords: classical orthogonal polynomials, Jacobi polynomials, little q-Jacobi polynomials.

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1. Introduction

The Askey scheme [13], [14] provides a list of all known "classical" orthogonal polynomials. The term "classical" means that the orthogonal polynomials satisfy a three-term recurrence relation

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x), \quad P_{-1} = 0, \; P_0 = 1 \tag{1.1}$$

and that they are also eigenfunctions

$$L P_n(x) = \lambda_n P_n(x) \tag{1.2}$$

of an operator L which acts on the variable x. The operator L can be either a second-order differential operator (in this case we obtain the purely classical polynomials: Jacobi, Laguerre, Hermite) or a second-order difference operator on a uniform or non-uniform grid. The latter case leads to Wilson (q=1) and Askey-Wilson ($q \neq 1$) polynomials and their descendents.

Bannai and Ito [4] proved a general theorem (which generalizes a famous theorem of Leonard [16], dealing only with polynomials orthogonal on a finite set of points) stating that all such orthogonal polynomials should coincide with the Askey-Wilson polynomials or their specializations. They also found a "missing" case of the Askey scheme corresponding to the limit $q = -1$ of the q-Racah polynomials (see also [19]). The example of Bannai and Ito corresponds to polynomials orthogonal on a finite set of points. Hence in this case the operator L is merely a finite-dimensional 3-diagonal matrix which corresponds to the case of "Leonard pairs" (see, e.g. [19]).

It is hence sensible to investigate other possibilities as q approaches -1 in the Askey scheme. Of course, the limit $q = 1$ is well studied and classified (see, e.g. [13]). The limit $q = -1$ however, has not been explored much. In [1] Askey and Ismail have studied the limit $q = -1$ for the q-ultraspherical polynomials, but in this case, the operator L disappears in the limit $q = -1$.

Here we show that there is a very simple class of polynomials which can be obtained from the little q-Jacobi polynomials in the limit $q = -1$. Under appropriate choice of the parameters, the operator L survives in the limit $q = -1$. The polynomials thus obtained are indeed classical: they satisfy the eigenvalue equation (1.2). But in contrast to the case of pure classical polynomials (like Jacobi polynomials), the operator L is a combination of a differential operator of first order and of the reflection operator R (see formula (2.14) in the next section). Operators of this type are known as Dunkl operators [9]. So far operators of Dunkl type were used typically to transform one polynomial family into another. In the one-dimensional case, the Dunkl operator T_μ is defined as [9], [6]

$$T_\mu f(x) = f'(x) + \mu \frac{f(x) - f(-x)}{x} \tag{1.3}$$

where μ is a deformation parameter (when $\mu = 0$ the Dunkl operator becomes the ordinary derivative operator).

Clearly, the Dunkl operator (1.3) reduces the degree of any polynomial by 1. Hence, there are no polynomials which are eigenfunctions of the operator T_μ.

Our operator (1.2), in contrast, preserves the linear space of polynomials of given degree. In the following we construct the general solution of the eigenvalue equation (2.15) and obtain explicit expression for the corresponding little -1 Jacobi polynomials $P_n^{(-1)}(x)$.

We also show that the polynomials $P_n^{(-1)}(x)$ are Dunkl-classical, i.e. $T_\mu P_n^{(-1)}(x) = Q_{n-1}(x)$, where $Q_n(x)$ is another set of little -1 Jacobi polynomials (with different parameters)

2. Limit of the little q-Jacobi polynomials as $q \to -1$

The little q-Jacobi polynomials are defined through the recurrence coefficients

$$u_n = A_{n-1}C_n, \quad b_n = A_n + C_n, \tag{2.1}$$

where A_n, C_n are given by

$$A_n = q^n \frac{(1-aq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})},\ C_n = aq^n \frac{(1-q^n)(1-bq^n)}{(1-abq^{2n+1})(1-abq^{2n})},$$

They have the following simple expression in terms of the basic hypergeometric function

$$P_n(x) = \kappa_n \ {}_2\Phi_1 \left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \Big| q; qx \right) \tag{2.2}$$

with a normalization factor κ_n to ensure that they are monic.

They satisfy the orthogonality relation

$$\sum_{s=0}^{\infty} \frac{(bq;q)_s}{(q;q)_s} (aq)^s P_n(q^s) P_m(q^s) = h_n \ .\delta_{nm} \tag{2.3}$$

The moments corresponding to this weight function are

$$c_n = \frac{(aq;q)_n}{(abq^2;q)_n}, \tag{2.4}$$

where $(x;q)_n = (1-x)(1-qx)\dots(1-q^{n-1}x)$ is standard notation for the q-shifted factorials (Pochhammer q-symbol).

There is a q-difference equation of the form

$$a(bq - x^{-1})(P_n(qx) - P_n(x)) + (1 - x^{-1})(P_n(q^{-1}x) - P_n(x)) = \lambda_n P_n(x), \tag{2.5}$$

where

$$\lambda_n = (q^{-n} - 1)(1 - abq^{n+1}) \tag{2.6}$$

If $a = q^\alpha, b = q^\beta$ then in the limit $q \to 1$ we get the ordinary Jacobi polynomials with parameters α, β.

There is, however, another nontrivial limit if one puts

$$q = -e^{\varepsilon},\ a = -e^{\varepsilon\alpha},\ b = -e^{\varepsilon\beta} \tag{2.7}$$

and take the limit $\varepsilon \to 0$. This is the limit $q = -1$ of the little q-Jacobi polynomials.

A direct calculation shows that in this limit, we have the recurrence coefficients

$$u_n = \frac{(n + (1-\theta_n)\alpha)(n+\beta+\theta_n\alpha)}{(2n+\alpha+\beta)^2}, \quad b_n = (-1)^n \frac{(2n+1)\alpha + \alpha\beta + \alpha^2 + (-1)^n\beta}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}, \tag{2.8}$$

where

$$\theta_n = \frac{1+(-1)^n}{2}$$

is the characteristic function of even numbers.

The corresponding moments are obtained directly from the moments (2.4):

$$c_{2n} = c_{2n-1} = \frac{(\alpha/2+1/2)_n}{(\alpha/2+\beta/2+1)_n}, \quad n = 1, 2, 3, \dots, \tag{2.9}$$

where $(x)_n = x(x+1)\dots(x+n-1)$ is the ordinary Pochhammer symbol (shifted factorial).

Using this explicit expression for the moments, we can recover the weight function $w(x)$ for the resulting orthogonal polynomials. It is easily verified that

$$w(x) = \kappa|x|^\alpha(1-x^2)^{(\beta-1)/2}(1+x), \tag{2.10}$$

where

$$\kappa = \frac{\Gamma(\alpha/2+\beta/2+1)}{\Gamma(\beta/2+1/2)\Gamma(\alpha/2+1/2)}.$$

Indeed, we have (using the ordinary Euler B-integral)

$$\int_{-1}^{1} w(x)x^n dx = c_n, \quad n = 0, 1, 2, \dots,$$

where c_n is given by (2.9). The coefficient κ is chosen to provide the standard normalization condition $c_0 = 1$. Under the obvious conditions $\alpha > -1$, $\beta > -1$, the weight function $w(x)$ is positive, all moments c_n are well defined and the moment problem is positive definite, i.e. $\Delta_n > 0$ for all $n = 0, 1, 2, \dots$.

Consider the form of the q-difference equation (2.5) in this limit. We divide both sides of (2.5) by ε and introduce the operator L_ε which acts on any polynomial $f(x)$ as

$$L_\varepsilon f(x) = a\varepsilon^{-1}(bq - x^{-1})(f(qx) - f(x)) + \varepsilon^{-1}(1-x^{-1})(f(q^{-1}x) - f(x)) \tag{2.11}$$

(the parameters q, a, b depends on ε as in (2.7)). For monomials $f = x^n$ we have in the limit $\varepsilon = 0$

$$L_0 x^n = \xi_n x^n + \eta_n x^{n-1}, \tag{2.12}$$

where

$$\xi_n = 2(-1)^{n+1}n + (1-(-1)^n)(\alpha+\beta+1), \quad \eta_n = 2(-1)^n n - (1-(-1)^n)\alpha \tag{2.13}$$

This allows one to present the operator L_0 in the form

$$L_0 = 2(1-x)\partial_x R + (\alpha+\beta+1-\alpha x^{-1})(1-R), \tag{2.14}$$

where R is the reflection operator $Rf(x) = f(-x)$.

Thus we have that our polynomials are classical: they satisfy the eigenvalue equation

$$L_0 P_n(x) = \lambda_n P_n(x), \tag{2.15}$$

where

$$\lambda_n = \begin{cases} -2n & \text{if } n \text{ is even} \\ 2(\alpha+\beta+n+1) & \text{if } n \text{ is odd} \end{cases} \tag{2.16}$$

But in contrast to true classical polynomial the operator L_0 is not purely differential: it contains the reflection operator R.

In the next section we construct the general solution of the eigenvalue problem (2.15) in terms of the Gauss hypergeometric functions

3. Different forms of R-differential equation

Consider the eigenvalue equation

$$L_0 F(x) = \lambda F(x), \tag{3.1}$$

where L_0 has the expression (2.14). The operator L_0 is a differential operator of the first order containing the reflection operator R. Notice the important property of the operator L_0: it preserves any linear space of polynomials of degrees $\le N$ for any $N = 1, 2, 3, \ldots$. Hence the operator L_0 behaves like the classical hypergeometric operator: for any $n = 0, 1, 2, \ldots$ there exists a polynomial eigenvalue solution $L_0 P_n(x) = \lambda_n P_n(x)$ for an appropriate set of eigenvalues λ_n.

In order to find an explicit expression for these polynomials, we need first to solve the eigenvalue problem (3.1) with arbitrary λ.

We can arrive at a pure differential equation (without the operator R) by a standard procedure. Let us present the function $F(x)$ as a superposition of the even and odd parts:

$$F(x) = f(x) + g(x),$$

where $f(-x) = f(x),\ g(-x) = -g(x)$. The functions $f(x)$ and $g(x)$ are determined uniquely from $F(x)$. The operator R acts on these functions as: $Rf(x) = f(x),\ Rg(x) = -g(x)$. This allows one to rewrite the eigenvalue equation (3.1) in the form

$$(1-x)f'(x) + (x-1)g'(x) + (1+\alpha+\beta-\alpha/x)g(x) = \lambda(f(x)+g(x))/2. \tag{3.2}$$

Consider also the associated eigenvalue equation

$$RL_0 F(x) = \lambda RF(x) \tag{3.3}$$

which is obtained from (3.1) by application of the operator R. This equation can be presented in the form

$$(1+x)f'(x) + (1+x)g'(x) + (1+\alpha+\beta+\alpha/x)g(x) = \lambda(g(x)-f(x))/2 \tag{3.4}$$

Adding and subtracting, we obtain the simpler system of equations

$$
\begin{aligned}
&f'(x) + xg'(x) + (1+\alpha+\beta)g(x) = \lambda g(x)/2,\\
&xf'(x) + g'(x) + \alpha g(x)/x = -\lambda f(x)/2
\end{aligned} \tag{3.5}
$$

This presents the eigenvalue equation (3.2) equivalently as a system of two linear differential equations of first order. We can eliminate the function $g(x)$ from this system:

$$
g(x) = \frac{2(x^2-1)f'(x) + \lambda x f(x)}{2(\beta+1)-\lambda} \tag{3.6}
$$

and obtain a second-order differential equation for $f(x)$:

$$
4x(x^2-1)f''(x) + 4((\alpha+\beta+3)x^2-\alpha)f'(x) + \lambda x(2(\alpha+\beta)+4-\lambda)f(x) = 0 \tag{3.7}
$$

By a change of the variable $x \to x^2$ one can reduce equation (3.7) to the Gauss hypergeometric equation thus obtaining the general solution:

$$
f(x) = C_1\, {}_2F_1\left(\begin{matrix}\frac{\lambda}{4}, \frac{\alpha+\beta}{2}+1-\frac{\lambda}{4}\\ \frac{\alpha+1}{2}\end{matrix}; x^2\right) + C_2\, x^{1-\alpha}{}_2F_1\left(\begin{matrix}\frac{\lambda+2-2\alpha}{4}, \frac{2\beta+6-\lambda}{4}\\ \frac{3-\alpha}{2}\end{matrix}; x^2\right), \tag{3.8}
$$

where C_1, C_2 are arbitrary constants. But by construction, the function $f(x)$ must be even for all values of the parameters α, β. This is possible only if $C_2 = 0$ and the solution for $f(x)$ contains only one undetermined constant:

$$
f(x) = C_1\, {}_2F_1\left(\begin{matrix}\frac{\lambda}{4}, \frac{\alpha+\beta}{2}+1-\frac{\lambda}{4}\\ \frac{\alpha+1}{2}\end{matrix}; x^2\right) \tag{3.9}
$$

Quite similarly, we can eliminate the function $f(x)$ using:

$$
f(x) = \frac{2x(x^2-1)g'(x) + ((2\alpha+2\beta+2-\lambda)x^2-2\alpha)g(x)}{\lambda}
$$

Then one obtains a second-order differential equation for $g(x)$:

$$
4x^2(x^2-1)g''(x) + 4x((3+\alpha+\beta)x^2-\alpha)g'(x) + (4\alpha + (2+\lambda)(2\alpha+2\beta+2-\lambda)x^2)g(x) = 0
$$

Again the same change of the independent variable $x \to x^2$ leads to the Gauss hypergeometric equation with general solution

$$
g(x) = A_1\, x\, {}_2F_1\left(\begin{matrix}1+\frac{\lambda}{4}, \frac{\alpha+\beta}{2}+1-\frac{\lambda}{4}\\ \frac{\alpha+3}{2}\end{matrix}; x^2\right) + A_2\, x^{-\alpha}{}_2F_1\left(\begin{matrix}\frac{\lambda+2-2\alpha}{4}, \frac{2\beta+2-\lambda}{4}\\ \frac{1-\alpha}{2}\end{matrix}; x^2\right), \tag{3.10}
$$

with arbitrary constants A_1, A_2. The function $g(x)$ should be odd for all values of the parameter α, hence $A_2 = 0$ and we obtain the solution

$$
g(x) = A_1\, x\, {}_2F_1\left(\begin{matrix}1+\frac{\lambda}{4}, \frac{\alpha+\beta}{2}+1-\frac{\lambda}{4}\\ \frac{\alpha+3}{2}\end{matrix}; x^2\right) \tag{3.11}
$$

(The same result can be obtained directly from formula (3.6) if the function $f(x)$ is given explicitly as (3.9))

Now we have the general solution of equation (3.1) in the form

$$F(x) = f(x) + g(x) = C(\lambda)\, {}_2F_1\left({\frac{\lambda}{4}, \frac{\alpha+\beta}{2} + 1 - \frac{\lambda}{4} \atop \frac{\alpha+1}{2}} ; x^2 \right) + A(\lambda)\, x\, {}_2F_1\left({1 + \frac{\lambda}{4}, \frac{\alpha+\beta}{2} + 1 - \frac{\lambda}{4} \atop \frac{\alpha+3}{2}} ; x^2 \right), \qquad (3.12)$$

where the parameters $C(\lambda), A(\lambda)$ may depend on λ (as well as the parameters α, β). This dependence may be recovered if one substitutes expressions (3.9) and (3.11) for $f(x)$ and $g(x)$ into equations (3.2) or (3.4). Clearly, only the ratio $A(\lambda)/C(\lambda)$ is an essential parameter, so we can put $C(\lambda) = 1$ without loss of generality. A simple calculation yields $A = -\frac{\lambda}{2(1+\alpha)}$. We thus have the following result: the function

$$F(x) = {}_2F_1\left({\frac{\lambda}{4}, \frac{\alpha+\beta}{2} + 1 - \frac{\lambda}{4} \atop \frac{\alpha+1}{2}} ; x^2 \right) - \frac{\lambda}{2(1+\alpha)}\, x\, {}_2F_1\left({1 + \frac{\lambda}{4}, \frac{\alpha+\beta}{2} + 1 - \frac{\lambda}{4} \atop \frac{\alpha+3}{2}} ; x^2 \right), \qquad (3.13)$$

is a solution of the eigenvalue equation (3.1). The general solution of this equation is obtained by multiplying (3.13) by an arbitrary constant.

We would like to get polynomial solutions, i.e. find eigenvalues $\lambda_n,\ n = 0, 1, 2, \ldots$ such that $F_n(x)$ is a polynomial in x of exact degree n. It is easily seen that polynomial solutions are possible only if either $\lambda = -4n,\ n = 0, 1, 2, \ldots$ or $\lambda = 2(\alpha + \beta + 2 + 2n),\ n = 0, 1, 2, \ldots$. If $\lambda = -4n$, the first term in (3.13) is a polynomial of degree $2n$ whereas the second term is a polynomial of degree $2n - 1$. Hence for all $\lambda_n = -4n$ we will have polynomials $F_n(x)$ of the even degree $2n$. If $\lambda = 2(\alpha + \beta + 2 + 2n)$ then the first term in (3.13) is a polynomial of degree $2n$, while the second term is a polynomial of degree $2n + 1$. Hence for all $\lambda_n = 2(\alpha + \beta + 2 + 2n),\ n = 0, 1, 2, \ldots$ we obtain polynomials $F_n(x)$ having odd degree $2n + 1$. This solves the problem, and we thus have the following explicit expression.

If n is even then

$$P_n^{(-1)}(x) = \kappa_n \left[{}_2F_1\left({-\frac{n}{2}, \frac{n+\alpha+\beta+2}{2} \atop \frac{\alpha+1}{2}} ; x^2 \right) + \frac{nx}{\alpha+1}\, {}_2F_1\left({1 - \frac{n}{2}, \frac{n+\alpha+\beta+2}{2} \atop \frac{\alpha+3}{2}} ; x^2 \right) \right] \qquad (3.14)$$

If n is odd then

$$P_n^{(-1)}(x) = \kappa_n \left[{}_2F_1\left({\frac{1-n}{2}, \frac{n+\alpha+\beta+1}{2} \atop \frac{\alpha+1}{2}} ; x^2 \right) - \frac{(\alpha+\beta+n+1)x}{\alpha+1}\, {}_2F_1\left({\frac{1-n}{2}, \frac{n+\alpha+\beta+3}{2} \atop \frac{\alpha+3}{2}} ; x^2 \right) \right], \qquad (3.15)$$

where κ_n is an appropriate normalization factor to ensure that the polynomial $P_n^{(1)}(x) = x^n + O(x^{n-1}$ is monic. (We need not its explicit expression).

4. Relation with the symmetric Jacobi polynomials

The symmetric Jacobi polynomials were introduced by Chihara [7]. They can be defined as follows.

Let $P_n^{(\xi,\eta)}(x)$ be the Jacobi polynomials

$$P_n^{(\xi,\eta)}(x) = \kappa_n\, {}_2F_1\left({-n, n+\xi+\eta+1 \atop \xi+1} ; x \right) \qquad (4.1)$$

which are orthogonal on the interval $[0,1]$

$$\int_0^1 P_n^{(\xi,\eta)}(x)P_m^{(\xi,\eta)}(x)x^\xi(1-x)^\eta = h_n\delta_{nm}$$

(κ_n is a factor needed for the polynomials $P_n^{\xi,\eta}(x)$ to be monic).

We can introduce symmetric polynomials $S_n^{(\xi,\eta)}(x)$ by the formulas

$$S_{2n}^{(\xi,\eta)}(x) = P_n^{(\xi,\eta)}(x^2), \quad S_{2n+1}^{(\xi,\eta)}(x) = xP_n^{(\xi+1,\eta)}(x^2) \tag{4.2}$$

Then it is easily verified that the polynomials $S_n(x)$ satisfy the symmetric property $S_n(-x) = (-1)^n S_n(x)$ and are orthogonal on the interval $[-1,1]$

$$\int_{-1}^1 S_n^{(\xi,\eta)}(x)S_m^{(\xi,\eta)}(x)w(x) = h_n\delta_{nm} \tag{4.3}$$

with the weight function

$$w(x) = |x|^{2\xi+1}(1-x^2)^\eta \tag{4.4}$$

(Note that the original definition [7] by Chihara of the symmetric Jacobi polynomials $S_n(x)$ is slightly different. This is not essential for our purposes).

If $\xi = -1/2$, the polynomials $S_m^{(-1/2,\eta)}(x)$ have the weight function $w(x) = (1-x^2)^\eta$ and hence can be identified with the classical ultraspherical (Gegenbauer) polynomials. This is why the polynomials $S_n^{\xi,\eta}(x)$ are sometimes called generalized Gegenbauer polynomials [5].

Starting from the polynomials $S_n^{\xi,\eta}(x)$ we can perform the Christoffel transform [17]

$$\tilde{S}_n^{\xi,\eta}(x) = \frac{S_{n+1}^{\xi,\eta}(x) - A_n S_n^{\xi,\eta}(x)}{x+1}, \tag{4.5}$$

where

$$A_n = \frac{S_{n+1}^{\xi,\eta}(-1)}{S_n^{\xi,\eta}(-1)}$$

The polynomials $\tilde{S}_n^{\xi,\eta}(x)$ are again orthogonal on the interval $[-1,1]$ with the weight function $\tilde{w}(x)$ obtained from (4.4) as

$$\tilde{w}(x) = w(x)(x+1) = |x|^{2\xi+1}(1-x^2)^\eta(1+x) \tag{4.6}$$

(we do not take into account the normalization condition which is not essential for our purposes).

The Christoffel transform (4.5) can be presented in explicit form because the coefficients A_n are expressible in terms of the hypergeometric function ${}_2F_1(z)$ with argument $z=1$ (this is seen from formulas (4.2) and (4.1)).

Hence from (4.5), we have an explicit expression for the polynomials $\tilde{S}_n(x)$ as a linear combination of two hypergeometric functions. The polynomials $\tilde{S}_n(x)$ are not symmetric, i.e. they satisfy the general 3-term recurrence relation

$$\tilde{S}_{n+1}(x) + \tilde{b}_n\tilde{S}_n(x) + \tilde{u}_n\tilde{S}_{n-1}(x) = x\tilde{S}_n(x), \tag{4.7}$$

where the coefficients $\tilde{b}_n, \tilde{u}_n$ can be expressed in terms of the coefficients A_n using the properties of the Christoffel transform [17]. Hence these coefficients are also explicit.

One can express the weight function (4.6) in an equivalent form

$$\tilde{w}(x) = |x|^{2\xi+1}(1-x^2)^{\eta+1}(1-x)^{-1}$$

which corresponds to the Geronimus transformation [22]:

$$\tilde{S}_n(x) = S_n^{\xi,\eta+1}(x) - B_n S_{n-1}^{\xi,\eta+1}(x) \tag{4.8}$$

with some coefficients B_n.

We thus have that the same polynomials $\tilde{S}_n(x)$ can be obtained from the generalized Gegenbauer polynomials $S_n^{\xi,\eta}(x)$ by either Christoffel or Geronimus transformations.

In [8] these polynomials were presented in the form (4.8); the recurrence coefficients $\tilde{b}_n$, $\tilde{u}_n$ were derived as well in [8].

The comparison of the weight functions (4.6) and (2.10) leads to the conclusion that the little -1 Jacobi polynomials $P_n^{(-1)}(x)$ coincide with polynomials $\tilde{S}_n^{(\xi,\eta)}$, where

$$\xi = \frac{\alpha-1}{2}, \quad \eta = \frac{\beta-1}{2}.$$

We thus identified our "classical" polynomials $P_n^{(-1)}(x)$ with the "nonsymmetric" generalized Gegenbauer proposed by L.Chihara and T.Chihara [8].

Explicitly we have from (4.8)

$$P_n^{(-1)}(x) = S_n^{(\frac{\alpha-1}{2},\frac{\beta-1}{2})}(x) - B_n S_{n-1}^{(\frac{\alpha-1}{2},\frac{\beta-1}{2})}(x), \tag{4.9}$$

where

$$B_n = \frac{2n + (1-(-1)^n)\alpha}{2(\alpha+\beta+2n)}.$$

The same polynomials were also considered in [2], [3] from another point of view.

We would like to stress that other basic properties of these polynomials , e.g. the existence of a Dunkl-type operator L_0 providing the eigenvalue problem (2.15), as well as the origin of these polynomials as a limit case (q=-1) of the little q-Jacobi polynomials had not been identified.

5. The Dunkl-classical property

All "classical" orthogonal polynomials satisfy an important characteristic condition: they are "covariant" with respect to a "derivative" operator $\mathcal{D}$:

$$\mathcal{D}P_n(x) = [n]\tilde{P}_{n-1}(x), \tag{5.1}$$

where $[n]$ is a specific function of n depending on the choice of the operator $\mathcal{D}$, $\tilde{P}_n(x)$ is another set of "classical" orthogonal polynomials, and the operator D possesses the basic property of reducing the degree any polynomial by one.

Well known examples of the operator $\mathcal{D}$ are:

(i) the derivative operator $\mathcal{D} = \partial_x$;

(ii) the difference operator $\mathcal{D}f(x) = f(x+1) - f(x)$;

(iii) the q-derivative operator: $\mathcal{D}f(x) = \frac{f(xq)-f(x)}{(q-1)x}$

(iv) the Askey-Wilson operator $\mathcal{D}f(x(s)) = \frac{f(x(s+1/2))-f(x(s-1/2))}{x(s+1/2)-x(s-1/2)}$.

In the last case the function $x(s)$ is either trigonometric $x(s) = a_1q^s + a_2q^{-s} + a_0$ or quadratic $x(s) = a_2s^2 + a_1s + a_0$ with some constants a_i.

Recently it was recognized that apart from these operators there is one more operator which generates "classical" orthogonal polynomials. This operator is the Dunkl operator T_μ (1.3). Namely, in [6] it was shown that the only *symmetric* orthogonal polynomials $P_n(x)$ satisfying the property

$$T_\mu P_n(x) = [n]_\mu \tilde{P}_{n-1}(x), \quad [n]_\mu = n + (1-(-1)^n)\mu \tag{5.2}$$

are the generalized Hermite or the generalized Gegenbauer polynomials. Recall that symmetric orthogonal polynomials are defined by the property $P_n(-x) = (-1)^n P_n(x)$. The generalized Hermite polynomials $H_n^{(\mu)}(x)$ [7] are symmetric orthogonal polynomials which are orthogonal on whole real line with the weight function

$$w(x) = |x|^{2\mu}\exp(-x^2)$$

When $\mu = 0$ (i.e. in the case when the Dunkl operator T_μ becomes the derivative operator ∂_x) the generalized Hermite polynomials become the ordinary Hermite polynomials.

The generalized Gegenbauer polynomials $S_n^{(\xi,\eta)}(x)$ [7], [5] are orthogonal on the interval $[-1,1]$ with the weight function (4.4). The generalized Gegenbauer polynomials satisfy the Dunkl-classical property (5.2) with $\mu = \xi + 1/2$

In both cases it is assumed that $\mu > -1/2$.

In the present case we have correspondingly the following simple but important result

Proposition 1 *The little -1 Jacobi polynomials $P_n^{(-1)}(x)$ satisfy the Dunkl-classical property (5.2) with $\mu = \alpha/2$, where the polynomials $\tilde{P}_n(x)$ are again little -1 Jacobi polynomials with parameters* $(\alpha, \beta+2)$.

The proof of this proposition follows easily from the explicit formula (4.9) and from the fact that the generalized Gegenbauer polynomials $S_n^{(\xi,\eta)}(x)$ satisfy the Dunkl-classical propertry (5.2) [6]:

$$T_\mu S_n^{(\xi,\eta)}(x) = [n]_\mu S_{n-1}^{(\xi,\eta+1)}(x), \quad \mu = \xi + 1/2$$

In contrast to the assumptions of [6], the little -1 Jacobi polynomials are not symmetric. Hence we perhaps obtained the first example of Dunkl-classical orthogonal polynomials beyond the family of symmetric polynomials. The problem of finding all such orthogonal polynomials is an interesting open question.

All known families of "classical" orthogonal polynomials possess not only lowering operators like (5.1) but also raising operators Θ with the property

$$\Theta P_n(x) = \nu_{n+1} Q_{n+1}(x), \tag{5.3}$$

where the polynomials $Q_n(x)$ belong to the same family of classical orthogonal polynomials (albeit with different parameters).

In the case of the little -1 Jacobi polynomials it is directly verified that the operator Θ does exists and has the expression

$$\Theta f(x) = (x^2-1)f'(x) + \frac{\alpha(x-1)^2}{2x} f(-x) + \left((\beta+\alpha/2)x - 1 - \frac{\alpha}{2x}\right) f(x) \tag{5.4}$$

Given (5.4) property (5.3) holds with

$$\nu_{n+1} = n + \beta + \frac{1-(-1)^n}{2}\alpha = \beta + [n]_\mu, \quad \mu = \alpha/2$$

and $Q_n(x)$ the same monic little -1 Jacobi polynomials with parameters $(\alpha, \beta-2)$.

The generalized Hermite and Gegenbauer polynomials can be obtained from the ordinary Hermite and Gegenbauer polynomials through the acting of the Dunkl intertwining operator [9], [6].

Recall that the Dunkl intertwining operator V_μ acts on the space of polynomials by the formulas [9], [10]

$$V_\mu x^n = \sigma_n x^n, \quad \sigma_{2n-1} = \sigma_{2n} = \frac{(1/2)_n}{(\mu+1/2)_n}.$$

and is realized by the following integral representation [9]

$$V_\mu(f(x)) = \frac{\Gamma(\mu+1/2)}{\Gamma(\mu)\Gamma(1/2)} \int_{-1}^{1} f(xt)(1-t)^{\mu-1}(1+t)^\mu$$

It preserves the space of polynomials and has the fundamental intertwining property

$$T_\mu V_\mu = V_\mu \partial_x, \tag{5.5}$$

From this property it is possible to obtain the following result (see Proposition **2** below). Assume that the monic polynomials $P_n(x)$ and $Q_n(x)$ are related as

$$P_n'(x) = nQ_{n-1}(x). \tag{5.6}$$

Let us construct the monic polynomials $\tilde{P}_n(x) = \sigma_n^{-1} V_\mu P_n(x)$, $\tilde{Q}_n(x) = \sigma_n^{-1} V_\mu Q_n(x)$. Then these polynomials are correspondingly related:

$$T_\mu \tilde{P}_n(x) = [n]_\mu \tilde{Q}_{n-1}(x) \tag{5.7}$$

In particular, if all polynomials $P_n(x), Q_n(x), \tilde{P}_n(x), \tilde{Q}_n(x)$ are orthogonal then the operator V_μ allows to obtain Dunkl-classical polynomials (defined by property (5.7)) from ordinary classical polynomials (defined by property (5.6)).

In [6] it was shown that the generalized Gegenbauer polynomials $S_n^{(\xi,\eta)}(x)$ can be obtained from the ordinary Gegenbauer polynomials $S_n^{(-1/2,\eta+\mu)}(x)$ by the action of the intertwining operator V_μ:

$$S_n^{(\xi,\eta)}(x) = \sigma_n^{-1} V_\mu S_n^{(-1/2,\eta+\mu)}(x), \quad \mu = \xi + 1/2 \tag{5.8}$$

(A similar property for the generalized Hermite polynomials was obtained by Dunkl [9], [10]).

Introduce now the ordinary monic Jacobi polynomials $P_n^{(\xi,\eta)}(x)$ by the formula

$$P_n^{(\xi,\eta)}(x) = \frac{2^n(\xi+1)_n}{(\xi+\eta+n+1)_n}\, {}_2F_1\left({-n, n+\xi+\eta+1 \atop \xi+1}; \frac{1-x}{2}\right) \tag{5.9}$$

Notice that this definition differs from (4.1) by an affine transformation of the argument. The polynomials (5.9) coincide with standard Jacobi polynomials orthogonal on the interval $[-1,1]$[13].

We have the following

Proposition 2 *The little -1 Jacobi polynomials (3.14), (3.15) can be obtained from the Jacobi polynomials (5.9) by the action of the Dunkl intertwining operator*

$$P_n^{(-1)}(x) = \sigma_n^{-1} V_\mu P_n^{(\xi,\xi+1)}(x), \quad \xi = \frac{\alpha+\beta-1}{2},\ \mu = \frac{\alpha}{2} \tag{5.10}$$

The proof of this proposition is based on formula (4.9) and property (5.8).

6. Askey-Wilson algebra relations for exceptional polynomials

The Askey-Wilson polynomials satisfy the so-called AW(3)-algebra [21], [19]. Among different equivalent forms of this algebra we choose the following one, which possesses an obvious symmetry with respect to all 3 operators (see, e.g. [12]):

$$XY - qYX = \mu_3 Z + \omega_3, \quad YZ - qZY = \mu_1 X + \omega_1, \quad ZX - qXZ = \mu_2 Y + \omega_2 \tag{6.1}$$

Here q is a fixed parameter corresponding to the "base" parameter in q-hypergeometric functions for the Askey-Wilson polynomials [13]. The pairs of operators (X,Y), (Y,Z) and (Z,X) play the role of "Leonard pairs" (see [19], [12]).

The Casimir operator

$$Q = (q^2-1)XYZ + \mu_1 X^2 + \mu_2 q^2 Y^2 + \mu_3 Z^2 + (q+1)(\omega_1 X + \omega_2 q Y + \omega_3 Z) \tag{6.2}$$

commutes with all operators X, Y, Z.

The constants $\omega_i,\ i = 1,2,2$ (together with the value of the Casimir operator Q) define representations of the $AW(3)$ algebra (see [21] for details).

In the case of the little q-Jacobi operator, the realization of the AW(3) algebra is given by the operators

$$X = g(L+1+qab), \quad Y = x \tag{6.3}$$

where

$$g = \frac{1}{(q^2-1)\sqrt{ab}}$$

(the arithmetic meaning of the square root is assumed), and the operator L coincides with the difference eigenvalue operator for the little q-Jacobi polynomials in lhs of (2.5), i.e.

$$Lf(x) = a(bq - x^{-1})(f(qx) - f(x)) + (1-x^{-1})(f(q^{-1}x) - f(x)) \tag{6.4}$$

We then have the $AW(3)$ relations

$$XY - qYX = Z + \omega_3, \quad YZ - qZY = 0, \quad ZX - qXZ = Y + \omega_2, \tag{6.5}$$

where

$$\omega_2 = -\frac{b+1}{b(q+1)}, \quad \omega_3 = -\frac{a+1}{\sqrt{ab}(q+1)}.$$

The Casimir operator

$$Q = (q^2-1)XYZ + q^2Y^2 + Z^2 + (q+1)(\omega_2 qY + \omega_3 Z) \tag{6.6}$$

takes the value

$$Q = -b^{-1}$$

These relations survive in the limit $q = -1$.

Indeed, let us define the operators

$$X = \frac{L_0}{2} - \frac{1+\alpha+\beta}{2}, \quad Y = x, \quad Z = (x-1)R, \tag{6.7}$$

where L_0 is the operator defined by (2.14) and the operator Y is multiplication by x.

Then it is elementary to verify that the operators X, Y, Z satisfy the relations

$$XY + YX = Z + \alpha, \quad YZ + ZY = 0, \quad ZX + XZ = Y + \beta \tag{6.8}$$

which corresponds to the $AW(3)$ algebra with parameters $q = -1$, $\omega_3 = \alpha$, $\omega_1 = 0$, $\omega_2 = \beta$.

It is easily verified that the Casimir operator commuting with X, Y, Z is

$$Q = Y^2 + Z^2. \tag{6.9}$$

In the case of of the realization (6.7) of the operators X, Y, Z, the Casimir operator becomes the identity operator:

$$Q = I \tag{6.10}$$

Note that relations (6.8) can be considered as an anticommutator version of some Lie algebra. Like in the case of an ordinary Lie algebra, the Casimir operator is quadratic in the operators Y, Z (the cubic part XYZ disappears in both "classical" limits $q = \pm 1$).

The "canonical" representation of the algebra (6.8) is obtained in the basis $P_n(x)$ of orthogonal polynomials. In this basis the operator X is diagonal

$$XP_n(x) = \frac{\lambda_n + 1 + \alpha + \beta}{2} P_n(x)$$

while the operator Y is 3-diagonal

$$YP_n(x) = P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x)$$

The monomial basis $M_n = x^n$ provides another convenient representation, where the lower- and upper- triangular operators X, Y are

$$XM_n = \xi_n M_n + \eta_n M_{n-1}, \quad YM_n = M_{n+1},$$

with the coefficients ξ_n, η_n given by (2.13)

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References

[1] R. A. Askey and M. E. H. Ismail, *Recurrence relations, continued fractions and orthogonal polynomials*, Mem. AMS, **49** (1984), No. 300, 1–108.

[2] M. J. Atia, *An example of non-symmetric semi-classical form of class s=1. Generalization of a case of Jacobi sequence*. Int. J. Math. and Math. Sci. **24** (10) (2000), 673–689.

[3] M.J.Atia, *Some generalized Jacobi polynomials*, OPSFA 2009, Proceedings (to be published).

[4] E. Bannai and T. Ito, *Algebraic Combinatorics I: Association Schemes*. 1984. Benjamin & Cummings, Mento Park, CA.

[5] S. Belmehdi, *Generalized Gegenbauer polynomials*, J. Comput. Appl Math. **133** (2001), 195-205.

[6] Y. Ben Cheikh and M.Gaied, *Characterization of the Dunkl-classical symmetric orthogonal polynomials*, Appl. Math. and Comput. **187**, (2007) 105–114.

[7] T. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, NY, 1978.

[8] L.M.Chihara,T.S.Chihara, *A class of nonsymmetric orthogonal polynomials*. J. Math. Anal. Appl. **126** (1987), 275–291.

[9] C.F.Dunkl, *Integral kernels with reflection group invariance*. Canadian Journal of Mathematics, **43** (1991) 1213–1227.

[10] C.F.Dunkl, *Orthogonal polynomials and reflection groups*, in: Special functions 2000: current perspective and future directions, NATO Science Series, 111–128, 2001.

[11] Ya.L.Geronimus, *On polynomials orthogonal with respect to to the given numerical sequence and on Hahn's theorem*, Izv.Akad.Nauk, **4** (1940), 215-228 (in Russian).

[12] T.Ito, P.Terwilliger, *Double Affine Hecke Algebras of Rank 1 and the Z_3-Symmetric Askey-Wilson Relations*, SIGMA **6** (2010), 065, 9 pages.

[13] R.Koekoek, R.Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Report no. 98-17, Delft University of Technology, 1998.

[14] R. Koekoek,P. Lesky, R. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their Q-analogues*, Springer-Verlag, 2010.

[15] K.H. Kwon, L.L. Littlejohn and B.H. Yoo, *New characterizations of classical orthogonal polynomials*, Indag. Mathem., N.S., **7** (1996), 199–213.

[16] D.Leonard, *Orthogonal Polynomials, Duality and Association Schemes*, SIAM J. Math. Anal. **13** (1982) 656–663.

[17] V.Spiridonov, L.Vinet and A.Zhedanov, *Spectral transformations, self-similar reductions and orthogonal polynomials*, J.Phys. A: Math. and Gen. **30** (1997), 7621–7637.

[18] G. Szegő, Orthogonal Polynomials, fourth edition, AMS, 1975.

[19] P. Terwilliger, *Two linear transformations each tridiagonal with respect to an eigenbasis of the other.* Linear Algebra Appl. **330** (2001) 149–203.

[20] L.Vinet and A.Zhedanov, *Generalized Bochner theorem: characterization of the Askey-Wilson polynomials*, J.Comp.Appl.Math., **211** (2008) 45 – 56.

[21] A. S. Zhedanov. *"Hidden symmetry" of Askey-Wilson polynomials*, Teoret. Mat. Fiz. **89** (1991) 190–204. (English transl.: Theoret. and Math. Phys. **89** (1991), 1146–1157).

[22] A.S. Zhedanov, *Rational spectral transformations and orthogonal polynomials*, J. Comput. Appl. Math. **85**, no. 1 (1997), 67–86.