OPERATOR ALGEBRA QUANTUM GROUPS OF UNIVERSAL GAUGE GROUPS

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Abstract. In this paper, we quantize universal gauge groups such as $SU(\infty)$, in the σ-C^*-algebra setting. More precisely, we propose a concise definition of σ-C^*-quantum groups and explain the concept here. At the same time, we put this definition in the mathematical context of countably compactly generated groups as well as C^*-compact quantum groups.

If H is a compact and Hausdorff topological group, then the C^*-algebra of all continuous functions $\mathrm{C}(H)$ admits a comultiplication map $\Delta : \mathrm{C}(H) \to \mathrm{C}(H)\hat{\otimes}\mathrm{C}(H)$ arising from the multiplication in H. This observation motivated Woronowicz (see, for instance, [12]), amongst others such as Soibelman [10], to introduce the notion of a *C^*-compact quantum group* in the setting of operator algebras as a unital C^*-algebra with a coassociative comultiplication, satisfying a few other conditions. If the group H is only locally compact then the situation becomes significantly more difficult. One of the reasons is that the multiplication map $m : H \times H \to H$ is no longer a proper map and one needs to introduce multiplier algebras of C^*-algebras to obtain a comultiplication, see for instance, Kustermans-Vaes [5], and the excellent and thorough introduction to this theory [6]. In the sequel we show that if $H = \varinjlim_n H_n$ is a countably compactly generated group, i.e., $H_n \subset H_{n+1}$ compact and Hausdorff topological groups for all $n \in \mathbb{N}$ and H is the direct limit, then a story similar to the compact group case goes through using the general framework of σ-C^*-algebras as systematically developed by Phillips [8, 9], motivated by some earlier work by Arveson, Mallios, Voiculescu, amongst others. There is a clean formulation of, what we call, *σ-C^*-quantum groups*, which are noncommutative generalizations of $\mathrm{C}(H)$. Examples of such groups are $U(\infty) = \varinjlim_n U(n)$, $SU(\infty) = \varinjlim_n SU(n)$, where $U(n)$ (resp. $SU(n)$) are the unitary (resp. special unitary) groups. They are also known in the physics literature as *universal gauge groups*, see Harvey-Moore [4] and Carey-Mickelsson [3]. Such spaces are not locally compact and hence the existing literature on quantum groups cannot handle them. Moreover, locally compact groups that are not compact, are also not countably compactly generated. We also discuss in detail the interesting example of quantum versions of the universal special unitary group, $C(SU_q(\infty))$.

A *pro C^*-algebra* is an inverse limit of C^*-algebras and $*$-homomorphisms, where the inverse limit is constructed inside the category of all topological $*$-algebras and continuous $*$-homomorphisms. For the general theory of topological $*$-algebras one may refer to, for instance, [7]. The topology of a pro C^*-algebra is necessarily complete and Hausdorff. It is not a C^*-algebra in general; it would be so if, for instance, the directed set is finite.

2010 *Mathematics Subject Classification.* 58B32, 58B34, 46L65.

Key words and phrases. σ-C^*-quantum groups, universal gauge groups, operator algebras.

Acknowledgements. Both authors gratefully acknowledge support under the Australian Research Council's *Discovery Projects* funding scheme.

If the directed set is countable, then the inverse limit is called a *σ-C^*-algebra*. One can choose a linearly directed cofinal subset inside any countable directed set and the passage to a cofinal subset does not change the inverse limit. Therefore, we shall always identify a σ-C^*-algebra $A \cong \varprojlim_n A_n$, where $n \in \mathbb{N}$. The inverse limit could have also been constructed inside the category of C^*-algebras; however, the two results will not agree. For instance, if $H = \varinjlim_n H_n$ as above, then the inverse limit $\varprojlim_n \mathrm{C}(H_n)$ inside the category of topological $*$-algebras is $\mathrm{C}(H)$, whereas that inside the category of C^*-algebras is $\mathrm{C}_b(H)$, i.e., the norm bounded functions on H. It is known that $\mathrm{C}_b(H) \cong \mathrm{C}(\beta H)$, where βH is the Stone–Čech compactification of H. Therefore, if one wants to model a space via its algebra of all continuous functions then the former inverse limit is the appropriate one. *Henceforth, the inverse limits are always constructed inside the category of topological $*$-algebras.* It is known that any $*$-homomorphism between two pro C^*-algebras is automatically continuous, provided the domain is a σ-C^*-algebra (see Theorem 5.2. of [8]). Furthermore, the category of commutative and unital σ-C^*-algebras with unital $*$-homomorphisms (automatically continuous) is contravariantly equivalent to the category of countably compactly generated and Hausdorff spaces with continuous maps via the functor $X \mapsto \mathrm{C}(X)$ (see Proposition 5.7. of [8]). If $A \cong \varprojlim_n A_n$, $B \cong \varprojlim_n B_n$ are two σ-C^*-algebras, then the minimal tensor product is defined to be $A\hat{\otimes}_{\min}B = \varprojlim_n A_n\hat{\otimes}_{\min}B_n$. *Henceforth, $A\hat{\otimes}B$ will always denote the minimal or spatial tensor product between σ-C^*-algebras.*

If H is a countably compactly generated and Hausdorff topological group, although the multiplication map $m : H \times H \to H$ is not proper, we get an induced comultiplication map $m^* : \mathrm{C}(H) \to \mathrm{C}(H \times H) \cong \mathrm{C}(H)\hat{\otimes}\mathrm{C}(H)$, which will be coassociative owing to the associativity of m. Motivated by the definition of Woronowicz (see also Definition 1 of [5]), we propose:

Definition. *A unital σ-C^*-algebra A is called a σ-C^*-quantum group if there is a unital $*$-homomorphism $\Delta : A \to A\hat{\otimes}A$ which satisfies coassociativity, i.e., $(\Delta\hat{\otimes}\mathrm{id})\Delta = (\mathrm{id}\hat{\otimes}\Delta)\Delta$ and such that the linear spaces $\Delta(A)(A\hat{\otimes}1)$ and $\Delta(A)(1\hat{\otimes}A)$ are dense in $A\hat{\otimes}A$.*

Lemma. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable inverse system of C^*-algebras and let $B_n \subset A_n$ be a dense subset for all n. Then $\varprojlim_n B_n$ is a dense subset of the σ-C^*-algebra $\varprojlim_n A_n$.*

Proof. This is the Corollary to Proposition 9 in §4-4 of [2]. □

Example. *Let $\{A_n, \theta_n : A_n \to A_{n-1}\}_{n\in\mathbb{N}}$ be a countable inverse system of C^*-compact quantum groups with θ_n surjective for all n. Furthermore, let us assume that the comultiplication homomorphisms Δ_n form a morphism of inverse systems of C^*-algebras $\{\Delta_n\} : \{A_n\} \to \{A_n\hat{\otimes}A_n\}$. Then $(A, \Delta) = (\varprojlim_n A_n, \varprojlim_n \Delta_n)$ is a σ-C^*-quantum group. Indeed, the density of the linear spaces $\Delta(A)(A\hat{\otimes}1)$ and $\Delta(A)(1\hat{\otimes}A)$ inside $A\hat{\otimes}A$ follow from the above Lemma.*

If G is set of generators and R is a set of admissible relations (see Definition 1.1. of [1]) one can always form a universal C^*-algebra $C^*(G, R)$. For instance, the universal C^*-algebra generated by one generator x, subject to the relation $x^*x = 1 = xx^*$, is isomorphic to $\mathrm{C}(S^1)$. Let $\{(G_i, R_i)\}_{i\in\mathbb{N}}$ be a countable family of admissible generators and relations, so that $C^*(G_i, R_i)$ exist for all i. Let us further assume that all the relations R_i are algebraic ($\|x\| \leqslant 1$ is not algebraic whereas $x^*x = 1$ is) and that there are surjective maps $\theta_i : G_i \to G_{i-1}$, so that the following diagrams commutes:

(1)
$$\begin{array}{ccc} G = \varprojlim_i G_i & \xrightarrow{p_i} & G_i \\ & \searrow^{p_{i-1}} & \downarrow{\scriptstyle \theta_i} \\ & & G_{i-1}. \end{array}$$

Morever, we require that $\theta_i(G_i)$ is compatible with R_{i-1} for all $i \geqslant 1$ in the following sense: Let $\mathbb{C}[G_i \coprod G_i^*]$ denote the free complex algebra on the generators $G_i \coprod G_i^*$. Here $\coprod$ denotes disjoint union and $G_i^* = \{g^* \,|\, g \in G_i\}$ (formal adjoints). Since the relations are algebraic, each R_i is a subset of $\mathbb{C}[G_i \coprod G_i^*]$. The maps θ_i extend uniquely to a complex algebra homomorphism $\theta_i : \mathbb{C}[G_i \coprod G_i^*] \to \mathbb{C}[G_{i-1} \coprod G_{i-1}^*]$. We require that $\theta_i(R_i) = R_{i-1}$. The surjective maps θ_i induce surjective $*$-homomorphisms $\theta_i : C^*(G_i, R_i) \to C^*(G_{i-1}, R_{i-1})$ and $\{C^*(G_i, R_i), \theta_i\}_{i \in \mathbb{N}}$ forms a countable inverse system of C^*-algebras. We may form the inverse limit σ-C^*-algebra $\varprojlim_i C^*(G_i, R_i)$. Setting $R = \varprojlim_i R_i$ with canonical projection maps $p_i : R \to R_i$, one observes that (G, R) is a weakly admissible set of generators and relations (see Definition 1.3.4 and Example 1.3.5.(1) of [9]), so that one may construct the universal pro C^*-algebra $C^*(G, R)$ (see Proposition 1.3.6. of [9]). The maps p_i induce surjective $*$-homomorphisms $p_i : C^*(G, R) \to C^*(G_i, R_i)$ and Equation (1) says that they are compatible with the inverse system. Consequently, there is a canonical induced continuous $*$-homomorphism $\eta : C^*(G, R) \to \varprojlim_i C^*(G_i, R_i)$.

Theorem. *The $*$-homomorphism $\eta : C^*(G, R) \to \varprojlim_i C^*(G_i, R_i)$ is an isomorphism.*

Proof. Let us first show that $\varprojlim_i C^*(G_i, R_i)$ is a universal representation of (G, R), i.e., there is a map $\iota : G \to \varprojlim_i C^*(G_i, R_i)$ such that $\iota(G)$ satisfies R inside $\varprojlim_i C^*(G_i, R_i)$ and given any map $\rho : G \to B$ (B a pro C^*-algebra) such that $\rho(G)$ satisfies R inside B, there is a unique continuous $*$-homomorphism $\kappa : \varprojlim_i C^*(G_i, R_i) \to B$ making the following diagram commute:

(2)
$$\begin{array}{ccc} G & \xrightarrow{\iota} & \varprojlim_i C^*(G_i, R_i) \\ & \searrow^{\rho} & \downarrow{\scriptstyle \kappa} \\ & & B. \end{array}$$

The map $\iota : G \to \varprojlim_i C^*(G_i, R_i)$ is defined as $g \mapsto \{p_i(g)\}$. It follows from Equation (1) that $\{p_i(g)\}$ is a coherent system of elements in $\varprojlim_i C^*(G_i, R_i)$. Let $F(G)$ (resp. $F(G_i)$) denote the free nonunital complex $*$-algebra generated by $G \coprod G^*$ (resp $G_i \coprod G_i^*$). Then $C^*(G, R)$ (resp. $C^*(G_i, R_i)$) is defined via a certain Hausdorff completion of $F(G)$ (resp. $F(G_i)$) with respect to representations in pro C^*-algebras (resp. C^*-algebras) satisfying R (resp. R_i). By the above Lemma it suffices to define κ on coherent sequences of the form $\{w_i\}$, where w_i is an finite $\mathbb{C}$-linear combination of words in $G_i \coprod G_i^*$, which extends uniquely to a continuous function on the Hausdorff completions. There is a unique choice for $\kappa(w_i)$ forced by the compatibility requirement, i.e., $\kappa(w_i) = \rho(w_i)$. Note that ρ extends uniquely to a $*$-homomorphism $F(G) \to B$. By construction κ is a $*$-homomorphism and it is continuous since $\varprojlim_i C^*(G_i, R_i)$ is a σ-C^*-algebra. Setting $B = C^*(G, R)$ in the above diagram one finds a $*$-homomorphism, which can be checked to be the inverse of η. $\square$

Remark. *All matrix C^*-compact quantum groups considered, for instance, in* [11, 12]*, such that the relations put a bound on the norms of the generators, are of the form $C^*(G,R)$, where (G,R) is an admissible set of generators and relations.*

Our next goal is to outline the construction of quantum universal special unitary group, $C(SU_q(\infty))$. First recall that for $q \in (0,1)$, the C^*-algebra $C(SU_q(n))$ is the universal unital C^*-algebra generated by n^2 elements $\{u_{ij} : i,j = 1,\dots,n\}$ which satisfy the following relations

$$\sum_{k=1}^{n} u_{ik}u_{jk}^* = \delta_{ij}, \qquad \sum_{k=1}^{n} u_{ki}^* u_{kj} = \delta_{ij} \tag{3}$$

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n} E_{i_1 i_2 \cdots i_n} u_{j_1 i_1}\cdots u_{j_n i_n} = E_{j_1 j_2 \cdots j_n} \tag{4}$$

where

$$E_{i_1 i_2 \cdots i_n} \quad := \quad \begin{cases} 0 & \text{whenever} \quad i_1, i_2, \cdots, i_n \text{ are not distinct;} \\ (-q)^{\ell(i_1,i_2,\cdots,i_n)}. & \end{cases}$$

Here $\ell(\sigma)$ denotes its length of a permutation σ on $\{1,2,\cdots,n\}$. The C^*-algebra $C(SU_q(n))$ has a C^*-compact quantum group structure with the comultiplication Δ given by

$$\Delta(u_{ij}) := \sum_k u_{ik} \otimes u_{kj}.$$

Denoting the generators of $C(SU_q(n-1))$ by v_{ij}, the map $\theta_n : C(SU_q(n)) \to C(SU_q(n-1))$ defined by

$$\theta_n(u_{ij}) \quad := \quad \begin{cases} v_{ij} & \text{if } 1 \le i,j \le n-1, \\ \delta_{ij} & \text{otherwise.} \end{cases} \tag{5}$$

is a surjective unital C^*-algebra homomorphism such that the following diagram commutes

$$\begin{array}{ccc} C(SU_q(n)) & \xrightarrow{\quad \Delta_n \quad} & C(SU_q(n))\hat{\otimes} C(SU_q(n)) \\ \Big\downarrow{\scriptstyle \theta_n} & & \Big\downarrow{\scriptstyle \theta_n \hat{\otimes} \theta_n} \\ C(SU_q(n-1)) & \xrightarrow{\quad \Delta_{n-1} \quad} & C(SU_q(n-1))\hat{\otimes} C(SU_q(n-1)) \end{array} \tag{6}$$

One can verify this assertion by a routine calculation on the generators. Consequently, for $n \geqslant 2$ the families $\{C(SU_q(n)), \theta_n\}$ and $\{C(SU_q(n))\hat{\otimes}C(SU_q(n)), \theta_n\hat{\otimes}\theta_n\}$ form countable inverse systems of C^*-algebras and $\{\Delta_n\} : \{C(SU_q(n))\} \to \{C(SU_q(n))\hat{\otimes}C(SU_q(n))\}$ becomes a morphism of inverse systems of C^*-algebras. We may form the inverse limit σ-C^*-algebra $\varprojlim_n C(SU_q(n))$, which we denote by $C(SU_q(\infty))$. Then $C(SU_q(\infty))$ is a σ-C^*-quantum group, since it is the inverse limit of C^*-compact quantum groups, where the comultiplication Δ on $C(SU_q(\infty))$ is defined as $\Delta = \varprojlim_n \Delta_n$ (see the Example above).

An immediate application of the above Theorem enables us to describe $C(SU_q(\infty))$ explicitly in terms of generators and relations.

Corollary. *Let $G = \{u_{ij}\}_{i,j\in\mathbb{N}}$ and R denote the inverse limit of the relations in equations* (3) *and* (4) *for all $n \geqslant 2$. Let $p_n : G \to G_n$ be the canonical map, where $G_n = \{u_{ij}\}_{1\le i,j\le n}$. By construction, $p_n(R) = R_n$, where R_n denotes the relations in equations* (3) *and* (4)*. Then*

$$C(SU_q(\infty)) = C^*(G, R).$$

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