ON THE MORDELL–WEIL GROUP OF THE ELLIPTIC CURVE $y^2 = x^3 + n$

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ABSTRACT. We study an infinite family of Mordell curves (i.e. the elliptic curves in the form $y^2 = x^3 + n$, $n \in \mathbb{Z}$) over \mathbb{Q} with three explicit integral points. We show that the points are independent in certain cases. We describe how to compute bounds of the canonical heights of the points. Using the result we show that any pair in the three points can always be a part of a basis of the free part of the Mordell–Weil group.

1. INTRODUCTION

Let E be an elliptic curve over a number field K. It is known that the set of rational points E(K) is a finitely generated abelian group by the Mordell–Weil theorem. If the absolute value of the discriminant of E is not large, we can practically use Cremona's program 'mwrank'. However there is no known algorithm which determines the structure of E(K) even if $K = \mathbb{Q}$. The difficulties come from the free part of the group. We are interested in the families of elliptic curves of which we can at least partially determine the structure of the Mordell–Weil group, that is, the families which have explicit points which can be in a system of generators of the Mordell–Weil group. In the paper [3] Duquesne considered an infinite family of elliptic curves in the form $y^2 = x^3 - nx$. He showed that the curves in the family have two explicit integral points which can always be in a system of generators. Recently, the first author and Terai ([4]) generalized Duquesne's theorem on generators and showed that the same is true for infinitely many binary forms n = n(k, l) in $\mathbb{Z}[k, l]$. In this paper we consider an infinite family of elliptic curves in the form of $y^2 = x^3 + n$ with three explicit integral points.

Let a, b be integers and

(1.1)
$$E_{a,b}: y^2 = x^3 + a^6 + 16b^6$$

the elliptic curve over \mathbb{Q} . We put

(1.2) $P_1 = (-a^2, 4b^3), P_2 = (2ab, a^3 + 4b^3), P_3 = (-2ab, a^3 - 4b^3).$

Then it is easy to see that they are in $E_{a,b}(\mathbb{Q})$. In this paper we prove the following theorem.

Theorem 1.3. Assume that a, b are relatively prime integers with $a, b \ge 3$ such that $a^6 + 16b^6$ is square-free, ab is odd and b is divisible by 3 but not by 9. Then the rank of the Mordell–Weil group $E_{a,b}(\mathbb{Q})$ is at least 3 and any pair of two points $\{P_i, P_j\}$ $(i = 1, 2, 3, i \ne j)$ can always be in a system of generators of $E_{a,b}(\mathbb{Q})$.

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Remark 1.4. If *n* is square-free and not equal to 1, the elliptic curve $y^2 = x^3 + n$ has no rational torsion points by [6, Theorem 5.3]. Therefore P_1 , P_2 , P_3 are non-torsion in the situation of Theorem 1.3.

We prove Theorem 1.3 along similar lines to Duquesne's ([3]). First we compute bounds of the canonical heights of P_i 's (i = 1, 2, 3). For that purpose we use the decomposition of the canonical height into the sum of local heights. The nonarchimedean part of local heights is computed by Silverman's algorithm. To compute the archimedean part of local heights, we use Tate's series. Then the archimedean part is in the form $\log |x(P_i)| + \frac{1}{4} \sum_{n=0}^{\infty} 4^{-n} \log |z(2^n P_i)|$, where z(P) is a certain function. But to do this, it is necessary that the x-coordinates of the points on the curve are away from zero. To deal with the difficulty, we shift the elliptic curve in the direction of x-axis. We set the shifting width by functions of a, b such as $2a^2 + 4b^2$, $3a^2 + 4b^2$. Then the computation of the bounds comes down to a maximum-minimum problem of elementary functions. Further we find in our case z(P) above is bounded independently of a, b and P. Thanks to this, we obtain an upper bound and a lower bound whose difference is a constant.

On the other hand we find a uniform lower bound of the canonical height. It is a bound independent of the points on the elliptic curve. This is computed by Cohen's algorithm.

Next using those bounds we estimate the lattice indices of $\{P_i, P_j\}$ $(i, j = 1, 2, 3, i \neq j)$ (for the definition of the lattice index see Section 5). The key theorem is Siksek's theorem which comes from the theory of quadratic forms.

The goal of the proof of Theorem 1.3 is to show that the lattice indices equal 1. By the estimete using Siksek's theorem we can show that the lattice indices are less than 5. On the other hand we prove that the lattice indices are not divisible by 2, 3 by an argument of the descent. This completes the proof.

The organization of this paper is as follows. In Section 2 we review basic notations of elliptic curves. We also review the canonical height and the local height function. In Section 3 we compute bounds of the canonical heights of P_1, P_2, P_3 . In Section 4 we compute an uniform lower bound of the canonical height. In Section 5 we estimate the lattice indices by applying Siksek's theorem to the results of Sections 3 and 4. In Section 6 we prove that the lattice indices are not divisible by 2, 3 by an argument of the descent. Then we complete the proof of Theorem 1.3. Further we prove that the family of the elliptic curves satisfying the condition of Theorem 1.3 is an infinite family. Finally in Section 7 we compute the bounds of z(P), which are used in Section 3.

2. Preliminaries

The standard symbols \mathbb{Q} , \mathbb{R} , \mathbb{C} and \mathbb{Z} will denote respectively the set of rational, real and complex numbers and the rational integers. We denote the discrete valuation on \mathbb{Z} at the prime p by $v_p(\cdot)$. We denote the set of all places of a number field K by M_K .

Throughout this paper, we assume that $a, b \in \mathbb{Z}$, $a, b \geq 3$, gcd(a, b) = 1 and $m = a^6 + 16b^6$.

As usual we write the Weierstrass equation for elliptic curves ${\cal E}$ over a number field K as

(2.1)
$$E: y^2 + a_1 x y + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \ (a_1, a_2, a_3, a_4, a_6 \in K).$$

Since the characteristic of K is not equal to 2, by completing the square of the lefthand side we have

(2.2)
$$(2y + a_1x + a_3)^2 = 4x^3 + b_2x^2 + 2b_4x + b_6,$$

where

(2.3)
$$b_2 = a_1^2 + 4a_2, \quad b_4 = 2a_4 + a_1a_3, \quad b_6 = a_3^2 + 4a_6, \\ b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2.$$

Further, by putting

$$c_4 = b_2^2 - 24b_4, \quad c_6 = -b_2^2b_8 + 36b_2b_4 - 216b_6,$$

we have

$$\{108(2y+a_1x+a_3)\}^2 = (36x+3b_2)^3 - 27c_4(36x+3b_2) - 54c_6.$$

We also define the discriminant of E as

(2.4)
$$\Delta = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6$$

Using the form (2.3), we can write

(2.5)
$$x(2P) = \frac{x^4 - b_4 x^2 - 2b_6 x - b_8}{4x^3 + b_2 x^2 + 2b_4 x + b_6}$$

for $P = (x, y) \in E$.

Next we define the canonical height, which is a powerful tool to consider the arithmetic of elliptic curves. Let E be an elliptic curve over \mathbb{Q} and $P = (x, y) \in E(\mathbb{Q})$. If x = n/d and gcd(n, d) = 1, we define the naïve height of P by $h(P) = \max\{\log |n|, \log |d|\}$ ([12, p. 202]) and the canonical height of P by

$$\hat{h}(P) = \lim_{n \to \infty} \frac{h(2^n P)}{4^n}$$

([12, p. 248]).

Remark 2.6. In our definition the value of \hat{h} is twice of those in [12], [1] and [11].

The canonical height has the following properties.

- $\hat{h}(P) = 0$ if and only if P is a torsion point.
- $\hat{h}(kP) = k^2 \hat{h}(P)$ for all $P \in E(\mathbb{Q})$ and all $k \in \mathbb{Z}$.
- \hat{h} is a quadratic form on E.

For details see also [12, Chapter VIII Section 9].

Our computations of the canonical height is done by using the local height. We recall the existence of the local height function as follows.

Theorem 2.7. (Néron, Tate, [11, p. 341]) Let K be a number field, v a place and K_v its completion respect to an absolute value $|\cdot|_v$. Let E be the elliptic curve over K given by (2.1). Then there exists a unique function $\hat{\lambda}_v : E(K_v) \setminus O \to \mathbb{R}$ which has the following three properties.

(1) For all $P \in E(K_v)$ with $2P \neq O$,

$$\hat{\lambda}_{v}(2P) = 4\hat{\lambda}_{v}(P) - 2\log|2y(P) + a_{1}x(P) + a_{3}|_{v}$$

(2) The limit $\lim_{\substack{P \to O \\ v \text{-adic}}} (\hat{\lambda}_v(P) - \log |x(P)|_v)$ exists.

(3) $\hat{\lambda}_v$ is bounded on any v-adic open subset of $E(K_v)$ disjoint from O.

The function λ_v above is called the *local height function*. If we have to specify the elliptic curve, we may use the notation such as $\hat{\lambda}_{E,v}$. The canonical height can be decomposed as the sum of local heights. The sum of the local heights for all archimedean (resp. non-archimedean) places is called the archimedean (resp. nonarchimedean) part of the canonical height and denoted by $\hat{h}_f(P)$ (resp. $\hat{h}_{\infty}(P)$). We only consider the case $K = \mathbb{Q}$ and in this situation,

(2.8)
$$\hat{h}(P) = \hat{h}_f(P) + \hat{h}_\infty(P) = \sum_{p:\text{prime}} \hat{\lambda}_p(P) + \hat{\lambda}_\infty(P).$$

Let $d \in K$ and

$$E': (y')^2 + a_1'x'y' + a_3'y' = (x')^3 + a_2'(x')^2 + a_4'x' + a_6'$$

the elliptic curve obtained by making the substitution

(2.9)
$$x' = x + d, \ y' = y$$

in (2.1). Then

(2.10)
$$a_1' = a_1, \ a_2' = a_2 - 3d, \ a_3' = a_3 - da_1, a_4' = a_4 - 2da_2 + 3d^2, \ a_6' = a_6 - da_4 + d^2a_2 - d^3.$$

Now let $P \in E(K_v)$ and $P' = (x(P) + d, y(P)) \in E'(K_v)$. It is clear that the map $E(K_v) \ni P \mapsto P' \in E'(K_v)$ is a group isomorphism.

Lemma 2.11. In the situation above, we have $\hat{\lambda}_{E,v}(P) = \hat{\lambda}_{E',v}(P')$.

Proof. To see this, it is sufficient to show that the function $f : E'(K_v) \to \mathbb{R}$ defined by $f(P') = \hat{\lambda}_{E,v}(P)$ satisfies the three properties of $\hat{\lambda}_v$ in Theorem 2.7.

The property (1) follows from the equality

$$2y' + a_1'x' + a_3' = 2y + a_1(x+d) + a_3 - da_1 = 2y + a_1x + a_3.$$

For the property (2), we have

$$\lim_{\substack{P' \to O' \\ v - \text{adic}}} \{ f(P') - \log |x'(P')|_v \} = \lim_{\substack{P \to O \\ v - \text{adic}}} \{ \hat{\lambda}_{E,v}(P) - \log |x(P)|_v + d|_v \}$$
$$= \lim_{\substack{P \to O \\ v - \text{adic}}} \left\{ \hat{\lambda}_{E,v}(P) - \log |x(P)|_v - \log \left| \frac{x(P) + d}{x(P)} \right|_v \right\}$$
$$= \lim_{\substack{P \to O \\ v - \text{adic}}} \left\{ \hat{\lambda}_{E,v}(P) - \log |x(P)|_v - \log \left| 1 + \frac{d}{x(P)} \right|_v \right\}$$
$$= \lim_{\substack{P \to O \\ v - \text{adic}}} \{ \hat{\lambda}_{E,v}(P) - \log |x(P)|_v \}.$$

The property (3) is clearly satisfied.

3. Computing the canonical height

Let $E_{a,b}$ be the elliptic curve (1.1) and P_1, P_2, P_3 the rational points on $E_{a,b}$ defined in (1.2).

Proposition 3.1. If ab is odd, $v_3(b) = 1$ and m is square-free, then the canonical heights of the points P_1 , P_2 , P_3 have the following bounds

$$\frac{1}{3}\log m - 0.7441 < \hat{h}(P_1) < \frac{1}{3}\log m + 0.5409,$$

$$\frac{1}{3}\log m - 0.7579 < \hat{h}(P_2) < \frac{1}{3}\log m + 1.0515,$$

$$\frac{1}{3}\log m - 0.5113 < \hat{h}(P_3) < \frac{1}{3}\log m + 0.5665.$$

Proof of Proposition 3.1. We use the decomposition (2.8) to estimate the canonical height. We first estimate the archimedean part $\hat{h}_{\infty}(P) = \hat{\lambda}_{\infty}(P)$ by using Tate's series with Silverman's shifting trick ([11]).

Let E be the elliptic curve defined by (2.1). For $P \in E(\mathbb{R})$, we put

(3.2)

$$t = t(P) := 1/x(P),$$

$$z = z(P) := 1 - b_4 t^2 - 2b_6 t^3 - b_8 t^4,$$

$$w = w(P) := 4t + b_2 t^2 + 2b_4 t^3 + b_6 t^4,$$

where b_2 , b_4 , b_6 , b_8 are as in (2.3). Note that we have x(2P) = z(P)/w(P). By the property of the local height (Theorem 2.7 (1)) we have

$$\hat{\lambda}_{\infty}(2P) = 4\hat{\lambda}_{\infty}(P) - 2\log|2y(P) + a_1x(P) + a_3|.$$

Then using (2.2), we have

$$\begin{aligned} \hat{\lambda}_{\infty}(2P) - \log|x(2P)| &= 4\hat{\lambda}_{\infty}(P) - 2\log|2y(P) + a_1x(P) + a_3| - \log|x(2P)| \\ &= 4\hat{\lambda}_{\infty}(P) - \log|4x(P)^3 + b_2x(P)^2 + 2b_4x(P) + b_6| \\ &- \log|z(P)/w(P)| \\ &= 4\{\hat{\lambda}_{\infty}(P) - \log|x(P)|\} - \log|z(P)|. \end{aligned}$$

Putting $\mu(P) := \hat{\lambda}_{\infty}(P) - \log |x(P)|,$

$$\mu(2P) = 4\mu(P) - \log|z(P)|.$$

So if we ignore the convergence, we have

$$\mu(P) = \frac{1}{4} \sum_{n=0}^{\infty} 4^{-n} \log |z(2^n P)|.$$

In fact, by Tate's theorem ([11, Theorem 1.2]) if there is $\epsilon > 0$ such that $|x(P)| > \epsilon$ for all $P \in E(\mathbb{R})$, then for any $P \in E(\mathbb{R})$, $\log |z(2^n P)|$ is bounded independently of n and therefore

$$\hat{\lambda}_{\infty}(P) = \log |x(P)| + \frac{1}{4} \sum_{n=0}^{\infty} 4^{-n} \log |z(2^n P)|.$$

For $d \in \mathbb{Q}$ and $P \in E(\mathbb{R})$, the point P' = (x(P) + d, y(P)) is on the curve (3.3) $E' : (y')^2 + a_1'x'y' + a_3'y' = (x')^3 + a_2'(x')^2 + a_4'x' + a_6',$ where

$$a_1' = a_1, \ a_2' = a_2 - 3d, \ a_3' = a_3 - da_1,$$

 $a_4' = a_4 - 2da_2 + 3d^2, \ a_6' = a_6 - da_4 + d^2a_2 - d^3$

as we saw in (2.10). We similarly put

(3.4)

$$t' = t'(P') := 1/x'(P'),$$

$$z' = z'(P') := 1 - b'_{4}(t')^{2} - 2b'_{6}(t')^{3} - b'_{8}(t')^{4},$$

$$w' = w'(P') := 4t' + b'_{2}(t')^{2} + 2b'_{4}(t')^{3} + b'_{6}(t')^{4},$$

where b'_2 , b'_4 , b'_6 , b'_8 are the values obtained by replacing a_1, \ldots, a_6 by a'_1, \ldots, a'_6 in (2.3).

The reason why we make this substitution is that we obtain the Weierstrass model to which we can apply Tate's theorem above. We call this the *shifting trick* following Silverman.





Now let $E = E_{a,b}$. If $P \in E(\mathbb{R})$, then $x(P) \geq -m^{1/3}$. So if we take d such that $d > m^{1/3}$, then $x'(P') = x(P) + d \geq -m^{1/3} + d > 0$. Therefore the assumption of Tate's result is satisfied and we have

$$\hat{\lambda}_{E',\infty}(P') = \log |x'(P')| + \frac{1}{4} \sum_{n=0}^{\infty} 4^{-n} \log |z'(2^n P')|.$$

On the other hand, by Lemma 2.11 $\hat{\lambda}_{E',\infty}(P') = \hat{\lambda}_{E,\infty}(P)$. So we have

$$\hat{\lambda}_{\infty}(P) = \hat{\lambda}_{E,\infty}(P) = \log |x'(P')| + \frac{1}{4} \sum_{n=0}^{\infty} 4^{-n} \log |z'(2^n P')|.$$

Let us compute $\hat{\lambda}_{\infty}(P_2)$ by this formula, taking $d = 2a^2 + 4b^2$. Then the condition $d > m^{1/3}$ is clearly satisfied. So we have

(3.5)
$$\hat{\lambda}_{\infty}(P_2) = \log |x'(P_2')| + \frac{1}{4} \sum_{n=0}^{\infty} 4^{-n} \log |z'(2^n P_2')|.$$

We put X = a/b and compute $x'(P'_2)$, $z'(P'_2)$, $z'(2P'_2)$, $z'(4P'_2)$. Since $x(P_2) = 2ab$, the following hold.

- $x'(P'_2) = 2ab + 2a^2 + 4b^2$ (see (1.2) for the coordinate of P_2)
- $z'(P'_2) = (X^8 2X^7 + 2X^6 + 8X^5 + 2X^4 + 16X^3 + 16X^2 32X + 32)/(2X^8 + 8X^7 + 28X^6 + 56X^5 + 98X^4 + 112X^3 + 112X^2 + 64X + 32)$
- $z'(2P'_2) = (X^{32} + 4X^{31} + 2X^{30} 32X^{29} + 124X^{28} 64X^{27} + 144X^{26} + 864X^{25} + 288X^{24} + 1344X^{23} + 9408X^{22} + 2688X^{21} + 8256X^{20} + 34176X^{19} + 16512X^{18} + 59904X^{17} + 237600X^{16} + 119808X^{15} + 132096X^{14} + 546816X^{13} + 264192X^{12} + 344064X^{11} + 2408448X^{10} + 688128X^9 + 589824X^8 + 3538944X^7 + 1179648X^6 2097152X^5 + 8126464X^4 4194304X^3 + 1048576X^2 + 4194304X + 2097152) /(2X^{32} 16X^{31} + 64X^{30} 96X^{29} 96X^{28} + 864X^{27} 1376X^{26} + 256X^{25} + 6800X^{24} 13120X^{23} + 16640X^{22} + 35200X^{21} 70400X^{20} + 141184X^{19} + 121472X^{18} 217600X^{17} + 878624X^{16} + 123904X^{15} 570368X^{14} + 2603008X^{13} 1312768X^{12} 753664X^{11} + 6422528X^{10} 6127616X^9 + 1884160X^8 + 8126464X^7 12845056X^6 + 8912896X^5 + 5767168X^4 16777216X^3 + 16777216X^2 8388608X + 2097152)$
- $z'(4P'_2) = (X^{128} 8X^{127} + 2X^{126} + 384X^{125} + \dots + 38685626227668133590597632)$ / $(2X^{128} + 32X^{127} + 208X^{126} + 448X^{125} + \dots + 38685626227668133590597632)$

In the computation of $z'(P'_2)$, $z'(2P'_2)$, $z'(4P'_2)$, we used the software PARI/GP (Version 2.3.4) [9]. The commands

allocatemem(64000000);

E=ellinit([0,0,0,0,m]); Ed=ellchangecurve(E,[1,-d,0,0]); zd=1-Ed.b4*(x+d)^-2-2*Ed.b6*(x+d)^-3-Ed.b8*(x+d)^-4; zd_a=subst(zd,m,a^6+16*b^6); zd_b=subst(zd_a,d,2*a^2+4*b^2);

zd_1P2p=subst(zd_b,x,2*a*b); zd_1P2=subst(subst(zd_1P2p,a,X),b,1) zd_2P2p=subst(zd_b,x,ellpow(E,[2*a*b,a^3+4*b^3],2)[1]); zd_2P2=subst(subst(zd_2P2p,a,X),b,1) zd_4P2p=subst(zd_b,x,ellpow(E,[2*a*b,a^3+4*b^3],4)[1]); zd_4P2=subst(subst(zd_4P2p,a,X),b,1)

compute them.

Since $x'(P'_2)^3/m$, $z'(P'_2)$, $z'(2P'_2)$, $z'(4P'_2)$ are functions of X, by elementary calculus we can compute their maximum and minimum. So we can find the following bounds.

(3.6)

$$\frac{1}{3}\log(4m) < \log x'(P_2') < \frac{1}{3}\log(57.2218701m),$$

$$-0.6637015 < 4^{-1}\log z'(P_2') < 0,$$

$$-0.0433217 < 4^{-2}\log z'(2P_2') < 0.1396289,$$

$$-0.0363430 < 4^{-3}\log z'(4P_2') \le 0.$$

7

For example, to have the bounds of $\log x'(P'_2)$ as above it suffices to show

$$4 < \frac{x'(P_2')^3}{m} \left(= \frac{(2X^2 + 2X + 4)^3}{X^6 + 16} \right) < 57.2218701$$

We type (the above codes are needed)

```
f2p=(2*a*b+2*a^2+4*b^2)^3/(a^6+16*b^6);
f2=subst(subst(f2p,a,X),b,1);
df2=deriv(f2,X);
df2n=numerator(df2);
fac2=factor(df2n*1.)
```

to find the factorization of the numerator of $((2X^2 + 2X + 4)^3/(X^6 + 16))'$. We do not write the output here, but it turns out that the fourth root $(X = 1.6484223 \cdots)$ is the only positive root. Then $x'(P'_2)^3/m = 57.22187008 \cdots$ by the following command.

subst(f2,X,-subst(fac2[4,1],X,0))

Since $\lim_{X\to 0} (2X^2 + 2X + 4)^3 / (X^6 + 16) = 4$ and $\lim_{X\to\infty} (2X^2 + 2X + 4)^3 / (X^6 + 16) = 8$, we have the bounds for $\log x'(P'_2)$ as above.

To compute the bounds of $z'(P'_2)$, $z'(2P'_2)$, $z'(4P'_2)$, we proceed similarly. Note that if only a, b are real numbers, $d = 2a^2 + 4b^2 > m^{1/3}$ is satisfied. Then $\log |z'(2^n P'_2)|$ has a finite value by Tate's theorem. So the denominators of $z'(P'_2)$, $z'(2P'_2)$, $z'(4P'_2)$ do not have real roots. For $z'(P'_2)$, we type

dzd_1P2=deriv(zd_1P2); dzd_1P2n=numerator(dzd_1P2); facdz_1P2=factor(dzd_1P2n*1.)

to find the factorizations of the derivative of $z'(P'_2)$. Then we substitute all the values of the positive roots to $4^{-1} \log z'(P'_2)$ by the commands

4^{-1*log(subst(zd_1P2,X,-subst(facdz_1P2[□₁,1],X,0)))}

where $\Box_1 = 4$ because only the fourth root is positive. Since $\lim_{X\to 0} 4^{-1} \log z'(P_2) = 0$ and $\lim_{X\to\infty} 4^{-1} \log z'(P_2) = 4^{-1} \log(1/2) = -0.173286795\cdots$, we have the bounds for $4^{-1} \log z'(P_2)$.

For $z'(2P'_2)$ and $z'(4P'_2)$ we type

```
dzd_2P2=deriv(zd_2P2);
dzd_2P2n=numerator(dzd_2P2);
facdz_2P2=factor(dzd_2P2n*1.)
4^-2*log(subst(zd_2P2,X,-subst(facdz_2P2[□<sub>2</sub>,1],X,0)))
dzd_4P2=deriv(zd_4P2);
dzd_4P2n=numerator(dzd_4P2);
facdz_4P2=factor(dzd_4P2n*1.)
4^-3*log(subst(zd_4P2,X,-subst(facdz_4P2[□<sub>3</sub>,1],X,0)))
```

where $\Box_2 = 6, 7, 8$ and $\Box_3 = 8, 9, 12, 13, 16$ since they are all the indices which correspond to the positive roots in factorizations up to multiplicity. Since $\lim_{X\to 0} \log z'(2P'_2) = \lim_{X\to 0} \log z'(4P'_2) = 0$, $\lim_{X\to\infty} 4^{-2} \log z'(2P'_2) = 4^{-2} \log(1/2) = -0.0433216\cdots$ and $\lim_{X\to\infty} 4^{-3} \log z'(4P'_2) = 4^{-3} \log(1/2) = -0.0108304\cdots$, by comparing the values obtained we have the bounds for $4^{-2} \log z'(2P'_2)$ and $4^{-3} \log z'(4P'_2)$.

For the estimate of the remaining terms $z'(2^n P'_2)$ $(n \ge 3)$ we use the following two lemmas, which we shall prove in Section 7.

Lemma 3.7. Let $d = 2a^2 + 4b^2$ or $d = 3a^2 + 4b^2$. Then z'(P') < 120.531634 for any $P \in E_{a,b}(\mathbb{R})$.

Lemma 3.8. (1) If $d = 2a^2 + 4b^2$, then 0.062326 < z'(P') for any $P \in E_{a,b}(\mathbb{R})$. (2) If $d = 3a^2 + 4b^2$, then 0.038068 < z'(P') for any $P \in E_{a,b}(\mathbb{R})$.

Remark 3.9. In general there is Silverman's bound of z'(P') ([11, Lemma 4.1]), which gives a bound dependent on a, b. In our case we find that there is a bound of z'(P') independent of a, b.

We continue the proof of Proposition 3.1. Since $(1/4) \sum_{n=3}^{\infty} 4^{-n} = 1/192$, we have

(3.10)
$$\frac{1}{192}\log(0.062326) < \frac{1}{4}\sum_{n=3}^{\infty} 4^{-n}\log z'(2^n P_2') < \frac{1}{192}\log(120.531634).$$

By (3.5), (3.6) and (3.10), we have

$$\frac{1}{3}\log m - 0.295724 < \hat{\lambda}_{\infty}(P_2) < \frac{1}{3}\log m + 1.513566.$$

To compute the non-archimedean part $\hat{h}_f(P_2)$, we use Lemma 3.19, which is proved in the next subsection. Recall $P_2 = (2ab, a^3 + 4b^3)$. So α, β, δ in Lemma 3.19 correspond to $2ab, a^3 + 4b^3$, 1 respectively. Therefore

$$\hat{h}_f(P_2) = -\frac{2}{3}\log 2.$$

Since $\hat{h}(P_2) = \hat{\lambda}_{\infty}(P_2) + \hat{h}_f(P_2)$, we have

$$\frac{1}{3}\log m - 0.7579 < \hat{h}(P_2) < \frac{1}{3}\log m + 1.0515.$$

We can estimate $\hat{h}(P_1)$, $\hat{h}(P_3)$ similarly by taking $d = 3a^2 + 4b^2$, $2a^2 + 4b^2$ respectively.

Remark 3.11. The shifting width is not necessary to be $3a^2 + 4b^2$, $2a^2 + 4b^2$. We choose the width which give good enough bounds. We do not have an idea to determine the width which give the best bound.

3.1. Non-archimedean part. In this subsection we compute the non-archimedean part of the canonical height, which was required in the proof of Proposition 3.1. To do this, we use [11, THEOREM 5.2]. But the Weierstrass equation of the elliptic curve to which we apply this theorem needs to be minimal at p to compute $\hat{\lambda}_p$. Let $n \in \mathbb{Z}$ be sixth power free and E the elliptic curve $y^2 = x^3 + n$. Then the Weierstrass equation of E is global minimal if and only if $n \not\equiv 16 \pmod{64}$ ([2, Corollary 5.6.4]). Since in Theorem 1.3 we assume that $a^6 + 16b^6$ is square-free, we may assume that

 $n \neq 16 \pmod{64}$, and that $y^2 = x^3 + n$ is global minimal. Further we may assume that $v_2(n) = 0$, $v_3(n) = 0$. The reason is explained by the next lemma.

Lemma 3.12. Let $a, b \in \mathbb{Z}$ with gcd(a, b) = 1. (1) If $n = a^6 + 16b^6$, then

 $v_3(n) = 0.$

(2) If $n = a^6 + 16b^6$ and $n \not\equiv 16 \pmod{64}$, then

$$v_2(n) = 0.$$

Proof. (1) For any $k \in \mathbb{Z}$, we have $(3k \pm 1)^2 = 9k^2 \pm 6k + 1 \equiv 1 \pmod{3}$. So since gcd(a,b) = 1, we have $(a^6, b^6) \equiv (1,0)$, (1,1) or $(0,1) \pmod{3}$. In all the cases $n \not\equiv 0 \pmod{3}$.

(2) We assume that $v_2(n) \neq 0$, and deduce a contradiction. Since $v_2(n) \neq 0$, $a \equiv 0 \pmod{2}$. Since gcd(a,b) = 1, $b \not\equiv 0 \pmod{2}$. Since b is odd, b^3 is odd and we can write $b^3 = 2k + 1$. Then we have $n = a^6 + 16(2k + 1)^2 \equiv 64k^2 + 64k + 16 \equiv 16 \pmod{64}$, since $a^6 \equiv 0 \pmod{64}$. This is a contradiction.

In the next two lemmas we compute $\hat{\lambda}_2(P)$, $\hat{\lambda}_3(P)$ assuming that $v_2(n) = 0$, $v_3(n) = 0$ respectively.

Lemma 3.13. Let $n \not\equiv 16 \pmod{64}$ be sixth power free and E the elliptic curve $y^2 = x^3 + n$ over \mathbb{Q} . Let $P = (\alpha/\delta^2, \beta/\delta^3)$ $(\alpha, \beta, \delta \in \mathbb{Z}, \delta > 0, \gcd(\alpha, \delta) = \gcd(\beta, \delta) = 1)$ be a rational point on E. Assume $v_2(n) = 0$. If $v_2(\alpha) = 0$, then $\hat{\lambda}_2(P) = 2v_2(\delta) \log 2$. If $v_2(\alpha) \neq 0$, then $\hat{\lambda}_2(P) = -\frac{2}{3} \log 2$.

Proof. Since $n \not\equiv 16 \pmod{64}$, $y^2 = x^3 + n$ is global minimal. So we compute $\hat{\lambda}_2(P)$ following the algorithm ([11, p.354, SUBROUTINE in THEOREM 5.2]).

For the general Weierstrass equation (2.1) and a point P on it, we put x := x(P), y := y(P). Further we define A, B, C, Λ for P as follows.

(3.14)
$$A := v_p (3x^2 + 2a_2x + a_4 - a_1y), \ B := v_p (2y + a_1x + a_3),$$
$$C := v_p (3x^4 + b_2x^3 + 3b_4x^2 + 3b_6x + b_8),$$
$$\Lambda := \hat{\lambda}_p(P) / \log p.$$

This is the same definition as in the algorithm but the value of Λ is twice of that in the algorithm. Recall that in our definition the value of the canonical height is twice of that in [11].

For our elliptic curve, since $a_1 = a_2 = a_3 = a_4 = 0$, $b_2 = b_4 = b_8 = 0$ and $b_6 = 4n$, we have

(3.15)
$$A = v_p \left(\frac{3\alpha^2}{\delta^4}\right), \ B = v_p \left(\frac{2\beta}{\delta^3}\right), \ C = v_p \left(\frac{3\alpha(\alpha^3 + 4n\delta^6)}{\delta^8}\right).$$

Note that $c_4 = 0$ (i.e. $v_p(c_4) \neq 0$). This condition has an effect in the algorithm.

On this condition, by the algorithm we have

(3.16)
$$\Lambda = \begin{cases} 2 \max\left\{0, -\frac{1}{2}v_p(\alpha/\delta^2)\right\} & \text{if } A \le 0 \text{ or } B \le 0\\ -\frac{2B}{3} & \text{if } A, \ B > 0, \ C \ge 3B\\ -\frac{C}{4} & \text{if } A, \ B > 0, \ C < 3B \end{cases}$$

Now we consider the case of p = 2. If $v_2(\alpha) = 0$, then $A \leq 0$ and by (3.16)

$$\hat{\lambda}_2(P) = \Lambda \log 2 = 2 \max\left\{0, -\frac{1}{2}v_2(\alpha/\delta^2)\right\} \cdot \log 2 = 2v_2(\delta)\log 2.$$

We assume that $v_2(\alpha) \neq 0$. Then $v_2(\delta) = 0$, since $gcd(\alpha, \delta) = 1$. So A, B > 0. Since P is on E, we have the equation $n\delta^6 = \beta^2 - \alpha^3$. Since $v_2(n) = 0, v_2(\beta) = 0$. So $B = v_2(2\beta) = 1$ and $C = v_2(\alpha) + v_2(\alpha^3 + 4n\delta^6) \geq 1 + 2 = 3$. So $C \geq 3B$, and by (3.16)

$$\hat{\lambda}_2(P) = \Lambda \log 2 = -\frac{2B}{3} \log 2 = -\frac{2}{3} \log 2.$$

Lemma 3.17. We consider the situation of Lemma 3.13. Assume $v_3(n) = 0$. If $v_3(\beta) = 0$, then $\hat{\lambda}_3(P) = 2v_3(\delta) \log 3$. If $v_3(\beta) \neq 0$, then $\hat{\lambda}_3(P) = -\frac{1}{2} \log 3$.

Proof. We compute $\hat{\lambda}_3(P)$ following (3.15), (3.16) for p = 3.

If $v_3(\beta) = 0$, then $B \leq 0$ and by (3.16)

$$\hat{\lambda}_3(P) = \Lambda \log 3 = 2 \max \left\{ 0, -\frac{1}{2} v_3(\alpha/\delta^2) \right\} \cdot \log 3 = 2v_3(\delta) \log 3.$$

The last equality is as follows. If $v_3(\delta) = 0$, then $\max\left\{0, -\frac{1}{2}v_3(\alpha/\delta^2)\right\} = 0$. So $\max\left\{0, -\frac{1}{2}v_3(\alpha/\delta^2)\right\} = v_3(\delta)$. If $v_3(\delta) \neq 0$, then since $\gcd(\alpha, \delta) = 1, v_3(\alpha) = 0$. So $\max\left\{0, -\frac{1}{2}v_3(\alpha/\delta^2)\right\} = v_3(\delta)$.

We assume that $v_3(\beta) \neq 0$. Then $v_3(\delta) = 0$, since $gcd(\beta, \delta) = 1$. So $B = v_3(2\beta/\delta^3) = v_3(\beta) > 0$ and $A = v_3(3\alpha^2/\delta^4) = v_3(3\alpha^2) > 0$. Since P is on E, $n\delta^6 = \beta^2 - \alpha^3$. Since $v_3(n) = 0$, $v_3(\alpha) = 0$. Using the equality $\alpha^3 + 4n\delta^6 = \beta^2 + 3n\delta^6$,

$$C = v_3(3\alpha) + v_3(\alpha^3 + 4n\delta^6) = v_3(3\alpha) + v_3(\beta^2 + 3n\delta^6) = 1 + 1 = 2.$$

So we have 3B > C. By (3.16)

$$\hat{\lambda}_3(P) = \Lambda \log 3 = -\frac{C}{4} \log 3 = -\frac{1}{2} \log 3.$$

Lemma 3.18. Let $n \in \mathbb{Z}$ be square-free and E the elliptic curve $y^2 = x^3 + n$ over \mathbb{Q} . Let $P = (\alpha/\delta^2, \beta/\delta^3)$ $(\alpha, \beta, \delta \in \mathbb{Z}, \delta > 0, \operatorname{gcd}(\alpha, \delta) = \operatorname{gcd}(\beta, \delta) = 1)$ be a rational point on E. We assume that $p \neq 2, 3$. Then $\hat{\lambda}_p(P) = 2v_p(\delta) \log p$. *Proof.* We compute $\hat{\lambda}_p(P)$ following (3.15), (3.16). At first if $v_p(\alpha) = 0$ or $v_p(\beta) = 0$, then since δ is an integer, $A \leq 0$ or $B \leq 0$. So

$$\hat{\lambda}_p(P) = \Lambda \log p = 2 \max\left\{0, -\frac{1}{2}v_p(\alpha/\delta^2)\right\} \cdot \log p = 2v_p(\delta)\log p.$$

The last equality follows from the same reason as that in the proof of Lemma 3.17.

Next we assume that $v_p(\alpha) > 0$ and $v_p(\beta) > 0$. Then $v_p(\delta) = 0$ because $gcd(\alpha, \delta) = 1$. Since $v_p(\beta^2 - \alpha^3) > 1$ and $n\delta^6 = \beta^2 - \alpha^3$, we have $v_p(n\delta^6) > 1$. But *n* is square-free, $v_p(n) = 0$ or 1. So this case does not happen.

By the previous four lemmas, we have the following lemma.

Lemma 3.19. Let $n \in \mathbb{Z}$ be square-free and E the elliptic curve $y^2 = x^3 + n$ over \mathbb{Q} . Let $P = (\alpha/\delta^2, \beta/\delta^3)$ $(\alpha, \beta, \delta \in \mathbb{Z}, \delta > 0, \operatorname{gcd}(\alpha, \delta) = \operatorname{gcd}(\beta, \delta) = 1)$ be a rational point on E. Then the non-archimedean part of the canonical height of P is as follows:

$$\hat{h}_f(P) = 2\log\delta + \lambda'_2(P) + \lambda'_3(P),$$

where

$$\lambda_{2}'(P) = \begin{cases} 0 & (v_{2}(\alpha) = 0), \\ -\frac{2}{3}\log 2 & (v_{2}(\alpha) \neq 0), \end{cases}$$
$$\lambda_{3}'(P) = \begin{cases} 0 & (v_{3}(\beta) = 0), \\ -\frac{1}{2}\log 3 & (v_{3}(\beta) \neq 0). \end{cases}$$

Proof.

$$\begin{split} \hat{h}_{f}(P) &= \hat{\lambda}_{2}(P) + \hat{\lambda}_{3}(P) + \sum_{p \neq 2,3} \hat{\lambda}_{p}(P) \\ &= \hat{\lambda}_{2}(P) + \hat{\lambda}_{3}(P) + \sum_{p \neq 2,3} 2v_{p}(\delta) \log p \\ &= \hat{\lambda}_{2}(P) - 2v_{2}(\delta) \log 2 + \hat{\lambda}_{3}(P) - 2v_{3}(\delta) \log 3 + 2\log \prod_{p} p^{v_{p}(\delta)} \\ &= \hat{\lambda}_{2}(P) - 2v_{2}(\delta) \log 2 + \hat{\lambda}_{3}(P) - 2v_{3}(\delta) \log 3 + 2\log \delta. \end{split}$$

By Lemmas 3.13 and 3.17 we see that $\hat{\lambda}_2(P) - 2v_2(\delta) \log 2$ and $\hat{\lambda}_3(P) - 2v_3(\delta) \log 3$ are nothing but $\lambda'_2(P)$ and $\lambda'_3(P)$ respectively.

4. Uniform lower bound

In this section we compute a uniform lower bound of the canonical height (Proposition 4.3), that is a lower bound of the canonical height independent of $P \in E(\mathbb{Q})$.

Proposition 4.1. Let $n \in \mathbb{Z}$ and let E be the elliptic curve $y^2 = x^3 + n$ over \mathbb{Q} . Let $P = (\alpha/\delta^2, \beta/\delta^3) \ (\alpha, \beta, \delta \in \mathbb{Z}, \ \delta > 0, \ \gcd(\alpha, \delta) = \gcd(\beta, \delta) = 1)$ be a rational point on E. We assume that n > 0. Then we have

$$\hat{\lambda}_{\infty}(P) > \frac{1}{12}\log n + \frac{1}{2}\log \left|\frac{\beta}{\delta^3}\right| + 0.31494685.$$

Proof. Recall that in our definition the value of the canonical height is twice of that in [1]. By Algorithm 7.5.7 [1] and (2.2)

(4.2)
$$\hat{\lambda}_{\infty}(P) = \frac{1}{16} \log \left| \frac{\Delta}{q} \right| + \frac{1}{4} \log \left(\frac{\omega_1 y(P)^2}{2\pi} \right) - \frac{1}{2} \log \left| \theta \right|,$$

where $q = \exp(2\pi i\omega_2/\omega_1)$, $\theta = \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \sin \{2\pi (2n+1)\operatorname{Re}(z_P)/\omega_1\}$, Δ is the discriminant of E, ω_1 and ω_2 are periods of E such that $\omega_1 > 0$, $\operatorname{Im}(\omega_2) > 0$ and $\operatorname{Re}(\omega_2/\omega_1) = -1/2$ and z_P is the elliptic logarithm of P. Recall that z_P is the complex number in $\{t_1\omega_1 + t_2\omega_2 : 0 \leq t_1, t_2 \leq 1\}$ such that $\wp(z_P) = x(P)$ and $\wp'(z_P) = 2y(P)$, where \wp is the Weierstrass \wp -function.

Note that q is a real number since

$$q = \exp\left(2\pi i \frac{\omega_2}{\omega_1}\right) = \exp\left(2\pi i \left(-\frac{1}{2} + i \operatorname{Im}\left(\frac{\omega_2}{\omega_1}\right)\right)\right)$$
$$= \exp\left(-\pi i - 2\pi \operatorname{Im}\left(\frac{\omega_2}{\omega_1}\right)\right) = -\exp\left(-2\pi \operatorname{Im}\left(\frac{\omega_2}{\omega_1}\right)\right).$$

By Definition 7.4.6 and Algorithm 7.4.7 in [1]

$$\omega_1 = \frac{2\pi}{\text{AGM}(2\sqrt[4]{3}n^{\frac{1}{6}}, \sqrt{2\sqrt{3}-3}n^{\frac{1}{6}})} = n^{-\frac{1}{6}} \cdot \frac{2\pi}{\text{AGM}(2\sqrt[4]{3}, \sqrt{2\sqrt{3}-3})},$$

where $AGM(\cdot, \cdot)$ is the arithmetic geometric mean. So if we let ω'_1 , ω'_2 be the periods of the elliptic $y^2 = x^3 + 1$, then we have $\omega_1 = n^{-\frac{1}{6}} \times \omega'_1$. It turns out that $\omega'_1 = 4.206546315 \cdots$. This can be done by PARI/GP (Version 2.3.4) ([9]) as follows.

E1=ellinit([0,0,0,0,1]); E1.omega

Similarly by [1, Algorithm 7.4.7], we have

$$\omega_2/\omega_1 = -\frac{1}{2} + \frac{i}{2} \frac{\text{AGM}(2\sqrt[4]{3}n^{\frac{1}{6}}, \sqrt{2\sqrt{3}} + 3n^{\frac{1}{6}})}{\text{AGM}(2\sqrt[4]{3}n^{\frac{1}{6}}, \sqrt{2\sqrt{3}} - 3n^{\frac{1}{6}})}$$
$$= -\frac{1}{2} + \frac{i}{2} \frac{\text{AGM}(2\sqrt[4]{3}, \sqrt{2\sqrt{3}} + 3)}{\text{AGM}(2\sqrt[4]{3}, \sqrt{2\sqrt{3}} - 3)} = \omega_2'/\omega_1'$$

and so it turns out that $q = -0.163033534\cdots$ by PARI/GP as follows(the above commands are needed).

-exp(-2*Pi*imag(E1.omega[2]/E1.omega[1]))

Substituting these values and $\Delta = -432n^2$ in (4.2), we have

$$\begin{aligned} \hat{\lambda}_{\infty}(P) &= \frac{1}{16} \log \left| \frac{432n^2}{q} \right| + \frac{1}{4} \log \left(\frac{n^{-\frac{1}{6}} \omega_1' \beta^2}{2\pi \delta^6} \right) - \frac{1}{2} \log |\theta| \\ &> \frac{1}{16} \log \left| \frac{432n^2}{0.163033535} \right| + \frac{1}{4} \log \left(\frac{4.206546315n^{-\frac{1}{6}} \beta^2}{2\pi \delta^6} \right) - \frac{1}{2} \log |1.167385748| \\ &= \frac{1}{12} \log n + \frac{1}{2} \log \left| \frac{\beta}{\delta^3} \right| + 0.3149468597 \cdots \end{aligned}$$

by the trivial bound $|\theta| < 1 + |q| + |q|^3 + |q|^6 + |q|^{10} + |q|^{15} + |q|^{21} + \dots < 1 + |q| + |q|^3 + |q|^6 + \frac{|q|^{10}}{1 - |q|^5} = 1.16738574713\cdots$

Proposition 4.3. Let n be a positive, square-free integer and E the elliptic curve $y^2 = x^3 + n$. If P is a rational, non-torsion point on E, then

(4.4)
$$\hat{h}(P) > \frac{1}{12}\log n - 0.147152.$$

Proof. By Lemmas 3.13, 3.17, 3.19 and Proposition 4.1, we have

$$\hat{h}(P) = \hat{h}_f(P) + \hat{\lambda}_{\infty}(P)$$

$$> 2\log \delta + \lambda_2'(P) + \lambda_3'(P) + \frac{1}{12}\log n + \frac{1}{2}\log \left|\frac{\beta}{\delta^3}\right| + 0.31494685$$

$$= \frac{1}{2}\log \delta + \lambda_2'(P) + \left\{\lambda_3'(P) + \frac{1}{2}\log|\beta|\right\} + \frac{1}{12}\log n + 0.31494685$$

$$\ge \frac{1}{12}\log n - \frac{2}{3}\log 2 + 0.31494685 = \frac{1}{12}\log n - 0.1471512\cdots,$$

$$\hat{I} \in \mathbb{Z} \text{ and } \lambda_2'(P) + \frac{1}{2}\log|\beta| \ge 0.$$

since $\delta \in$ \mathbb{Z} and $\lambda'_3(P) + \frac{1}{2} \log |\beta| \ge 0$

5. Estimate of the lattice index

Let E be an elliptic curve of rank $r(\geq 2)$ defined over a number filed K. Let $Q_1, Q_2, ..., Q_s$ $(s \leq r)$ be independent points in E(K). Then there exist generators $G_1, G_2, ..., G_r$ of the free part of E(K) such that $Q_1, Q_2, ..., Q_s \in \mathbb{Z}G_1 + \mathbb{Z}G_2 + \cdots + \mathbb{Z}G_s$ by the elementary divisor theory. The index of the subgroup $\mathbb{Z}Q_1 + \mathbb{Z}Q_2 + \cdots + \mathbb{Z}Q_s$ in $\mathbb{Z}G_1 + \mathbb{Z}G_2 + \cdots + \mathbb{Z}G_s$ is called the *lattice index* of $\{Q_1, Q_2, \dots, Q_s\}$. We put

$$\langle Q_i, Q_j \rangle = \frac{1}{2} \left(\hat{h}(Q_i + Q_j) - \hat{h}(Q_i) - \hat{h}(Q_j) \right),$$

$$R(Q_1, Q_2, ..., Q_s) = \det \left(\langle Q_i, Q_j \rangle \right)_{1 \le i, j \le s}.$$

It is known that the canonical height h is a positive definite quadratic form on $E(K)/E(K)_{\text{tors}}$. When we identify $E(K)/E(K)_{\text{tors}} \simeq \mathbb{Z}G_1 + \mathbb{Z}G_2 + \cdots + \mathbb{Z}G_r$ as \mathbb{Z} -modules, \hat{h} is the quadratic form defined by the symmetric matrix $(\langle G_i, G_j \rangle)_{1 \leq i,j \leq r}$.

Let $f(\mathbf{x}) = \sum_{i,j=1}^{n} f_{i,j} x_i x_j$ be a positive definite symmetric quadratic form. Then it is known that there exists a constant γ_n called the *Hermite constant* such that

$$\inf_{\mathbf{m}\in\mathbb{Z}^r\setminus\{0\}}f(\mathbf{m})\leq\gamma_n\det(f_{i,j}).$$

For example,

$$\gamma_1^1 = 1, \ \gamma_2^2 = 4/3, \ \gamma_3^3 = 2, \ \gamma_4^4 = 4, \ \dots$$

In this section we estimate the lattice index. For this we use the following theorem of Siksek.

Theorem 5.1. ([10, Theorem 3.1]) Let E be an elliptic curve of rank $r (\geq 2)$ defined over a number field K. Let $Q_1, Q_2, ..., Q_s$ ($s \leq r$) be independent points in E(K) and ν the lattice index of $\{Q_1, Q_2, ..., Q_s\}$. Suppose that $\lambda > 0$ is a constant such that any point $P \in E(K)$ of infinite order satisfies $\hat{h}(P) > \lambda$. Then

$$\nu \leq R(Q_1, Q_2, ..., Q_s)^{1/2} (\gamma_s / \lambda)^{s/2}.$$

Proposition 5.2. Assume that $m = a^6 + 16b^6$ is square-free, ab is odd and the discrete valuation $v_3(b)$ equals 1. If $m > 6.38 \times 10^{22}$ (this is true for either a > 6321 or b > 3982), the lattice indices of $\{P_1, P_2\}$, $\{P_2, P_3\}$, $\{P_3, P_1\}$ are less than 5. If m > 19088 (this is always true), the lattice indices of $\{P_1, P_2\}$, $\{P_2, P_3\}$, $\{P_2, P_3\}$, $\{P_3, P_1\}$ are less than 7.

Proof. In this situation P_1, P_2, P_3 are independent by Proposition 6.7 in the next section. Let $\lambda = \frac{1}{12} \log m - 0.147152$. Then $\hat{h}(P) > \lambda$ for any non-torsion point $P \in E_{a,b}(\mathbb{Q})$. Now by Theorem 5.1, it suffices to show that $R(P_i, P_j)^{1/2} (\gamma_2/\lambda)^{2/2}$ is less than 5 or 7, when $m > 6.38 \times 10^{22}$ or m > 19088 respectively for $i \neq j$ (i, j = 1, 2, 3). Since

$$R(P_2, P_3) = \hat{h}(P_2)\hat{h}(P_3) - \frac{1}{4}\left\{\hat{h}(P_2 + P_3) - \hat{h}(P_2) - \hat{h}(P_3)\right\}^2,$$

we have

$$\left\{ R(P_2, P_3)^{1/2} (\gamma_2/\lambda)^{2/2} \right\}^2 = \frac{4}{3} \frac{\hat{h}(P_2)\hat{h}(P_3) - \frac{1}{4} \left\{ \hat{h}(P_2 + P_3) - \hat{h}(P_2) - \hat{h}(P_3) \right\}^2}{\lambda^2} < \frac{4}{3} \frac{\hat{h}(P_2)\hat{h}(P_3)}{\lambda^2} < \frac{4}{3} \frac{(\frac{1}{3}\log m + 1.0515)(\frac{1}{3}\log m + 0.5665)}{(\frac{1}{12}\log m - 0.147152)^2}.$$

The last inequality follows from Propositions 3.1 and 4.3. By elementary calculus we see that the last bound is less than 25 if $m > 6.38 \times 10^{22}$, less than 49 if m > 19088 and decreasing if $m > e^2$.

Since the upper bound of $\hat{h}(P_1)$ given in Proposition 3.1 is less than those of $\hat{h}(P_2)$ and $\hat{h}(P_3)$, the cases of $\{P_1, P_2\}, \{P_3, P_1\}$ are clear.

6. INDEPENDENCE OF P_1, P_2, P_3

In this section we show that in the situation of Proposition 5.2, P_1 , P_2 , P_3 are independent and the lattice index of $\{P_i, P_j\}$ $(i \neq j)$ is not divisible by 2, 3.

Lemma 6.1. Let $n \in \mathbb{Z}$ and let E be the elliptic curve $y^2 = x^3 + n$ over \mathbb{Q} and $Q \in E(\mathbb{Q}) \setminus E(\mathbb{Q})_{\text{tors}}$. We write $x(Q) = u/s^2$ with gcd(u, s) = 1. Then $Q \notin 2E(\mathbb{Q})$ in either of the following cases:

- (1) n is odd, $u \not\equiv 0 \pmod{8}$ and s is odd,
- (2) $n \equiv 1 \pmod{9}, u \equiv 2 \pmod{3}$ and $s \not\equiv 0 \pmod{3}$.

Proof. We assume that there exists $R = (w/t^2, z/t^3) \in E(\mathbb{Q})$ with gcd(w, t) = 1 such that Q = 2R and deduce a contradiction. By (2.5) or the following PARI/GP commands,

En=ellinit([0,0,0,0,n]); ellpow(En,[w/t²,z/t³],2)[1]

we have $x(2R) = (9w^4 - 8wz^2)/(4z^2t^2)$ and so $u/s^2 = (9w^4 - 8wz^2)/(4z^2t^2)$. On the other hand $(z/t^3)^2 = (w/t^2)^3 + n$ since R is on E. Eliminating z,

(6.2)
$$s^2 w (w^3 - 8nt^6) = 4ut^2 (w^3 + nt^6)$$

(1) If n and s are odd, then w is even by (6.2). Further t is odd since gcd(w, t) = 1. Then $v_2(w(w^3 - 8nt^6)) \ge 5$ (note that if $v_2(w) = 1$, $w^3 - 8nt^6 = 8 \times$ even). So $v_2(4ut^2(w^3 + nt^6)) \ge 5$ and therefore $v_2(u) \ge 3$. This is a contradiction since $u \ne 0$ (mod 8).

(2) Assume that $n \equiv 1 \pmod{9}$, $u \equiv 2 \pmod{3}$ and $s \not\equiv 0 \pmod{3}$. Note that if $x \not\equiv 0 \pmod{3}$, then $x^2 \equiv 1 \pmod{9}$ (so modulo 3 also).

Assume $w \equiv 0 \pmod{3}$. Then $t \not\equiv 0 \pmod{3}$ since gcd(w, t) = 1. So the left hand side of $(6.2) \equiv 0 \pmod{3}$ and the right hand side of $(6.2) \not\equiv 0 \pmod{3}$. This is a contradiction.

Assume $w \equiv 1 \pmod{3}$. If $t \equiv 0 \pmod{3}$, then the left hand side of $(6.2) \equiv 1 \pmod{3}$ and the right hand side of $(6.2) \equiv 0 \pmod{3}$. This is a contradiction.

If $t \not\equiv 0 \pmod{3}$, then the left hand side of $(6.2) \equiv 2 \pmod{3}$ and the right hand side of $(6.2) \equiv 1 \pmod{3}$. This is a contradiction.

Assume $w \equiv -1 \pmod{3}$. If $t \equiv 0 \pmod{3}$, then the left hand side of $(6.2) \not\equiv 0 \pmod{3}$ and the right hand side of $(6.2) \equiv 0 \pmod{3}$. This is a contradiction.

Note that $w^3 \equiv -1 \pmod{9}$.

If $t \not\equiv 0 \pmod{3}$, then $w^3 - 8nt^6 \equiv 0 \pmod{9}$ and $w^3 + nt^6 \equiv 0 \pmod{9}$. So we can write $w^3 - 8nt^6 = 9W_1$, $w^3 + nt^6 = 9W_2$. Then by (6.2) we have $s^2w \cdot 9W_1 \equiv 4ut^2 \cdot 9W_2 \pmod{27}$. So $s^2wW_1 \equiv 4ut^2W_2 \pmod{3}$. Therefore $-W_1 \equiv -W_2 \pmod{3}$. On the other hand $9W_2 - 9W_1 = 9nt^6$ and so $W_2 - W_1 = nt^6 \not\equiv 0 \pmod{3}$. This is a contradiction.

Remark 6.3. Assume that we can write $x(Q) = u/s^2 = u'/s'^2$ $(u', s' \in \mathbb{Z}$ and not necessarily gcd(u', s') = 1). So u|u' and s|s'. Then if $u' \neq 0 \pmod{8}$, $u \neq 0 \pmod{8}$. If s' is odd, s is odd. If $s' \neq 0 \pmod{3}$, $s \neq 0 \pmod{3}$. If $u' \equiv 2 \pmod{3}$, $u \equiv 2 \pmod{3}$, $u \equiv 2 \pmod{3}$, $u \equiv (s'/s)^2 u$ and $s'/s \neq 0 \pmod{3}$.

So it is not necessary to assume gcd(u, s) = 1 in Lemma 6.1.

Lemma 6.4. Let $n \in \mathbb{Z}$ and let E be the elliptic curve $y^2 = x^3 + n$ over \mathbb{Q} and $Q \in E(\mathbb{Q}) \setminus E(\mathbb{Q})_{\text{tors}}$. We write $x(Q) = u/s^2$ with gcd(u, s) = 1. Then $Q \notin 3E(\mathbb{Q})$ in either of the following cases:

- (1) n is odd and u is even,
- (2) $n \equiv 1 \pmod{9}, u \equiv 1 \pmod{3}$ and $v_3(s) = 1$.

Proof. We assume that there exists $R = (w/t^2, z/t^3) \in E(\mathbb{Q})$ with gcd(w, t) = 1 such that Q = 3R and deduce a contradiction. By the following PARI/GP commands

16

En=ellinit([0,0,0,0,n]); ellpow(En,[w/t²,z/t³],3)[1]

we have $x(3R) = (64z^6 - 144w^3z^4 + 81w^9)/9t^2w^2(4z^2 - 3w^3)^2$ and so $u/s^2 = (64z^6 - 144w^3z^4 + 81w^9)/9t^2w^2(4z^2 - 3w^3)^2$. On the other hand $(z/t^3)^2 = (w/t^2)^3 + n$ since R is on E. Eliminating z,

(6.5)
$$s^{2} \left\{ (w^{3} + 4nt^{6})^{3} - 2^{2}3^{3}nw^{6}t^{6} \right\} = 3^{2}uw^{2}t^{2}(w^{3} + 4nt^{6})^{2}.$$

(1) If u is even, then s is odd since gcd(u, s) = 1. Then since $(w^3 + 4nt^6)^3 - 2^2 3^3 nw^6 t^6$ is even, w must be even. So t is odd since gcd(w, t) = 1. Since n is odd, $v_2(w^3 + 4nt^6) = 2$ and therefore v_2 (the left hand side of (6.5)) = 6. On the other hand v_2 (the right hand side of (6.5)) ≥ 7 .

(2) If $v_3(s) = 1$, we can write $s = 3s' \ (s' \not\equiv 0 \pmod{3})$. So we have

(6.6)
$$s^{\prime 2} \left\{ (w^3 + 4nt^6)^3 - 2^2 3^3 n w^6 t^6 \right\} = u w^2 t^2 (w^3 + 4nt^6)^2.$$

Now we show $wt \not\equiv 0 \pmod{3}$. Assume that $wt \equiv 0 \pmod{3}$. Then since the each side of $(6.6) \equiv 0 \pmod{3}$, we have $(w^3 + 4nt^6)^3 - 2^2 3^3 n w^6 t^6 \equiv 0 \pmod{3}$. So $w^3 + 4nt^6 \equiv 0 \pmod{3}$. But this does not happen since gcd(w, t) = 1 and $n \equiv 1 \pmod{9}$. So we see $wt \not\equiv 0 \pmod{3}$.

Now if we assume that $w \equiv -1 \pmod{3}$, then $w^3 + 4nt^6 \equiv -1 + 4t^6 \equiv 3 \pmod{9}$. So $v_3(w^3 + 4nt^6) = 1$. Then v_3 (the left hand side of (6.6)) ≥ 3 and v_3 (the right hand side of (6.6)) = 2. This is a contradiction.

If we assume that $w \equiv 1 \pmod{3}$, then $w^3 + 4nt^6 \equiv -1 \pmod{3}$. Then seeing (6.6) modulo 3, we have $u \equiv -1 \pmod{3}$. This is a contradiction.

Proposition 6.7. We assume that $m = a^6 + 16b^6$ is square-free, ab is odd and the discrete valuation $v_3(b)$ equals 1. Then P_1 , P_2 , P_3 , $P_1 + P_2$, $P_2 + P_3$, $P_1 + P_3$, $P_1 + P_2 + P_3 \notin 2E_{a,b}(\mathbb{Q})$ and P_1 , P_2 , P_3 , $P_1 \pm P_2$, $P_2 \pm P_3$, $P_1 \pm P_3$, $P_1 + P_2 \pm P_3$, $P_1 - P_2 \pm P_3 \notin 3E_{a,b}(\mathbb{Q})$. In particular, P_1 , P_2 , P_3 are independent and the lattice indices of $\{P_1, P_2, P_3\}$, $\{P_1, P_2\}$, $\{P_2, P_3\}$, $\{P_3, P_1\}$ are not divisible by 2 nor 3.

Proof. To ease the notation, we put $E = E_{a,b}$. We have

$$x(P_1) = -a^2, \ x(P_2) = 2ab, \ x(P_3) = -2ab,$$

$$\begin{aligned} x(P_1 + P_2) &= \frac{2a(a^3 + a^2b - 2ab^2 - 4b^3)}{(a + 2b)^2}, \\ x(P_1 - P_2) &= \frac{2(a^4 - 3a^3b + 6a^2b^2 - 8ab^2 + 8b^4)}{a^2}, \\ x(P_1 + P_3) &= \frac{2(a^4 + 3a^3b + 6a^2b^2 + 8ab^3 + 8b^4)}{a^2}, \\ x(P_1 - P_3) &= \frac{2a(a^3 - a^2b - 2ab^2 + 4b^3)}{(a - 2b)^2}, \\ x(P_2 + P_3) &= \frac{4b^4}{a^2}, \ x(P_2 - P_3) = \frac{a^4}{(2b)^2}, \\ x(P_1 + P_2 + P_3) &= \frac{2a(a^5 + 4a^4b + 8a^3b^2 + 12a^2b^3 + 14ab^4 + 8b^5)}{(a^2 + 2ab + 2b^2)^2} \end{aligned}$$

17

 \square

$$x(P_1 - P_2 - P_3) = \frac{2a(a^5 - 4a^4b + 8a^3b^2 - 12a^2b^3 + 14ab^4 - 8b^5)}{(a^2 - 2ab + 2b^2)^2}.$$

Note that $m = a^6 + 16b^6 \equiv 1 \pmod{9}$ since $v_3(b) = 1$ and gcd(a, b) = 1. As we saw in Remark 6.3, we can use Lemma 6.1 without the assumption that the *x*-coordinate is an irreducible fraction. Note that *m* in this proposition corresponds to *n* in Lemma 6.1.

We see that $P_1 + P_2 \notin 2E(\mathbb{Q})$ by Lemma 6.1(2) since $2a(a^3 + a^2b - 2ab^2 - 4b^3) \equiv 2a^4 \equiv 2 \pmod{3}$ and $a + 2b \equiv a \not\equiv 0 \pmod{3}$. Similarly $P_1 + P_3 \notin 2E(\mathbb{Q})$ by Lemma 6.1(2). It is clear that $P_1, P_2, P_3, P_2 + P_3, P_1 + P_2 + P_3 \notin 2E(\mathbb{Q})$ by Lemma 6.1(1).

If there is a rational point R such that $P_1 = 3R$, then $\hat{h}(P_1) = 9\hat{h}(R)$. But by Proposition 4.4 we have $9\hat{h}(R) > 9(\frac{1}{12}\log m - 0.147152) > \frac{1}{3}\log m + 0.5409 > \hat{h}(P_1)$ for $m \ge 88$, which is a contradiction. So $P_1 \notin 3E(\mathbb{Q})$.

Since $a^4/(2b)^2$ is an irreducible fraction, by Lemma 6.4(2) $P_2 - P_3 \notin 3E(\mathbb{Q})$. By computations we have

 $\begin{aligned} x(2P_1 - 2P_2 - P_3) \\ &= a(-6144b^{17} + 34816ab^{16} - 101376a^2b^{15} + 204544a^3b^{14} - 320128a^4b^{13} + 409472a^5b^{12} - 439840a^6b^{11} + 403168a^7b^{10} - 318248a^8b^9 + 217216a^9b^8 - 128160a^{10}b^7 + 65072a^{11}b^6 - 28152a^{12}b^5 + 10200a^{13}b^4 - 3006a^{14}b^3 + 684a^{15}b^2 - 108a^{16}b + 9a^{17})/b^2(2b - a)^2(16b^6 - 40ab^5 + 56a^2b^4 - 46a^3b^3 + 28a^4b^2 - 12a^5b + 3a^6)^2. \end{aligned}$

We denote the numerator by U' and the denominator by S'^2 . Further we write $U'/S'^2 = U/S^2$ as an irreducible fraction since it is an x-coordinate of an elliptic curve. Since $v_3(9a^{17}) = 2$ and the orders of other terms of U' is greater than 2, $v_3(U') = 2$. In S', $v_3(b^2) = 2$, $v_3(3a^6) = 1$ and other factors are not divisible by 3. So $v_3(S'^2) = 4$. Therefore, $v_3(S) = 1$ and U'' := U'/9, S'' := S'/9 are integers. Clearly $U''/S''^2 = U/S^2$. Since $U'' \equiv a^{18} \equiv 1 \pmod{3}$, $U \equiv 1$ by the same argument as in Remark 6.3. So $2P_1 - 2P_2 - P_3 \notin 3E(\mathbb{Q})$ by Lemma 6.4(2). Therefore $P_1 - P_2 + P_3 = -(2P_1 - 2P_2 - P_3) + 3(P_1 - P_2) \notin 3E(\mathbb{Q})$. We have

$$\begin{split} &x(2P_1+2P_2+P_3) \\ &= (4096b^{18}+24576ab^{17}+71680a^2b^{16}+135680a^3b^{15}+188160a^4b^{14}+204800a^5b^{13}+181632a^6b^{12}+133536a^7b^{11}+83488a^8b^{10}+48472a^9b^9+30720a^{10}b^8+22464a^{11}b^7+16496a^{12}b^6+10584a^{13}b^5+5496a^{14}b^4+2178a^{15}b^3+612a^{16}b^2+108a^{17}b+9a^{18})/a^2b^2(48b^6+128ab^5+156a^2b^4+114a^3b^3+56a^4b^2+18a^5b+3a^6)^2 \end{split}$$

and by the same argument as above, we have $2P_1 + 2P_2 + P_3 \notin 3E(\mathbb{Q})$ by Lemma 6.4(2). Therefore $P_1 + P_2 - P_3 = -(2P_1 + 2P_2 + P_3) + 3(P_1 + P_2) \notin 3E(\mathbb{Q})$. We see that P_2 , P_3 , $P_1 \pm P_2$, $P_2 + P_3$, $P_1 \pm P_3$, $P_1 + P_2 + P_3$, $P_1 - P_2 - P_3 \notin 3E(\mathbb{Q})$ by Lemma 6.4(1), since the denominators of the x-coordinates of them are all odd.

Next we prove the latter assertion of the proposition. By the elementary divisor theory there are generators $G_1, \ldots, G_r \in E(\mathbb{Q})$ and $M \in M_3(\mathbb{Z})$ such that

(6.8)
$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = M \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}.$$

Note that the lattice index of $\{P_1, P_2, P_3\}$ equals $|\det M|$. Let p be a rational prime. We have

(6.9)
$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \equiv \bar{M} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \pmod{pE(\mathbb{Q})},$$

where \overline{M} is the image of M in $M_3(\mathbb{Z}/p\mathbb{Z})$. We assume that there exists $A \in \operatorname{GL}_3(\mathbb{Z}/p\mathbb{Z})$ such that $A\overline{M}$ has the row $[\overline{0} \ \overline{0} \ \overline{0}]$ and deduce a contradiction. Since we may assume that the first row is $[\overline{0} \ \overline{0} \ \overline{0}]$, by the left multiplication of A on (6.9) we have

(6.10)
$$\begin{bmatrix} k_1 P_1 + k_2 P_2 + k_3 P_3 \\ * \\ * \end{bmatrix} \equiv \begin{bmatrix} \bar{0} & \bar{0} & \bar{0} \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \pmod{pE(\mathbb{Q})},$$

where $[k_1 \ k_2 \ k_3]$ is the first row of A. But the former assertion of this proposition implies that $k_1P_1 + k_2P_2 + k_3P_3 \notin pE(\mathbb{Q}) \ (p = 2, 3)$ for any $(k_1, k_2, k_3) \in (\mathbb{Z}/p\mathbb{Z})^3 \setminus (\bar{0}, \bar{0}, \bar{0})$. This is a contradiction. Therefore det M is not congruent to 0 modulo 2 or modulo 3.

By the same argument as above, the cases of $\{P_1, P_2\}, \{P_2, P_3\}, \{P_3, P_1\}$ follow. \Box

Remark 6.11. By the same reason as above, if we verify that P_1 , P_2 , P_3 , $P_1 \pm P_2$, $P_2 \pm P_3$, $P_3 \pm P_1$, $P_1 \pm 2P_2$, $P_2 \pm 2P_3$, $P_3 \pm 2P_1 \notin 5E_{a,b}(\mathbb{Q})$, we can prove that the lattice indices of $\{P_1, P_2\}$, $\{P_2, P_3\}$, $\{P_3, P_1\}$ are not divisible by 5. Note that $P \notin 5E(\mathbb{Q})$ amounts to $kP \notin 5E(\mathbb{Q})$ ($k = \pm 1, \pm 2$). For $3 \le a \le 6321$, $3 \le b \le 3982$ we can verify this by the software Magma ([7]). We give the code for this in Appendix A.

Now we can finish the proof of our main theorem.

Proof of Theorem 1.3. For a > 6321, b > 3982 by Propositions 5.2, 6.7 the lattice indices of $\{P_1, P_2\}$, $\{P_2, P_3\}$, $\{P_3, P_1\}$ equal 1. For $5 \le a \le 6321, 3 \le b \le 3982$ by Propositions 5.2, 6.7 and Remark 6.11 the lattice indices of $\{P_1, P_2\}$, $\{P_2, P_3\}$, $\{P_3, P_1\}$ equal 1. This completes the proof of Theorem 1.3.

We prove that there are infinitely many (a, b)'s which satisfy the condition of Theorem 1.3.

Lemma 6.12. The set

$$S := \left\{ m = a^{6} + 16b^{6} \in \mathbb{Z} \mid a, b \in \mathbb{Z}, m : \text{square-free} \\ v_{2}(ab) = 0, v_{3}(b) = 1 \right\}$$

is an infinite set.

Proof. We put

$$S_0 := \left\{ m = (2k+3l)^6 + 16(6k-9l)^6 \in \mathbb{Z} \ \middle| \ k, l \in \mathbb{Z}, \ m : \text{square-free} \right\}.$$

For $(2k+3l)^6 + 16(6k-9l)^6$ being square-free it is necessary that $v_3(k) = v_2(l) = 0$. Hence S_0 is a subset of S. From the Theorem of Greaves ([5, THEOREM]) we see that S_0 is an infinite set, since $(2x+3y)^6 + 16(6x-9y)^6 = 8503785y^6 - 34009308xy^5 + 16(6x-9y)^6 = 8503785y^6 - 34009308xy^6 + 16(6x-9y)^6 + 16($ $56691900x^2y^4-50384160x^3y^3+25196400x^4y^2-6717888x^5y+746560x^6$ is an irreducible polynomial over \mathbb{Z} . This is verified by the command factor of the software Maple ([8]). Therefore S is an infinite set.

7. Uniform bounds of z'(P')

We use the notation of (3.3), (3.4). In this section we prove Lemmas 3.7 and 3.8, which were used in Proposition 3.1 to give bounds of z'(P') independent of $P \in E_{a,b}(\mathbb{Q})$.

Although the following computations can be done by manually except the numerical evaluations, we included Maple commands to carry out all the steps at end of this section.

Proof of Lemma 3.7. Let x = x(P). In this case since E' is $y^2 = (x - d)^3 + m$, we have $a'_1 = a'_3 = 0$, $a'_2 = -3d$, $a'_4 = 3d^2$, $a'_6 = m - d^3$, $b'_4 = 6d^2$, $b'_6 = 4m - 4d^3$, $b'_8 = 3d^4 - 12dm$. By (3.4)

$$z'(P') = 1 - \frac{6d^2}{(x+d)^2} - 2\frac{4m - 4d^3}{(x+d)^3} - \frac{3d^4 - 12dm}{(x+d)^4} = \frac{x^4 + 4dx^3 - 8mx + 4dm}{(x+d)^4}.$$

Since $x^3 + m = y^2 \ge 0$, $x \ge -m^{1/3}$. Note that $d > m^{1/3}$, since $(2a^2 + 4b^2)^3 - (a^6 + 16b^6) = 48b^6 + 96a^2b^4 + 48a^4b^2 + 7a^6 > 0$. If x > 0

$$\frac{x^4 + 4dx^3 - 8mx + 4dm}{(x+d)^4} \le \frac{x^4}{(x+d)^4} + \frac{4dx^3}{(x+d)^4} + \frac{4dm}{(x+d)^4} \\ < 1 + 4 + 4 = 9.$$

If x < 0

$$\frac{x^4 + 4dx^3 - 8mx + 4dm}{(x+d)^4} = \frac{x^3(x+4d)}{(x+d)^4} + \frac{-8mx + 4dm}{(x+d)^4}$$
$$< \frac{-8mx + 4dm}{(x+d)^4} < \frac{8m^{\frac{4}{3}} + 4dm}{(-m^{\frac{1}{3}} + d)^4}.$$

Assume $d = 2a^2 + 4b^2$. Putting $Y = (a/b)^2$ yields

$$\frac{8m^{\frac{4}{3}} + 4dm}{(-m^{\frac{1}{3}} + d)^4} = \frac{8(Y^3 + 16)\left((Y^3 + 16)^{\frac{1}{3}} + Y + 2\right)}{\left((Y^3 + 16)^{\frac{1}{3}} - 2Y - 4\right)^4}$$

We denote the right hand side by $g_{2,4}(Y)$. Then

$$\frac{d}{dY}g_{2,4}(Y) = -\frac{48 \left(Y^2 - 8\right) \left(2Y \left(Y^3 + 16\right)^{\frac{2}{3}} + 4 \left(Y^3 + 16\right)^{\frac{2}{3}} + 3Y^3 + 48\right)}{\left(Y^3 + 16\right)^{\frac{2}{3}} \left(\left(Y^3 + 16\right)^{\frac{1}{3}} - 2Y - 4\right)^5}.$$

Note that $(Y^3 + 16)^{\frac{1}{3}} - 2Y - 4 < 0$, since $d - m^{1/3} > 0$. So $g_{2,4}(Y)$ has a minimum at $Y = \sqrt{8}$ and a maximum at $Y = -\sqrt{8}$. Therefore

$$g_{2,4}(Y) \le \max\left\{\lim_{Y\to 0} g_{2,4}(Y), \lim_{Y\to\infty} g_{2,4}(Y)\right\}.$$

Since $g_{2,4}(0) = 120.53163357\cdots$, $\lim_{Y\to\infty} g_{2,4}(Y) = 16$, we have $z'(P') = g_{2,4}(Y) < 120.531634$.

The case $d = 3a^2 + 4b^2$ is similar and we have z'(P') < 120.531634.

Proof of Lemma 3.8. We use the notation at the beginning of the proof of Lemma 3.7. Let x = x(P), u = x/d and $u_0 = -m^{1/3}/d$. Then $u \ge u_0 > -1$, since $x \ge -m^{1/3} > -d$. Putting $Y = (a/b)^2$ with substitution $d = 2a^2 + 4b^2$ yields

$$\begin{aligned} z'(P') &= \frac{x^4 + 4dx^3 - 8mx + 4dm}{(x+d)^4} = \frac{d^4u^4 + 4d^4u^3 - 8dmu + 4dm}{(du+d)^4} \\ &= \frac{2u^4(Y^3 + 6Y^2 + 12Y + 8) + 8u^3(Y^3 + 6Y^2 + 12Y + 8) - 2u(Y^3 + 16) + Y^3 + 16}{2(u+1)^4(Y+2)^3}. \end{aligned}$$

We denote the last function by f(u, Y). Computing the derivatives, we have

$$\begin{split} \frac{\partial f}{\partial Y} &= -\frac{3(2u-1)(Y^2-8)}{(u+1)^4(Y+2)^4},\\ \frac{\partial f}{\partial u} &= 3\frac{(4u^2Y^3+uY^3-Y^3+24u^2Y^2+48u^2Y+32u^2+16u-16)}{(u+1)^5(Y+2)^3}\\ &= 12(Y^3+6Y^2+12Y+8)\frac{(u-u_1)(u-u_2)}{(u+1)^5(Y+2)^3}, \end{split}$$

where

$$\begin{split} u_1 &= -\frac{\sqrt{17Y^6 + 96Y^5 + 192Y^4 + 416Y^3 + 1536Y^2 + 3072Y + 2304} + Y^3 + 16}{8Y^3 + 48Y^2 + 96Y + 64},\\ u_2 &= \frac{\sqrt{17Y^6 + 96Y^5 + 192Y^4 + 416Y^3 + 1536Y^2 + 3072Y + 2304} - Y^3 - 16}{8Y^3 + 48Y^2 + 96Y + 64}. \end{split}$$

Clearly $u_1 < 0 < u_2$ for any fixed Y > 0, and the graph f(u, Y) has the either of the two forms below. Therefore $f(u, Y) \ge \min\{f(u_0, Y), f(u_2, Y)\}$.





FIGURE 4.

At first we consider $f(u_0, Y)$. Since $d = 2a^2 + 4b^2$,

$$f(u_0, Y) = f(-m^{\frac{1}{3}}/d, Y) = \frac{d^4u^4 + 4d^4u^3 - 8dmu + 4dm}{(du+d)^4} \bigg|_{x=-m^{\frac{1}{3}}/d}$$
$$= \frac{9m^{\frac{4}{3}}}{(d-m^{\frac{1}{3}})^4} = \frac{9(Y^3 + 16)^{\frac{4}{3}}}{(2Y+4-(Y^3+16)^{\frac{1}{3}})^4} \ge 0.75725080\cdots.$$

The last inequality follows from elementary calculus.

Next we consider $f(u_2, Y)$.

$$\frac{df(u_2, Y)}{dY} = \frac{\partial f}{\partial u}\Big|_{u=u_2} \cdot \frac{du_2}{dY} + \frac{\partial f}{\partial Y}\Big|_{u=u_2} \cdot \frac{dY}{dY}$$
$$= \frac{\partial f}{\partial Y}\Big|_{u=u_2}$$
$$= -\frac{3(2u_2 - 1)(Y^2 - 8)}{(u_2 + 1)^4(Y + 2)^4}.$$

Now

 $\begin{aligned} &2u_2 - 1\\ &= \frac{2\sqrt{17Y^6 + 96Y^5 + 192Y^4 + 416Y^3 + 1536Y^2 + 3072Y + 2304} - 2Y^3 - 32}{8Y^3 + 48Y^2 + 96Y + 64} - 1\\ &= \frac{2\sqrt{17Y^6 + 96Y^5 + \dots} - (10Y^3 + 48Y^2 + 96Y + 96)}{8Y^3 + 48Y^2 + 96Y + 64} < 0, \end{aligned}$

since

$$\left(2\sqrt{17Y^6 + 96Y^5 + \cdots}\right)^2 - (10Y^3 + 48Y^2 + 96Y + 96)^2$$

= $-32Y^6 - 576Y^5 - 3456Y^4 - 9472Y^3 - 12288Y^2 - 6144Y < 0.$

So $f(u_2, Y) \ge f(u_2(\sqrt{8}), \sqrt{8})$. Since $f(u_2(\sqrt{8}), \sqrt{8}) = 0.06232685 \cdots$, $f(u_2, Y) > 0.062326$. Therefore z'(P') = f(u, Y) > 0.062326.

The case $d = 3a^2 + 4b^2$ is similar and we have $z'(P') > 0.03806854\cdots$.

The commands for those computations by Maple are as follows. The commands for Lemma 3.7.

```
(8*m^(4/3)+4*d*m)/(x+d)^4;
subs(x=-m^(1/3),%);
subs(m=a^6+16*b^6,d=2*a^2+4*b^2,%);
g24:=simplify(subs(a=X,b=1,%)):
g24:=factor(simplify(subs(X=sqrt(Y),%)));
dg24:=factor(simplify(diff(%,Y)));
evalf(subs(Y=0,g24),16);
value(Limit(g24,Y=infinity));
```

```
(8*m^(4/3)+4*d*m)/(x+d)^4;
subs(x=-m^(1/3),%);
subs(m=a^6+16*b^6,d=3*a^2+4*b^2,%);
g34:=simplify(subs(a=X,b=1,%)):
```

```
g34:=factor(simplify(subs(X=sqrt(Y),%)));
dg34:=factor(simplify(diff(%,Y)));
evalf(subs(Y=0,g34),16);
value(Limit(g34,Y=infinity));
```

The commands for Lemma 3.8.

```
(1) d = 2a^2 + 4b^2
   (x^{4+4*d*x^{3}-8*m*x+4*d*m})/(x+d)^{4};
   subs(x=u*d.%);
   subs(m=a^6+16*b^6,d=2*a^2+4*b^2,%);
   subs(a=X,b=1,\%):
   subs(X=sqrt(Y),%);
   f24:=factor(simplify(%));
   df24Y:=factor(diff(f24,Y));
   df24u:=factor(diff(f24,u));
   sol24:=solve(numer(df24u),u);
   f24a:=subs(u=sol24[1],f24):
   evalf(subs(Y=sqrt(8),f24a),16);
   9*m^{(4/3)/(d-m^{(1/3)})^4};
   subs(m=a^6+16*b^6,d=2*a^2+4*b^2,%);
   subs(a=X,b=1,\%):
   ff24:=factor(simplify(subs(X=sqrt(Y),%)));
   factor(diff(%,Y));
```

```
evalf(subs(Y=sqrt(8),ff24),16);
```

```
(2) d = 3a^2 + 4b^2
```

```
(x^4+4*d*x^3-8*m*x+4*d*m)/(x+d)^4;
subs(x=u*d,%);
subs(m=a^6+16*b^6,d=3*a^2+4*b^2,%);
subs(a=X,b=1,%):
subs(X=sqrt(Y),%);
f34:=factor(simplify(%));
df34Y:=factor(diff(f34,Y));
df34u:=factor(diff(f34,u));
sol34:=solve(numer(df34u),u);
f34a:=subs(u=sol34[1],f34):
evalf(subs(Y=sqrt(12),f34a),16);
```

```
9*m^(4/3)/(d-m^(1/3))^4;
subs(m=a^6+16*b^6,d=3*a^2+4*b^2,%);
subs(a=X,b=1,%):
ff34:=factor(simplify(subs(X=sqrt(Y),%)));
factor(diff(%,Y));
evalf(subs(Y=sqrt(12),ff34),16);
```

APPENDIX A.

This is the code for Magma used in Remark 6.11. By this code we can verify that $P_1, P_2, P_3, P_1 \pm P_2, P_2 \pm P_3, P_3 \pm P_1, P_1 \pm 2P_2, P_2 \pm 2P_3, P_3 \pm 2P_1 \notin 5E_{a,b}(\mathbb{Q})$ for $5 \leq a \leq 6321, 3 \leq b \leq 3982$ in the situation of Theorem 1.3.

```
for a in [5..6321] do
for b in [3..3982] do
if IsOdd(a*b) then
if Valuation(b, 3) eq 1 then
if Gcd(a,b) eq 1 then
m:=a^6+16*b^6;
if m lt 6.381*10^22 then
if IsSquarefree(m) then
E:=EllipticCurve([0,0,0,0,m]);
P1:=E![-a<sup>2</sup>,4*b<sup>3</sup>,1];
P2:=E![2*a*b,a^3+4*b^3,1];
P3:=E![-2*a*b,a^3-4*b^3,1];
P4:=P1+P2;
P5:=P2+P3;
P6:=P3+P1;
P7:=P1-P2;
P8:=P2-P3;
P9:=P3-P1;
POa:=P1+2*P2;
POb:=P1-2*P2;
POc:=P2+2*P3;
POd:=P2-2*P3;
POe:=P3+2*P1;
POf:=P3-2*P1;
if
{
<
DivisionPoints(P1,5), DivisionPoints(P2,5), DivisionPoints(P3,5),
DivisionPoints(P4,5), DivisionPoints(P5,5), DivisionPoints(P6,5),
DivisionPoints(P7,5), DivisionPoints(P8,5), DivisionPoints(P9,5),
DivisionPoints(P0a,5), DivisionPoints(P0b,5), DivisionPoints(P0c,5),
DivisionPoints(POd,5), DivisionPoints(POe,5), DivisionPoints(POf,5)
>
} ne
{
} then
print a, b;
end if;
end if;
end if;
```

end if; end if; end if; end for; end for;

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25