ON FUZZY-Γ-IDEALS OF Γ-ABEL-GRASSMANN'S GROUPOIDS

*Madad Khan and Faisal

Department of Mathematics
COMSATS Institute of Information Technology
Abbottabad, Pakistan

***E-mail: madadmath@yahoo.com, yousafzaimath@yahoo.com**

Abstract. In this paper, we have introduced the notion of Γ-fuzzification in Γ-AG-groupoids which is in fact the generalization of fuzzy AG-groupoids. We have studied several properties of an intra-regular Γ-AG**-groupoids in terms of fuzzy Γ-left (right, two-sided, quasi, interior, generalized bi-, bi-) ideals. We have proved that all fuzzy Γ-ideals coincide in intra-regular Γ-AG**-groupoids. We have also shown that the set of fuzzy Γ-two-sided ideals of an intra-regular Γ-AG**-groupoid forms a semilattice structure.

Keywords. Γ-AG-groupoid, intra-regular Γ-AG**-groupoid and fuzzy Γ-ideals.

1. Introduction

Abel-Grassmann's groupoid (AG-groupoid) is the generalization of semigroup theory with wide range of usages in theory of flocks [12]. The fundamentals of this non-associative algebraic structure was first discovered by M. A. Kazim and M. Naseeruddin in 1972 [4]. AG-groupoid is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup. It is interesting to note that an AG-groupoid with right identity becomes a commutative monoid [8].

After the introduction of fuzzy sets by L. A. Zadeh [17] in 1965, there have been a number of generalizations of this fundamental concept. A. Rosenfeld [13] gives the fuzzification of algebraic structures and give the concept of fuzzy subgroups. The ideal of fuzzification in semigroup was first introduced by N. Kuroki [6].

The concept of a Γ-semigroup has been introduced by M. K. Sen [14] in 1981 as follows: A nonempty set S is called a Γ-semigroup if $x\alpha y \in S$ and $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. A Γ-semigroup is the generalization of semigroup.

In this paper we characterize Γ-AG**-groupoids by the properties of their fuzzy Γ-ideals and generalize some results. A Γ-AG-groupoid is the generalization of AG-groupoid. Let S and Γ be any nonempty sets. If there exists a mapping $S \times \Gamma \times S \to S$ written as (x, α, y) by $x\alpha y$, then S is called a Γ-AG-groupoid if $x\alpha y \in S$ such that the following Γ-left invertive law holds for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$

$$(x\alpha y)\beta z = (z\alpha y)\beta x. \tag{1}$$

2000 *Mathematics Subject Classification.* 20M10, 20N99.

A Γ-AG-groupoid also satisfies the Γ-medial law for all $w, x, y, z \in S$ and $\alpha, \beta, \gamma \in \Gamma$

$$(w\alpha x)\beta(y\gamma z) = (w\alpha y)\beta(x\gamma z). \tag{2}$$

Note that if a Γ-AG-groupoid contains a left identity, then it becomes an AG-groupoid with left identity.

A Γ-AG-groupoid is called a Γ-AG**-groupoid if it satisfies the following law for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$

$$x\alpha(y\beta z) = y\alpha(x\beta z). \tag{3}$$

A Γ-AG**-groupoid also satisfies the Γ-paramedial law for all $w, x, y, z \in S$ and $\alpha, \beta, \gamma \in \Gamma$

$$(w\alpha x)\beta(y\gamma z) = (z\alpha y)\beta(x\gamma w). \tag{4}$$

2. Preliminaries

The following definitions are available in [15].

Let S be a Γ-AG-groupoid, a non-empty subset A of S is called a Γ-AG-subgroupoid if $a\gamma b \in A$ for all a, $b \in A$ and $\gamma \in \Gamma$ or if $A\Gamma A \subseteq A$.

A subset A of a Γ-AG-groupoid S is called a Γ-left (right) ideal of S if $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$) and A is called a Γ-two-sided-ideal of S if it is both a Γ-left ideal and a Γ-right ideal.

A subset A of a Γ-AG-groupoid S is called a Γ-generalized bi-ideal of S if $(A\Gamma S)\,\Gamma A \subseteq A$.

A sub Γ-AG-groupoid A of a Γ-AG-groupoid S is called a Γ-bi-ideal of S if $(A\Gamma S)\,\Gamma A \subseteq A$.

A subset A of a Γ-AG-groupoid S is called a Γ-interior ideal of S if $(S\Gamma A)\,\Gamma S \subseteq A$.

A subset A of a Γ-AG-groupoid S is called a Γ-quasi-ideal of S if $S\Gamma A \cap A\Gamma S \subseteq A$.

A fuzzy subset f of a given set S is described as an arbitrary function $f : S \longrightarrow [0,1]$, where $[0,1]$ is the usual closed interval of real numbers.

Now we introduce the following definitions.

Let f and g be any fuzzy subsets of a Γ-AG-groupoid S, then the Γ-product $f \circ_\Gamma g$ is defined by

$$(f \circ_\Gamma g)(a) = \begin{cases} \bigvee\limits_{a=b\alpha c} \{f(b) \wedge g(c)\}, & \text{if } \exists\, b, c \in S \ni \ a = b\alpha c \text{ where } \alpha \in \Gamma. \\ 0, & \text{otherwise.} \end{cases}$$

A fuzzy subset f of a Γ-AG-groupoid S is called a fuzzy Γ-AG-subgroupoid if $f(x\alpha y) \geq f(x) \wedge f(y)$ for all x, $y \in S$ and $\alpha \in \Gamma$.

A fuzzy subset f of a Γ-AG-groupoid S is called fuzzy Γ-left ideal of S if $f(x\alpha y) \geq f(y)$ for all x, $y \in S$ and $\alpha \in \Gamma$.

A fuzzy subset f of a Γ-AG-groupoid S is called fuzzy Γ-right ideal of S if $f(x\alpha y) \geq f(x)$ for all x, $y \in S$ and $\alpha \in \Gamma$.

A fuzzy subset f of a Γ-AG-groupoid S is called fuzzy Γ-two-sided ideal of S if it is both a fuzzy Γ-left ideal and a fuzzy Γ-right ideal of S.

A fuzzy subset f of a Γ-AG-groupoid S is called fuzzy Γ-generalized bi-ideal of S if $f((x\alpha y)\beta z) \geq f(x) \wedge f(z)$, for all x, y and $z \in S$ and $\alpha, \beta \in \Gamma$.

A fuzzy Γ-AG-subgroupoid f of a Γ-AG-groupoid S is called fuzzy Γ-bi-ideal of S if $f((x\alpha y)\beta z) \geq f(x) \wedge f(z)$, for all x, y and $z \in S$ and $\alpha, \beta \in \Gamma$.

A fuzzy subset f of a Γ-AG-groupoid S is called fuzzy Γ-interior ideal of S if $f((x\alpha y)\beta z) \geq f(y)$, for all x, y and $z \in S$ and $\alpha, \beta \in \Gamma$.

A fuzzy subset f of a Γ-AG-groupoid S is called fuzzy Γ-interior ideal of S if $(f \circ_\Gamma S) \cap (S \circ_\Gamma f) \subseteq f$.

3. Γ-fuzzification in Γ-AG-groupoids

Example 1. *Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The following multiplication table shows that S is an AG-groupoid and also an AG-band.*

.	1	2	3	4	5	6	7	8	9
1	1	4	7	3	6	8	2	9	5
2	9	2	5	7	1	4	8	6	3
3	6	8	3	5	9	2	4	1	7
4	5	9	2	4	7	1	6	3	8
5	3	6	8	2	5	9	1	7	4
6	7	1	4	8	3	6	9	5	2
7	8	3	6	9	2	5	7	4	1
8	2	5	9	1	4	7	3	8	6
9	4	7	1	6	8	3	5	2	9

Clearly S is non-commutative and non-associative because $23 \neq 32$ and $(42)3 \neq 4(23)$.

Let $\Gamma = \{\alpha, \beta\}$ and define a mapping $S \times \Gamma \times S \to S$ by $a\alpha b = a^2 b$ and $a\beta b = ab^2$ for all $a, b \in S$. Then it is easy to see that S is a Γ-AG-groupoid and also a Γ-AG-band. Note that S is non-commutative and non-associative because $9\alpha 1 \neq 1\alpha 9$ and $(6\alpha 7)\beta 8 \neq 6\alpha(7\beta 8)$.

Example 2. *Let $\Gamma = \{1, 2, 3\}$ and define a mapping $\mathbb{Z} \times \Gamma \times \mathbb{Z} \to \mathbb{Z}$ by $a\beta b = b - \beta - a - \beta - z$ for all $a, b, z \in \mathbb{Z}$ and $\beta \in \Gamma$, where "$-$" is a usual subtraction of integers. Then $\mathbb{Z}$ is a Γ-AG-groupoid. Indeed*

$$\begin{aligned}(a\beta b)\gamma c &= (b - \beta - a - \beta - z)\gamma c = c - \gamma - (b - \beta - a - \beta - z) - \gamma - z \\ &= c - \gamma - b + \beta + a + \beta + z - \gamma - z = c - b + 2\beta + a - 2\gamma.\end{aligned}$$

and

$$\begin{aligned}(c\beta b)\gamma a &= (b - \beta - c - \beta - z)\gamma a = a - \gamma - (b - \beta - c - \beta - z) - \gamma - z \\ &= a - \gamma - b + \beta + c + \beta + z - \gamma - z = a - 2\gamma - b + 2\beta + c \\ &= c - b + 2\beta + a - 2\gamma.\end{aligned}$$

Which shows that $(a\beta b)\gamma c = (c\beta b)\gamma a$ for all $a, b, c \in \mathbb{Z}$ and $\beta, \gamma \in \Gamma$. This example is the generalizaion of a Γ-AG-groupoid given by T. Shah and I. Rehman (see [15]).

Example 3. *Assume that S is an AG-groupoid with left identity and let $\Gamma = \{1\}$. Define a mapping $S \times \Gamma \times S \to S$ by $x1y = xy$ for all $x, y \in S$, then S is a Γ-AG-groupoid. Thus we have seen that every AG-groupoid is a Γ-AG-groupoid for $\Gamma = \{1\}$, that is, Γ-AG-groupoid is the generalization of AG-groupoid. Also S is a Γ-AG^{**}-groupoid because $x1(y1z) = y1(x1z)$ for all $x, y, z \in S$.*

Example 4. *Let S be an AG-groupoid and $\Gamma = \{1\}$. Define a mapping $S \times \Gamma \times S \to S$ by $x1y = xy$ for all $x, y \in S$, then we know that S is a Γ-AG-groupoid . Let L be a left ideal of an AG-groupoid S, then $S\Gamma L = SL \subseteq L$. Thus L is Γ-left ideal* *of S. This shows that every Γ-left ideal of Γ-AG-groupoid is a generalization of a left ideal in an AG-groupoid (for suitable Γ). Similarly all the fuzzy Γ-ideals are the generalizations of fuzzy ideals.*

By keeping the generalization, the proof of Lemma 1 and Theorem 1 are same as in [5].

Lemma 1. *Let f be a fuzzy subset of a Γ-AG-groupoid S, then $S \circ_\Gamma f = f$.*

Theorem 1. *Let S be a Γ-AG-groupoid, then the following properties hold in S.*

(*i*) $(f \circ_\Gamma g) \circ_\Gamma h = (h \circ_\Gamma g) \circ_\Gamma f$ for all fuzzy subsets f, g and h of S.
(*ii*) $(f \circ_\Gamma g) \circ_\Gamma (h \circ_\Gamma k) = (f \circ_\Gamma h) \circ_\Gamma (g \circ_\Gamma k)$ for all fuzzy subsets f, g, h and k of S.

Theorem 2. *Let S be a Γ-AG^{**}-groupoid, then the following properties hold in S.*

(*i*) $f \circ_\Gamma (g \circ_\Gamma h) = g \circ_\Gamma (f \circ_\Gamma h)$ for all fuzzy subsets f, g and h of S.
(*ii*) $(f \circ_\Gamma g) \circ_\Gamma (h \circ_\Gamma k) = (k \circ_\Gamma h) \circ_\Gamma (g \circ_\Gamma f)$ for all fuzzy subsets f, g, h and k of S.

Proof. (i) : Assume that x is an arbitrary element of a Γ-AG**-groupoid S and let $\alpha, \beta \in \Gamma$. If x is not expressible as a product of two elements in S, then $(f \circ_\Gamma (g \circ_\Gamma h))(x) = 0 = (g \circ_\Gamma (f \circ_\Gamma h))(x)$. Let there exists y and z in S such that $x = y\alpha z$, then by using (3), we have

$$\begin{aligned}
((f \circ_\Gamma g) \circ_\Gamma (h \circ_\Gamma k))(x) &= \bigvee_{x=y\alpha z} \{(f \circ_\Gamma g)(y) \wedge (h \circ_\Gamma k)(z)\} \\
&= \bigvee_{x=y\alpha z} \left\{ \bigvee_{y=p\beta q} \{f(p) \wedge g(q)\} \wedge \bigvee_{z=u\gamma v} \{h(u) \wedge k(v)\} \right\} \\
&= \bigvee_{x=(p\beta q)\alpha(u\gamma v)} \{f(p) \wedge g(q) \wedge h(u) \wedge k(v)\} \\
&= \bigvee_{x=(v\beta u)\alpha(q\gamma p)} \{k(v) \wedge h(u) \wedge_\Gamma g(q) \wedge f(p)\} \\
&= \bigvee_{x=m\alpha n} \left\{ \bigvee_{m=v\beta u} \{k(v) \wedge h(u)\} \wedge \bigvee_{n=q\gamma p} \{g(q) \wedge f(p)\} \right\} \\
&= \bigvee_{x=m\alpha n} \{(k \circ_\Gamma h)(m) \wedge (g \circ_\Gamma f)(n)\} \\
&= ((k \circ_\Gamma h) \circ_\Gamma (g \circ_\Gamma f))(x).
\end{aligned}$$

If z is not expressible as a product of two elements in S, then $(f \circ_\Gamma (g \circ_\Gamma h))(x) = 0 = (g \circ_\Gamma (f \circ_\Gamma h))(x)$. Hence, $(f \circ_\Gamma (g \circ_\Gamma h))(x) = (g \circ_\Gamma (f \circ_\Gamma h))(x)$ for all x in S.

(ii) : If any element x of S is not expressible as product of two elements in S at any stage, then, $((f \circ_\Gamma g) \circ_\Gamma (h \circ_\Gamma k))(x) = 0 = ((k \circ_\Gamma h) \circ_\Gamma (g \circ_\Gamma f))(x)$. Assume that $\alpha, \beta, \gamma \in \Gamma$ and let there exists y, z in S such that $x = y\alpha z$, then by using (4), we have

$$\begin{aligned}
\left((f\circ_\Gamma g)\circ_\Gamma(h\circ_\Gamma k)\right)(x) &= \bigvee_{x=y\alpha z}\left\{(f\circ_\Gamma g)(y)\wedge(h\circ_\Gamma k)(z)\right\}\\
&= \bigvee_{x=y\alpha z}\left\{\bigvee_{y=p\beta q}\{f(p)\wedge g(q)\}\wedge\bigvee_{z=u\gamma v}\{h(u)\wedge k(v)\}\right\}\\
&= \bigvee_{x=(p\beta q)\alpha(u\gamma v)}\{f(p)\wedge g(q)\wedge h(u)\wedge k(v)\}\\
&= \bigvee_{x=(v\beta u)\alpha(q\gamma p)}\{k(v)\wedge h(u)\wedge_\Gamma g(q)\wedge f(p)\}\\
&= \bigvee_{x=m\alpha n}\left\{\bigvee_{m=v\beta u}\{k(v)\wedge h(u)\}\wedge\bigvee_{n=q\gamma p}\{g(q)\wedge f(p)\}\right\}\\
&= \bigvee_{x=m\alpha n}\{(k\circ_\Gamma h)(m)\wedge(g\circ_\Gamma f)(n)\}\\
&= \left((k\circ_\Gamma h)\circ_\Gamma(g\circ_\Gamma f)\right)(x).
\end{aligned}$$

□

By keeping the generalization, the proof of the following two lemma's are same as in [7].

Lemma 2. *Let f be a fuzzy subset of a Γ-AG-groupoid S, then the following properties hold.*

(*i*) f is a fuzzy Γ-AG-subgroupoid of S if and only if $f\circ_\Gamma f\subseteq f$.
(*ii*) f is a fuzzy Γ-left (right) ideal of S if and only if $S\circ_\Gamma f\subseteq f$ $(f\circ_\Gamma S\subseteq f)$.
(*iii*) f is a fuzzy Γ-two-sided ideal of S if and only if $S\circ_\Gamma f\subseteq f$ and $f\circ_\Gamma S\subseteq f$.

Lemma 3. *Let f be a fuzzy Γ-AG-subgroupoid of a Γ-AG-groupoid S, then f is a fuzzy Γ-bi-ideal of S if and only if $(f\circ_\Gamma S)\circ_\Gamma f\subseteq f$.*

Lemma 4. *Let f be any fuzzy Γ-right ideal and g be any fuzzy Γ-left ideal of Γ-AG-groupoid S, then $f\cap g$ is a fuzzy Γ-quasi ideal of S.*

Proof. It is easy to observe the following

$$((f\cap g)\circ_\Gamma S)\cap(S\circ_\Gamma(f\cap g))\subseteq(f\circ_\Gamma S)\cap(S\circ_\Gamma g)\subseteq f\cap g.$$

□

Lemma 5. *Every fuzzy Γ-quasi ideal of a Γ-AG-groupoid S is a fuzzy Γ-AG-subgroupoid of S.*

Proof. Let f be any fuzzy Γ-quasi ideal of S, then $f\circ_\Gamma f\subseteq f\circ_\Gamma S$, and $f\circ_\Gamma f\subseteq S\circ_\Gamma f$, therefore

$$f\circ_\Gamma f\subseteq f\circ_\Gamma S\cap S\circ_\Gamma f\subseteq f.$$

Hence f is a fuzzy Γ-AG-subgroupoid of S. □

A fuzzy subset f of a Γ-AG-groupoid S is called Γ-idempotent, if $f\circ_\Gamma f=f$.

Lemma 6. *In a Γ-AG-groupoid S, every Γ-idempotent fuzzy Γ-quasi ideal is a fuzzy Γ-bi-ideal of S.*

Proof. Let f be any fuzzy Γ-quasi ideal of S, then by lemma 5, f is a fuzzy Γ-AG-subgroupoid. Now by using (2), we have

$$(f \circ_\Gamma S) \circ_\Gamma f \subseteq (S \circ_\Gamma S) \circ_\Gamma f \subseteq S \circ_\Gamma f$$

and

$$\begin{aligned}(f \circ_\Gamma S) \circ_\Gamma f &= (f \circ_\Gamma S) \circ_\Gamma (f \circ_\Gamma f) = (f \circ_\Gamma f) \circ_\Gamma (S \circ_\Gamma f) \\ &\subseteq f \circ_\Gamma (S \circ_\Gamma S) \subseteq f \circ_\Gamma S.\end{aligned}$$

This implies that $(f \circ_\Gamma S) \circ_\Gamma f \subseteq (f \circ_\Gamma S) \cap (S \circ_\Gamma f) \subseteq f$. Hence by lemma 3, f is a fuzzy Γ-bi-ideal of S. □

Lemma 7. *In a Γ-AG-groupoid S, each one sided fuzzy Γ-(left, right) ideal is a fuzzy Γ-quasi ideal of S.*

Proof. It is obvious. □

Corollary 1. *In a Γ-AG-groupoid S, every fuzzy Γ-two- sided ideal of S is a fuzzy Γ-quasi ideal of S.*

Lemma 8. *In a Γ-AG-groupoid S, each one sided fuzzy Γ-(left, right) ideal of S is a fuzzy Γ-generalized bi-ideal of S.*

Proof. Assume that f be any fuzzy Γ-left ideal of S. Let a, b, $c \in S$ and let $\alpha, \beta \in \Gamma$. Now by using (1), we have $f((a\alpha b)\beta c) \geq f((c\alpha b)\beta a) \geq f(a)$ and $f((a\alpha b)\beta c) \geq f(c)$. Thus $f((a\alpha b)\beta c) \geq f(a) \wedge f(c)$. Similarly in the case of fuzzy Γ-right ideal. □

Lemma 9. *Let f or g is a Γ-idempotent fuzzy Γ-quasi ideal of a Γ-AG^{**}-groupoid S, then $f \circ_\Gamma g$ or $g \circ_\Gamma f$ is a fuzzy Γ-bi-ideal of S.*

Proof. Clearly $f \circ g$ is a fuzzy Γ-AG-subgroupoid. Now using lemma 3, (1), (4) and (2), we have

$$\begin{aligned}((f \circ_\Gamma g) \circ_\Gamma S) \circ_\Gamma (f \circ_\Gamma g) &= ((S \circ_\Gamma g) \circ_\Gamma f) \circ_\Gamma (f \circ_\Gamma g) \\ &\subseteq ((S \circ_\Gamma S) \circ_\Gamma f) \circ_\Gamma (f \circ_\Gamma g) \\ &= (S \circ_\Gamma f) \circ_\Gamma (f \circ_\Gamma g) \\ &= (g \circ_\Gamma f) \circ_\Gamma (f \circ_\Gamma S) \\ &= ((f \circ_\Gamma S) \circ_\Gamma f) \circ_\Gamma g \subseteq (f \circ_\Gamma g).\end{aligned}$$

Similarly we can show that $g \circ f$ is a fuzzy Γ-bi-ideal of S. □

Lemma 10. *The product of two fuzzy Γ-left (right) ideal of a Γ-AG^{**}-groupoid S is a fuzzy Γ-left (right) ideal of S.*

Proof. Let f and g be any two fuzzy Γ-left ideals of S, then by using (3), we have

$$S \circ_\Gamma (f \circ_\Gamma g) = f \circ_\Gamma (S \circ_\Gamma g) \subseteq f \circ_\Gamma g.$$

Let f and g be any two fuzzy Γ-right ideals of S, then by using (2), we have

$$(f \circ_\Gamma g) \circ_\Gamma S = (f \circ_\Gamma g) \circ_\Gamma (S \circ_\Gamma S) = (f \circ_\Gamma S) \circ_\Gamma (g \circ_\Gamma S) \subseteq f \circ_\Gamma g.$$

□

4. Γ-fuzzification in intra-regular Γ-AG**-groupoids

An element a of a Γ-AG-groupoid S is called an intra-regular if there exists x, $y \in S$ and $\beta, \gamma, \xi \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$ and S is called intra-regular if every element of S is intra-regular.

Note that in an intra-regular Γ-AG-groupoid S, we can write $S \circ_\Gamma S = S$.

Example 5. *Let $S = \{1, 2, 3, 4, 5\}$ be an AG-groupoid with the following multiplication table.*

.	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	d	e	c
d	a	b	c	d	e
e	a	b	e	c	d

Let $\Gamma = \{1\}$ and define a mapping $S \times \Gamma \times S \to S$ by $x1y = xy$ for all $x, y \in S$, then S is a Γ-AG**-groupoid because $(x1y)1z = (z1y)1x$ and $x1(y1z) = y1(x1z)$ for all $x, y, z \in S$. Also S is an intra-regular because $a = (b1(a1a))1a$, $b = (c1(b1b))1d$, $c = (c1(c1c))1d$, $d = (c1(d1d))1e$, $e = (c1(e1e))1c$.

It is easy to observe that $\{a, b\}$ is a Γ-two-sided ideal of an intra-regular Γ-AG**-groupoid S.

It is easy to observe that in an intra-regular Γ-AG-groupoid S, the following holds

$$S = S\Gamma S. \tag{5}$$

Lemma 11. *A fuzzy subset f of an intra-regular Γ-AG-groupoid S is a fuzzy Γ-right ideal if and only if it is a fuzzy Γ-left ideal.*

Proof. Assume that f is a fuzzy Γ-right ideal of S. Since S is an intra-regular Γ-AG-groupoid, so for each $a \in S$ there exist $x, y \in S$ and $\beta, \xi, \gamma \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$. Now let $\alpha \in \Gamma$, then by using (1), we have

$$f(a\alpha b) = f(((x\beta(a\xi a))\gamma y)\alpha b) = f((b\gamma y)\alpha(x\beta(a\xi a))) \geq f(b\gamma y) \geq f(b).$$

Conversely, assume that f is a fuzzy Γ-left ideal of S. Now by using (1), we have

$$\begin{aligned} f(a\alpha b) &= f(((x\beta(a\xi a))\gamma y)\alpha b) = f((b\gamma y)\alpha(x\beta(a\xi a))) \\ &\geq f(x\beta(a\xi a)) \geq f(a\xi a) \geq f(a). \end{aligned}$$

□

Theorem 3. *Every fuzzy Γ-left ideal of an intra-regular Γ-AG**-groupoid S is Γ-idempotent.*

Proof. Assume that f is a fuzzy Γ-left ideal of S, then clearly $f \circ_\Gamma f \subseteq S \circ_\Gamma f \subseteq f$. Since S is an intra-regular Γ-AG-groupoid, so for each $a \in S$ there exist $x, y \in S$ and $\beta, \gamma, \xi \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$. Now let $\alpha \in \Gamma$, then by using (3) and (1), we have

$$a = (x\beta(a\xi a))\gamma y = (a\beta(x\xi a))\gamma y = (y\beta(x\xi a))\gamma a.$$

Thus, we have

$$\begin{aligned} (f \circ_\Gamma f)(a) &= \bigvee_{a=(y\beta(x\xi a))\gamma a} \{f(y\beta(x\xi a)) \wedge f(a)\} \geq f(y\beta(x\xi a)) \wedge f(a) \\ &\geq f(a) \wedge f(a) = f(a). \end{aligned}$$

□

Corollary 2. *Every fuzzy* Γ*-two-sided ideal of an intra-regular* Γ-AG^{**}*-groupoid* S *is* Γ*-idempotent.*

Theorem 4. *In an intra-regular* Γ-AG^{**}*-groupoid* S, $f \cap g = f \circ_\Gamma g$ *for every fuzzy* Γ*-right ideal* f *and every fuzzy* Γ*-left ideal* g *of* S.

Proof. Assume that S is intra-regular Γ-AG**-groupoid. Let f and g be any fuzzy Γ-right and fuzzy Γ-left ideal of S, then

$$f \circ_\Gamma g \subseteq f \circ_\Gamma S \subseteq f \text{ and } f \circ_\Gamma g \subseteq S \circ_\Gamma g \subseteq g \text{ which implies that } f \circ_\Gamma g \subseteq f \cap g.$$

Since S is an intra-regular, so for each $a \in S$ there exist $x, y \in S$ and $\beta, \xi, \gamma \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$. Now let $\psi \in \Gamma$, then by using (3), (1), (5) and (4), we have

$$\begin{aligned} a &= (x\beta(a\xi a))\gamma y = (a\beta(x\xi a))\gamma y = (y\beta(x\xi a))\gamma a \\ &= ((u\psi v)\beta(x\xi a))\gamma a = ((a\psi x)\beta(v\xi u))\gamma a. \end{aligned}$$

Therefore, we have

$$\begin{aligned} (f \circ_\Gamma g)(a) &= \bigvee_{a=((a\psi x)\beta(v\xi u))\gamma a} \{f((a\psi x)\beta(v\xi u)) \wedge g(a)\} \\ &\geq f((a\psi x)\beta(v\xi u)) \wedge g(a) \\ &\geq f(a) \wedge g(a) = (f \cap g)(a). \end{aligned}$$

□

Corollary 3. *In an intra-regular* Γ-AG^{**}*-groupoid* S, $f \cap g = f \circ_\Gamma g$ *for every fuzzy* Γ*-right ideal* f *and* g *of* S.

Theorem 5. *The set of fuzzy* Γ*-two-sided ideals of an intra-regular* Γ-AG^{**}*-groupoid* S *forms a semilattice structure with identity* S.

Proof. Let $\mathbb{I}_\Gamma$ be the set of fuzzy Γ-two-sided ideals of an intra-regular Γ-AG**-groupoid S and f, g and $h \in \mathbb{I}_\Gamma$, then clearly $\mathbb{I}_\Gamma$ is closed and by corollary 2 and corollary 3, we have $f = f \circ_\Gamma f$ and $f \circ_\Gamma g = f \cap g$, where f and g are fuzzy Γ-two-sided ideals of S. Clearly $f \circ_\Gamma g = g \circ_\Gamma f$, and now by using (1), we get $(f \circ_\Gamma g) \circ_\Gamma h = (h \circ_\Gamma g) \circ_\Gamma f = f \circ_\Gamma (g \circ_\Gamma h)$. Also by using (1) and lemma 1, we have

$$f \circ_\Gamma S = (f \circ_\Gamma f) \circ_\Gamma S = (S \circ_\Gamma f) \circ_\Gamma f = f \circ_\Gamma f = f.$$

□

A fuzzy Γ-two-sided ideal f of a Γ-AG-groupoid S is said to be a Γ-strongly irreducible if and only if for fuzzy Γ-two-sided ideals g and h of S, $g \cap h \subseteq f$ implies that $g \subseteq f$ or $h \subseteq f$.

The set of fuzzy Γ-two-sided ideals of a Γ-AG-groupoid S is called a Γ-totally ordered under inclusion if for any fuzzy Γ-two-sided ideals f and g of S either $f \subseteq g$ or $g \subseteq f$.

A fuzzy Γ-two-sided ideal h of a Γ-AG-groupoid S is called a Γ-fuzzy prime ideal of S, if for any fuzzy Γ-two-sided f and g of S, $f \circ_\Gamma g \subseteq h$, implies that $f \subseteq h$ or $g \subseteq h$.

Theorem 6. *In an intra-regular* Γ-AG^{**}*-groupoid* S, *a fuzzy* Γ*-two-sided ideal is* Γ*-strongly irreducible if and only if it is* Γ*-fuzzy prime.*

Proof. It follows from corollory 3. □

Theorem 7. *Every fuzzy* Γ*-two-sided ideal of an intra-regular* Γ*-*AG^{**}*-groupoid* S *is* Γ*-fuzzy prime if and only if the set of fuzzy* Γ*-two-sided ideals of* S *is* Γ*-totally ordered under inclusion.*

Proof. It follows from corollory 3. □

Theorem 8. *For a fuzzy subset* f *of an intra-regular* Γ*-*AG^{**}*-groupoid, the following statements are equivalent.*

(i) f is a fuzzy Γ-two-sided ideal of S.
(ii) f is a fuzzy Γ-interior ideal of S.

Proof. $(i) \Rightarrow (ii)$: Let f be any fuzzy Γ-two- sided ideal of S, then obviously f is a fuzzy Γ-interior ideal of S.

$(ii) \Rightarrow (i)$: Let f be any fuzzy Γ-interior ideal of S and a, $b \in S$. Since S is an intra-regular Γ-AG-groupoid, so for each $a, b \in S$ there exist $x, y, u, v \in S$ and $\beta, \xi, \gamma, \delta, \psi, \eta \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$ and $b = (u\delta(b\psi b))\eta v$. Now let $\alpha, \in \Gamma$, thus by using (1), (3) and (2), we have

$$\begin{aligned} f(a\alpha b) &= f\left(\left(x\beta(a\xi a)\right)\gamma y\right)\alpha b) = f\left(\left(b\gamma y\right)\alpha\left(x\beta(a\xi a)\right)\right) \\ &= f\left(\left(b\gamma y\right)\alpha\left(a\beta\left(x\xi a\right)\right)\right) = f\left(\left(b\gamma a\right)\alpha\left(y\beta\left(x\xi a\right)\right)\right) \geq f\left(a\right). \end{aligned}$$

Also by using (3), (4) and (2), we have

$$\begin{aligned} f(a\alpha b) &= f\left(a\alpha\left(\left(u\delta(b\psi b)\right)\eta v\right)\right) = f\left(\left(u\delta(b\psi b)\right)\alpha\left(a\eta v\right)\right) = f\left(\left(b\delta(u\psi b)\right)\alpha\left(a\eta v\right)\right) \\ &= f\left(\left(v\delta a\right)\alpha\left((u\psi b)\eta b\right)\right) = f\left((u\psi b)\alpha\left(\left(v\delta a\right)\eta b\right)\right) \geq f(b). \end{aligned}$$

Hence f is a fuzzy Γ-two- sided ideal of S. □

Theorem 9. *A fuzzy subset* f *of an intra-regular* Γ*-*AG^{**}*-groupoid is fuzzy* Γ*-two-sided ideal if and only if it is a fuzzy* Γ*-quasi ideal.*

Proof. Let f be any fuzzy Γ-two-sided ideal of S, then obviously f is a fuzzy Γ-quasi ideal of S.

Conversely, assume that f is a fuzzy Γ-quasi ideal of S, then by using corollory 2 and (4), we have

$$f \circ_\Gamma S = (f \circ_\Gamma f) \circ_\Gamma (S \circ_\Gamma S) = (S \circ_\Gamma S) \circ_\Gamma (f \circ_\Gamma f) = S \circ_\Gamma f.$$

Therefore $f \circ_\Gamma S = (f \circ_\Gamma S) \cap (S \circ_\Gamma f) \subseteq f$. Thus f is a fuzzy Γ-right ideal of S and by lemma 11, f is a fuzzy Γ-left ideal of S. □

Theorem 10. *For a fuzzy subset* f *of an intra-regular* Γ*-*AG^{**}*-groupoid* S*, the following conditions are equivalent.*

(i) f is a fuzzy Γ-bi-ideal of S.
(ii) f is a fuzzy Γ-generalized bi-ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Let f be any fuzzy Γ-bi-ideal of S, then obviously f is a fuzzy Γ-generalized bi-ideal of S.

$(ii) \Rightarrow (i)$: Let f be any fuzzy Γ-generalized bi-ideal of S, and a, $b \in S$. Since S is an intra-regular Γ-AG-groupoid, so for each $a \in S$ there exist $x, y \in S$ and

$\beta, \gamma, \xi \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$. Now let $\alpha, \delta, \in \Gamma$, then by using (5), (4), (2) and (3), we have

$$\begin{aligned} f(a\alpha b) &= f(((x\beta(a\xi a))\gamma y)\alpha b) = f(((x\beta(a\xi a))\gamma(u\delta v))\alpha b) \\ &= f(((v\beta u)\gamma((a\xi a)\delta x))\alpha b) = f(((a\xi a)\gamma((v\beta u)\delta x))\alpha b) \\ &= f(((x\gamma(v\beta u))\delta(a\xi a))\alpha b) = f((a\delta((x\gamma(v\beta u))\xi a))\alpha b) \geq f(a) \wedge f(b). \end{aligned}$$

Therefore f is a fuzzy bi-ideal of S. □

Theorem 11. *For a fuzzy subset f of an intra-regular Γ-AG^{**}-groupoid S, the following conditions are equivalent.*

(i) f is a fuzzy Γ-two- sided ideal of S.
(ii) f is a fuzzy Γ-bi-ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Let f be any fuzzy Γ-two- sided ideal of S, then obviously f is a fuzzy Γ-bi-ideal of S.

$(ii) \Rightarrow (i)$: Let f be any fuzzy Γ-bi-ideal of S. Since S is an intra-regular Γ-AG-groupoid, so for each $a, b \in S$ there exist $x, y, u, v \in S$ and $\beta, \xi, \gamma, \delta, \psi, \eta \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$ and $b = (u\delta(b\psi b))\eta v$. Now let $\alpha \in \Gamma$, then by using (1), (4), (2) and (3), we have

$$\begin{aligned} f(a\alpha b) &= f(((x\beta(a\xi a))\gamma y)\alpha b) = f((b\gamma y)\alpha(x\beta(a\xi a))) \\ &= f(((a\xi a)\gamma x)\alpha(y\beta b)) = f(((y\beta b)\gamma x)\alpha(a\xi a)) \\ &= f((a\gamma a)\,\alpha(x\xi(y\beta b))) = f(((x\xi(y\beta b))\gamma a)\,\alpha a) \\ &= f(((x\xi(y\beta b))\gamma((x\beta(a\xi a))\gamma y))\alpha a) \\ &= f(((x\beta(a\xi a))\gamma((x\xi(y\beta b))\gamma y))\alpha a) \\ &= f(((y\beta(x\xi(y\beta b)))\gamma((a\xi a)\gamma x))\alpha a) \\ &= f(((a\xi a)\gamma((y\beta(x\xi(y\beta b)))\gamma x))\alpha a) \\ &= f(((x\xi(y\beta(x\xi(y\beta b))))\gamma\,(a\gamma a))\alpha a) \\ &= f((a\gamma\,((x\xi(y\beta(x\xi(y\beta b))))\gamma a))\alpha a) \\ &\geq f(a) \wedge f(a) = f(a). \end{aligned}$$

Now by using (3), (4) and (1), we have

$$\begin{aligned} f(a\alpha b) &= f(a\alpha(u\delta(b\psi b))\eta v) = f((u\delta(b\psi b))\alpha(a\eta v)) = f((v\delta a)\alpha((b\psi b)\eta u)) \\ &= f((b\psi b)\alpha((v\delta a)\eta u)) = f((((v\delta a)\eta u)\psi b)\,\alpha b) \\ &= f((((v\delta a)\eta u)\psi((u\delta(b\psi b))\eta v))\,\alpha b) \\ &= f(((u\delta(b\psi b))\psi(((v\delta a)\eta u)\eta v))\,\alpha b) \\ &= f(((v\delta((v\delta a)\eta u))\psi((b\psi b)\eta u))\,\alpha b) \\ &= f(((b\psi b)\psi((v\delta((v\delta a)\eta u))\eta u))\,\alpha b) \\ &= f(((u\psi(v\delta((v\delta a)\eta u)))\psi(b\eta b))\,\alpha b) \\ &= f((b\psi((u\psi(v\delta((v\delta a)\eta u)))\eta b))\,\alpha b) \\ &\geq f\,(b) \wedge f\,(b) = f\,(b)\,. \end{aligned}$$

□

Theorem 12. *Let f be a subset of an intra-regular Γ-AG^{**}-groupoid S, then the following conditions are equivalent.*

(i) f is a fuzzy Γ-bi-ideal of S.
(ii) $(f \circ_\Gamma S) \circ_\Gamma f = f$ and $f \circ_\Gamma f = f$.

Proof. $(i) \Longrightarrow (ii)$: Let f be a fuzzy Γ-bi-ideal of an intra-regular Γ-AG**-groupoid S. Let $a \in A$, then there exists $x, y \in S$ and $\beta, \gamma, \xi \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$. Now let $\alpha \in \Gamma$, then by using (3), (1), (5), (4) and (2), we have

$$\begin{aligned}
a &= (x\beta(a\xi a))\gamma y = (a\beta(x\xi a))\gamma y = (y\beta(x\xi a))\gamma a = (y\beta(x\xi((x\beta(a\xi a))\gamma y)))\gamma a \\
&= ((u\alpha v)\beta(x\xi((a\beta(x\xi a))\gamma y)))\gamma a = ((((a\beta(x\xi a))\gamma y)\alpha x)\beta(v\gamma u))\gamma a \\
&= (((x\gamma y)\alpha(a\beta(x\xi a)))\beta(v\gamma u))\gamma a = ((a\alpha((x\gamma y)\beta(x\xi a)))\beta(v\gamma u))\gamma a \\
&= ((a\alpha((x\gamma x)\beta(y\xi a)))\beta(v\gamma u))\gamma a = (((v\alpha u)\beta((x\gamma x)\beta(y\xi a)))\gamma a)\gamma a \\
&= (((v\alpha u)\beta((x\gamma x)\beta(y\xi((x\beta(a\xi a))\gamma(u\alpha v)))))\gamma a)\gamma a \\
&= (((v\alpha u)\beta((x\gamma x)\beta(y\xi((v\beta u)\gamma((a\xi a)\alpha x)))))\gamma a)\gamma a \\
&= (((v\alpha u)\beta((x\gamma x)\beta(y\xi((a\xi a)\gamma((v\beta u)\alpha x)))))\gamma a)\alpha a \\
&= (((v\alpha u)\beta((x\gamma x)\beta((a\xi a)\xi(y\gamma((v\beta u)\alpha x)))))\gamma a)\gamma a \\
&= (((v\alpha u)\beta((a\xi a)\beta((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))))\gamma a)\gamma a \\
&= (((a\xi a)\beta((v\alpha u)\beta((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))))\gamma a)\gamma a \\
&= (((((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))\xi(v\alpha u))\beta(a\beta a))\gamma a)\gamma a \\
&= ((a\beta((((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))\xi(v\alpha u))\beta a))\gamma a)\gamma a = (p\gamma a)\gamma a
\end{aligned}$$

where $p = a\beta((((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))\xi(v\alpha u))\beta a)$. Therefore

$$\begin{aligned}
((f \circ_\Gamma S) \circ_\Gamma f)(a) &= \bigvee_{a=(pa)a} \{(f \circ_\Gamma S)(pa) \wedge f(a)\} \geq \bigvee_{pa=pa} \{f(p) \circ_\Gamma S(a)\} \wedge f(a) \\
&\geq \{f(a\beta((((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))\xi(v\alpha u))\beta a)) \wedge S(a)\} \wedge f(a) \\
&\geq f(a) \wedge 1 \wedge f(a) = f(a).
\end{aligned}$$

Now by using (3), (1), (5)and (4), we have

$$\begin{aligned}
a &= (x\beta(a\xi a))\gamma y = (a\beta(x\xi a))\gamma y = (y\beta(x\xi a))\gamma a \\
&= (y\beta(x\xi((x\beta(a\xi a))\gamma(u\alpha v))))\gamma a = (y\beta(x\xi((v\beta u)\gamma((a\xi a)\alpha x))))\gamma a \\
&= (y\beta(x\xi((a\xi a)\gamma((v\beta u)\alpha x))))\gamma a = (y\beta((a\xi a)\xi(x\gamma((v\beta u)\alpha x))))\gamma a \\
&= ((a\xi a)\beta(y\xi(x\gamma((v\beta u)\alpha x))))\gamma a = (((x\gamma((v\beta u)\alpha x))\xi y)\beta(a\xi a))\gamma a \\
&= (a\beta((x\gamma((v\beta u)\alpha x))\xi y))\gamma a = (a\beta p)\gamma a
\end{aligned}$$

where $p = ((x\gamma((v\beta u)\alpha x))\xi y)$. Therefore

$$\begin{aligned}
((f \circ_\Gamma S) \circ f)(a) &= \bigvee_{a=(a\beta p)\gamma a} \{(f \circ_\Gamma S)(a\beta p) \wedge f(a)\} \\
&= \bigvee_{a=(a\beta p)\gamma a} \left(\bigvee_{a\beta p=a\beta p} f(a) \wedge S(p) \right) \wedge f(a) \\
&= \bigvee_{a=(a\beta p)\gamma a} \{f(a) \wedge 1 \wedge f(a)\} = \bigvee_{a=(a\beta p)\gamma a} f(a) \wedge f(a) \\
&\leq \bigvee_{a=(a\beta p)\gamma a} f((a\beta((x\gamma((v\beta u)\alpha x))\xi y))\gamma a) = f(a).
\end{aligned}$$

Thus $(f \circ_\Gamma S) \circ_\Gamma f = f$. As we have shown that

$$a = ((a\beta((((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))\xi(v\alpha u))\beta a))\gamma a)\gamma a.$$

Let $a = p\gamma a$ where $p = (a\beta((((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))\xi(v\alpha u))\beta a))\gamma a$. Therefore

$$\begin{aligned}(f \circ_\Gamma f)(a) &= \bigvee_{a=p\gamma a} \{f((a\beta((((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))\xi(v\alpha u))\beta a))\gamma a) \wedge f(a)\} \\ &\geq f(a) \wedge f(a) \wedge f(a) = f(a).\end{aligned}$$

Now by using lemma 2, we get $f \circ_\Gamma f = f$.

$(ii) \Longrightarrow (i)$: Assume that f is a fuzzy subset of an intra-regular Γ-AG**-groupoid S and let $\beta, \gamma \in \Gamma$, then

$$\begin{aligned}f((x\beta a)\gamma y) &= ((f \circ_\Gamma S) \circ_\Gamma f)((x\beta a)\gamma y) = \bigvee_{(x\beta a)\gamma y=(x\beta a)\gamma y} \{(f \circ_\Gamma S)(xa) \wedge f(y)\} \\ &\geq \bigvee_{x\beta a=x\beta a} \{f(x) \wedge S(a)\} \wedge f(y) \geq f(x) \wedge 1 \wedge f(y) = f(x) \wedge f(z).\end{aligned}$$

Now by using lemma 2, f is fuzzy Γ-bi-ideal of S. □

Theorem 13. *Let f be a subset of an intra-regular Γ-AG^{**}-groupoid S, then the following conditions are equivalent.*

(i) f is a fuzzy Γ-interior ideal of S.
(ii) $(S \circ_\Gamma f) \circ_\Gamma S = f$.

Proof. $(i) \Longrightarrow (ii)$: Let f be a fuzzy Γ-bi-ideal of an intra-regular Γ-AG**-groupoid S. Let $a \in A$, then there exists $x, y \in S$ and $\beta, \gamma, \xi \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$. Now let $\alpha \in \Gamma$, then by using (3), (5), (4)and (1), we have

$$\begin{aligned}a &= (x\beta(a\xi a))\gamma y = (a\beta(x\xi a))\gamma y = ((u\alpha v)\beta(x\xi a))\gamma y \\ &= ((a\alpha x)\beta(v\xi u))\gamma y = (((v\xi u)\alpha x)\beta a)\gamma y.\end{aligned}$$

Therefore

$$\begin{aligned}((S \circ_\Gamma f) \circ_\Gamma S)(a) &= \bigvee_{a=(((v\xi u)\alpha x)\beta a)\gamma y} \{(S \circ_\Gamma f)(((v\xi u)\alpha x)\beta a) \wedge S(y)\} \\ &\geq \bigvee_{((v\xi u)\alpha x)\beta a=((v\xi u)\alpha x)\beta a} \{(S((v\xi u)\alpha x) \wedge f(a)\} \wedge 1 \\ &\geq 1 \wedge f(a) \wedge 1 = f(a).\end{aligned}$$

Now again

$$\begin{aligned}((S \circ_\Gamma f) \circ_\Gamma S)(a) &= \bigvee_{a=(x\beta(a\xi a))\gamma y} \{(S \circ_\Gamma f)(x\beta(a\xi a)) \wedge S(y)\} \\ &= \bigvee_{a=(x\beta(a\xi a))\gamma y} \left(\bigvee_{(x\beta(a\xi a))=(x\beta(a\xi a))} S(x) \wedge f(a\xi a) \right) \wedge S(y) \\ &= \bigvee_{a=(x\beta(a\xi a))\gamma y} \{1 \wedge f(a\xi a) \wedge 1\} = \bigvee_{a=(x\beta(a\xi a))\gamma y} f(a\xi a) \\ &\leq \bigvee_{a=(x\beta(a\xi a))\gamma y} f((x\beta(a\xi a))\gamma y) = f(a).\end{aligned}$$

Hence it follows that $(S \circ_\Gamma f) \circ_\Gamma S = f$.

$(ii) \Longrightarrow (i)$: Assume that f is a fuzzy subset of an intra-regular Γ-AG**-groupoid S and let $\beta, \gamma \in \Gamma$, then

$$\begin{aligned} f((x\beta a)\gamma y) &= ((S \circ_\Gamma f) \circ_\Gamma S)((x\beta a)\gamma y) = \bigvee_{(x\beta a)\gamma y=(x\beta a)\gamma y} \{(S \circ_\Gamma f)(x\beta a) \wedge S(y)\} \\ &\geq \bigvee_{x\beta a=x\beta a} \{(S(x) \wedge f(a)\} \wedge S(y) \geq f(a). \end{aligned}$$

□

Lemma 12. *Let f be a fuzzy subset of of an intra-regular Γ-AG^{**}-groupoid S, then $S \circ_\Gamma f = f = f \circ_\Gamma S$.*

Proof. Let f be a fuzzy Γ-left ideal of an intra-regular Γ-AG**-groupoid S and let $a \in S$, then there exists $x, y \in S$ and $\beta, \gamma, \xi \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$.

Let $\alpha \in \Gamma$, then by using (5), (4) and (3), we have

$$\begin{aligned} a &= (x\beta(a\xi a))\gamma y = (x\beta(a\xi a))\gamma(u\alpha v) = (v\beta u)\gamma((a\xi a)\alpha x) = (a\xi a)\gamma((v\beta u)\alpha x) \\ &= (x\xi(v\beta u))\gamma(a\alpha a) = a\gamma((x\xi(v\beta u))\alpha a). \end{aligned}$$

Therefore

$$\begin{aligned} (f \circ_\Gamma S)(a) &= \bigvee_{a=a\gamma((x\xi(v\beta u))\alpha a)} \{f(a) \wedge S((x\xi(v\beta u))\alpha a)\} \\ &= \bigvee_{a=a\gamma((x\xi(v\beta u))\alpha a)} \{f(a) \wedge 1\} \\ &= \bigvee_{a=a((x(ye))a)} f(a) = f(a). \end{aligned}$$

The rest of the proof can be followed from lemma 1. □

Theorem 14. *For a fuzzy subset f of an intra-regular Γ-AG^{**}-groupoid S, the following conditions are equivalent.*

(i) f is a fuzzy Γ-left ideal of S.
(ii) f is a fuzzy Γ-right ideal of S.
(iii) f is a fuzzy Γ-two-sided ideal of S.
(iv) f is a fuzzy Γ-bi-ideal of S.
(v) f is a fuzzy Γ-generalized bi- ideal of S.
(vi) f is a fuzzy Γ-interior ideal of S.
(vii) f is a fuzzy Γ-quasi ideal of S.
$(viii)$ $S \circ_\Gamma f = f = f \circ_\Gamma S$.

Proof. $(i) \Longrightarrow (viii)$: It follows from lemma 12.

$(viii) \Longrightarrow (vii)$: It is obvious.

$(vii) \Longrightarrow (vi)$: Let f be a fuzzy Γ-quasi ideal of an intra-regular Γ-AG**-groupoid S and let $a \in S$, then there exists $b, c \in S$ and $\beta, \gamma, \xi \in \Gamma$ such that $a = (b\beta(a\xi a))\gamma c$. Let $\delta, \eta \in \Gamma$, then by using (3), (4) and (1), we have

$$\begin{aligned} (x\delta a)\eta y &= (x\delta(b\beta(a\xi a))\gamma c)\eta y = ((b\beta(a\xi a))\delta(x\gamma c))\eta y \\ &= ((c\beta x)\delta((a\xi a)\gamma b))\eta y = ((a\xi a)\delta((c\beta x)\gamma b))\eta y \\ &= (y\delta((c\beta x)\gamma b))\eta(a\xi a) = a\eta((y\delta((c\beta x)\gamma b))\xi a) \end{aligned}$$

and from above

$$\begin{aligned}(x\delta a)\eta y &= (y\delta((c\beta x)\gamma b))\eta(a\xi a) = (a\delta a)\eta(((c\beta x)\gamma b)\xi y) \\ &= ((((c\beta x)\gamma b)\xi y)\delta a)\eta a.\end{aligned}$$

Now by using lemma 12, we have

$$\begin{aligned}f((x\delta a)\eta y) &= ((f\circ_\Gamma S)\cap(S\circ_\Gamma f))((x\delta a)\eta y) \\ &= (f\circ_\Gamma S)((x\delta a)\eta y)\wedge(S\circ_\Gamma f)((x\delta a)\eta y).\end{aligned}$$

Now

$$(f\circ_\Gamma S)((x\delta a)\eta y) = \bigvee_{(x\delta a)\eta y = a\eta((y\delta((c\beta x)\gamma b))\xi a)} \{f(a)\wedge S_\delta((y\delta((c\beta x)\gamma b))\xi a)\} \geq f(a)$$

and

$$(S\circ_\Gamma f)((x\delta a)\eta y) = \bigvee_{(x\delta a)\eta y = ((((cx)b)y)a)a} \{S((((c\beta x)\gamma b)\xi y)\delta a)\wedge f(a)\} \geq f(a).$$

This implies that $f((x\delta a)\eta y) \geq f(a)$ and therefore f is a fuzzy Γ-interior ideal of S.

$(vi) \Longrightarrow (v)$: It follows from theorems 8, 11 and 10.

$(v) \Longrightarrow (iv)$: It follows from theorem 10.

$(iv) \Longrightarrow (iii)$: It follows from theorem 11.

$(iii) \Longrightarrow (ii)$: It is obvious and $(ii) \Longrightarrow (i)$ can be followed from lemma 11. □

References

[1] P. Holgate, Groupoids satisfying a simple invertive law, The Math. Stud., 1 − 4, 61 (1992), 101 − 106.

[2] J. Ježek and T. Kepka, A note on medial division groupoids, Proc. Amer. Math. Soc., 2, 119 (1993), 423 − 426.

[3] J. Ježek and T. Kepka, The equational theory of paramedial cancellation groupoids, Czech. Math. J., 50, 125 (2000), 25 − 34.

[4] M. A. Kazim and M. Naseeruddin, On almost semigroups, The Alig. Bull. Math., 2 (1972), 1 − 7.

[5] M. N. A. Khan, Fuzzification in Abel-Grassmann's groupoids, MS. Thesis, 2009.

[6] N. Kuroki, Fuzzy bi-ideals in semigroups, Comment. Math. Univ. St. Pauli, 27 (1979), 17−21.

[7] J. N. Mordeson, D. S. Malik and N. Kuroki, *Fuzzy semigroups*, Springer-Verlag, Berlin Germany, 2003.

[8] Q. Mushtaq and S. M. Yusuf, On LA-semigroups, The Alig. Bull. Math., 8 (1978), 65 − 70.

[9] Q. Mushtaq and S. M. Yusuf, On LA-semigroup defined by a commutative inverse semigroup, Math. Bech., 40 (1988), 59 − 62.

[10] Q. Mushtaq and M. Khan, M-systems in LA-semigroups, Souteast Asian Bulletin of Mathematics, 33 (2009), 321 − 327.

[11] Q. Mushtaq and M. Khan, Semilattice decomposition of locally associative AG**-groupoids, Algeb. Colloq., 16, 1 (2009), 17 − 22.

[12] M. Naseeruddin, Some studies in almost semigroups and flocks, Ph.D., thesis, Aligarh Muslim University, Aligarh, India, 1970.

[13] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl., 35 (1971), 512 − 517.

[14] M. K. Sen, On Γ-semigroups, Proceeding of International Symposium on Algebra and Its Applications, Decker Publication, New York, (1981), 301 − 308.

[15] T. Shah and I. Rehman, On M-systems in Γ-AG-groupoids, Proc. Pakistan Acad. Sci., 47(1) : 33 − 39. 2010.

[16] N. Stevanović and P. V. Protić, Composition of Abel-Grassmann's 3-bands, Novi Sad, J. Math., 2, 34 (2004), 175 − 182.

[17] L. A. Zadeh, Fuzzy sets, Inform. Control, 8 (1965), 338 − 353.