

SOLITON-LIKE SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATION WITH VARIABLE QUADRATIC HAMILTONIANS

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ABSTRACT. We construct soliton-like solutions for the nonlinear Schrödinger equation with variable quadratic Hamiltonians in a unified form by using a complete integrability of generalized harmonic oscillators. Most of linear (hypergeometric, Bessel) and a few of nonlinear (Jacobian elliptic, Painlevé II transcendental) classical special functions of mathematical physics are linked together through these solutions. Examples include bright and dark solitons, Jacobi elliptic and Painlevé II transcendental solutions.

1. INTRODUCTION

Advances of the past decades in nonlinear optics, Bose–Einstein condensates, propagation of soliton waves in plasma physics and in other fields of nonlinear science have involved a detailed study of nonlinear Schrödinger equations (see, for example, [8], [29], [34], [65], [67] and references therein). In the theory of Bose–Einstein condensation [15], [45], from a general point of view, the dynamics of gases of cooled atoms in a magnetic trap at very low temperatures can be described by an effective equation for the condensate wave function known as the Gross–Pitaevskii (or nonlinear Schrödinger) equation [26], [27], [30] and [44]. Experimental observations of dark and bright solitons [9], [28] and bright soliton trains [29], [53] in the presence of harmonic confinement have generated a considerable research interest in this area.

The propagation of optical pulse inside a real fiber optics is also well described by nonlinear Schrödinger equation for the envelope of wave functions travelling inside the fiber [3], [8], [23]. A class of self-similar solutions that exists for physically realistic dispersion and nonlinearity profiles in a fiber with anomalous group velocity dispersion is found in [31], [32], [40], [50], [51], which suggests a method of pulse compression and a model of steady-state asynchronous laser mode locking [41].

Integration techniques of the nonlinear Schrödinger equation include Painlevé analysis [10], [34], [60], Hirota method [25], [34], Lax method [34], [36], [67], inverse scattering transform and Hamiltonian approach [1], [2], [20], [21], [43] among others. We elaborate on results of recent papers [6], [8], [18], [24], [29], [31], [32], [33], [47], [53], [57], [62], [63], [64], [65] on construction of exact solitary wave solutions of the nonlinear Schrödinger equation with variable quadratic Hamiltonians (see also [56] and [67]). In this Letter, a unified form of these soliton-like solutions is presented thus combining advances of the soliton theory with a complete integrability of generalized harmonic oscillators. Examples include bright and dark solitons, Jacobi elliptic and Painlevé II transcendental solutions for solitary wave profiles.

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2. SOLITON-LIKE SOLUTIONS

The nonlinear Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = H\psi + g\psi + h |\psi|^2 \psi, \quad (2.1)$$

where the variable Hamiltonian H is an arbitrary quadratic form of operators $p = -i\partial/\partial x$ and x , namely,

$$i\psi_t = -a(t) \psi_{xx} + b(t) x^2 \psi - ic(t) x\psi_x - id(t) \psi + g(x, t) \psi + h(t) |\psi|^2 \psi \quad (2.2)$$

(a, b, c, d are suitable real-valued functions of time only) has the following soliton-like solutions

$$\begin{aligned} \psi(x, t) &= \frac{e^{i\phi}}{\sqrt{\mu(t)}} \exp(i(\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2)) \\ &\times F(\beta(t)x + 2\gamma(t)y) \end{aligned} \quad (2.3)$$

(ϕ is a real constant, y is a parameter and $\mu, \alpha, \beta, \gamma$ are real-valued functions of time only given by equations (2.13)–(2.19) below), provided that

$$g = g_0 a(t) \beta^2(t) (\beta(t)x + 2\gamma(t)y)^m, \quad h = h_0 a(t) \beta^2(t) \mu(t) \quad (2.4)$$

(g_0 and h_0 are constants and $m = 0, 1$). As we shall see in the next section, these conditions control a delicate balance between the linear Hamiltonian, dispersion and nonlinearity in the Schrödinger equation (2.2) thus making an existence of the soliton-like solution with damping possible in the presence of variable quadratic potentials (cf. [56]).

Here, the soliton profile function $F(z)$ of a single travelling wave-type argument $z = \beta x + 2\gamma y$ satisfies the ordinary nonlinear differential equation of the form

$$F''(z) = g_0 z^m F(z) + h_0 F^3(z). \quad (2.5)$$

If $m = 0$, with the help of an integrating factor,

$$\left(\frac{dF}{dz}\right)^2 = C_0 + g_0 F^2 + \frac{1}{2} h_0 F^4 \quad (C_0 \text{ is a constant}), \quad (2.6)$$

which can be solved in terms of Jacobian elliptic functions [4], [19], [34], [59]. When $m = 1$, equation (2.5) leads to Painlevé II transcendents [2], [34].

The variable phase is given in terms of solutions of the following system of ordinary differential equations:

$$\frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^2 = 0, \quad (2.7)$$

$$\frac{d\beta}{dt} + (c + 4a\alpha)\beta = 0, \quad (2.8)$$

$$\frac{d\gamma}{dt} + a\beta^2 = 0 \quad (2.9)$$

(see Ref. [11] and the next section for more details), where the standard substitution

$$\alpha = \frac{1}{4a(t)} \frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)} \quad (2.10)$$

reduces the Riccati equation (2.7) to the second order linear equation

$$\mu'' - \tau(t) \mu' + 4\sigma(t) \mu = 0 \quad (2.11)$$

with

$$\tau(t) = \frac{a'}{a} - 2c + 4d, \quad \sigma(t) = ab - cd + d^2 + \frac{d}{2} \left(\frac{a'}{a} - \frac{d'}{d} \right). \quad (2.12)$$

It is worth noting that in the soliton-like solution under consideration (2.3) the linear and nonlinear factors are essentially separated, namely, the nonlinear part is represented only by the profile function F of a single travelling wave variable $z = \beta x + 2\gamma y$ as solution of the nonlinear equation (2.5). The initial value problem for the system (2.7)–(2.9), which corresponds to the linear Schrödinger equation with a variable quadratic Hamiltonian (generalized harmonic oscillators [7], [16], [22], [61], [66]), can be explicitly solved in terms of solutions of our characteristic equation (2.11) as follows [11], [13], [54], [55]:

$$\mu(t) = 2\mu(0) \mu_0(t) (\alpha(0) + \gamma_0(t)), \quad (2.13)$$

$$\alpha(t) = \alpha_0(t) - \frac{\beta_0^2(t)}{4(\alpha(0) + \gamma_0(t))}, \quad (2.14)$$

$$\beta(t) = -\frac{\beta(0)\beta_0(t)}{2(\alpha(0) + \gamma_0(t))} = \frac{\beta(0)\mu(0)}{\mu(t)} \lambda(t), \quad (2.15)$$

$$\gamma(t) = \gamma(0) - \frac{\beta^2(0)}{4(\alpha(0) + \gamma_0(t))}, \quad (2.16)$$

where

$$\alpha_0(t) = \frac{1}{4a(t)} \frac{\mu'_0(t)}{\mu_0(t)} - \frac{d(t)}{2a(t)}, \quad (2.17)$$

$$\beta_0(t) = -\frac{\lambda(t)}{\mu_0(t)}, \quad \lambda(t) = \exp\left(-\int_0^t (c(s) - 2d(s)) ds\right), \quad (2.18)$$

$$\gamma_0(t) = \frac{1}{2\mu_1(0)} \frac{\mu_1(t)}{\mu_0(t)} + \frac{d(0)}{2a(0)} \quad (2.19)$$

provided that μ_0 and μ_1 are the standard solutions of equation (2.11) corresponding to the following initial conditions $\mu_0(0) = 0$, $\mu'_0(0) = 2a(0) \neq 0$ and $\mu_1(0) \neq 0$, $\mu'_1(0) = 0$. (Formulas (2.17)–(2.19) correspond to Green's function of generalized harmonic oscillators; see, for example, [11], [13], [17], [38], [54], [55] and references therein for more details.)

The continuity with respect to initial data,

$$\lim_{t \rightarrow 0^+} \alpha(t) = \alpha(0), \quad \lim_{t \rightarrow 0^+} \beta(t) = \beta(0), \quad \lim_{t \rightarrow 0^+} \gamma(t) = \gamma(0), \quad (2.20)$$

has been established in [54] for suitable smooth coefficients of the linear Schrödinger equation. Thus the soliton-like solution (2.3) evolves to the future $t > 0$ starting from the following initial data:

$$\begin{aligned} \psi(x, 0) &= \lim_{t \rightarrow 0^+} \psi(x, t) \\ &= \frac{e^{i\phi}}{\sqrt{\mu(0)}} \exp(i(\alpha(0)x^2 + \beta(0)xy + \gamma(0)y^2)) \\ &\quad \times F(\beta(0)x + 2\gamma(0)y), \end{aligned} \quad (2.21)$$

where ϕ , $\mu(0)$, $\alpha(0)$, $\beta(0)$, $\gamma(0)$ and y are arbitrary real parameters.

3. SKETCH OF THE PROOF

Following [11] (see also [39] and [47]), we are looking for exact solutions of the form

$$\psi = A(x, t) e^{iS(x, t)}, \quad S(x, t) = \alpha(t) x^2 + \beta(t) xy + \gamma(t) y^2 \quad (3.1)$$

(y is a parameter). Substituting into (2.2) and taking the imaginary part,

$$A_t + ((4a\alpha + c)x + 2a\beta y) A_x + (2\alpha a + d) A = 0. \quad (3.2)$$

For the real part, equating coefficients of all admissible powers of $x^m y^n$ with $m + n = 2$, one gets our system of ordinary differential equations (2.7)–(2.9) of the corresponding linear Schrödinger equation with the unique solution (2.13)–(2.19) already obtained in Refs. [11], [54], [55] and/or elsewhere. In addition, an auxiliary nonlinear equation of the form

$$aA_{xx} = gA + hA^3 \quad (3.3)$$

appears as a contribution from the last two terms. With the help of (2.8) and (2.10) our equation (3.2) can be rewritten as

$$A_t - \left(\frac{\beta'}{\beta} x - 2a\beta y \right) A_x + \frac{1}{2} \frac{\mu'}{\mu} A = 0. \quad (3.4)$$

Looking for a travelling wave solution with damping of the form

$$A = A(x, t) = \frac{1}{\sqrt{\mu(t)}} F(z), \quad z = c_0(t)x + c_1(t)y, \quad (3.5)$$

one gets

$$c'_0 x + c'_1 y = \left(\frac{\beta'}{\beta} x - 2a\beta y \right) c_0 \quad (3.6)$$

with $c_0 = \beta$ and $c_1 = 2\gamma$ (or $z = \beta x + 2\gamma y$). Then equation (3.3) takes the form

$$\frac{d^2}{dz^2} F(z) = \frac{g}{a\beta^2} F(z) + \frac{h}{a\beta^2 \mu} F^3(z), \quad (3.7)$$

which must have all coefficients depending on z only in order to preserve a self-similar profile of the travelling wave with damping. This results in the required equation (2.5) under the balancing conditions (2.4) and our proof is complete.

4. SUMMARY AND EXAMPLES

A brief description of the method under consideration is as follows. In order to obtain soliton-like solutions (2.3) explicitly, say in terms of elementary and/or transcendental functions, one has to solve, in general, the nonlinear equation (2.5) for the profile function $F(z)$ in terms of Jacobian elliptic functions [4], [19], [34], [46], [59] (some elementary solutions are also available), when $m = 0$, or in terms of Painlevé II transcendents, when $m = 1$ (it is known that if $m > 1$, this equation does not have the Painlevé property [2], [34]). In addition, one has to solve the linear characteristic equation (2.11), which has a variety of solutions in terms of elementary and special (hypergeometric, Bessel) functions [5], [37], [42], [46], [58]. Many elementary solutions of the corresponding linear Schrödinger equation for generalized harmonic oscillators are known explicitly (see, for example, [11], [12], [13], [14], [17], [38], [55], [61], [66] and references therein). Then, the linear part allows to determine the travelling wave argument $z = \beta x + 2\gamma y$ and the damping factor $\mu^{-1/2}$ of the

soliton-like solution (2.3). Our balancing conditions (2.4) control dispersion and nonlinearity in the original Schrödinger equation (2.2), which is crucial for the soliton existence.

4.1. Nonlinear Part. When $m = 0$, equation (2.5) is integrated to the first order equation (2.6) and (the corresponding initial value problem) can be solved in terms of elliptic integrals and Jacobian (doubly) periodic elliptic functions [4], [19], [34]. Some soliton configurations are

$$F(z) = \left(\frac{g_0 + \sqrt{g_0^2 - 2C_0 h_0}}{-h_0} \right)^{1/2} \times \text{cn} \left((g_0^2 - 2C_0 h_0)^{1/4} z, \left(\frac{g_0 + \sqrt{g_0^2 - 2C_0 h_0}}{2\sqrt{g_0^2 - 2C_0 h_0}} \right)^{1/2} \right), \quad (4.1)$$

if $h_0 < 0$ and

$$F(z) = \left(\frac{-g_0 + \sqrt{g_0^2 - 2C_0 h_0}}{h_0} \right)^{1/2} \times \text{sn} \left(\left(\frac{C_0 h_0}{-g_0 + \sqrt{g_0^2 - 2C_0 h_0}} \right) z, \left(\frac{g_0 - \sqrt{g_0^2 - 2C_0 h_0}}{g_0 + \sqrt{g_0^2 - 2C_0 h_0}} \right)^{1/2} \right), \quad (4.2)$$

if $g_0 < 0$. Here, $\text{cn}(u, k)$ and $\text{sn}(u, k)$ are the Jacobi elliptic functions [4], [19], [59]. Familiar special cases include the *bright* soliton:

$$F(z) = \sqrt{\frac{2g_0}{-h_0}} \frac{1}{\cosh(\sqrt{g_0}z)} \quad (4.3)$$

with $C_0 = 0$ in (4.1) and the *dark* soliton:

$$F(z) = \sqrt{\frac{-g_0}{h_0}} \tanh \left(\sqrt{\frac{-g_0}{2}} z \right) \quad (4.4)$$

with $C_0 = g_0^2/(2h_0)$ in (4.2), when $\text{cn}(u, 1) = 1/\cosh u$ and $\text{sn}(u, 1) = \tanh u$, respectively (the real period tends to infinity). More details can be found in Refs. [4], [19], [32], [59] and/or elsewhere.

If $m = 1$, the substitution $F(z) = g_0^{1/3} \sqrt{2/h_0} u(\zeta)$ and $\zeta = z g_0^{1/3}$ transforms (2.5) into the second Painlevé equation,

$$u'' = \zeta u + 2u^3, \quad (4.5)$$

whose solutions are discussed in [2], [34] (see also references therein).

4.2. Linear Part. Generalized harmonic oscillators [7], [16], [22], [61], [66], which correspond to the Schrödinger equation with variable quadratic Hamiltonians, are very-well studied in quantum mechanics (see also [11], [12], [13], [14], [17], [35], [38], [55] and references therein for a general approach and known elementary and transcendental solutions).

4.3. Examples. Combination of linear and nonlinear parts together by our formula (2.3) results in numerous explicit soliton-like solutions for corresponding nonlinear Schrödinger equations. It is worth noting that in this approach most of linear and some of nonlinear classical special functions of mathematical physics are linked together through these solutions.

4.3.1. *Nonlinear Optics.* In the simplest case,

$$i\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x^2} + g\psi + h|\psi|^2\psi, \quad (4.6)$$

one gets [11], [40], [41]

$$\alpha(t) = \frac{\alpha_0}{1 - 4\alpha_0 t}, \quad \beta(t) = \frac{\beta_0}{1 - 4\alpha_0 t}, \quad (4.7)$$

$$\gamma(t) = \gamma_0 + \frac{\beta_0^2 t}{1 - 4\alpha_0 t}, \quad \mu(t) = \mu_0(1 - 4\alpha_0 t)$$

($\mu_0, \alpha_0, \beta_0, \gamma_0$ are constants) and

$$z = \frac{\beta_0 x + 2(\gamma_0 + (\beta_0^2 - 4\alpha_0\gamma_0)t)y}{1 - 4\alpha_0 t}, \quad (4.8)$$

$$g(x, t) = -\frac{g_0\beta_0^2}{(1 - 4\alpha_0 t)^2} z^m \quad (m = 0, 1), \quad (4.9)$$

$$h(t) = -\frac{h_0\mu_0\beta_0^2}{1 - 4\alpha_0 t}. \quad (4.10)$$

(Traditionally, $\alpha_0 = 0$ and $m = 0$ with $\psi = \chi \exp(ig_0\beta_0^2 t)$ [34], [67].)

The case $b = c = 0$,

$$i\psi_t = -a\psi_{xx} - id\psi + g\psi + h|\psi|^2\psi, \quad (4.11)$$

is of interest in fiber optics. The substitution $\psi = \chi e^{-\lambda}$, $\lambda(t) = \int_0^t d(s) ds$ results in

$$i\frac{\partial\chi}{\partial t} = -a\frac{\partial^2\chi}{\partial x^2} + g\chi + he^{-2\lambda}|\chi|^2\chi, \quad (4.12)$$

which, of course, can be solved by the method under consideration, but the standard change of time variable,

$$\tau = \tau(0) - \int_0^t a(s) ds, \quad (4.13)$$

transforms this equation into the previous one. Details are left to the reader (see also [6], [31], [32] and [51], where this simple observation has been omitted).

4.3.2. *Bose–Einstein Condensation.* The Gross–Pitaevskii equation for a zero-temperature Bose–Einstein condensate of atoms, confined in a cylindrical trap $V_0(x, y) = m\omega_\perp^2(x^2 + y^2)/2$, and a time-dependent harmonic confinement, which can be either attractive or repulsive, along the z direction $V_1(z, t) = m\omega_0^2(t)z^2/2$, is given by [6], [15], [45], [49], [52]:

$$i\hbar\frac{\partial\Psi(\mathbf{r}, t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta + U|\Psi(\mathbf{r}, t)|^2 + V + i\frac{\eta(t)}{2} \right) \Psi(\mathbf{r}, t), \quad (4.14)$$

where $U = 4\pi\hbar^2 a_s(t)/m$, $V = V_0(x, y) + V_1(z, t)$ and the condensate interaction with the normal atomic cloud through three-body interaction is phenomenologically incorporated by a gain or loss term $\eta(t)$. If the interaction energy of atoms is much less than the kinetic energy in the transverse direction, then [49]:

$$\Psi(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi a_B a_\perp}} \psi\left(\frac{z}{a_\perp}, \omega_\perp t\right) \exp\left(-i\omega_\perp t - \frac{x^2 + y^2}{2a_\perp^2}\right) \quad (4.15)$$

and equation (4.14) reduces to the following one-dimensional nonlinear Schrödinger equation in dimensionless units:

$$i\psi_t = -\frac{1}{2}\psi_{\xi\xi} + \frac{1}{2}\omega^2(t)\xi^2\psi + i\frac{\delta(t)}{2}\psi + \varepsilon(t)|\psi|^2\psi. \quad (4.16)$$

Here, $\varepsilon(t) = 2a_s(t)/a_B$, $\omega^2(t) = \omega_0^2(t)/\omega_\perp^2$, $\delta(t) = \eta(t)/(\hbar\omega_\perp)$, $a_\perp = (\hbar/m\omega_\perp)^{1/2}$ and a_B is the Bohr radius.

By the substitution

$$\psi = \chi \exp\left(\int_0^t (\delta(s) - ig(s)) ds\right) \quad (4.17)$$

with the function $g(t)$ given by (4.21) below, equation (4.16) can be transformed into the form (2.2) with the following coefficients

$$a = \frac{1}{2}, \quad b = \frac{1}{2}\omega^2(t), \quad c = d = 0, \quad h = \varepsilon(t) \exp\left(2\int_0^t \delta ds\right). \quad (4.18)$$

Then our characteristic equation (2.11) take the form

$$\mu'' + \omega^2(t)\mu = 0, \quad (4.19)$$

which describes the motion of a classical oscillator with variable frequency [37]. Choosing the standard solutions $\mu_0(t)$ and $\mu_1(t)$ with $\mu_0(0) = 0$, $\mu_0'(0) = 2a(0) \neq 0$ and $\mu_1(0) \neq 0$, $\mu_1'(0) = 0$, one can use formulas (2.13)–(2.19) with $c = d = 0$ in order to solve the linear problem. This gives the soliton travelling-wave variable $z = \beta x + 2\gamma y$ and the following balancing conditions:

$$\varepsilon(t) = \frac{\beta^2(0)\mu^2(0)}{2\mu(t)} \exp\left(-2\int_0^t \delta(s) ds\right), \quad (4.20)$$

$$g(t) = \frac{\beta^2(0)\mu^2(0)}{2\mu^2(t)}, \quad (4.21)$$

when $m = 0$ [6].

Other examples are left to the reader (see [6], [8], [18], [24], [29], [31], [32], [57], [64], [65], [67] and references therein).

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