Infinite disorder scaling of random quantum magnets in three and higher dimensions

István A. Kovács[1, 2, *] and Ferenc Iglói[2, 3, †]

[1]*Department of Physics, Loránd Eötvös University,*
H-1117 Budapest, Pázmány P. s. 1/A, Hungary
[2]*Research Institute for Solid State Physics and Optics, H-1525 Budapest, P.O.Box 49, Hungary*
[3]*Institute of Theoretical Physics, Szeged University, H-6720 Szeged, Hungary*
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Using a very efficient numerical algorithm of the strong disorder renormalization group method we have extended the investigations about the critical behavior of the random transverse-field Ising model in three and in higher dimensions. In all studied cases an infinite disorder quantum critical point is identified, which ensures that the applied method is asymptotically correct and the calculated critical exponents tend to the exact values for large scales. We have shown that the critical exponents are independent of the form of the disorder and they are smooth functions of the dimensionality.

Introduction Quantum phase transitions are among the fundamental problems of modern physics, the properties of which are studied in solid state physics, quantum field-theory, quantum information and statistical mechanics[1]. These transitions take place at $T = 0$ temperature, i.e. in the ground state of the quantum system by varying a control parameter, such as the strength of a transverse field. One basic question in this field of research is how quenched disorder influences the properties of quantum phases and phase transitions. In this respect quantum spin glasses and the glass transition are particularly interesting[2]. This latter problem theoretically is very challenging, since the corresponding quantum state is the result of an interplay between quantum and disorder fluctuations, strong correlations and frustration.

One of the paradigmatic models of random quantum magnets with a discrete symmetry is the random transverse-field Ising model (RTIM), which is defined by the Hamiltonian:

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} \sigma_i^x \sigma_j^x - \sum_i h_i \sigma_i^z . \tag{1}$$

Here the $\sigma_i^{x,z}$ are Pauli-matrices and i, j denote sites of a lattice (or a graph). Experimentally the RTIM is realized by the compound $LiHo_xY_{1-x}F_4$ and at a concentration, $x = 0.167$, a spin glass phase was observed[3]. In this compound there is a dipole-coupling between the Ising spins, thus the interaction is long-ranged. Applying a magnetic field H_t transverse to the Ising axis results in a transverse field of strength, $h_i = H_t^2$. In the theoretical investigations the interactions in the RTIM are generally assumed to be short-ranged, thus the first sum in Eq.(1) runs over nearest neighbors. Furthermore the J_{ij} couplings and the h_i transverse fields are independent random numbers, which are taken from the distributions, $p(J)$ and $q(h)$, respectively.

Detailed theoretical results about the RTIM are known in one dimension (1D) due to a complete analytical solution of a renormalization group (RG) treatment[4]. The RG results are expected to be asymptotically ex-

act in the vicinity of the critical point (and also in the Griffiths-phase, as long as dynamical singularities are concerned[5]), which is indeed demonstrated by a comparison with independent analytical[6, 7] and numerical[8, 9] works. One important observation, that the critical properties of the 1D model are governed by an infinite disorder fixed point (IDFP), in which the strength of disorder growths without limit during renormalization[10] and thus become dominant over quantum fluctuations.

The IDFP scenario is found to be valid for the 2D RTIM, too, as observed in numerical RG studies[11–17] and in Monte Carlo (MC) simulations[18]. The calculated critical exponents are in agreement with the MC results about the 2D random contact process[19], which is a simple nonequilibrium model of spreading infections. The d-dimensional random contact process is expected to be in the same universality class[20] as the RTIM, at least for strong enough disorder.

In three dimensions, which is connected to real quantum magnets, no quantitative results are known, so far. Analysis of the numerical RG trajectories lead to the conclusion[11], that the critical behavior in this case is probably controlled by an IDFP, but no estimates about the critical exponents are available. For even higher dimensions it is unclear, if the IDFP scenario stays valid for any finite value of d, or there is some upper critical dimension, d^c, so that for $d > d^c$ the critical behavior is of conventional disorder type. We note that the large-d limit of the problem is relevant for models with long-range interactions, such as in $LiHo_xY_{1-x}F_4$.

In this Letter we aim to extend the investigations about the critical behavior of the RTIM in three and in higher dimensions. Our studies are based on an improved numerical algorithm of the strong disorder RG (SDRG) procedure, which makes possible to obtain accurate results in 3D and 4D lattices. We have also studied the large-d limit of the problem, which is realized by Erdős-Rényi (ER) random graphs[21] with a finite coordination number.

SDRG procedure In the calculation we used the SDRG procedure[22], which has been introduced by Ma, Dasgupta and Hu[23]. At each step of the renormalization the largest local term in the Hamiltonan (either a coupling or a transverse field) is eliminated and new terms are generated between remaining sites by second-order perturbation method. After decimating a strong coupling the two connected spins form a spin cluster having an additive moment, μ, which is placed in an effective transverse field. After decimating a large transverse field the actual spin is eliminated and new effective couplings are generated between each pair of spins being nearest neighbors to the decimated site. If at one step two parallel couplings appear between two neighboring sites the maximum of them is taken. Application of this "maximum rule" is exact at an IDFP and results in simplifications of the RG procedure. Here we have developed an optimized algorithm, which needs $t \sim \mathcal{O}(N \log N + E)$ time to renormalize a cluster with N sites and E edges up to the last spin, irrespective of the dimension and topology of the cluster[24]. In this algorithm terms in the Hamiltonian are decimated in descending order in energy and we have applied the following theorem for transverse field decimation. According to this theorem for a decimated site, i, there is always one relevant neighboring site, j, so that after decimating i only those renormalized couplings should be created, which connect j with its new neighboring sites[25]. All the couplings which start from other nearest neighbors of i (and does not end at j) are irrelevant and need not be created. In this way during one RG step not only the number of sites is reduced (by one), but the number of couplings, as well. Using this algorithm we avoid to generate almost fully connected clusters, which is the main drawback of the naïve implementation of the method in higher dimensions[17], having a performance: $t \sim \mathcal{O}(N^3)$.

Calculation of critical parameters In the actual calculation we have renormalized finite clusters up to the last site and we kept the energy scale, ϵ, as well as the total moment, μ. (The way how to deduce these quantities in the RG procedure are described in Ref.[17].) In 3D and 4D the clusters have a hypercubic shape with a linear length, L, while for the ER model a graph consists of N sites and of $kN/2$ edges being in random positions. In the actual calculation we have $k = 3$, so that the graph is percolating, and some control measurements are also performed with graphs having $k = 4$. The sizes of the largest systems we studied are shown in Table I. In order to check universality and to control the disorder dependence of the estimates we have used two different form of randomness. Both have the same uniform distribution of the couplings: $p(J) = \Theta(J)\Theta(1 - J)$ ($\Theta(x)$ being the Heaviside step-function), which are ferromagnetic[26]. For the 'box-h' disorder also the transverse fields are uniformly distributed: $q(h) = \frac{1}{h_b}\Theta(h)\Theta(h_b - h)$, whereas for

the 'fixed-h' model we have a constant transverse field: $h_i = h_f$. We used the logarithmic variable, $\theta = \log(h_b)$ or $\theta = \log(h_f)$, as a quantum control parameter. We have checked that the computational time to renormalize an $N = 10^3$ ($N = 10^6$) cluster is typically ~ 0.015 (~ 50) second (in a 2.4GHz processor), which does not depend on the dimension and the topology of the cluster. The numbers of realizations used in the calculations were typically 40000 but even for the largest sizes we have at least 10000 samples.

During the calculation we have basically followed the same strategy as used in the 2D problem[17].

(i) First, we have calculated sample dependent pseudo-critical points, $\theta_c(N)$, by a variant of the doubling method. In this procedure[17] we glue together two identical copies of the random sample by surface couplings[27] and perform the renormalization for different values of θ. In the locally ordered phase, $\theta < \theta_c(N)$, there is an effective spin cluster, in which equivalent sites in the two copies are present. On the contrary, in the locally disordered phase, $\theta > \theta_c(N)$, there is no such a *correlation cluster*.
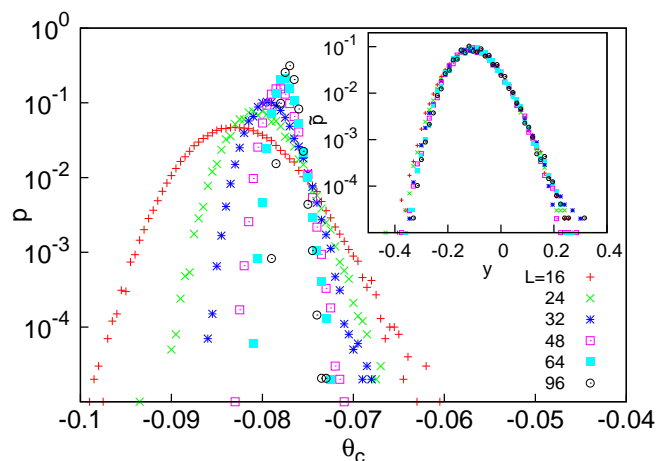

FIG. 1: (Color online) Distribution of the pseudo-critical points, $\theta_c(N)$, for various sizes, $N = L^3$, for fixed-h randomness for the 3D model. In the inset the scaled distributions are shown as a function of $y = (\theta_c(N) - \theta_c)N^{1/d\nu}$, see the text.

(ii) In the second step we have studied the size-dependence of the distributions of the pseudo-critical points, which is illustrated in Fig.1 for the 3D model. From the scaling of the width, $\Delta\theta_c(N) \sim N^{-1/d\nu_w}$, and from the scaling of the mean value: $|\theta_c - \overline{\theta_c(N)}| \sim N^{-1/d\nu_s}$ we have obtained the critical exponents, ν_w and ν_s, respectively. We have calculated size-dependent effective exponents by two-point fits[17], which are then extrapolated. The effective exponents for the 3D model are shown in the inset of Fig.2 for the two different randomnesses. As in this example we have generally observed that the extrapolated critical exponents are universal, i.e.

randomness independent. Estimates of the exponents are presented in Table I, together with the values of the *true* critical points, $\theta_c^{(b)}$ and $\theta_c^{(f)}$, for the two randomnesses, respectively.
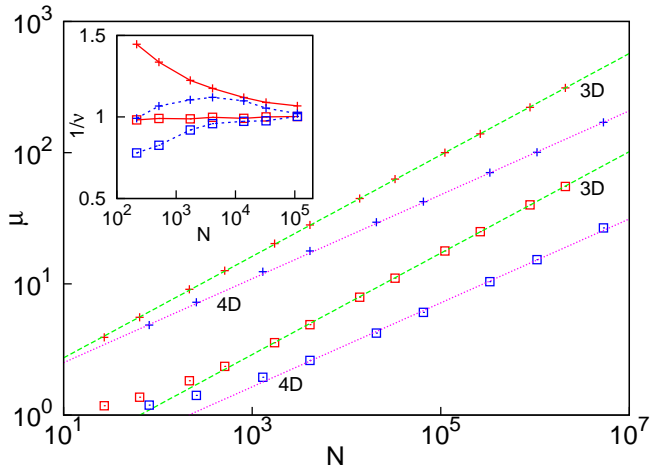

FIG. 2: (Color online) Average moment of the correlation cluster at the critical point vs. the size of the system in a log-log plot for the 3D and 4D models for two types of randomness (box-h □, fixed-h: +). The slope of the straight lines is given by: $d_f/d = 0.387$ and $d_f/d = 0.32$ for 3D and 4D, respectively. Inset: Finite-size effective exponents, ν_s (blue - dashed) and ν_w (red - full), for the 3D model for two types of randomness.

(iii) In the third step we have renormalized the systems at the critical points and studied the scaling behavior of the moment of the correlation cluster, μ_N, as well as that of the log-energy parameter, $\gamma_N = -\log \epsilon_N$. The average moment is found to scale as: $\mu_N \sim N^{d_f/d}$, where d_f is the fractal dimension of the correlation cluster. We illustrate this relation in Fig.2 for the 3D and the 4D models, in which μ_N is shown as a function of N in a log-log scale. Indeed, for not too small systems, $N > 1000$, the points are very well on straight lines, the slope of which being the same for the two different randomnesses for the same d. From the cluster moment the magnetization is calculated as, $m = \mu_N/N$, thus we have the scaling relations: $x/d = \beta/(d\nu) = 1 - d_f/d$. Estimates for the exponents x/d, which are calculated through two-point fits are shown in Table I.

The distribution of the log-energy parameter, γ_N, is shown in the upper panel of Fig.3, for the different models. As a clear indication of infinite disorder scaling the width of the distribution is increasing with N. In 3D and 4D the appropriate scaling variable is $\tilde{\gamma} = (\gamma_N - \gamma_0)N^{-\psi/d}$, as illustrated with the data collapse in the lower panel of Fig.3. The critical exponent ψ has been calculated from the optimal collapse of the distributions, as well as from two-point fits of the mean values, which are presented in Table I. The ER random graphs are infinite dimensional objects and in this case the broadening of the distribution of the log-energy pa-
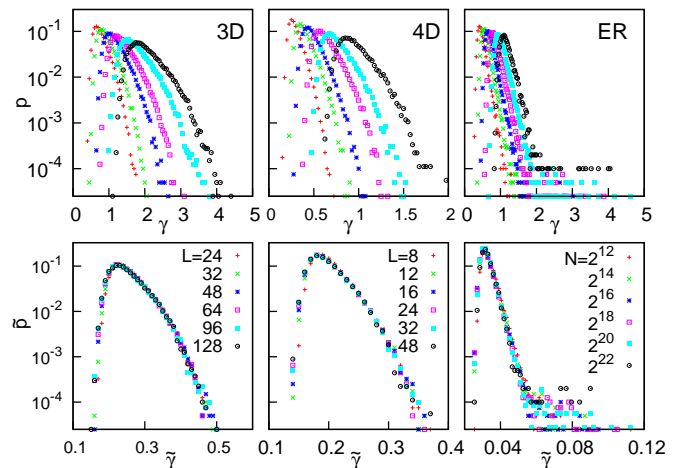

FIG. 3: (Color online) Distribution of the log-energy parameters at the critical point in 3D, 4D and in the ER random graph for fixed-h randomness and for different sizes (upper panel). In the lower panel the scaled distributions are shown, as described in the text.

TABLE I: Critical properties of the RTIM in three and four dimensions and in ER random graphs. N_{max} denotes the number of spins in the largest finite systems used in the RG calculation.

	3D	4D	ER
N_{max}	128^3	48^4	2^{22}
$\theta_c^{(b)}$	2.5305(10)	3.110(5)	2.775(2)
$\theta_c^{(f)}$	$-0.07627(2)$	$-0.04698(10)$	$-0.093(1)$
$d\nu_w$	2.90(15)	3.30(15)	7.(2)
$d\nu_s$	2.96(5)	2.96(15)	5.(1)
x/d	0.613(5)	0.68(3)	0.81(4)
ψ	0.46(2)	0.46(2)	–

rameter is found to scale with $\log N$. As a good scaling combination we have here $\tilde{\gamma} = (\gamma_N - \gamma_0)(\log N/N_0)^{-\omega}$, which is illustrated with the data collapse in the lower panel of Fig.3, with an exponent $\omega = 1.3(2)$.

Discussion Our numerical RG results indicate that the critical behavior of the random transverse-field Ising model in three and four dimensions as well as in the ER random graph is controlled by infinite disorder fixed points. This fact justifies the use of the SDRG method and ensures that the calculated numerical results about the critical exponents tend to be asymptotically correct for large sizes. Since the ER random graph represents the large-dimensional limit of the problem, infinite disorder scaling is expected to be valid at any dimensions. The critical exponents presented in Table I are found to be disorder independent, thus the IDFP-s are expected to be attractive, at least for strong enough disorder. Singularities of the thermodynamic quantities at small temperatures involve these exponents. For example the susceptibility and the specific heat behave

as: $\chi(T) \sim (\log T)^{(d-2x)/\psi}/T$ and $C_V(T) \sim (\log T)^{d/\psi}$, respectively[10, 22]. The analogous expressions for the ER random graph are: $\log[T\chi(T)] \sim (1-2x/d)(\log T)^{1/\omega}$ and $\log[C_V(T)] \sim (\log T)^{1/\omega}$.

The IDFP-s in Table I control the critical behavior of the spinglass transition[26], as well as that of a class of random quantum systems having an order parameter with discrete symmetry, such as random quantum Potts[28] and clock models[29]. Nonequilibrium phase transitions, such as the contact process with (strong) disorder[20] also belong to this class of universality. The critical exponents in Table I show a smooth variation with the dimension and indicate that the large-d limit is not singular. For a given d the correlation length critical exponents, ν_s and ν_w, agree with each other, within the error of the calculation. These satisfy the rigorous bound[30], $\nu \geq 2/d$, and are in agreement with the scaling theory at conventional random fixed points[31, 32]. Interestingly, the exponent ψ is found very close to $1/2$ for any considered finite dimension[33]. This fact can be explained with our observation, that the low-energy excitations in any dimension are quasi-1D objects and the energy scale can be obtained by renormalizing these objects practically independently of the rest of the system. This leads to approximately the same type of linear-size dependence of the energy in any dimension.

The SDRG investigations presented in this Letter can be extended in several directions. Here we mention the calculation of the entanglement entropy[14–16, 34] in these systems, as well as study of the dynamical singularities in the disordered and ordered Griffiths phases[22].

* Electronic address: ikovacs@szfki.hu
† Electronic address: igloi@szfki.hu

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[24] The algorithm described in Ref.[17] has the same performance in 2D, but it is considerably slower in higher dimensions.

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[26] At an IDFP frustration does not matter, thus quantum spinglass and random quantum ferromagnet has the same fixed point for strong enough disorder.

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