Holomorphic Realization of Unitary Representations of Banach–Lie Groups

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To J. Wolf on the occasion of his 75th birthday

Abstract

In this paper we explore the method of holomorphic induction for unitary representations of Banach–Lie groups. First we show that the classification of complex bundle structures on homogeneous Banach bundles over complex homogeneous spaces of real Banach–Lie groups formally looks as in the finite dimensional case. We then turn to a suitable concept of holomorphic unitary induction and show that this process preserves commutants. In particular, holomorphic induction from irreducible representations leads to irreducible ones. Finally we develop criteria to identify representations as holomorphically induced ones and apply these to the class of so-called positive energy representations. All this is based on extensions of Arveson's concept of spectral subspaces to representations on Fréchet spaces, in particular on spaces of smooth vectors.
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Introduction

This paper is part of a long term project concerned with a systematic approach to unitary representations of Banach–Lie groups in terms of conditions on spectra in the derived representation. A unitary representation $\pi\colon G \to \mathrm{U}(\mathcal{H})$ is said to be *smooth* if the subspace $\mathcal{H}^\infty$ of smooth vectors is dense. This is automatic for continuous representations of finite dimensional groups but not in general (cf. [Ne10a]). For any smooth unitary representation, the *derived representation*

$$\mathtt{d}\pi\colon \mathfrak{g} = \mathbf{L}(G) \to \mathrm{End}(\mathcal{H}^\infty), \quad \mathtt{d}\pi(x)v := \frac{d}{dt}\Big|_{t=0} \pi(\exp tx)v$$

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carries significant information in the sense that the closure of the operator $\mathtt{d}\pi(x)$ coincides with the infinitesimal generator of the unitary one-parameter group $\pi(\exp tx)$. We call $(\pi, \mathcal{H})$ *semibounded* if the function

$$s_\pi \colon \mathfrak{g} \to \mathbb{R} \cup \{\infty\}, \quad s_\pi(x) := \sup\big(\mathrm{Spec}(i\mathtt{d}\pi(x))\big) = \sup\{\langle i\mathtt{d}\pi(x)v, v\rangle \colon v \in \mathcal{H}^\infty, \|v\| = 1\}$$

is bounded on the neighborhood of some point in $\mathfrak{g}$ (cf. [Ne08, Lemma 5.7]). Then the set W_π of all such points is an open invariant convex cone in the Lie algebra $\mathfrak{g}$. We call π *bounded* if s_π is bounded on some 0-neighborhood, which is equivalent to π being norm continuous (cf. Proposition 1.1). All finite dimensional unitary representations are bounded and many of the unitary representations appearing in physics are semibounded (cf. [Ne10c]).

One of our goals is a classification of the irreducible semibounded representations and the development of tools to obtain direct integral decompositions of semibounded representations. To this end, realizations of unitary representations in spaces of holomorphic sections of vector bundles turn out to be extremely helpful. Clearly, the bundles in question should be allowed to have fibers which are infinite dimensional Hilbert spaces, and to treat infinite dimensional groups, we also have to admit infinite dimensional base manifolds. The main point of the present paper is to provide effective methods to treat unitary representations of Banach–Lie groups in spaces of holomorphic sections of homogeneous Hilbert bundles.

In Section 1 we explain how to parametrize the holomorphic structures on Banach vector bundles $\mathbb{V} = G \times_H V$ over a Banach homogeneous space $M = G/H$ associated to a norm continuous representation (ρ, V) of the isotropy group H. The main result of Section 1 is Theorem 1.6 which generalizes the corresponding results for the finite dimensional case by Tirao and Wolf ([TW70]). As in finite dimensions, the complex bundle structures are specified by "extensions" $\beta \colon \mathfrak{q} \to \mathfrak{gl}(V)$ of the differential $\mathtt{d}\rho \colon \mathfrak{h} \to \mathfrak{gl}(V)$ to a representation of the complex subalgebra $\mathfrak{q} \subseteq \mathfrak{g}_\mathbb{C}$ specifying the complex structure on M in the sense that $T_H(M) \cong \mathfrak{g}_\mathbb{C}/\mathfrak{q}$ (cf. [Bel05]). The main point in [TW70] is that the homogeneous space G/H need not be realized as an open G-orbit in a complex homogeneous space of a complexification $G_\mathbb{C}$, which is impossible if the subgroup of $G_\mathbb{C}$ generated by $\exp \mathfrak{q}$ is not closed. In the Banach context, two additional difficulties appear: The Lie algebra $\mathfrak{g}_\mathbb{C}$ may not be integrable in the sense that it does not belong to any Banach–Lie group ([GN03]) and, even if $G_\mathbb{C}$ exists and the subgroup $Q := \langle \exp \mathfrak{q}\rangle$ is closed, it need not be a Lie subgroup so that there is no natural construction of a manifold structure on the quotient space G/Q. As a consequence, the strategy of the proof in [TW70] can not be used for Banach Lie groups. Another difficulty of the infinite dimensional context is that there is no general existence theory for solutions of $\overline{\partial}$-equations (see in particular [Le99]).

The next step, carried out in Section 2, is to analyze Hilbert subspaces of the space $\Gamma(\mathbb{V})$ of holomorphic sections of $\mathbb{V}$ on which G acts unitarily. In this context (ρ, V) is a bounded unitary representation. The most regular Hilbert spaces with this property are those that we call *holomorphically induced from* (ρ, β). They contain (ρ, V) as an H-subrepresentation satisfying a compatibility condition with respect to β. Here we show that if the subspace $V \subseteq \mathcal{H}$ is invariant under the commutant $B_G(\mathcal{H}) = \pi(G)'$, then restriction to V yields an isomorphism of the von Neumann algebra $B_G(\mathcal{H})$ with a suitably defined commutant $B_{H,\mathfrak{q}}(V)$ of (ρ, β) (Theorem 2.12). This has remarkable consequences. One is that the representation of G on $\mathcal{H}$ is irreducible (multiplicity free, discrete, type I) if and only if the representation of $(H, \mathfrak{q})$ on V has this property. The second main result in Section 2 is a criterion for a unitary representation $(\pi, \mathcal{H})$ of G to be holomorphically induced (Theorem 2.17).

Section 3 is devoted to a description of environments in which Theorem 2.17 applies naturally. Here we consider an element $d \in \mathfrak{g}$ which is *elliptic* in the sense that the one-parameter group $e^{\mathbb{R}\,\mathrm{ad}\,d}$ of automorphisms of $\mathfrak{g}$ is bounded. This is equivalent to the existence of an invariant compatible norm. Suppose that 0 is isolated in $\mathrm{Spec}(\mathrm{ad}\,d)$. Then the subgroup $H := Z_G(d) = \{g \in G\colon \mathrm{Ad}(g)d = d\}$ is a Lie subgroup and the homogeneous space G/H carries a natural complex manifold structure. A smooth unitary representation $(\pi, \mathcal{H})$ of G is said to be of *positive energy* if the selfadjoint operator $-i\mathtt{d}\pi(d)$ is bounded from below. Note that this is in particular the case if π is semibounded with $d \in W_\pi$. Any positive energy representation is generated as a G-representation by the closed subspace $V := \overline{(\mathcal{H}^\infty)^{\mathfrak{p}^-}}$, where $\mathfrak{q} = \mathfrak{p}^+ \rtimes \mathfrak{h}_\mathbb{C} \subseteq \mathfrak{g}_\mathbb{C}$ is the complex subalgebra defining the complex structure on G/H and $\mathfrak{p}^- := \overline{\mathfrak{p}^+}$. If the H-representation (ρ, V) is bounded, then $(\pi, \mathcal{H})$ is holomorphically induced by (ρ, β), where β is determined by $\beta(\mathfrak{p}^+) = \{0\}$ (Theorem 3.7). In Theorem 3.14 we further show that, under these assumptions, π is semibounded with $d \in W_\pi$. These results are rounded off by Theorem 3.12 which shows that, if π is semibounded with $d \in W_\pi$, then π is a direct sum of holomorphically induced representations and if, in addition, π is irreducible, then V coincides with the minimal eigenspace of $-i\mathtt{d}\pi(d)$ and the representation of H on this space is automatically bounded. For all this we use refined analytic tools based on the fact that the space $\mathcal{H}^\infty$ of smooth vectors is a Fréchet space on which G acts smoothly ([Ne10a]) and the $\mathbb{R}$-action on $\mathcal{H}^\infty$ defined by $\pi_d(t) := \pi(\exp td)$ is equicontinuous. These properties permit us to use a suitable generalization of Arveson's spectral theory, developed in Appendix A.

In [Ne11] we shall use the techniques developed in the present paper to obtain a complete descriptions of semibounded representations for the class of hermitian Lie groups. One would certainly like to extend the tools developed here to Fréchet–Lie groups such as diffeomorphism groups and groups of smooth maps. Here a serious problem is the construction of holomorphic vector bundle structures on associated bundles, and we do not know how to extend this beyond the Banach context, especially because of the non-existing solution theory for $\overline{\partial}$-equations (cf. [Le99]).

Several of our results have natural predecessors in more restricted contexts. In [BR07] Beltiţă and Ratiu study holomorphic Hilbert bundles over Banach manifolds M and consider the endomorphism bundle $B(\mathbb{V})$ over $M \times \overline{M}$ (using a different terminology). They also relate Hilbert subspaces $\mathcal{H}$ of $\Gamma(\mathbb{V})$ to reproducing kernels, which in this context are sections of $B(\mathbb{V})$ satisfying a certain holomorphy condition which under the assumption of local boundedness of the kernel is equivalent to holomorphy as a section of $B(\mathbb{V})$ ([BR07, Thm. 4.2]; see also [BH98, Thm. 1.4] and [MPW97] for the case of finite dimensional bundles over finite dimensional manifolds and [Od92] for the case of line bundles). Beltiţă and Ratiu use this setup to realize certain representations of a C^*-algebra $\mathcal{A}$, which define bounded unitary representations of the unitary group $G := \mathrm{U}(\mathcal{A})$, in spaces of holomorphic sections of a bundle over a homogeneous space of the unit group $\mathcal{A}^\times$ on which $\mathrm{U}(\mathcal{A})$ acts transitively ([BR07, Thm. 5.4]). These homogeneous spaces are of the form G/H, where H is the centralizer of some hermitian element $a \in \mathcal{A}$ with finite spectrum, so that the realization of these representations by holomorphic sections could also be derived from our Corollary 3.9. This work has been continued by Beltiţă with Galé in a different direction, focusing on complexifications of real homogeneous spaces instead of invariant complex structures on the real spaces ([BG08]).

In [Bo80] Boyer constructs irreducible unitary representations of the Hilbert–Lie group $\mathrm{U}_2(\mathcal{H}) = \mathrm{U}(\mathcal{H}) \cap (\mathbf{1} + B_2(\mathcal{H}))$ via holomorphic induction from characters of the diagonal subgroup. They live in spaces of holomorphic sections on homogeneous spaces of the complexified group $\mathrm{GL}_2(\mathcal{H}) = \mathrm{GL}(\mathcal{H}) \cap (\mathbf{1} + B_2(\mathcal{H}))$, which are restricted versions of flag manifolds carrying strong Kähler structures.

In the present paper we only deal with representations associated to homogeneous vector bundles. To understand branching laws for restrictions of representations to subgroups, one should also study situations where the group G does not act transitively on the base manifold. For finite dimensional holomorphic vector bundles, this has been done extensively by T. Kobayashi who obtained powerful criteria for representations in Hilbert spaces of holomorphic sections to be multiplicity free (cf. [KoT05], [KoT06]). It would be interesting to explore the extent to which Kobayashi's technique of visible actions on complex manifolds can be extended to Banach manifolds.

Notation: For a group G we write $\mathbf{1}$ for the neutral element and $\lambda_g(x) = gx$, resp., $\rho_g(x) = xg$ for left multiplications, resp., right multiplications. We write $\mathfrak{g}_\mathbb{C}$ for the complexification of a real Lie algebra $\mathfrak{g}$ and $\overline{x + iy} := x - iy$ for the complex conjugation on $\mathfrak{g}_\mathbb{C}$, which is an antilinear Lie algebra automorphism. For two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, we write $B(\mathcal{H}_1, \mathcal{H}_2)$ for the space of continuous (=bounded) linear operators $\mathcal{H}_1 \to \mathcal{H}_2$ and $B_p(\mathcal{H})$, $1 \leq p < \infty$, for the space of Schatten class operators $A \colon \mathcal{H} \to \mathcal{H}$ of order p, i.e., A is compact with $\mathrm{tr}((A^*A)^{p/2}) < \infty$.

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Contents

1 Holomorphic Banach bundles

To realize unitary representations of Banach–Lie groups in spaces of holomorphic sections of Hilbert bundles, we first need a parametrization of holomorphic bundle structures on given homogeneous vector bundles for real Banach–Lie groups. As we shall see in Theorem 1.6 below, formulated appropriately, the corresponding results from the finite dimensional case (cf. [TW70]) can be generalized to the Banach context.

The following observation provides some information on the assumptions required on the isotropy representation of a Hilbert bundle. It is a slight modification of results from [Ne09, Sect. 3].

Proposition 1.1. *Let $(\pi, \mathcal{H})$ be a unitary representation of the Banach–Lie group G for which all vectors are smooth. Then $\pi\colon G \to \mathrm{U}(\mathcal{H})$ is a morphism of Lie groups, hence in particular norm continuous, i.e., a bounded representation.*

Proof. Our assumption implies that, for each $x \in \mathfrak{g}$, the infinitesimal generator $\mathtt{d}\pi(x)$ of the unitary one-parameter group $\pi_x(t) := \pi(\exp_G(tx))$ is everywhere defined, hence a bounded operator because its graph is closed.

Therefore the derived representation leads to a morphism of Banach–Lie algebras $\mathtt{d}\pi\colon \mathfrak{g} \to B(\mathcal{H})$. Since the function s_π is a sup of a set of continuous linear functionals, it is lower semi-continuous. Hence the function

$$x \mapsto \|\mathtt{d}\pi(x)\| = \max(s_\pi(x), s_\pi(-x))$$

is a lower semi-continuous seminorm and therefore continuous because $\mathfrak{g}$ is barreled (cf. [Bou07, §III.4.1]). We conclude that the set of all linear functional $\langle \mathtt{d}\pi(\cdot)v, v\rangle$, $v \in \mathcal{H}^\infty$ a unit vector, is equicontinuous in $\mathfrak{g}'$, so that the assertion follows from [Ne09, Thm. 3.1]. □

We now turn to the case where M is a Banach homogeneous space. Let G be a Banach–Lie group with Lie algebra $\mathfrak{g}$ and $H \subseteq G$ be a split Lie subgroup, i.e., the Lie algebra $\mathfrak{h}$ of H has a closed complement in $\mathfrak{g}$, for which the coset space $M := G/H$ carries the structure of a complex manifold such that the projection $q_M : G \to G/H$ is a smooth H-principal bundle and G acts on M by holomorphic maps. Let $m_0 = q_M(\mathbf{1}) \in M$ be the canonical base point and $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ be the kernel of the complex linear extension of the map $\mathfrak{g} \to T_{m_0}(G/H)$ to $\mathfrak{g}_{\mathbb{C}}$, so that $\mathfrak{q}$ is a closed subalgebra of $\mathfrak{g}_{\mathbb{C}}$ invariant under $\mathrm{Ad}(H)$ (cf. [Bel05, Thm. 15]). We call $\mathfrak{q}$ the *subalgebra defining the complex structure on $M = G/H$* because specifying $\mathfrak{q}$ means to identify $T_{m_0}(G/H) \cong \mathfrak{g}/\mathfrak{h}$ with the complex Banach space $\mathfrak{g}_{\mathbb{C}}/\mathfrak{q}$ and thus specifying the complex structure on M.

Remark 1.2. If the Banach–Lie group G acts smoothly by isometric bundle automorphisms on the holomorphic Hilbert bundle $\mathbb{V}$ over $M = G/H$, then the action of the stabilizer group H on $V := \mathbb{V}_{m_0}$ is smooth, so that Proposition 1.1 shows that it defines a bounded unitary representation $\rho\colon H \to \mathrm{U}(V)$.

If $\sigma_M\colon G \times M \to M$ denotes the corresponding action on M and $\dot\sigma_M\colon \mathfrak{g} \to \mathcal{V}(M)$ the derived action, then, for the closed subalgebra

$$\mathfrak{q} := \{x \in \mathfrak{g}_{\mathbb{C}} \colon \dot\sigma_M(x)(m_0) = 0\},$$

the representation $\beta\colon \mathfrak{q} \to B(V)$ is given by a continuous bilinear map

$$\widehat{\beta}\colon \mathfrak{q} \times V \to V, \quad (x, v) \mapsto \beta(x)v.$$

This means that β is a continuous morphism of Banach–Lie algebras.

This observation leads us to the following structures.

Definition 1.3. Let $H \subseteq G$ be a Lie subgroup and $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ be a closed subalgebra containing $\mathfrak{h}_{\mathbb{C}}$. If $\rho\colon H \to \mathrm{GL}(V)$ is a norm continuous representation on the Banach space V, then a morphism $\beta\colon \mathfrak{q} \to \mathfrak{gl}(V)$ of complex Banach–Lie algebras is said to be an *extension of* ρ if

$$\mathtt{d}\rho = \beta|_{\mathfrak{h}} \quad \text{and} \quad \beta(\mathrm{Ad}(h)x) = \rho(h)\beta(x)\rho(h)^{-1} \quad \text{for} \quad h \in H, x \in \mathfrak{q}. \tag{1}$$

Definition 1.4. (a) If $q\colon \mathbb{V} = G \times_H V \to M$ is a homogeneous vector bundle defined by the norm continuous representation $\rho\colon H \to \mathrm{GL}(V)$, we associate to each section $s\colon M \to \mathbb{V}$ the function $\widehat{s}\colon G \to V$ specified by $s(gH) = [g, \widehat{s}(g)]$. A function $f\colon G \to V$ is of the form $\widehat{s}$ for a section of $\mathbb{V}$ if and only if

$$f(gh) = \rho(h)^{-1} f(g) \quad \text{for} \quad g \in G, h \in H. \tag{2}$$

We write $C(G,V)_\rho$, resp., $C^\infty(G,V)_\rho$ for the continuous, resp., smooth functions satisfying (2).

(b) We associate to each $x \in \mathfrak{g}_{\mathbb{C}}$ the left invariant differential operator on $C^\infty(G,V)$ defined by

$$(L_x f)(g) := \frac{d}{dt}\Big|_{t=0} f(g \exp(tx)) \quad \text{for} \quad x \in \mathfrak{g}.$$

By complex linear extension, we define the operators

$$L_{x+iy} := L_x + iL_y \quad \text{for} \quad z = x + iy \in \mathfrak{g}_{\mathbb{C}}, x, y \in \mathfrak{g}.$$

(c) For any extension β of ρ, we write $C^\infty(G,V)_{\rho,\beta}$ for the subspace of those elements $f \in C^\infty(G,V)_\rho$ satisfying, in addition,

$$L_x f = -\beta(x) f \quad \text{for} \quad x \in \mathfrak{q}. \tag{3}$$

Remark 1.5. For $x \in \mathfrak{g}$, $h \in H$ and any smooth function φ defined on an open right H-invariant subset of G, we have

$$(L_x\varphi)(gh) = \frac{d}{dt}\Big|_{t=0} \varphi\big(g \exp(t\,\mathrm{Ad}(h)x)h\big) = (L_{\mathrm{Ad}(h)x}(\varphi \circ \rho_h))(g),$$

so that we obtain for each $x \in \mathfrak{g}_{\mathbb{C}}$ and $h \in H$ the relation

$$(L_x\varphi) \circ \rho_h = L_{\mathrm{Ad}(h)x}(\varphi \circ \rho_h). \tag{4}$$

The proof of the following theorem is very much inspired by [TW70].

Theorem 1.6. *Let $M = G/H$, V be a complex Banach space and $\rho\colon H \to \mathrm{GL}(V)$ be a norm continuous representation. Then, for any extension $\beta\colon \mathfrak{q} \to \mathfrak{gl}(V)$ of ρ, the associated bundle $\mathbb{V} := G \times_H V$ carries a unique structure of a holomorphic vector bundle over M, which is determined by the characterization of the holomorphic sections $s\colon M \to \mathbb{V}$ as those for which $\widehat{s} \in C^\infty(G, V)_{\rho,\beta}$. Any such holomorphic bundle structure is G-invariant in the sense that G acts on $\mathbb{V}$ by holomorphic bundle automorphism. Conversely, every G-invariant holomorphic vector bundle structure on $\mathbb{V}$ is obtained from this construction.*

Proof. **Step 1:** Let β be an extension of ρ and $E \subseteq \mathfrak{g}$ be a closed subspace complementing $\mathfrak{h}$. Then $E \cong \mathfrak{g}/\mathfrak{h} \cong T_{m_0}(M)$ implies the existence of a complex structure I_E on E for which $E \to \mathfrak{g}/\mathfrak{h}$ is an isomorphism of complex Banach spaces. Therefore the I_E-eigenspace decomposition

$$E_{\mathbb{C}} = E_+ \oplus E_-, \quad E_\pm = \ker(I_E \mp i\mathbf{1}),$$

is a direct decomposition into closed subspaces. The quotient map $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}/\mathfrak{q} \cong \mathfrak{g}/\mathfrak{h}$ is surjective on E_+ and annihilates E_-. Now $\mathfrak{r} := E_+$ is a closed complex complement of $\mathfrak{q} = \mathfrak{h}_{\mathbb{C}} \oplus E_-$ in $\mathfrak{g}_{\mathbb{C}}$ ([Bel05, Thm. 15]).

Pick open convex 0-neighborhoods $U_{\mathfrak{r}} \subseteq \mathfrak{r}$ and $U_{\mathfrak{q}} \subseteq \mathfrak{q}$ such that the BCH multiplication $*$ defines a biholomorphic map $\mu\colon U_{\mathfrak{r}} \times U_{\mathfrak{q}} \to U, (x, y) \mapsto x * y$, onto the open 0-neighborhood $U \subseteq \mathfrak{g}_{\mathbb{C}}$ and that $*$ defines an associative multiplication defined on all triples of elements of U.

Step 2: On the 0-neighborhood $U \subseteq \mathfrak{g}_{\mathbb{C}}$, we consider the holomorphic function

$$F\colon U \to \mathrm{GL}(V), \quad F(x * y) := e^{-\beta(y)}.$$

Let $U_{\mathfrak{g}} \subseteq U$ be an open 0-neighborhood which is mapped by $\exp_G$ diffeomorphically onto an open $\mathbf{1}$-neighborhood U_G of G. Then we consider the smooth function

$$f\colon U_G \to \mathrm{GL}(V), \quad \exp_G z \mapsto F(z).$$

For $w \in \mathfrak{g}$ and $z \in U_{\mathfrak{g}}$, the BCH product $z * tw \in \mathfrak{g}_{\mathbb{C}}$ is defined if t is small enough, and we have

$$(L_w f)(\exp_G z) = \frac{d}{dt}\Big|_{t=0} F(z * tw) = \mathtt{d}F(z)\mathtt{d}\lambda_z^*(0)w, \tag{5}$$

where $\mathtt{d}\lambda_z^*(0)\colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ is the differential of the multiplication map $\lambda_z^*(x) = z * x$ in 0. As (5) is complex linear in $w \in \mathfrak{g}_{\mathbb{C}}$, it follows that

$$(L_w f)(\exp_G z) = \mathtt{d}F(z)\mathtt{d}\lambda_z^*(0)w$$

hold for every $w \in \mathfrak{g}_{\mathbb{C}}$, hence in particular for $w \in \mathfrak{q}$. For $w \in \mathfrak{q}$ and $z = \mu(x, y)$, we thus obtain

$$\begin{aligned}\mathtt{d}F(z)\mathtt{d}\lambda_z^*(0)w &= \frac{d}{dt}\Big|_{t=0} F(x * y * tw) = \frac{d}{dt}\Big|_{t=0} e^{-\beta(y*tw)} = \frac{d}{dt}\Big|_{t=0} e^{-t\beta(w)}e^{-\beta(y)} \\ &= -\beta(w)e^{-\beta(y)}.\end{aligned}$$

We conclude that

$$(L_w f)(g) = -\beta(w) f(g) \quad \text{for} \quad g \in U_G.$$

In particular, we obtain $f(gh) = \rho(h)^{-1} f(g)$ for $g \in U_G, h \in H_0$.

Step 3: Since H is a complemented Lie subgroup, there exists a connected submanifold $Z \subseteq G$ containing $\mathbf{1}$ for which the multiplication map $Z \times H \to G, (x,h) \mapsto xh$ is a diffeomorphism onto an open subset of G. Shrinking U_G, we may therefore assume that $U_G = U_Z U_H$ holds for a connected open $\mathbf{1}$-neighborhood U_Z in Z and a connected open $\mathbf{1}$-neighborhood U_H in H. Then

$$\widetilde{f}(zh) := \rho(h)^{-1} f(z) \quad \text{for} \quad z \in U_Z, h \in H,$$

defines a smooth function $\widetilde{f} \colon U_Z H \to \mathrm{GL}(V)$. That it extends f follows from the fact that $u = zh \in U_G$ with $z \in U_Z$ and $h \in U_H$ implies $h \in H_0$, so that

$$\widetilde{f}(u) = \rho(h)^{-1} f(z) = f(zh) = f(u).$$

For $w \in \mathfrak{q}$, formula (4) leads to

$$\begin{aligned}(L_w \widetilde{f})(zh) &= (L_{\mathrm{Ad}(h)w}(\widetilde{f} \circ \rho_h))(z) = L_{\mathrm{Ad}(h)w}(\rho(h)^{-1}\widetilde{f})(z) = L_{\mathrm{Ad}(h)w}(\rho(h)^{-1} f)(z) \\ &= -\rho(h)^{-1}\beta(\mathrm{Ad}(h)w) f(z) = -\beta(w)\rho(h)^{-1} f(z) = -\beta(w)\widetilde{f}(zh).\end{aligned}$$

Therefore $\widetilde{f}$ satisfies

$$L_w \widetilde{f} = -\beta(w)\widetilde{f} \quad \text{for} \quad w \in \mathfrak{q}. \tag{6}$$

Step 4: For $m \in M$ we choose an element $g_m \in G$ with $g_m.m_0 = m$ and put $U_m := g q_M(U_Z)$, so that

$$G^{U_m} := q_M^{-1}(U_m) = g_m U_Z H.$$

On this open subset of G, the function

$$F_m \colon G^{U_m} \to \mathrm{GL}(V), \quad F_m(g) := \widetilde{f}(g_m^{-1} g)$$

satisfies

(a) $F_m(gh) = \rho(h)^{-1} F_m(g)$ for $g \in G^{U_m}$, $h \in H$.

(b) $L_w F_m = -\beta(w) F_m$ for $w \in \mathfrak{q}$.

(c) $F_m(g_m) = \mathbf{1} = \mathrm{id}_V$.

Next we note that the function F_m defines a smooth trivialization

$$\varphi_m \colon U_m \times V \to \mathbb{V}|_{U_m} = G^{U^m} \times_H V, \quad (gH, v) \mapsto [g, F_m(g)v].$$

The corresponding transition functions are given by

$$\varphi_{m,n} \colon U_{m,n} := U_m \cap U_n \to \mathrm{GL}(V), \quad gH \mapsto \widetilde{\varphi}_{m,n}(g) := F_m(g)^{-1} F_n(g).$$

To verify that these transition functions are holomorphic, we have to show that the functions $\widetilde{\varphi}_{m,n} \colon G^{U_{m,n}} \to B(V)$ are annihilated by the differential operators L_w, $w \in \mathfrak{q}$. This follows easily from the product rule and (b):

$$\begin{aligned} L_w(F_m^{-1}F_n) &= L_w(F_m^{-1})F_n + F_m^{-1}L_w(F_n) = -F_m^{-1}L_w(F_m)F_m^{-1}F_n + F_m^{-1}L_w(F_n) \\ &= F_m^{-1}(\beta(w)F_mF_m^{-1} - \beta(w))F_n = F_m^{-1}(\beta(w) - \beta(w))F_n = 0. \end{aligned}$$

We conclude that the transition functions $(\varphi_{m,n})_{m,n\in M}$ define a holomorphic vector bundle atlas on $\mathbb{V}$.

Step 5: To see that this holomorphic structure is determined uniquely by β, pick an open connected subset $U \subseteq M$ containing m_0 on which the bundle is holomorphically trivial and the trivialization is specified by a smooth function $F \colon G^U \to \mathrm{GL}(V)$. Then $L_wF = -\beta(w)F$ implies that $\beta(w) = -(L_wF)(\mathbf{1})F(\mathbf{1})^{-1}$ is determined uniquely by the holomorphic structure on $\mathbb{V}$.

Step 6: The above construction also shows that, for any holomorphic vector bundle structure on $\mathbb{V}$, for which G acts by holomorphic bundle automorphisms, we may consider

$$\beta \colon \mathfrak{q} \to B(V), \quad \beta(w) := -(L_wF)(\mathbf{1})F(\mathbf{1})^{-1} \tag{7}$$

for a local trivialization given by $F \colon G^U \to \mathrm{GL}(V)$, where $m_0 \in U$. Then β is a continuous linear map. To see that it does not depend on the choice of F, we note that, for any other trivialization $\widetilde{F} \colon G^U \to \mathrm{GL}(V)$, the function $F^{-1} \cdot \widetilde{F}$ factors through a holomorphic function on U, so that

$$0 = L_w(F^{-1}\widetilde{F})(\mathbf{1}) = F(\mathbf{1})^{-1}\Big(- (L_wF)(\mathbf{1})F(\mathbf{1})^{-1} + (L_w\widetilde{F})(\mathbf{1})\widetilde{F}(\mathbf{1})^{-1}\Big)\widetilde{F}(\mathbf{1}).$$

We conclude that the right hand side of (7) does not depend on the choice of F. Applying this to functions of the form $F_g(g') := F(gg')$, defined on $g^{-1}G^U$, we obtain in particular

$$-\beta(w) = -(L_wf)(\mathbf{1})F(\mathbf{1})^{-1} = (L_wF_g)(\mathbf{1})F_g(\mathbf{1})^{-1} = (L_wF)(g)F(g)^{-1},$$

so that $L_wF = -\beta(w)F$.

For $g = h \in H$, we obtain with (4)

$$\begin{aligned} -\beta(w) &= (L_wF_h)(\mathbf{1})F(h)^{-1} = (L_wF)(h)F(\mathbf{1})^{-1}\rho(h) \\ &= \rho(h)^{-1}(L_{\mathrm{Ad}(h)w}F)(\mathbf{1})F(\mathbf{1})^{-1}\rho(h) = -\rho(h)^{-1}\beta(\mathrm{Ad}(h)w)\rho(h). \end{aligned}$$

For $w \in \mathfrak{h}$, the relation $\beta(w) = \mathtt{d}\rho(w)$ follows immediately from the H-equivariance of F. To see that β is an extension of ρ, it now remains to verify that it is a homomorphism of Lie algebras. For $w_1, w_2 \in \mathfrak{q}$, we have

$$\begin{aligned} \beta([w_1, w_2])F &= -L_{[w_1,w_2]}F = L_{w_2}L_{w_1}F - L_{w_1}L_{w_2}F \\ &= -L_{w_2}\beta(w_1)F + L_{w_1}\beta(w_2)F = \beta(w_1)\beta(w_2)F - \beta(w_2)\beta(w_1)F, \end{aligned}$$

which shows that β is a homomorphism of Lie algebras. ☐

Example 1.7. We consider the special case where G is contained as a Lie subgroup in a complex Banach–Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and where $Q := \langle \exp \mathfrak{q} \rangle$ is a Lie subgroup of $G_{\mathbb{C}}$ with $Q \cap G = H$. Then the orbit mapping $G \to G_{\mathbb{C}}/Q, g \mapsto gQ$, induces an open embedding of $M = G/H$ as an open G-orbit in the complex manifold $G_{\mathbb{C}}/Q$.

In this case every holomorphic representation $\pi\colon Q \to \mathrm{GL}(V)$ defines an associated holomorphic Banach bundle $\mathbb{V} := G_{\mathbb{C}} \times_Q V$ over the complex manifold $G_{\mathbb{C}}/Q \cong G/H$.

Since the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ need not be integrable in the sense that it is the Lie algebra of a Banach–Lie group (cf. [GN03, Sect. 6]), the assumption $G \subseteq G_{\mathbb{C}}$ is not general enough to cover every situation. One therefore needs the general Theorem 1.6.

In [GN03, Sect. 6] one finds examples of simply connected Banach–Lie groups G for which $\mathfrak{g}_{\mathbb{C}}$ is not integrable. Here G is a quotient of a simply connected Lie group $\widehat{G}$ by a central subgroup $Z \cong \mathbb{R}$, $\widehat{G}$ has a simply connected universal complexification $\eta_{\widehat{G}}\colon \widehat{G} \to \widehat{G}_{\mathbb{C}}$, and the subgroup $\exp_{\widehat{G}_{\mathbb{C}}}(\mathfrak{z}_{\mathbb{C}})$ is not closed; its closure is a 2-dimensional torus.

Then the real Banach–Lie group G acts smoothly and faithfully by holomorphic maps on a complex Banach manifold M for which $\mathfrak{g}_{\mathbb{C}}$ is not integrable. It suffices to pick a suitable tubular neighborhood $\widehat{G} \times U \cong \widehat{G}\exp(U) \subseteq \widehat{G}_{\mathbb{C}}$, where $U \subseteq i\hat{\mathfrak{g}}$ is a convex 0-neighborhood. Then the quotient $M := \widehat{G}/Z \times (U + i\mathfrak{z})/i\mathfrak{z}$ is a complex manifold on which $G \cong \widehat{G}/Z$ acts faithfully, but $\mathfrak{g}_{\mathbb{C}}$ is not integrable.

2 Hilbert spaces of holomorphic sections

In this section we take a closer look at Hilbert spaces of holomorphic sections in $\Gamma(\mathbb{V})$ for holomorphic Hilbert bundles $\mathbb{V}$ constructed with the methods from Section 1. Here we are interested in Hilbert spaces with continuous point evaluations (to be defined below) on which G acts unitarily. This leads to the concept of a holomorphically induced representation, which for infinite dimensional fibers is a little more subtle than in the finite dimensional case. The first key result of this section is Theorem 2.12 relating the commutant of a holomorphically induced unitary G-representation $(\pi, \mathcal{H})$ to the commutant of the representation (ρ, β) of $(H, \mathfrak{q})$ on the fiber V. This result is complemented by Theorem 2.17 which is a recognition devise for holomorphically induced representations.

Definition 2.1. Let $q\colon \mathbb{V} \to M$ be a holomorphic Hilbert bundle on the complex manifold M. We write $\Gamma(\mathbb{V})$ for the space of holomorphic sections of $\mathbb{V}$. A Hilbert subspace $\mathcal{H} \subseteq \Gamma(\mathbb{V})$ is said to have *continuous point evaluations* if all the evaluation maps

$$\mathrm{ev}_m\colon \mathcal{H} \to \mathbb{V}_m, \quad s \mapsto s(m)$$

are continuous and the function $m \mapsto \|\,\mathrm{ev}_m\,\|$ is locally bounded.

If G is a group acting on $\mathbb{V}$ by holomorphic bundle automorphisms, G acts on $\Gamma(\mathbb{V})$ by

$$(g.s)(m) := g.s(g^{-1}m). \tag{8}$$

We call a Hilbert subspace $\mathcal{H} \subseteq \Gamma(\mathbb{V})$ with continuous point evaluations *G-invariant* if $\mathcal{H}$ is invariant under the action defined by (8) and the so obtained representation of G on $\mathcal{H}$ is unitary.

Remark 2.2. Let $q\colon \mathbb{V} \to M$ be a holomorphic Hilbert bundle over M. Then we can represent holomorphic sections of this bundle by holomorphic functions on the dual bundle $\mathbb{V}^*$ whose fiber $(\mathbb{V}^*)_m$ is the dual space $\mathbb{V}_m^*$ of $\mathbb{V}_m$. We thus obtain an embedding

$$\Psi\colon \Gamma(\mathbb{V}) \to \mathcal{O}(\mathbb{V}^*), \quad \Psi(s)(\alpha_m) = \alpha_m(s(m)), \tag{9}$$

whose image consists of holomorphic functions on $\mathbb{V}^*$ which are fiberwise linear.

If G is a group acting on $\mathbb{V}$ by holomorphic bundle automorphisms, then G also acts naturally by holomorphic maps on $\mathbb{V}^*$ via $(g.\alpha_m)(v_{g.m}) := \alpha_m(g^{-1}.v_{g.m})$ for $\alpha_m \in \mathbb{V}_m^*$. Therefore

$$\Psi(g.s)(\alpha_m) = \alpha_m(g.s(g^{-1}.m)) = (g^{-1}.\alpha_m)(s(g^{-1}.m)) = \Psi(s)(g^{-1}.\alpha_m)$$

implies that Ψ is equivariant with respect to the natural G-actions on $\Gamma(\mathbb{V})$ and $\mathcal{O}(\mathbb{V}^*)$.

2.1 Existence of analytic vectors

Lemma 2.3. *If $M = G/H$ is a Banach homogeneous space with a G-invariant complex structure and $\mathbb{V} = G \times_H V$ a G-equivariant holomorphic vector bundle over M defined by a pair (ρ, β) as in* Theorem 1.6*, then the G-action on $\mathbb{V}$ is analytic.*

Proof. The manifold $\mathbb{V}$ is a G-equivariant quotient of the product manifold $G \times V$ on which G acts analytically by left multiplications in the left factor. Since the quotient map $q\colon G \times V \to \mathbb{V}$ is a real analytic submersion, the action of G on $\mathbb{V}$ is also analytic. □

Definition 2.4. Let M be a complex manifold (modeled on a locally convex space) and $\mathcal{O}(M)$ the space of holomorphic complex-valued functions on M. We write $\overline{M}$ for the conjugate complex manifold. A holomorphic function $Q\colon M \times \overline{M} \to \mathbb{C}$ is said to be a *reproducing kernel* of a Hilbert subspace $\mathcal{H} \subseteq \mathcal{O}(M)$ if for each $w \in M$ the function $Q_w(z) := Q(z, w)$ is contained in $\mathcal{H}$ and satisfies

$$\langle f, Q_z \rangle = f(z) \quad \text{for} \quad z \in M, f \in \mathcal{H}.$$

Then $\mathcal{H}$ is called a *reproducing kernel Hilbert space* and since it is determined uniquely by the kernel Q, it is denoted $\mathcal{H}_Q$ (cf. [Ne00, Sect. I.1]).

Now let G be a group and $\sigma\colon G \times M \to M, (g, m) \mapsto g.m$ be a smooth right action of G on M by holomorphic maps. Then $(g.f)(m) := f(g^{-1}.m)$ defines a unitary representation of G on a reproducing kernel Hilbert space $\mathcal{H}_Q \subseteq \mathcal{O}(M)$ if and only if the kernel Q is *invariant*:

$$Q(g.z, g.w) = Q(z, w) \quad \text{for} \quad z, w \in M, g \in G$$

([Ne00, Rem. II.4.5]). In this case we call $\mathcal{H}_Q$ a G-invariant reproducing kernel Hilbert space.

Lemma 2.5. *Let G be a Banach–Lie group acting analytically via*

$$\sigma\colon G \times M \to M, \quad (g, m) \mapsto \sigma_g(m)$$

by holomorphic maps on the complex manifold M.

(a) *Let $\mathcal{H} \subseteq \mathcal{O}(M)$ be a reproducing kernel Hilbert space whose kernel Q is a G-invariant holomorphic function on $M \times \overline{M}$. Then the elements $(Q_m)_{m \in M}$ representing the evaluation functionals in $\mathcal{H}$ are analytic vectors for the representation of G, defined by $\pi(g)f := f \circ \sigma_g^{-1}$.*

(b) *Let $\mathbb{V} \to M$ be a holomorphic G-homogeneous Hilbert bundle and $\mathcal{H} \subseteq \Gamma(\mathbb{V})$ be a G-invariant Hilbert space with continuous point evaluations. Then every vector of the form $\mathrm{ev}_m^* v$, $m \in M$, $v \in \mathbb{V}_m$, is analytic for the G-action on $\mathcal{H}$.*

Proof. (a) Since Q is holomorphic on $M \times \overline{M}$, it is in particular real analytic. For $m \in M$, we have

$$\langle \pi(g)Q_m, Q_m \rangle = (\pi(g)Q_m)(m) = Q_m(g^{-1}m) = Q(g^{-1}m, m),$$

and since Q and the G-action are real analytic, this function is analytic on M. Now [Ne10b, Thm. 5.2] implies that Q_m is an analytic vector.

(b) As in Remark 2.2, we realize $\Gamma(\mathbb{V})$ by holomorphic functions on the dual bundle $\mathbb{V}^*$. We thus obtain a reproducing kernel Hilbert space $\mathcal{H}_Q := \Psi(\mathcal{H}) \subseteq \mathcal{O}(\mathbb{V}^*)$, and since Ψ is G-equivariant, the reproducing kernel Q is G-invariant. For $v \in \mathbb{V}_m$, evaluation in the corresponding element $\alpha_m := \langle \cdot, v \rangle \in \mathbb{V}_m^*$ is given by

$$s \mapsto \langle s(m), v \rangle = \langle \mathrm{ev}_m(s), v \rangle = \langle s, \mathrm{ev}_m^* v \rangle.$$

For the corresponding G-invariant kernel Q on $\mathbb{V}^*$ this means that $Q_{\alpha_m} = \mathrm{ev}_m^* v$, so that the assertion follows from (a). □

2.2 The endomorphism bundle and commutants

The goal of this section is Theorem 2.12 which connects the commutant of the G-representation on a Hilbert space $\mathcal{H}_V$ of a holomorphically induced representation with the corresponding representation (ρ, β) of $(H, \mathfrak{q})$ on V. The remarkable point is that, under natural assumptions, these commutants are isomorphic, so that both representations have the same decomposition theory. This generalizes an important result of S. Kobayashi concerning irreducibility criteria for the G-representation on $\mathcal{H}_V$ ([Ko68], [BH98, Thm. 2.5]).

Let $\mathbb{V} = G \times_H V \to M$ be a G-homogeneous holomorphic Hilbert bundle as in Theorem 1.6. Then the complex manifold $M \times \overline{M}$ is a complex homogeneous space $(G \times G)/(H \times H)$, where the complex structure is defined by the closed subalgebra $\mathfrak{q} \oplus \overline{\mathfrak{q}}$ of $\mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}$. On the Banach space $B(V)$ we consider the norm continuous representation of $H \times H$ by

$$\widetilde{\rho}(h_1, h_2)A = \rho(h_1)A\rho(h_2)^*$$

and the the corresponding extension $\widetilde{\beta}\colon \mathfrak{q} \oplus \overline{\mathfrak{q}} \to \mathfrak{gl}(B(V))$ by

$$\widetilde{\beta}(x_1, x_2)A := \beta(x_1)A + A\beta(\overline{x_2})^*.$$

We write $\mathbb{L} := (G \times G) \times_{H \times H} B(V)$ for the corresponding holomorphic Banach bundle over $M \times \overline{M}$ (Theorem 1.6).

Remark 2.6. For every $g \in G$, we have isomorphisms $V \to \mathbb{V}_{gH} = [g, V], v \mapsto [g, v]$. Accordingly, we have for every pair $(g_1, g_2) \in G \times G$ an isomorphism

$$\nu \colon B(V) \to B(\mathbb{V}_{g_2 H}, \mathbb{V}_{g_1 H}), \quad \nu(A)[g_2, v] \mapsto [g_1, Av].$$

This defines a map

$$\gamma \colon G \times G \times B(V) \to B(\mathbb{V}) := \bigcup_{m,n \in M} B(\mathbb{V}_m, \mathbb{V}_n), \quad \gamma(g_1, g_2, A)[g_2, v] = [g_1, Av].$$

For $h_1, h_2 \in H$, we then have

$$\begin{aligned} &\gamma(g_1 h_1, g_2 h_2, \widetilde{\rho}(h_1, h_2)^{-1} A)[g_2, v] = \gamma(g_1 h_1, g_2 h_2, \widetilde{\rho}(h_1, h_2)^{-1} A)[g_2 h_2, \rho(h_2)^{-1} v] \\ &= [g_1 h_1, \rho(h_1)^{-1} A v] = [g_1, Av] = \gamma(g_1, g_2, A)[g_2, v], \end{aligned}$$

so that γ factors through a bijection $\overline{\gamma} \colon \mathbb{L} \to B(\mathbb{V})$. This provides an interpretation of the bundle $\mathbb{L}$ as the *endomorphism bundle of* $\mathbb{V}$ (cf. [BR07]).

Since the group G acts (diagonally) on the bundle $\mathbb{L}$, it makes sense to consider G-invariant holomorphic sections.

Lemma 2.7. *The space $\Gamma(\mathbb{L})^G$ of G-invariant holomorphic sections of $\mathbb{L}$ has the property that the evaluation map*

$$\mathrm{ev} \colon \Gamma(\mathbb{L})^G \cong C^\infty(G \times G, B(V))_{\widetilde{\rho}, \widetilde{\beta}} \to B_H(V), \quad s \mapsto \widehat{s}(\mathbf{1}, \mathbf{1})$$

is injective.

Proof. If $\widehat{s}(\mathbf{1}, \mathbf{1}) = 0$, then the corresponding holomorphic section $s \in \Gamma(\mathbb{L})$ vanishes on the totally real submanifold

$$\Delta_M := \{(m, m) \colon m \in M\} = G(m_0, m_0) \subseteq M \times \overline{M},$$

and hence on all of $M \times \overline{M}$. For the function $\widehat{s} \colon G \times G \to B(V)$, we have for $h \in H$

$$\widehat{s}(\mathbf{1}, \mathbf{1}) = (h^{-1}.\widehat{s})(\mathbf{1}, \mathbf{1}) = \widehat{s}(h, h) = \rho(h)^{-1} \widehat{s}(\mathbf{1}, \mathbf{1}) \rho(h),$$

showing that $\widehat{s}(\mathbf{1}, \mathbf{1})$ lies in $B_H(V)$. □

Remark 2.8. In general, the holomorphic bundle $\mathbb{L}$ does not have any non-zero holomorphic section, although its restriction to the diagonal Δ_M has the trivial section R given by $R_{(m,m)} = \mathrm{id}_{\mathbb{V}_m}$ for every $m \in M$.

If $R \colon M \times \overline{M} \to \mathbb{L}$ is a holomorphic section, then we obtain for every $v \in V$ and $n \in M$ a holomorphic section $R_{n,v} \colon M \to \mathbb{V}, m \mapsto R(m, n)v$ which is non-zero in m if $R_{m,m} v \neq 0$. Therefore $\mathbb{V}$ has nonzero holomorphic sections if $\mathbb{L}$ does, and this is not always the case.

Example 2.9. (a) Let $(\pi, \mathcal{H})$ be a unitary representation of G and $\Psi\colon \mathcal{H} \to \Gamma(\mathbb{V})$ be G-equivariant such that the evaluation maps $\mathrm{ev}_m \circ \Psi\colon \mathcal{H} \to \mathbb{V}_m$ are continuous and $m \mapsto \|\mathrm{ev}_m\|$ is locally bounded. Then

$$R(m,n) := (\mathrm{ev}_m \Psi)(\mathrm{ev}_n \Psi)^* \in B(\mathbb{V}_n, \mathbb{V}_m)$$

defines a holomorphic section of $B(\mathbb{V})$. Since this assertion is local, it follows from the corresponding assertion for trivial bundles treated in [Ne00, Lemma A.III.9(iii)]. The corresponding smooth function

$$\widehat{R}\colon G \times G \to B(V), \quad \widehat{R}(g_1, g_2) = (\mathrm{ev}_{g_1} \Psi)(\mathrm{ev}_{g_2} \Psi)^*$$

is a G-invariant $B(V)$-valued kernel on $G \times G$ because

$$\mathrm{ev}_{gh} \Psi = \mathrm{ev}_h \Psi \circ \pi(g)^{-1} \quad \text{for} \quad g, h \in G.$$

We conclude that $\widehat{R} \in \Gamma(\mathbb{L})^G$, so that it is completely determined by

$$\widehat{R}(\mathbf{1}, \mathbf{1}) = (\mathrm{ev}_{\mathbf{1}} \Psi)(\mathrm{ev}_{\mathbf{1}} \Psi)^*$$

(Lemma 2.7). In particular, Ψ is completely determined by $\mathrm{ev}_{\mathbf{1}} \circ \Psi$. This follows also from its equivariance, which leads to $\Psi(v)(g) = \mathrm{ev}_{\mathbf{1}}(g^{-1}.\Psi(v)) = \mathrm{ev}_{\mathbf{1}} \Psi(\pi(g)^{-1}v)$.

(b) For $A, B \in B(V)$ commuting with $\rho(H)$ and $\beta(\mathfrak{q})$ and $\widehat{R} \in C^\infty(G \times G, B(V))_{\widetilde{\rho}, \widetilde{\beta}}$, the function

$$\widehat{R}^{A,B}\colon G \times G \to B(V), \quad \widehat{R}^{A,B}(g_1, g_2) = AR(g_1, g_2)B$$

is also contained in $C^\infty(G \times G, B(V))_{\widetilde{\rho}, \widetilde{\beta}}$, hence defines a holomorphic section of the bundle $\mathbb{L}$.

Definition 2.10. (a) For (ρ, β) as in Theorem 1.6 and the corresponding associated bundle $\mathbb{V} = G \times_H V$, a unitary representation $(\pi, \mathcal{H})$ of G is said to be *holomorphically induced from* (ρ, β) if there exists a realization $\Psi\colon \mathcal{H} \to \Gamma(\mathbb{V})$ as an invariant Hilbert space with continuous point evaluations whose kernel $R \in \Gamma(\mathbb{L})^G$ satisfies $\widehat{R}(\mathbf{1}, \mathbf{1}) = \mathrm{id}_V$.

(b) Since this condition determines R by Lemma 2.7, it also determines the reproducing kernel space $\Psi(\mathcal{H})$ and its norm. We conclude that, for every pair (ρ, β), there is at most one holomorphically induced unitary representation of G up to unitary equivalence. Accordingly, we call (ρ, β, V) *inducible* if there exists a corresponding holomorphically induced unitary representation of G.

If this is the case, then we use the isometric embedding $\mathrm{ev}_{\mathbf{1}}^*\colon V \to \mathcal{H}$ to identify V with a subspace of $\mathcal{H}$ and note that the evaluation map $\mathrm{ev}_{\mathbf{1}}\colon \mathcal{H} \to V$ corresponds to the orthogonal projection $p_V\colon \mathcal{H} \to V$.

Remark 2.11. (a) If $\mathcal{H}_V \hookrightarrow \Gamma(\mathbb{V})$ is a G-invariant Hilbert space on which the representation is holomorphically induced, then we obtain a G-invariant element $Q \in B(\mathbb{V})$ by

$$Q(m,n) := \mathrm{ev}_m \mathrm{ev}_n^* \in B(\mathbb{V}_n, \mathbb{V}_m),$$

and the relation $\mathrm{ev}_g = \mathrm{ev}_{\mathbf{1}} \circ \pi(g)^{-1} = p_V \circ \pi(g)^{-1}$ yields

$$\widehat{Q}(g_1, g_2) = \mathrm{ev}_{g_1} \mathrm{ev}_{g_2}^* = p_V \pi(g_1)^{-1} \pi(g_2) p_V = p_V \pi(g_1^{-1} g_2) p_V$$

and in particular $\widehat{Q}(\mathbf{1}, \mathbf{1}) = \mathrm{id}_V$.

(b) Suppose that $\mathcal{H} \subseteq \Gamma(\mathbb{V})$ is holomorphically induced. Since $\mathrm{ev}_{\mathbf{1}}$ is H-equivariant, the closed subspace $V \subseteq \mathcal{H}$ is H-invariant and the H-representation on this space is equivalent to (ρ, V), hence in particular bounded.

Moreover, $V \subseteq \mathcal{H}^\omega$ follows from Lemma 2.5(b). Since the evaluation maps $\mathrm{ev}_g \colon \mathcal{H} \to V$ separate the points, the analyticity of the elements $\mathrm{ev}_g^* v$ even implies that $\mathcal{H}^\omega$ is dense in $\mathcal{H}$.

For $x \in \mathfrak{q}$ and $s \in \mathcal{H}^\infty$ we have

$$\mathrm{ev}_{\mathbf{1}} \mathtt{d}\pi(x) s = (\mathtt{d}\pi(x) s)\widehat{}(\mathbf{1}) = -L_x \widehat{s}(\mathbf{1}) = \beta(x) \widehat{s}(\mathbf{1}) = \beta(x) \mathrm{ev}_{\mathbf{1}} s.$$

Therefore $\mathtt{d}\pi(\mathfrak{q})$ preserves the subspace $\mathcal{H}^\infty \cap V^\perp = \mathcal{H}^\infty \cap \ker(\mathrm{ev}_{\mathbf{1}})$. From $V \subseteq \mathcal{H}^\infty$, we derive

$$\mathcal{H}^\infty = V \oplus (V^\perp \cap \mathcal{H}^\infty),$$

so that the density of $\mathcal{H}^\infty$ in $\mathcal{H}$ implies that $V = (V^\perp \cap \mathcal{H}^\infty)^\perp$, and hence that this space is invariant under the restriction $\mathtt{d}\pi(\overline{x})$ of the adjoint $-\mathtt{d}\pi(x)^*$. For $s_j = \mathrm{ev}_{\mathbf{1}}^* v_j \in V$, $j = 1, 2$, we further obtain

$$\begin{aligned} \langle \mathtt{d}\pi(\overline{x}) s_1, s_2 \rangle &= -\langle \mathrm{ev}_{\mathbf{1}}^* v_1, \mathtt{d}\pi(x) s_2 \rangle = -\langle v_1, \mathrm{ev}_{\mathbf{1}} \mathtt{d}\pi(x) s_2 \rangle = -\langle v_1, \beta(x) \mathrm{ev}_{\mathbf{1}} s_2 \rangle \\ &= -\langle v_1, \beta(x) v_2 \rangle = -\langle \beta(x)^* v_1, v_2 \rangle, \end{aligned}$$

so that

$$\mathtt{d}\pi(\overline{x})|_V = -\beta(x)^*, \quad x \in \mathfrak{q}. \tag{10}$$

Finally we observe that

$$(\pi(G)V)^\perp = \{ s \in \mathcal{H} \colon (\forall g \in G)\, \widehat{s}(g) = 0 \} = \{0\}$$

implies that $\mathcal{H} = \overline{\mathrm{span}(\pi(G)V)}$.

If V is of the form $\overline{(\mathcal{H}^\infty)^{\mathfrak{n}}}$ for a subalgebra $\mathfrak{n} \subseteq \mathfrak{g}_{\mathbb{C}}$, then it is invariant under the commutant $B_G(V)$, but we do not know if this is always true for holomorphically induced representation. To make the following proposition as flexible as possible, we assume this naturality condition of V (cf. Remark 2.18 below).

Theorem 2.12. *Suppose that $(\pi, \mathcal{H}_V)$ is holomorphically induced from the representation (ρ, β) of $(H, \mathfrak{q})$ on V and that*

$$B_{H,\mathfrak{q}}(V) := \{ A \in B_H(V) \colon (\forall x \in \mathfrak{q})\, A\beta(x) = \beta(x) A, A^* \beta(x) = \beta(x) A^* \}$$

is the involutive commutant of $\rho(H)$ and $\beta(\mathfrak{q})$. If V is invariant under $B_G(V)$, then the map

$$R \colon B_G(\mathcal{H}_V) \to B_{H,\mathfrak{q}}(V), \quad A \mapsto A|_V$$

is an isomorphism of von Neumann algebras.

Proof. By assumption, every $A \in B_G(\mathcal{H}_V)$ preserves V, so that $A|_V$ can be identified with the operator $p_V A p_V \in B(V)$, where $p_V : \mathcal{H} \to V$ is the orthogonal projection. Clearly $A|_V$ commutes with each $\rho(h) = \pi(h)|_V$. It also preserves the subspace $\mathcal{H}^\infty$ on which it satisfies $\pi(x)A = A\pi(x)$ for every $x \in \mathfrak{g}_{\mathbb{C}}$. Therefore $A|_V$ commutes with $\mathsf{d}\pi(\overline{\mathfrak{q}})$ and hence with $\beta(\mathfrak{q})$ (cf. (10) in Remark 2.11).

Therefore R defines a homomorphism of von Neumann algebras. If $R(A) = 0$, then $AV = 0$ implies that $A\pi(G)V = \{0\}$, which leads to $A = 0$. Hence R is injective.

Since each von Neumann algebra is generated by orthogonal projections ([Dix96, Chap. 1, §1.2]) and images of von Neumann algebras under restriction maps are von Neumann algebras ([Dix96, Chap. 1, §2.1, Prop. 1]), we are done if we can show that every orthogonal projection in $B_{H,\mathfrak{q}}(V)$ is contained in the image of R. So let $P \in B_{H,\mathfrak{q}}(V)$. Then $V_1 := P(V)$ and $V_2 := (\mathbf{1} - P)(V)$ yields an $(H, \mathfrak{q})$-invariant orthogonal decomposition $V = V_1 \oplus V_2$.

Let $\widehat{Q} : G \times G \to B(V), (g_1, g_2) \mapsto p_V \pi(g_1^{-1} g_2) p_V$ be the natural kernel function defining the inclusion $\mathcal{H}_V \hookrightarrow \Gamma(\mathbb{V})$ and consider the G-invariant kernel

$$\widehat{R}(g_1, g_2) := P\widehat{Q}(g_1, g_2)(\mathbf{1} - P).$$

According to Example 2.9(b), it is contained in $C^\infty(G \times G, B(V))_{\widetilde{\rho},\widetilde{\beta}}$, hence defines an element $R \in \Gamma(\mathbb{L})^G$. In view of

$$\widehat{R}(\mathbf{1}, \mathbf{1}) = PQ(\mathbf{1}, \mathbf{1})(\mathbf{1} - P) = P(\mathbf{1} - P) = 0,$$

this section vanishes in the base point, hence on all of $M \times \overline{M}$ (Lemma 2.7). We conclude that

$$0 = \widehat{R}(g_1, g_2) = P p_V \pi(g_1^{-1} g_2) p_V (\mathbf{1} - P) = P\pi(g_1^{-1} g_2)(\mathbf{1} - P),$$

so that, for every $g \in G$, we have $P\pi(g)(\mathbf{1} - P) = 0$. This leads to $P\mathcal{H}_{V_2} = \{0\}$, and hence to $\mathcal{H}_{V_1} \perp \mathcal{H}_{V_2}$. We derive that $\mathcal{H}_V = \mathcal{H}_{V_1} \oplus \mathcal{H}_{V_2}$ is an orthogonal direct sum. Therefore the orthogonal projection $\widetilde{P} \in B_G(\mathcal{H}_V)$ onto $\mathcal{H}_{V_1}$ leaves V invariant and satisfies $\widetilde{P}|_V = P$. This proves that R is surjective. ☐

Remark 2.13. The preceding proof shows that, under the assumptions of Theorem 2.12, the range of the injective map

$$\mathrm{ev} : \Gamma(\mathbb{L})^G \to B_H(V), \quad R \mapsto \widehat{R}(\mathbf{1}, \mathbf{1})$$

contains $B_{H,\mathfrak{q}}(V)$. If $\beta(\mathfrak{q}) = \beta(\mathfrak{h}_{\mathbb{C}})$, then $B_{H,\mathfrak{q}}(V) = B_H(V)$, so that we obtain a linear isomorphism $\Gamma(\mathbb{L})^G \cong B_H(V)$.

The preceding theorem has quite remarkable consequences because it implies that the representations $(\pi, \mathcal{H}_V)$ and (ρ, V) decompose in the same way.

Corollary 2.14. *Suppose that $(\pi, \mathcal{H}_V)$ is holomorphically induced from (ρ, β), that V is $B_G(V)$-invariant, and that $\beta(\mathfrak{q}) = \beta(\mathfrak{h}_{\mathbb{C}})$, so that $B_H(V) = B_{H,\mathfrak{q}}(V)$. Then the G-representation $(\pi, \mathcal{H}_V)$ has any of the following properties if and only if the H-representation (ρ, V) does.*

(i) *Irreducibility.*

(ii) *Multiplicity freeness.*

(iii) *Type I, II or III.*

(iv) *Discreteness, i.e., being a direct sum of irreducible representations.*

Proof. (i) follows from the fact that, according to Schur's Lemma, irreducibility means that the commutant equals $\mathbb{C}\mathbf{1}$.

(ii) is clear because multiplicity freeness means that the commutant is commutative.

(iii) is clear because the type of a representation is defined as the type of its commutant as a von Neumann algebra.

(iv) That a unitary representation decomposes discretely means that its commutant is an ℓ^∞-direct sum of factors of type I. Hence the G representation on $\mathcal{H}_V$ has this property if only if the H-representation on V does. □

Corollary 2.14(i) is a version of S. Kobayashi's Theorem in the Banach context (cf. [Ko68]).

Problem 2.15. Theorem 2.12 should also be useful to derive direct integral decompositions of the G-representation on $\mathcal{H}_V$ from direct integral decompositions of the H-representation on V.

Suppose that the bounded unitary representation (ρ, V) of H is of type I and holomorphically inducible. Then it is a direct integral of irreducible representations (ρ_j, V_j). Are these irreducible representations of H also inducible?

Corollary 2.16. *Suppose that the G-representations $(\pi_1, \mathcal{H}_{V_1})$, resp., $(\pi_2, \mathcal{H}_{V_2})$ are holomorphically induced from the $(H, \mathfrak{q})$-representations (ρ_1, β_1, V_1), resp., (ρ_2, β_2, V_2). Then any unitary isomorphism $\gamma\colon V_1 \to V_2$ of $(H, \mathfrak{q})$-modules extends uniquely to a unitary equivalence $\widetilde{\gamma}\colon \mathcal{H}_{V_1} \to \mathcal{H}_{V_2}$.*

Proof. Since γ is $(H, \mathfrak{q})$-equivariant, we have a well-defined G-equivariant bijection

$$\widetilde{\gamma}\colon \Gamma(\mathbb{V}_1) \cong C^\infty(G, V_1)_{\rho_1,\beta_1} \to C^\infty(G, V_2)_{\rho_2,\beta_2}, \quad f \mapsto \gamma \circ f$$

obtained from a corresponding isomorphism $[(g, v)] \mapsto [(g, \gamma(v))]$ of holomorphic G-bundles. Therefore $\widetilde{\gamma}(\mathcal{H}_{V_1})$ is an invariant Hilbert space with continuous point evaluations. The corresponding G-invariant kernel $\widehat{Q} \in C^\infty(G \times G, B(V_2))$ satisfies

$$\widehat{Q}(\mathbf{1}, \mathbf{1}) = \gamma \circ \mathrm{id}_{V_1} \circ \gamma^* = \gamma\gamma^* = \mathrm{id}_{V_2}\,.$$

This implies that $\widetilde{\gamma}(\mathcal{H}_{V_1}) = \mathcal{H}_{V_2}$ and that the map $\widetilde{\gamma}\colon \mathcal{H}_{V_1} \to \mathcal{H}_{V_2}$ is unitary because both spaces have the same reproducing kernels (cf. Definition 2.10).

For $v \in V_1$, we further note that $(\widetilde{\gamma}\,\mathrm{ev}_{\mathbf{1}}^*\, v)(\mathbf{1}) = \gamma\,\mathrm{ev}_{\mathbf{1}}\,\mathrm{ev}_{\mathbf{1}}^*\, v = \gamma v$, so that $\widetilde{\gamma}$ extends γ, when considered as a map on the subspace $V_1 \cong \mathrm{ev}_{\mathbf{1}}^*\, V_1 \subseteq \mathcal{H}_1$. □

2.3 Realizing unitary representations by holomorphic sections

We conclude this section with a result that helps to realize certain subrepresentations of unitary representations in spaces of holomorphic sections. We continue in the setting of Section 1, where $M = G/H$ is a Banach homogeneous space with a complex structure defined by the subalgebra $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$.

From the discussion in Remark 2.11, we know that every holomorphically induced representation satisfies the assumptions (A1/2) in the theorem below, which is our main tool to prove that a given unitary representation is holomorphically induced.

Theorem 2.17. *Let $(\pi, \mathcal{H})$ be a continuous unitary representation of G and $V \subseteq \mathcal{H}$ be a closed subspace satisfying the following conditions:*

(A1) *V is H-invariant and the representation ρ of H on V is bounded. In particular, $\mathtt{d}\pi|_{\mathfrak{h}} \colon \mathfrak{h} \to \mathfrak{gl}(V)$ defines a continuous homomorphism of Banach–Lie algebras.*

(A2) *The exists a subspace $\mathcal{D}_V \subseteq V \cap \mathcal{H}^\infty$ dense in V which is invariant under $\mathtt{d}\pi(\overline{\mathfrak{q}})$, the operators $\mathtt{d}\pi(\overline{\mathfrak{q}})|_{\mathcal{D}_V}$ are bounded, and the so obtained representation of $\overline{\mathfrak{q}}$ on V defines a continuous morphism of Banach–Lie algebras*

$$\beta \colon \mathfrak{q} \to \mathfrak{gl}(V), \quad x \mapsto -(\mathtt{d}\pi(\overline{x})|_V)^*.$$

Then the following assertions hold:

(i) *β is an extension of ρ defining on $\mathbb{V} := G \times_H V$ the structure of a complex Hilbert bundle.*

(ii) *If $p_V \colon \mathcal{H} \to \mathcal{H}$ denotes the orthogonal projection to V, then*

$$\Phi \colon \mathcal{H} \to C(G, V)_\rho, \quad \Phi(v)(g) := p_V(\pi(g)^{-1}v)$$

maps $\mathcal{H}$ into $C^\infty(G, V)_{\rho,\beta} \cong \Gamma(\mathbb{V})$, and we thus obtain a G-equivariant unitary isomorphism of the closed subspace $\mathcal{H}_V := \overline{\operatorname{span} \pi(G)V}$ with the representation holomorphically induced from (ρ, β).

(iii) *$V \subseteq \mathcal{H}^\omega$.*

Proof. (i) For $x \in \mathfrak{h}$, we have $\beta(x) = -\mathtt{d}\rho(x)^* = \mathtt{d}\rho(x)$, and it is also easy to see that $\beta(\operatorname{Ad}(h)x) = \pi(h)\beta(x)\pi(h)^{-1}$ for $h \in H$ and $x \in \mathfrak{q}$. Therefore β is an extension of ρ, and we can use Theorem 1.6 to see that β defines the structure of a holomorphic Hilbert bundle on $\mathbb{V}$.

(ii) Clearly, $\Phi(\mathcal{H}^\infty) \subseteq C^\infty(G, V)_\rho$. For $v \in \mathcal{D}_V$, $w \in \mathcal{H}^\infty$ and $x \in \mathfrak{q}$, we further derive from (A2) that

$$\langle p_V(\mathtt{d}\pi(x)w), v\rangle = \langle \mathtt{d}\pi(x)w, v\rangle = \langle w, \mathtt{d}\pi(-\overline{x})v\rangle = \langle p_V(w), \mathtt{d}\pi(-\overline{x})v\rangle = \langle \beta(x)p_V(w), v\rangle,$$

so that the density of $\mathcal{D}_V$ in V implies

$$p_V \circ \mathtt{d}\pi(x) = \beta(x) \circ p_V|_{\mathcal{H}^\infty} \quad \text{for} \quad x \in \mathfrak{q}.$$

For $v \in \mathcal{H}^\infty$ and $x \in \mathfrak{q}$, we now obtain

$$\big(L_x \Phi(v)\big)(g) = -p_V(\mathtt{d}\pi(x)\pi(g)^{-1}v) = -\beta(x)p_V(\pi(g)^{-1}v) = -\beta(x)\Phi(v)(g). \quad (11)$$

This means that $\Phi(\mathcal{H}^\infty) \subseteq C^\infty(G,V)_{\rho,\beta}$. Writing $\Gamma_c(\mathbb{V})$ for the space of continuous sections of $\mathbb{V}$, we also obtain a map $\widetilde{\Phi}\colon \mathcal{H} \to \Gamma_c(\mathbb{V})$ which is continuous if $\Gamma_c(\mathbb{V})$ is endowed with the compact open topology. As $\Gamma(\mathbb{V})$ is closed in $\Gamma_c(\mathbb{V})$ with respect to this topology ([Ne01, Cor. III.12]), $\widetilde{\Phi}(\overline{\mathcal{H}^\infty}) \subseteq \Gamma(\mathbb{V})$, resp., $\Phi(\overline{\mathcal{H}^\infty}) \subseteq C^\infty(G,V)_{\rho,\beta}$.

Clearly $\Phi(v) = 0$ is equivalent to $v \bot \pi(G)V$, so that $(\ker\Phi)^\bot = \mathcal{H}_V$. Since (A2) implies that $V \subseteq \overline{\mathcal{H}^\infty}$, the same holds for $\mathrm{span}(\pi(G)V)$. This shows that

$$\Phi(\mathcal{H}) = \Phi(\mathcal{H}_V) = \Phi(\overline{\mathcal{H}^\infty}) \subseteq C^\infty(G,V)_{\rho,\beta}.$$

The corresponding kernel $\widehat{Q}\colon G \times G \to B(V)$ is given by

$$\widehat{Q}(g_1,g_2) = \mathrm{ev}_{g_1}\mathrm{ev}_{g_2}^* = p_V\pi(g_1)^{-1}(p_V \circ \pi(g_2)^{-1})^* = p_V\pi(g_1^{-1}g_2)p_V,$$

so that we have in particular $\widehat{Q}(\mathbf{1},\mathbf{1}) = \mathrm{id}_V$.

(iii) To see that V consists of analytic vectors, we may w.l.o.g. assume that $\mathcal{H} = \mathcal{H}_V$ and hence that $\mathcal{H} \subseteq \Gamma(\mathbb{V})$ is holomorphically induced and that $\Phi = \mathrm{id}_{\mathcal{H}}$. Let $\mathrm{ev}_{\mathbf{1}H}\colon \mathcal{H} \to \mathbb{V}_{\mathbf{1}H} \cong V$ be the evaluation map. The corresponding map

$$\mathrm{ev}_{\mathbf{1}}\colon C^\infty(G,V)_{\rho,\beta} \to V$$

is simply given by evaluation in $\mathbf{1} \in G$. Now $\mathrm{ev}_{\mathbf{1}}|_V\colon V \to V$ is the identity, so that $\mathrm{ev}_{\mathbf{1}}^*\colon V \to \mathcal{H}$ is simply the isometric inclusion. Hence the analyticity of $v = \mathrm{ev}_{\mathbf{1}}^* v = \mathrm{ev}_{\mathbf{1}H}^* v$ follows from Lemma 2.5. □

Remark 2.18. Suppose that there exist subalgebras $\mathfrak{p}^\pm \subseteq \mathfrak{g}_\mathbb{C}$ with $\mathfrak{q} = \mathfrak{h}_\mathbb{C} \rtimes \mathfrak{p}^+$ and $\mathrm{Ad}(H)\mathfrak{p}^\pm = \mathfrak{p}^\pm$. Then, for every unitary representation $(\pi,\mathcal{H})$ of G, the closed subspace $V := \overline{(\mathcal{H}^\infty)^{\mathfrak{p}^-}}$ is invariant under H and $B_G(\mathcal{H})$. If the representation (ρ,H) on V is bounded, then the dense subspace $\mathcal{D}_V := (\mathcal{H}^\infty)^{\mathfrak{p}^-}$ satisfies (A2) if we put $\beta(\mathfrak{p}^+) = \{0\}$. Theorem 2.17 now implies that $V \subseteq \mathcal{H}^\omega$, so that we see in particular that $V = (\mathcal{H}^\infty)^{\mathfrak{p}^-}$ is a closed subspace of $\mathcal{H}$. Moreover, Corollary 2.14 applies.

The following remark sheds some extra light on condition (A2).

Remark 2.19. Let $\mathcal{H} \subseteq \Gamma(\mathbb{V})$ be a G-invariant Hilbert subspace with continuous point evaluations, $m \in M$ and $V := \overline{\mathrm{im}(\mathrm{ev}_m^*)} \subseteq \mathcal{H}$. Then V is a G_m-invariant closed subspace of $\mathcal{H}$ and

$$V^\bot = \mathrm{im}(\mathrm{ev}_m^*)^\bot = \ker(\mathrm{ev}_m) = \{s \in \mathcal{H}\colon s(m) = 0\}.$$

The action of G on $\mathbb{V}$ by holomorphic bundle automorphisms leads to a homomorphism $\dot{\sigma}_\mathbb{V}\colon \mathfrak{g}_\mathbb{C} \to \mathcal{V}(\mathbb{V})$ and each $x \in \mathfrak{q}_m$ thus leads to a linear vector field $-\beta_m(x)$ on the fiber $\mathbb{V}_m$. Passing to derivatives in the formula $(g.s)(m) := g.s(g^{-1}m)$, we obtain for $x \in \mathfrak{q}_m$

$$(x.s)(m) = -\beta_m(x) \cdot s(m).$$

In particular, $V^\perp \cap \mathcal{H}^\infty$ is invariant under the derived action of $\mathfrak{q}_m$, so that one can expect that the adjoint operators coming from $\overline{\mathfrak{q}_m}$ act on $\mathbb{V}_m$

As we have seen in Lemma 2.5(b), the subspace $\mathrm{im}(\mathrm{ev}_m^*)$ of $\mathcal{H}$ consists of smooth vectors, so that $V \cap \mathcal{H}^\infty$ is dense in V.

Example 2.20. We consider the identical representation of $G = \mathrm{U}(\mathcal{H})$ on the complex Hilbert space $\mathcal{H}$. Let $\mathcal{K}$ be a closed subspace of $\mathcal{H}$. Then the subgroup $Q := \{g \in \mathrm{GL}(\mathcal{H}) \colon g\mathcal{K} = \mathcal{K}\}$ is a complex Lie subgroup of $\mathrm{GL}(\mathcal{H})$ and the Graßmannian $\mathrm{Gr}_\mathcal{K}(\mathcal{H}) := \mathrm{GL}(\mathcal{H})\mathcal{K} \cong \mathrm{GL}(\mathcal{H})/Q$ carries the structure of a complex homogeneous space on which the unitary group $G = \mathrm{U}(\mathcal{H})$ acts transitively and which is isomorphic to G/H for $H := \mathrm{U}(\mathcal{H})_\mathcal{K} \cong \mathrm{U}(\mathcal{K}) \oplus \mathrm{U}(\mathcal{K}^\perp)$.

Writing elements of $B(\mathcal{H})$ as (2×2)-matrices according to the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$, we have

$$\mathfrak{q} = \Big\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in B(\mathcal{K}), b \in B(\mathcal{K}^\perp, \mathcal{K}), d \in B(\mathcal{K}^\perp) \Big\},$$

and $\mathfrak{gl}(\mathcal{H}) = \mathfrak{q} \oplus \mathfrak{p}^-$ holds for $\mathfrak{p}^- = \Big\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} : c \in B(\mathcal{K}, \mathcal{K}^\perp) \Big\}$.

The representation of $\mathrm{U}(\mathcal{H})$ on $\mathcal{H}$ is bounded with $V := \mathcal{H}^{\mathfrak{p}^-} = \mathcal{K}^\perp$, and the representation of $H \cong \mathrm{U}(\mathcal{K}) \oplus \mathrm{U}(\mathcal{K}^\perp)$ on this space is bounded. In view of Theorem 2.17, the canonical extension $\beta \colon \mathfrak{q} \to \mathfrak{gl}(V), \beta(x) := (x^*|_V)^*$ now leads to a holomorphic vector bundle $\mathbb{V} := \mathrm{GL}(\mathcal{H}) \times_Q V \cong G \times_H V$ and a G-equivariant realization $\mathcal{H} \hookrightarrow \Gamma(\mathbb{V})$.

In this sense every Hilbert space can be realized as a space of holomorphic sections of a holomorphic vector bundle over any Graßmannian associated to $\mathcal{H}$. Note that $\mathrm{U}(\mathcal{H})_\mathcal{K} = \mathrm{U}(\mathcal{H})_V$ shows that $\mathrm{Gr}_\mathcal{K}(\mathcal{H}) \cong G/H$ can be identified in a natural way with $\mathrm{Gr}_V(\mathcal{H})$.

Remark 2.21. Let $(\pi, \mathcal{H})$ be a smooth unitary representation of the Lie group G and $V \subseteq \mathcal{H}$ be a closed H-invariant subspace. We then obtain a natural G-equivariant map $\eta \colon G/H \to \mathrm{Gr}_V(\mathcal{H}), gH \mapsto \pi(g)V$. If this map is holomorphic, then we can pull back the natural bundle $\mathbb{V} \to \mathrm{Gr}_V(\mathcal{H})$ from Example 2.20 and obtain a realization of $\mathcal{H}$ in $\Gamma(\eta^*\mathbb{V})$. This works very well if the representation $(\pi, \mathcal{H})$ is bounded because in this case $\pi \colon G \to \mathrm{U}(\mathcal{H})$ is a morphism of Banach–Lie groups, but if π is unbounded, then it seems difficult to verify that η is smooth, resp., holomorphic.

If (A1/2) in Theorem 2.17 are satisfied, then $V \subseteq \mathcal{H}^\omega \subseteq \mathcal{H}^\infty$ implies that, the operators $\mathtt{d}\pi(x)$, $x \in \mathfrak{g}$, are defined on V. Since they are closable, the graph of these restrictions is closed, which implies that the restrictions $\mathtt{d}\pi(x)|_V \colon V \to \mathcal{H}$ are continuous linear operators. We thus obtain a natural candidate for a tangent map

$$T_H(\eta) \colon T_H(G/H) \cong \mathfrak{g}/\mathfrak{h} \to T_V(\mathrm{Gr}_V(\mathcal{H})) \cong B(V, V^\perp), \quad x \mapsto (\mathbf{1} - p_V)\mathtt{d}\pi(x)p_V.$$

For the special case where $\dim V = 1$, we have $V = \mathbb{C}v_0$ for a smooth vector v_0, and since the projective orbit map $G \to \mathbb{P}(\mathcal{H}) \cong \mathrm{Gr}_V(\mathcal{H}), g \mapsto [\pi(g)v_0]$ is smooth, the induced map $\eta \colon G/H \to \mathbb{P}(\mathcal{H})$ is smooth as well. This construction is the key idea behind the theory of coherent state representations ([Od88], [Od92], [Li91], [Ne00], [Ne01]), where one uses a holomorphic map $\eta \colon G/H \to \mathbb{P}(\mathcal{H}^*)$ of a complex homogeneous space G/H to realize a unitary representation $(\pi, \mathcal{H})$ of G in the space of holomorphic sections of the line bundle $\eta^*\mathbb{L}$, where $\mathbb{L} \to \mathbb{P}(\mathcal{H}^*)$ is the canonical bundle on the dual projective space with $\Gamma(\mathbb{L}) \cong \mathcal{H}$.

3 Realizing positive energy representations

In this section we fix an element $d \in \mathfrak{g} = \mathbf{L}(G)$ for which the one-parameter group $e^{\mathbb{R}\,\mathrm{ad}\,d} \subseteq \mathrm{Aut}(\mathfrak{g})$ is bounded, i.e., preserves an equivalent norm. We call such elements *elliptic*. Then $H := Z_G(d)$ is a Lie subgroup and if 0 is isolated in $\mathrm{Spec}(D)$, then G/H carries a natural complex structure. The class of representations which one may expect to be realized by holomorphic sections of Hilbert bundles $\mathbb{V}$ over G/H is the class of *positive energy representations*, which is defined by the condition that the selfadjoint operator $-i\mathtt{d}\pi(d)$ is bounded below.

3.1 The splitting condition

Let $d \in \mathfrak{g}$ be an elliptic element. Then

$$H = Z_G(\exp \mathbb{R}d) = Z_G(d) = \{g \in G\colon\ \mathrm{Ad}(g)d = d\}$$

is a closed subgroup of G, not necessarily connected, with Lie algebra $\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(d) = \ker(\mathrm{ad}\,d)$. Since $\mathfrak{g}$ contains arbitrarily small $e^{\mathbb{R}\,\mathrm{ad}\,d}$-invariant 0-neighborhoods U, there exists such an open 0-neighborhood with $\exp_G(U) \cap H = \exp_G(U \cap \mathbf{L}(H))$. Therefore H is a Lie subgroup of G, i.e., a Banach–Lie group for which the inclusion $H \hookrightarrow G$ is a topological embedding.

Our assumption implies that $\alpha_t := e^{t\,\mathrm{ad}\,d}$ defines an equicontinuous one-paramter group of automorphisms of the complex Banach–Lie algebra $\mathfrak{g}_{\mathbb{C}}$. For $\delta > 0$, we consider the Arveson spectral subspace $\mathfrak{p}^+ := \mathfrak{g}_{\mathbb{C}}([\delta, \infty[)$ (cf. Definition A.5). Applying Proposition A.14 to the Lie bracket $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$, we see that $\mathfrak{p}^+$ is a closed complex subalgebra. For $f \in L^1(\mathbb{R})$, $\alpha(f) := \int_{\mathbb{R}} f(t)\alpha_t\,dt$ and $x \in \mathfrak{g}_{\mathbb{C}}$, the relations $\overline{\alpha(f)x} = \alpha(\overline{f})\overline{x}$ and $\widehat{\overline{f}}(\xi) = \overline{\widehat{f}(-\xi)}$ imply that $\mathfrak{p}^- := \overline{\mathfrak{p}^+} = \mathfrak{g}_{\mathbb{C}}(]-\infty, -\delta])$. To make the following constructions work, we assume the *splitting condition:*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{p}^-. \tag{SC}$$

In view of Lemma A.16, it is satisfied for some $\delta > 0$ if and only if 0 is isolated in $\mathrm{Spec}(\mathrm{ad}\,d)$.

Since $\mathrm{Ad}(H)$ commutes with $e^{\mathbb{R}\,\mathrm{ad}\,d}$, the closed subalgebras $\mathfrak{p}^{\pm} \subseteq \mathfrak{g}_{\mathbb{C}}$ are invariant under $\mathrm{Ad}(H)$ and $e^{\mathbb{R}\,\mathrm{ad}\,d}$. Now $\mathfrak{g} \cap (\mathfrak{p}^+ \oplus \mathfrak{p}^-)$ is a closed complement for $\mathfrak{h}$ in $\mathfrak{g}$, so that $M := G/H$ carries the structure of a Banach homogeneous space and $\mathfrak{q} := \mathfrak{h}_{\mathbb{C}} + \mathfrak{p}^+ \cong \mathfrak{p}^+ \rtimes \mathfrak{h}_{\mathbb{C}}$ defines a G-invariant complex manifold structure on M (cf. Section 1).

Remark 3.1. (a) For bounded derivations of compact L^*-algebras similar splitting conditions have been used by Beltiţă in [Bel03] to obtain Kähler polarizations of coadjoint orbits. In [Bel04] this is extended to bounded normal derivations of a complex Banach Lie algebra.

(b) If $\mathfrak{g}$ is a real Hilbert–Lie algebra, then one can use spectral measures to obtain natural complex structures on G/H even if the splitting condition is not satisfied, i.e., 0 need not be isolated in the spectrum of $\mathrm{ad}\,d$ ([BRT07, Prop. 5.4]).

Example 3.2. If α factors through an action of the circle group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, then the Peter–Weyl Theorem implies that the sum $\sum_{n\in\mathbb{Z}} \mathfrak{g}_{\mathbb{C}}^n$ of the corresponding eigenspaces $\mathfrak{g}_{\mathbb{C}}^n := \ker(\operatorname{ad} d - in\mathbf{1})$ is dense in $\mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{h}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^0$. Since the operator $\operatorname{ad} d$ is bounded, only finitely many $\mathfrak{g}_{\mathbb{C}}^n$ are non-zero, so that we actually have $\mathfrak{g}_{\mathbb{C}} = \sum_{n\in\mathbb{Z}} \mathfrak{g}_{\mathbb{C}}^n$. From $[\mathfrak{g}_{\mathbb{C}}^n, \mathfrak{g}_{\mathbb{C}}^m] \subseteq \mathfrak{g}_{\mathbb{C}}^{n+m}$ it follows that $\mathfrak{p}^\pm := \sum_{\pm n>0} \mathfrak{g}_{\mathbb{C}}^n$ are closed subalgebras for which $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{p}^-$ is direct. In this case the splitting condition is always satisfied and $\operatorname{Spec}(D) \subseteq i\mathbb{Z}$.

If, conversely, $d \in \mathfrak{g}$ is an element for which the complex linear extension of $\operatorname{ad} d$ to $\mathfrak{g}_{\mathbb{C}}$ is diagonalizable with finitely many eigenvalues in $i\mathbb{Z}$, then $e^{\mathbb{R} \operatorname{ad} d} \subseteq \operatorname{Aut}(\mathfrak{g}_{\mathbb{C}})$ is compact, hence preserves a compatible norm. An important special situation, where we have all this structure are hermitian Lie groups (cf. [Ne11]). In this case we simply have $\mathfrak{p}^\pm = \mathfrak{g}_{\mathbb{C}}^{\pm 1}$.

3.2 Positive energy representations

Lemma 3.3. *Let $\gamma\colon \mathbb{R} \to \mathrm{U}(\mathcal{H})$ be a strongly continuous unitary representation and $A = A^* = -i\gamma'(0)$ be its selfadjoint generator, so that $\gamma(t) = e^{itA}$ in terms of measurable functional calculus. Then the following assertions hold:*

(i) *For each $f \in L^1(\mathbb{R})$, we have $\gamma(f) = \widehat{f}(A)$, where $\widehat{f}(x) := \int_{\mathbb{R}} e^{ixy} f(y)\, dy$ is the Fourier transform of f.*

(ii) *Let $P\colon \mathfrak{B}(\mathbb{R}) \to B(\mathcal{H})$ be the unique spectral measure with $A = P(\mathrm{id}_{\mathbb{R}})$. Then, for every closed subset $E \subseteq \mathbb{R}$, the range $P(E)\mathcal{H}$ coincides with the Arveson spectral subspace $\mathcal{H}(E)$.*

Proof. Since the unitary representation $(\gamma, \mathcal{H})$ is a direct sum of cyclic representation, it suffices to prove the assertions for cyclic representations. Every cyclic representation of $\mathbb{R}$ is equivalent to the representation on some space $\mathcal{H} = L^2(\mathbb{R}, \mu)$, where μ is a Borel probability measure on $\mathbb{R}$ and $(\gamma(t)\xi)(x) = e^{itx}\xi(x)$ (see [Ne00, Thm. VI.1.11]).

(i) We have $(A\xi)(x) = x\xi(x)$, so that $(\widehat{f}(A)\xi)(x) = \widehat{f}(x)\xi(x)$. On the other hand, we have for $f \in L^1(\mathbb{R})$ in the space $\mathcal{H} = L^2(\mathbb{R}, \mu)$ the relation

$$(\gamma(f)\xi)(x) = \int_{\mathbb{R}} f(t) e^{itx} \xi(x)\, dt = \widehat{f}(x)\xi(x).$$

(ii) see [Ar74, p. 226]. □

Proposition 3.4. *Let $(\pi, \mathcal{H})$ be a smooth unitary representation of the Banach–Lie group G, $d \in \mathfrak{g}$ be elliptic, and $P\colon \mathfrak{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be the spectral measure of the unitary one-parameter group $\pi_d(t) := \pi(\exp_G td)$. Then the following assertions hold:*

(i) *$\mathcal{H}^\infty$ carries a Fréchet structure for which $\pi_d(t)_{t\in\mathbb{R}}$ defines a continuous equicontinuous action of $\mathbb{R}$ on $\mathcal{H}^\infty$. In particular, $\mathcal{H}^\infty$ is invariant under $\pi_d(f)$ for every $f \in L^1(\mathbb{R})$.*

(ii) *For every closed subset $E \subseteq \mathbb{R}$, we have $\mathcal{H}^\infty(E) = (P(E)\mathcal{H}) \cap \mathcal{H}^\infty$ for the corresponding spectral subspace.*

(iii) *For every open subset* $E \subseteq \mathbb{R}$, $(P(E)\mathcal{H}) \cap \mathcal{H}^\infty$ *is dense in* $P(E)\mathcal{H}^\infty$. *More precisely, there exists a sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $L^1(\mathbb{R})$ *for which* $\pi_d(f_n) \to P(E)$ *in the strong operator topology and* $\mathrm{supp}(\widehat{f_n}) \subseteq E$, *so that* $\pi_d(f_n)v \in \mathcal{H}^\infty \cap P(E)\mathcal{H}^\infty$ *for every* $v \in \mathcal{H}^\infty$.

(iv) *For closed subsets* $E, F \subseteq \mathbb{R}$, *the Arveson spectral subspaces* $\mathfrak{g}_\mathbb{C}(F)$ *satisfies*

$$\mathtt{d}\pi(\mathfrak{g}_\mathbb{C}(F))\big(\mathcal{H}^\infty \cap P(E)\mathcal{H}\big) \subseteq P(\overline{E+F})\mathcal{H}. \tag{12}$$

Proof. (i) We may w.l.o.g. assume that the norm on $\mathfrak{g}$ is invariant under $e^{\mathbb{R}\,\mathrm{ad}\,d}$. On $\mathcal{H}^\infty$ we consider the Fréchet topology defined by the seminorms

$$p_n(v) := \sup\{\|\mathtt{d}\pi(x_1)\cdots\mathtt{d}\pi(x_n)v\| \colon x_i \in \mathfrak{g}, \|x_i\| \leq 1\}$$

with respect to which the action of G on $\mathcal{H}^\infty$ is smooth (cf. [Ne10a, Thm. 4.4]). In particular, the bilinear map

$$\mathfrak{g}_\mathbb{C} \times \mathcal{H}^\infty \to \mathcal{H}^\infty, \quad (x, v) \mapsto \mathtt{d}\pi(x)v \tag{13}$$

is continuous because it can be obtained as a restriction of the tangent map of the G-action.

In view of the relation $\pi_d(t)\mathtt{d}\pi(x)\pi_d(t)^{-1} = \mathtt{d}\pi(e^{t\,\mathrm{ad}\,d}x)$ for $t \in \mathbb{R}, x \in \mathfrak{g}$, the isometry of $e^{t\,\mathrm{ad}\,d}$ on $\mathfrak{g}$ implies that the seminorms p_n on $\mathcal{H}^\infty$ are invariant under $\pi_d(\mathbb{R})$. Since the $\mathbb{R}$-action on $\mathcal{H}^\infty$ defined by the operators $\pi_d(t)$ is smooth, hence in particular continuous, we obtain with Definition A.5 an algebra homomorphism

$$\pi_d \colon (L^1(\mathbb{R}), *) \to \mathrm{End}(\mathcal{H}^\infty), \quad f \mapsto \int_\mathbb{R} f(t)\pi_d(t)\,dt$$

(cf. Definition A.5 below), and this implies (i).

(ii) Since $\mathcal{H}^\infty(E) = \mathcal{H}(E) \cap \mathcal{H}^\infty$ follows immediately from the definition of spectral subspaces (Remark A.6), this assertion is a consequence of Lemma 3.3(ii).

(iii) We write the open set E as an increasing union of compact subsets E_n and observe that $\bigcup_n P(E_n)\mathcal{H}$ is dense in $P(E)\mathcal{H}$. For every n, there exists a compactly supported function $h_n \in C_c^\infty(\mathbb{R},\mathbb{R})$ such that

$$\mathrm{supp}(h_n) \subseteq E, \quad 0 \leq h_n \leq 1, \quad \text{and} \quad h_n\big|_{E_n} = 1.$$

Let $f_n \in \mathcal{S}(\mathbb{R})$ with $\widehat{f_n} = h_n$. Then $\pi_d(f_n) = \widehat{f_n}(-i\gamma'(0)) = h_n(-i\gamma'(0))$ (Lemma 3.3(i)) and consequently

$$P(E_n)\mathcal{H} \subseteq \pi_d(f_n)\mathcal{H} \subseteq P(E)\mathcal{H}.$$

Therefore the subspace $\pi_d(f_n)\mathcal{H}^\infty$ of $\mathcal{H}^\infty$ is contained in $P(E)\mathcal{H}$. If $w = P(E)v$ for some $v \in \mathcal{H}^\infty$ then

$$\pi_d(f_n)w = \pi_d(f_n)P(E)v = \pi_d(f_n)v \in \mathcal{H}^\infty$$

and

$$\|\pi_d(f_n)w - w\|^2 = \|h_n(-i\pi_d'(0))w - w\|^2 \leq \|P(E\backslash E_n)w\|^2 \to 0$$

from which it follows that $\pi_d(f_n)w \to w$.

(iv) This follows from the continuity of (13), Proposition A.14 and (ii). ☐

Results of a similar type as Proposition 3.4(iv) and the more universal Proposition A.14 in the appendix are well known in the context of bounded operators (cf. [FV70], [Ra85], [Bel04, Prop. 1.1, Cor. 1.2]). Arveson also obtains variants for automorphism groups of operator algebras ([Ar74, Thm. 2.3]).

Remark 3.5. Combining Lemma 3.3(i) with Proposition 3.4(i), we derive that the subspace $\mathcal{H}^\infty$ of $\mathcal{H}$ is invariant under all operators $P(\widehat{f}) = \widehat{f}(-i\mathtt{d}\pi(d))$ for $f \in L^1(\mathbb{R})$. This implies in particular to the operators $P(h)$, $h \in \mathcal{S}(\mathbb{R})$, but not to the spectral projections $P(E)$. If $E \subseteq \mathbb{R}$ is open, Proposition 3.4(iii) provides a suitable approximate invariance.

The following proposition is of key importance for the following. It contains the main consequences of Arveson's spectral theory for the actions on $\mathfrak{g}_\mathbb{C}$ and $\mathcal{H}^\infty$.

Proposition 3.6. *If $d \in \mathfrak{g}$ is elliptic with 0 isolated in $\mathrm{Spec}(\mathrm{ad}\, d)$, then for any smooth positive energy representation $(\pi, \mathcal{H})$ of G, the H-invariant subspace $V := \overline{(\mathcal{H}^\infty)^{\mathfrak{p}^-}}$ satisfies $\mathcal{H} = \overline{\mathrm{span}(\pi(G)V)}$.*

Proof. First we show that $V \neq \{0\}$ whenever $\mathcal{H} \neq \{0\}$. Let

$$s := \inf(\mathrm{Spec}(-i\mathtt{d}\pi(d))) > -\infty.$$

For some $\varepsilon \in]0, \delta[$, we consider the closed subspace

$$W := P([s, s+\varepsilon[)\mathcal{H} = P(]s-\varepsilon, s+\varepsilon[)\mathcal{H}, \tag{14}$$

where $P \colon \mathfrak{B}(\mathbb{R}) \to B(\mathcal{H})$ is the spectral measure of π_d. Then Proposition 3.4 implies that $W^\infty := W \cap \mathcal{H}^\infty$ is dense in W and that

$$\mathtt{d}\pi(\mathfrak{p}^-)W^\infty \subseteq P(]-\infty, s+\varepsilon-\delta])\mathcal{H} = \{0\},$$

which leads to $\{0\} \neq W \subseteq V$.

Applying the preceding argument to the positive energy representation on the orthogonal complement of $\mathcal{H}_V := \overline{\mathrm{span}\, \pi(G)V}$, the relation $V \cap \mathcal{H}_V^\perp = \{0\}$ implies that $\mathcal{H}_V^\perp = \{0\}$, and hence that $\mathcal{H} = \mathcal{H}_V$. ☐

Theorem 3.7. *If $d \in \mathfrak{g}$ is elliptic with 0 isolated in $\mathrm{Spec}(\mathrm{ad}\, d)$ and $(\pi, \mathcal{H})$ is a smooth positive energy representation for which the H-representation $\rho(h) := \pi(h)|_V$ on $V := \overline{(\mathcal{H}^\infty)^{\mathfrak{p}^-}}$ is bounded, then $(\pi, \mathcal{H})$ is holomorphically induced from the representation (ρ, β) of $(H, \mathfrak{q})$ on V defined by $\beta(\mathfrak{p}^+) = \{0\}$. In particular, V consists of analytic vectors.*

Proof. Since $\pi(G)V$ spans a dense subspace of $\mathcal{H}$, i.e., $\mathcal{H} = \mathcal{H}_V$ (Proposition 3.6), the assertion follows from Remark 2.18. ☐

Remark 3.8. Since $\mathcal{H}_V \cong \mathcal{H}_{V'}$ as G-representations if and only if $V \cong V'$ as H-representations (cf. Corollary 2.16), the description of all G-representations of positive energy for which the H-representation (ρ, V) is bounded is equivalent to the determination of all bounded H-representations (ρ, V) for which (ρ, β, V) is inducible if we put $\beta(\mathfrak{p}^+) = \{0\}$.

Corollary 3.9. *If $d \in \mathfrak{g}$ is elliptic with 0 isolated in $\mathrm{Spec}(\mathrm{ad}\, d)$, then every bounded representation of G is holomorphically induced from the representation (ρ, β) of $(H, \mathfrak{q})$ on $V := \overline{(\mathcal{H}^\infty)^{\mathfrak{p}^-}}$ defined by $\beta(\mathfrak{p}^+) = \{0\}$.*

From Corollary 2.14 we obtain in particular:

Corollary 3.10. *A positive energy representation $(\pi, \mathcal{H})$ of G for which the representation (ρ, V) of H is bounded is a direct sum of irreducible ones if and only if (ρ, V) has this property.*

Example 3.11. The complex Banach–Lie algebra $\mathfrak{g}$ is called *weakly root graded* if there exists a finite reduced root system Δ such that $\mathfrak{g}$ contains the corresponding finite dimensional semisimple Lie algebra $\mathfrak{g}_\Delta$ and for some Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}_\Delta$, the Lie algebra $\mathfrak{g}$ is a direct sum of finitely many $\mathrm{ad}\, \mathfrak{h}$-eigenspaces.

Now suppose that $\mathfrak{g}$ is a real Banach–Lie algebra for which $\mathfrak{g}_\mathbb{C}$ is weakly root graded, that $\mathfrak{h}$ is invariant under conjugation and, for every $x \in \mathfrak{h} \cap i\mathfrak{g}$, the derivation $\mathrm{ad}\, x$ has real spectrum. Then the realization results for bounded unitary representations of G which follows from [MNS09, Thm. 5.1], applied to their holomorphic extensions $G_\mathbb{C} \to \mathrm{GL}(\mathcal{H})$, can be derived easily from Corollary 3.9.

The following theorem shows that, assuming that $(\pi, \mathcal{H})$ is semibounded with $d \in W_\pi$ permits us to get rid of the quite implicit assumption that the H-representation on V is bounded. It is an important generalization of Corollary 3.9 to semibounded representations.

Theorem 3.12. *Let $(\pi, \mathcal{H})$ be a semibounded unitary representation of the Banach–Lie group G and $d \in W_\pi$ be an elliptic element for which 0 is isolated in $\mathrm{Spec}(\mathrm{ad}\, d)$. We write $P \colon \mathfrak{B}(\mathbb{R}) \to B(\mathcal{H})$ for the spectral measure of the unitary one-parameter group $\pi_d(t) := \pi(\exp(td)$. Then the following assertions hold:*

(i) *The representation $\pi|_H$ of H is semibounded and, for each bounded measurable subset $B \subseteq \mathbb{R}$, the H-representation on $P(B)\mathcal{H}$ is bounded.*

(ii) *The representation $(\pi, \mathcal{H})$ is a direct sum of representations $(\pi_j, \mathcal{H}_j)$ for which there exist H-invariant subspaces $\mathcal{D}_j \subseteq (\mathcal{H}_j^\infty)^{\mathfrak{p}^-}$ for which the H-representation ρ_j on $V_j := \overline{\mathcal{D}_j}$ is bounded and $\mathrm{span}\big(\pi_j(G)V_j\big)$ is dense in $\mathcal{H}_j$. Then the representations $(\pi_j, \mathcal{H}_j)$ are holomorphically induced from (ρ_j, β_j, V), where $\beta_j(\mathfrak{p}^+) = \{0\}$.*

(iii) *If $(\pi, \mathcal{H})$ is irreducible and $s := \inf \mathrm{Spec}(-i\mathtt{d}\pi(d))$, then $P(\{s\})\mathcal{H} = \overline{(\mathcal{H}^\infty)^{\mathfrak{p}^-}}$ and $(\pi, \mathcal{H})$ is holomorphically induced from the bounded H-representation ρ on this space, extended by $\beta(\mathfrak{p}^+) = \{0\}$.*

Proof. (i) From $s_{\pi|_H} = s_\pi|_\mathfrak{h}$ it follows that $\pi|_H$ is semibounded with $d \in W_\pi \cap \mathfrak{h} \subseteq W_{\pi|_H}$. For every bounded measurable subset $B \subseteq \mathbb{R}$, the relation $d \in \mathfrak{z}(\mathfrak{h})$ entails that $P(B)\mathcal{H}$ is H-invariant. Let ρ_B denote the corresponding representation of H. Then the boundedness of $\mathtt{d}\rho_B(d)$ which commutes with $\mathtt{d}\rho(\mathfrak{h})$ implies that $W_{\rho_B} + \mathbb{R}d = W_{\rho_B}$, so that $d \in W_{\pi|_H} \subseteq W_{\rho_B}$ leads to $0 \in W_{\rho_B}$. This means that ρ_B is bounded.

(ii) We apply Zorn's Lemma to the ordered set of all pairwise orthogonal systems of closed G-invariant subspaces satisfying the required conditions. Therefore it suffices to show that if $\mathcal{H} \neq \{0\}$, then there exists a non-zero H-invariant subspaces $\mathcal{D} \subseteq (\mathcal{H}^\infty)^{\mathfrak{p}^-}$ for which the H-representation ρ on $\overline{\mathcal{D}}$ is bounded.

For $s := \inf \mathrm{Spec}(-i\mathtt{d}\pi(d))$ and $0 < \varepsilon < \delta$, we may take the space $\mathcal{D} := \mathcal{H}^\infty \cap P([s, s+\varepsilon[)\mathcal{H}$ from the proof of Proposition 3.6. As we have seen there, it is annihilated by $\mathtt{d}\pi(\mathfrak{p}^-)$ and the boundedness of the H-representation on its closure follows from (i).

That the representations $(\pi_j, \mathcal{H}_j)$ are holomorphically induced from the bounded H-representations on V_j follows from Theorem 2.17.

(iii) For $t > s$, let

$$\mathcal{D}_U := (\mathcal{H}^\infty)^{\mathfrak{p}^-} \cap P([s,t[)(\mathcal{H}^\infty)^{\mathfrak{p}^-} = (\mathcal{H}^\infty)^{\mathfrak{p}^-} \cap P(]s-\delta, t[)(\mathcal{H}^\infty)^{\mathfrak{p}^-}.$$

Claim 1: $\mathcal{D}_U$ is dense in $U := \overline{P([s,t[)(\mathcal{H}^\infty)^{\mathfrak{p}^-}}$.

Let $v \in (\mathcal{H}^\infty)^{\mathfrak{p}^-}$ and $w := P(]s-\delta, t[)v$. With Proposition 3.4(iii), we find a sequence $f_n \in L^1(\mathbb{R})$ for which $\pi_d(f_n)$ converges strongly to $P(]s-\delta, t[)$ and $\mathrm{supp}(\widehat{f_n}) \subseteq]s-\delta, t[$, so that $\pi_d(f_n)v \to w$ and

$$\pi_d(f_n)v = P(\widehat{f_n})v = P(]s-\delta, t[)P(\widehat{f_n})v \in P(]s-\delta, t[)\mathcal{H}^\infty.$$

Since $\mathfrak{p}^-$ is invariant under $e^{\mathrm{ad}\, d}$, the closed subspace $(\mathcal{H}^\infty)^{\mathfrak{p}^-}$ is invariant under $\pi_d(\mathbb{R})$, so that $\pi_d(f_n)v \in (\mathcal{H}^\infty)^{\mathfrak{p}^-}$. This proves Claim 1.

Claim 2: $(\pi, \mathcal{H})$ is holomorphically induced from the bounded H-representation (ρ, β, U), defined by $\beta(\mathfrak{p}^+) := \{0\}$.

From (i) we know that the H-representation on U is bounded and on the dense subspace $\mathcal{D}_U$ we have $\mathtt{d}\pi(\mathfrak{p}^-)\mathcal{D}_U = \{0\}$. Therefore (A1/2) in Theorem 2.17 are satisfied and this proves Claim 2.

Claim 3: $U = P(\{s\})\mathcal{H}$ for every $t > s$.

In view of Claim 2, Corollary 2.14 implies that the H-representation on U is irreducible. Since π_d commutes with H, it follows in particular that $\rho(\exp \mathbb{R}d) \subseteq \mathbb{T}\mathbf{1}$. The definition of s now shows that $U \subseteq P(\{s\})\mathcal{H}$.

For $0 < \varepsilon < \delta$ and $t < \delta + \varepsilon$, the proof of Proposition 3.6 implies that $\big(P([s,t[)\mathcal{H}\big) \cap \mathcal{H}^\infty$ is dense in $P([s,t[)\mathcal{H}$ and contained in $(\mathcal{H}^\infty)^{\mathfrak{p}^-}$, hence in $P([s,t[)(\mathcal{H}^\infty)^{\mathfrak{p}^-} \subseteq U$. We conclude in particular that $P(\{s\})\mathcal{H} \subseteq U$.

Claim 4: $U = \overline{(\mathcal{H}^\infty)^{\mathfrak{p}^-}}$.

From the definition of U it is clear that $U \subseteq \overline{(\mathcal{H}^\infty)^{\mathfrak{p}^-}}$. To see that we actually have equality, we note that Claim 2 shows that $P([s,t[)(\mathcal{H}^\infty)^{\mathfrak{p}^-} \subseteq P(\{s\})\mathcal{H} = U$ holds for every $t > s$. As $P([s,n]) \to P([s,\infty[) = \mathrm{id}$ holds pointwise, we obtain $(\mathcal{H}^\infty)^{\mathfrak{p}^-} \subseteq U$.

This completes the proof of (iii). □

Remark 3.13. For finite dimensional Lie groups the classification of irreducible semi-bounded unitary representations easily boils down to a situation where one can apply Theorem 3.12. Here $d \in \mathfrak{g}$ is a regular element whose centralizer $\mathfrak{h} = \mathfrak{t}$ is a compactly embedded Cartan subalgebra and the corresponding group $T = H$ is abelian and V

is one-dimensional (cf. [Ne00]). In this case 0 is trivially isolated in the finite set $\mathrm{Spec}(\mathrm{ad}\, d)$.

The following theorem provides a bridge between the seemingly weak positive energy condition and the much stronger semiboundedness condition.

Theorem 3.14. *Let $d \in \mathfrak{g}$ be elliptic with 0 isolated in $\mathrm{Spec}(\mathrm{ad}\, d)$. Then a smooth unitary representation $(\pi, \mathcal{H})$ of G for which the representation ρ of H on $\overline{(\mathcal{H}^\infty)^{\mathfrak{p}^-}}$ is bounded satisfies the positive energy condition*

$$\inf \mathrm{Spec}(-i\mathtt{d}\pi(d)) > -\infty \qquad (15)$$

if and only if π is semibounded with $d \in W_\pi$.

Proof. If π semibounded with $d \in W_\pi$, then we have in particular (15). It remains to show the converse if all the assumptions of the theorem are satisfied. Recall that the splitting condition (SC) is satisfied because 0 is isolated in $\mathrm{Spec}(\mathrm{ad}\, d)$. We note that the representation $\mathrm{ad}_{\mathfrak{p}^+}$ of $\mathfrak{h}$ on $\mathfrak{p}^+$ is bounded with $\mathrm{Spec}(\mathrm{ad}_{\mathfrak{p}^+}(-id)) \subseteq]0, \infty[$. Therefore the invariant cone

$$C := \{x \in \mathfrak{h} \colon \mathrm{Spec}(\mathrm{ad}_{\mathfrak{p}^+}(-ix)) \subseteq]0, \infty[\} \qquad (16)$$

is non-empty and open because it is the inverse image of the open convex cone

$$\{X \in \mathrm{Herm}(\mathfrak{p}^+) \colon \mathrm{Spec}(X) \subseteq]0, \infty[\}$$

(cf. [Up85, Thm. 14.31]) under the continuous linear map $\mathfrak{h} \to \mathfrak{gl}(\mathfrak{p}^+), x \mapsto \mathrm{ad}_{\mathfrak{p}^+}(-ix)$.

Let $x \in C$ and

$$s_\rho(x) := \sup(\mathrm{Spec}(i\mathtt{d}\rho(x)) = -\inf(\mathrm{Spec}(-i\mathtt{d}\rho(x)),$$

so that $V := \overline{(\mathcal{H}^\infty)^{\mathfrak{p}^-}} \subseteq \mathcal{H}^\infty$ (Theorem 3.7) is contained in the spectral subspace $\mathcal{H}^\infty([-s_\rho(x), \infty[)$ with respect to the one-parameter group $t \mapsto \pi(\exp tx)$. Since the map

$$\mathfrak{g}_{\mathbb{C}} \times \mathcal{H}^\infty \to \mathcal{H}^\infty, \quad (x, v) \mapsto \mathtt{d}\pi(x)v$$

is continuous bilinear and H-equivariant, we see with Proposition A.14 that the subspace $\mathcal{H}^\infty([-s_\rho(x), \infty[)$ of $\mathcal{H}^\infty$ is invariant under $\mathfrak{p}^+$.

For every $v \in V \subseteq \mathcal{H}^\omega$, the Poincaré–Birkhoff–Witt Theorem shows that it contains the subspace

$$U(\mathfrak{g}_{\mathbb{C}})v = U(\mathfrak{p}^+)U(\mathfrak{h}_{\mathbb{C}})U(\mathfrak{p}^-)v = U(\mathfrak{p}^+)U(\mathfrak{h}_{\mathbb{C}})v \subseteq U(\mathfrak{p}^+)V.$$

From $V \subseteq \mathcal{H}^\omega$ and $\mathcal{H} = \mathcal{H}_V$ it follows that $U(\mathfrak{g}_{\mathbb{C}})V$ is dense in $\mathcal{H}$, and hence that $\mathcal{H}^\infty([-s_\rho(x), \infty[) \subseteq \mathcal{H}([-s_\rho(x), \infty[)$ is dense in $\mathcal{H}$. We conclude that

$$s_\pi(x) = \sup(\mathrm{Spec}(i\mathtt{d}\pi(x))) = s_\rho(x) \quad \text{for} \quad x \in C. \qquad (17)$$

To see that $C \subseteq W_\pi$, it now suffices to show that $\mathrm{Ad}(G)C$ has interior points. Let $\mathfrak{p} := (\mathfrak{p}^+ + \mathfrak{p}^-) \cap \mathfrak{g}$ and note that this is a closed H-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$. The map $F \colon \mathfrak{h} \times \mathfrak{p} \to \mathfrak{g}, F(x, y) := e^{\mathrm{ad}\, y}x$ is smooth and

$$\mathtt{d}F(x, 0)(v, w) = [w, x] + v.$$

Since the operators $\operatorname{ad} x$, $x \in \mathfrak{h}$, preserve $\mathfrak{p}$, the operator $\mathtt{d}F(x,0)$ is invertible if and only if $\operatorname{ad} x \colon \mathfrak{p} \to \mathfrak{p}$ is invertible, and this is the case for any $x \in C$ because $\operatorname{ad} x|_{\mathfrak{p}^\pm}$ are invertible operators. This proves that C is contained in the interior of $F(C, \mathfrak{p})$, and hence that $C \subseteq W_\pi$. Therefore π is semibounded with $d \in W_\pi$. ☐

A Equicontinuous representations

In this appendix we first explain how a continuous representation $\pi \colon G \to \mathrm{GL}(V)$, i.e., a representation defining a continuous action of G on V, of the locally compact group G on the complete locally convex space V can be integrated to a representation of the group algebra $L^1(G)$, provided $\pi(G)$ is equicontinuous. If G is abelian, we use this to extend Arveson's concept of spectral subspaces to representations on complete locally convex spaces.

A.1 Equicontinuous groups

If (π, V) is a continuous representation of a compact group on a locally convex space, then $\pi(G)$ is an equicontinuous subgroup of $\mathrm{GL}(V)$ (cf. Lemma A.3) and if V is finite dimensional, then each equicontinuous subgroup of $\mathrm{GL}(V)$ has compact closure. Therefore, in the context of locally convex spaces, equicontinuous groups are natural analogs of compact groups.

Proposition A.1. *Let V be a locally convex space. For a subgroup $G \subseteq \mathrm{GL}(V)$, the following are equivalent:*

(1) *G is equicontinuous.*

(2) *For each 0-neighborhood $U \subseteq V$, the set $\bigcap_{g\in G} gU$ is a 0-neighborhood.*

(3) *There exists a basis of G-invariant absolutely convex 0-neighborhoods in V.*

(4) *The topology on V is defined by the set of G-invariant continuous seminorms.*

(5) *Each equicontinuous subset of the dual space V' is contained in a G-invariant equicontinuous subset.*

Proof. (1) ⇔ (2): The equicontinuity of G means that for each 0-neighborhood U there exists a 0-neighborhood W in V with $gW \subseteq U$ for each $g \in G$, i.e., $W \subseteq \bigcap_{g\in G} gU$.
(2) ⇔ (3) follows from the local convexity of V.
(3) ⇔ (4): For each G-invariant absolutely convex 0-neighborhood, the corresponding gauge functional is a G-invariant continuous seminorm and vice versa ([Ru73, Thm. 1.35]).
(3) ⇔ (5): First we note that a subset $K \subseteq V'$ is equicontinuous if and only if its polar

$$\widehat{K} := \{v \in V \colon \sup |\langle K, v\rangle| \leq 1\}$$

is a 0-neighborhood and, conversely, for each 0-neighborhood $U \subseteq V$, its polar set

$$\widehat{U} := \{\lambda \in V' \colon \sup |\lambda(U)| \leq 1\}$$

is equicontinuous. If (3) holds and $K \subseteq V'$ is equicontinuous, then $\widehat{K}$ is a 0-neighborhood and (3) implies the existence of a G-invariant absolutely convex 0-neighborhood U, contained in $\widehat{K}$. Then $\widehat{U} \supseteq \widehat{\widehat{K}} \supseteq K$ and the G-invariance of $\widehat{U}$ imply (5). If, conversely, (5) holds and $W \subseteq V$ is a 0-neighborhood, then $\widehat{W}$ is equicontinuous and (5) implies the existence of a G-invariant equicontinuous subset $K \supseteq \widehat{W}$. Then $\widehat{K} \subseteq W$ is an absolutely convex G-invariant 0-neighborhood in V. □

Remark A.2. (a) If V is a normed space, then a subgroup $G \subseteq \mathrm{GL}(V)$ is equicontinuous if and only if it is bounded. In fact, if $B_r(0)$ is the open ball or radius r, then the relation $GB_r(0) \subseteq B_s(0)$ is equivalent to $\|g\| \leq \frac{s}{r}$ for all $g \in G$. In view of the preceding proposition, this is equivalent to the existence of a G-invariant norm defining the topology.

(b) If $G \subseteq \mathrm{GL}(V)$ is equicontinuous, then each G-invariant continuous seminorm p on V defines a homomorphism $G \to \mathrm{Isom}(V_p)$, where V_p is the completion of the quotient space $V/p^{-1}(0)$ with respect to the norm induced by p. We thus obtain an embedding $G \hookrightarrow \prod_{i\in I} \mathrm{Isom}(V_{p_i})$, where $(p_i)_{i\in I}$ is a fundamental system of G-invariant continuous seminorms on V.

(c) If $G \subseteq \mathrm{GL}(V)$ is equicontinuous, then Proposition A.1(5) implies that each G-orbit in V' is equicontinuous.

Lemma A.3. *If $\alpha\colon G \to \mathrm{GL}(V)$ defines a continuous action of the compact group G on the locally convex space V, then $\alpha(G)$ is equicontinuous.*

Proof. We write $\sigma\colon G \times V \to V, (g, v) \mapsto \alpha(g)v$ for the action of G on V. Let $U \subseteq V$ be a convex balanced 0-neighborhood. Then $U_G := \bigcap_{g\in G} \alpha(g)(U)$ also is a zero neighborhood in V because $\sigma^{-1}(U) \subseteq G \times V$ is an open neighborhood of the compact subset $G \times \{0\}$. Hence there exists some open 0-neighborhood $W \subseteq V$ with $G \times W \subseteq \sigma^{-1}(U)$, and this means that $W \subseteq U_G$. Now we apply Proposition A.1. □

A.2 Arveson spectral theory on locally convex spaces

Definition A.4. A representation (π, V) of the group G on the locally convex space V is called *equicontinuous* if $\pi(G) \subseteq B(V)$ (the space of continuous linear operators on V) is an equicontinuous group of operators, which is equivalent to the existence of a family of G-invariant continuous seminorms defining the topology (cf. Proposition A.1).

Definition A.5. (a) Let (π, G) be an equicontinuous strongly continuous action of the locally compact group G on the complete complex locally convex space V.

We write $\mathcal{P}^G$ for the set of $\pi(G)$-invariant continuous seminorms on V. This set carries a natural order defined by $p \leq q$ if $p(v) \leq q(v)$ for every $v \in V$. Let V_p be the completion of $V/p^{-1}(0)$ with respect to p. For $p \leq q$ we then obtain a contractive map $\varphi_{pq}\colon V_q \to V_p$ of Banach spaces. Since V is complete, the natural map

$$V \to \varprojlim V_p := \Big\{(v_p) \in \prod_{p\in\mathcal{P}^G} V_p \colon (\forall p, q \in \mathcal{P}^G)\, p \leq q \Rightarrow \varphi_{pq}(v_q) = v_p\Big\} \subseteq \prod_{p\in\mathcal{P}^G} V_p$$

is a topological isomorphism, where the right hand side carries the product topology.

We now obtain on each V_p a continuous isometric representation (π_p, V_p) of G. Therefore each $f \in L^1(G)$ defines a bounded operator

$$\pi_p(f) := \int_G f(g)\pi_p(g)\, dg \in B(V_p)$$

where dg stands for a left Haar measure on G ([HR70, (40.26)]). Since these operators satisfy for $p \leq q$ the compatibility relation $\varphi_{pq} \circ \pi_q(f) = \pi_p(f)$, the product operator $(\pi_p(f))_{p\in\mathcal{P}^G}$ on $\prod_{p\in\mathcal{P}^G} V_p$ preserves the closed subspace $\varprojlim V_p \cong V$, so that we obtain a continuous linear operator $\pi(f) \in B(V)$. This leads to a representation

$$\pi\colon (L^1(G), *) \to B(V). \tag{18}$$

(b) If G is abelian and $\widehat{G} := \mathrm{Hom}(G, \mathbb{T})$ is its character group, then we have the Fourier transform

$$\mathcal{F}\colon L^1(G) \to C_0(\widehat{G}), \quad \mathcal{F}(f)(\chi) := \int_G \chi(g)f(g)\, dg.$$

We define the *spectrum of* (π, V) by

$$\mathrm{Spec}(V) := \mathrm{Spec}_\pi(V) := \{\chi \in \widehat{G}\colon (\forall f \in L^1(G))\, \pi(f) = 0 \Rightarrow \widehat{f}(\chi) = 0\},$$

i.e., the *hull* or *cospectrum* of the ideal $\ker \pi \trianglelefteq L^1(G)$. Accordingly, we define the *spectrum of an element* $v \in V$ by

$$\mathrm{Spec}(v) := \mathrm{Spec}_\pi(v) := \{\chi \in \widehat{G}\colon (\forall f \in L^1(G))\, \pi(f)v = 0 \Rightarrow \widehat{f}(\chi) = 0\},$$

which is the hull of the annihilator ideal of v. For a subset $E \subseteq \widehat{G}$, we now define the corresponding *Arveson spectral subspace*

$$V(E) := V(E, \pi) := \{v \in V\colon \mathrm{Spec}(v) \subseteq \overline{E}\}$$

(cf. [Ar74, Sect. 2], where this space is denoted $M^\pi(\overline{E})$; see also [Ta03, Ch. XI]).

Remark A.6. In [Ar74, p. 225] it is shown that

$$V(E) = \{v \in V\colon (\forall f \in L^1(G))\ \mathrm{supp}(\widehat{f}) \cap \overline{E} = \emptyset \Rightarrow \pi(f)v = 0\},$$

which implies in particular that $V(E)$ is a closed subspace which is clearly $\pi(G)$-invariant. Note that the condition $\mathrm{supp}(\widehat{f}) \cap \overline{E} = \emptyset$ means that $\widehat{f}$ vanishes on a neighborhood of $\overline{E}$.

Remark A.7. In [Jo82, Thm. 3.3] it is shown that for one-parameter groups of isometries of a Banach space X with infinitesimal generator A, the Arveson spectrum of an element $v \in X$ coincides with the complement of the maximal open subset $G \subseteq \mathbb{R}$ for which the map $z \mapsto (z - A)^{-1}v$ extends holomorphically to $(\mathbb{C} \setminus \mathbb{R}) \cup G$ (see also [Ta03, Ch. XI]).

Remark A.8. If $(E_j)_{j\in J}$ is a family of closed subsets of $\widehat{G}$, then

$$V(\bigcap_{j\in J} E_j) = \bigcap_{j\in J} V(E_j)$$

follows immediately from the definition.

Lemma A.9. ([Ar74, p. 226]) *For $\chi \in \widehat{G}$ we have*

$$V(\{\chi\}) = V_\chi(\pi) := \{v \in V \colon (\forall g \in G)\, \pi(g)v = \chi(g)v\}.$$

Definition A.10. Let $\mathfrak{g}$ be a complete locally convex Lie algebra and $x \in \mathfrak{g}$ be such that $\operatorname{ad} x$ generates a continuous equicontinuous one-parameter group $\pi\colon \mathbb{R} \to \operatorname{Aut}(\mathfrak{g})$ of automorphisms, i.e., π is strongly differentiable with $\pi'(0) = \operatorname{ad} x$. Then we obtain for each closed subset $E \subseteq \mathbb{R}$ a spectral subspace $\mathfrak{g}_{\mathbb{C}}(E)$.

Definition A.11. For a subset $E \subseteq \widehat{G}$, we also define

$$R(E) := R^\pi(E) := \overline{\operatorname{span}\{\pi(f)V \colon f \in L^1(G), \widehat{f} \in C_c(\widehat{G}), \operatorname{supp}(\widehat{f}) \subseteq E\}}.$$

Remark A.12. (a) Clearly $E_1 \subseteq E_2$ implies $R(E_1) \subseteq R(E_2)$ and Remark A.6 implies that $R(E) \subseteq V(E)$. Here we use that, for $f, h \in L^1(G)$, the conditions $\operatorname{supp}(\widehat{f}) \subseteq E$ and $\operatorname{supp}(\widehat{h}) \cap \overline{E} = \emptyset$ imply that $(h * f)\widehat{\ } = \widehat{h}\widehat{f} = 0$, and this leads to $h * f = 0$, which in turn shows that $\pi(h)\pi(f)v = \pi(hf)v = 0$.

As we shall see below, the following lemma is an important technical tool.

Lemma A.13. *For every closed subset $E \subseteq \widehat{G}$, the space $V(E)$ is the intersection of the subspaces $R(E + N)$, where N is an identity neighborhood in $\widehat{G}$.*

Proof. Since every identity neighborhood N in $\widehat{G}$ contains a compact one, it suffices to consider the spaces $R(E + N)$ for identity neighborhoods N which are compact. Hence the assertion follows from [Ar74, Prop. 2.2] if V is a Banach space.

We first show that $V(E) \subseteq R(E+N)$. We know already that this is the case if V is Banach. To verify it in the general case, let $v \in V(E)$ and U be a neighborhood of v. We have to show that U intersects $R(E+N)$. Let $p \in \mathcal{P}^G$ be a G-invariant seminorm for which the ε-ball $U^p_\varepsilon(v) = \{w \colon p(v - w) < \varepsilon\}$ is contained in U. Since the result holds for the G-representation (π_p, V_p) on the Banach space V_p and the image v_p of v in this space belongs to $V_p(E)$, the open ε-ball $U_\varepsilon(v_p)$ in V_p intersects $R(E + N, \pi_p)$, hence contains a finite sum $\sum_{j=1}^n \pi_p(f_j)w_j$ with $\operatorname{supp}(\widehat{f_j}) \subseteq E + N$ compact. Since the image of V in V_p is dense and $U_\varepsilon(v_p)$ is open, we may even assume that $w_j = v_{j,p}$ for some $v_j \in V$. Then

$$p\Big(v - \sum_{j=1}^n \pi(f_j)v_j\Big) = p\Big(v_p - \sum_{j=1}^n \pi_p(f_j)v_{j,p}\Big) < \varepsilon$$

implies that $\sum_{j=1}^n \pi(f_j)v_j \in U \cap R(E + N)$. We conclude that $v \in \overline{R(E + N)} = R(E + N)$.

We now arrive with Remarks A.8 and A.12 at

$$V(E) \subseteq \bigcap_N R(E+N) \subseteq \bigcap_N V(E+N) = V\Big(\bigcap_N (E+N)\Big) = V(E),$$

because all sets $E+N$ are closed. □

Proposition A.14. *Assume that (π_j, V_j), $j = 1,2,3$ are continuous equicontinuous representations of the locally compact abelian group G on the complete locally convex spaces V_j and that $\beta\colon V_1 \times V_2 \to V_3$ is a continuous equivariant bilinear map. Then we have for closed subsets $E_1, E_2 \subseteq \widehat{G}$ the relation*

$$\beta(V_1(E_1) \times V_2(E_2)) \subseteq V_3(E_1 + E_2).$$

Proof. Pick an identity neighborhood $U_0 \subseteq \widehat{G}$ and choose a symmetric $\mathbf{1}$-neighborhood U with $UU \subseteq U_0$. In view of Lemma A.13, the assertion follows, if we can show that

$$\beta(R^{\pi_1}(E_1 + U) \times R^{\pi_2}(E_2 + U)) \subseteq V_3(E_1 + E_2 + U_0).$$

To verify this relation, pick $v_1, v_2 \in V$ and $f_1, f_2 \in L^1(\mathbb{R})$ such that the functions $\widehat{f_j}$ are compactly supported with $\mathrm{supp}(\widehat{f_j}) \subseteq E_j + U$. We have to show that

$$\beta(\pi_1(f_1)v_1, \pi_2(f_2)v_2) \in V_3(E_1 + E_2 + U_0),$$

i.e., that for any $f_3 \in L^1(\mathbb{R})$ for which $\widehat{f_3}$ vanishes on a neighborhood of $\overline{E_1 + E_2 + U_0}$, we have

$$\pi_3(f_3)\beta(\pi_1(f_1)v_1, \pi_2(f_2)v_2) = 0.$$

This expression can easily be evaluated to

$$\begin{aligned}
&\pi_3(f_3)\beta(\pi_1(f_1)v_1, \pi_2(f_2)v_2)\\
&= \int_G \int_G \int_G f_3(g_3) f_1(g_1) f_2(g_2) \pi_3(g_3)\beta(\pi_1(g_1)v_1, \pi_2(g_2)v_2)\, dg_1 dg_2 dg_3\\
&= \int_G \int_G \int_G f_3(g_3) f_1(g_1) f_2(g_2) \beta(\pi_1(g_3 + g_1)v_1, \pi_2(g_3 + g_2)v_2)\, dg_1 dg_2 dg_3\\
&= \int_G \int_G \int_G f_3(g_3) f_1(g_1 - g_3) f_2(g_2 - g_3) \beta(\pi_1(g_1)v_1, \pi_2(g_2)v_2)\, dg_1 dg_2 dg_3\\
&= \int_G \int_G \int_G f_3(g_3) f_1(g_1 - g_3) f_2(g_2 - g_3) \beta(\pi_1(g_1)v_1, \pi_2(g_2)v_2)\, dg_3 dg_1 dg_2\\
&= \int_G \int_G (f_3 * F)(g_1, g_2) \beta(\pi_1(g_1)v_1, \pi_2(g_2)v_2)\, dg_3 dg_1 dg_2
\end{aligned}$$

for $F(g_1, g_2) := f_1(g_1) f_2(g_2)$ and

$$(f_3 * F)(g_1, g_2) = \int_G f_3(g) f_1(g_1 - g) f_2(g_2 - g)\, dg.$$

The Fourier transform of $f_3 * F \in L^1(G \times G)$ is given by

$$\begin{aligned}(f_3 * F)\widehat{\ }(\chi_1, \chi_2) &= \int_G \int_G \int_G f_3(g) f_1(g_1 - g) f_2(g_2 - g) \chi_1(g_1) \chi_2(g_2)\, dg\, dg_1\, dg_2 \\ &= \int_G f_3(g)\chi_1(g)\chi_2(g) \int_G \int_G f_1(g_1) f_2(g_2) \chi_1(g_1)\chi_2(g_2)\, dg_1\, dg_2\, dg \\ &= \widehat{f}_1(\chi_1)\widehat{f}_2(\chi_2)\widehat{f}_3(\chi_1 + \chi_2).\end{aligned}$$

By assumption, $\widehat{f}_3$ vanishes on

$$\mathrm{supp}(\widehat{f}_1) + \mathrm{supp}(\widehat{f}_2) \subseteq E_1 + E_2 + U + U \subseteq E_1 + E_2 + U_0.$$

Therefore $(f_3 * F)\widehat{\ } = 0$, and hence also $f_3 * F = 0$. This completes the proof. □

Remark A.15. Now we consider the case $G = \mathbb{R}$ and a strongly continuous representation (α, V) of $\mathbb{R}$ by isometries. We know already from Lemma A.9 that $V_0 := V(\{0\})$ is the set of fixed points. We would like to have a decomposition

$$V = V_+ \oplus V_0 \oplus V_-, \tag{19}$$

where $V_\pm$ are closed invariant subspaces satisfying

$$V_+ \subseteq V([\delta, \infty[) \quad \text{and} \quad V_- \subseteq V(]-\infty, -\delta])$$

for some $\delta > 0$.

We claim that this implies that the latter inclusions are equalities. In fact, for any three pairwise disjoint closed subsets E_1, E_2, E_3, it follows from the relation

$$V(E_1) + V(E_2) \subseteq V(E_1 \cup E_2)$$

and Remark A.8 that the sum $V(E_1) + V(E_2) + V(E_3)$ is direct. Our assumption (19) now implies that V satisfies

$$V(]-\infty, -\delta]) \oplus V(\{0\}) \oplus V([\delta, \infty[) = V. \tag{SC}$$

Since this sum is direct, it follows that

$$V_+ = V([\delta, \infty[) \quad \text{and} \quad V_- = V(]-\infty, -\delta]). \tag{20}$$

Condition (SC) is called the *splitting condition*.

Lemma A.16. *If $D := \alpha'(0)$ is a bounded operator, i.e., $\alpha\colon \mathbb{R} \to \mathrm{Aut}(V)$ is norm continuous, then the splitting condition* (SC) *is satisfied if and only if* 0 *is isolated in the spectrum of the hermitian operator D.*

Proof. First we recall from Remark A.7 that the Arveson spectrum $\mathrm{Spec}(V)$ coincides with the spectrum of D, which is a subset of $i\mathbb{R}$.

If 0 is isolated in $\mathrm{Spec}(D)$, then $\mathrm{Spec}(iD) \subseteq]-\infty, -\delta] \cup \{0\} \cup [\delta, \infty[$ for some $\delta > 0$, and the splitting condition follows from the existence of corresponding spectral projections defined by contour integrals (see also [Bel04, Thm. 1.4], where spectral subspaces are defined in terms of Dunford's local spectral theory.)

Suppose, conversely, that D is not invertibble and that the splitting condition is satisfied. Then $\mathrm{im}(D) = V_+ \oplus V_-$ is a closed subspace, $V_0 = \ker D$, and $V = \mathrm{im}(D) \oplus \ker(D)$. Hence [Si70] implies that 0 is isolated in $\mathrm{Spec}(D)$. □

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