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Analytic continuation of functions along parallel algebraic curves.

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Introduction

Classical lemma of Hartogs (see [1]), about continuation along fixed direction, conform, that if holomorphic in a domain

$$D \times V = D \times \{w \in C: \ |w| < r\} \subset C_z^n \times C_w$$

function $f(z,\ w)$, in each fixed $z \in D$ holomorphic in a ball $\{w \in C,\ |w| < r\}$, then it holomorphic on a collection of variables in $D \times V$. The Hartogs' lemma has a plenty generalizations which different on characterizations and direct verges to themes, connected with holomorphic continuation on fixed direction. A further results in this field are given in a works of Rothstein [2], M.V. Kazaryan [3], A.S.Sadullaev and E.M.Chirka [4], T.T.Tuychiev [5], S.A.Imomkulov and J.U.Khujamov [6], S.A.Imomkulov [7] and etc.

In this work we study a problem about analytic continuation along parallel algebraic curves.

Algebraic curve in $\mathbf{C^2}$ will be determining as a set of zero of some of polynomials:

$$A = \left\{(\xi,\eta) \in \mathbf{C^2}: \quad P\left(\xi,\eta\right) = 0\right\}.$$

A set of regular points $A^0$of algebraic curve A, open on A, and the set of critical points $A^c = A \backslash A^0$ is discrete set (see [1], [8],[9]).

The algebraic set call irreducible, if it is impossible to present in the view of joined algebraic sets, which differ from it.

Theorem. Let D is a domain from $\mathbf{C^n}$ and V is a domain from some of irreducible algebraic curve A. If a function $f(z,w)$ holomorphic in a domain $D \times V \subset C_z^n \times A$ and in each fixed ξ from some of nonpluripolar set $E \subset D$, a function $f(\xi,w)$ variable of w continuous till the function, holomorhpic on a whole A, excepting finite sets of singularities (from A^0), then $f(z,w)$ holomorphic extended in $(D \times A) \backslash S$, where S- some of analytic subset $D \times A$.

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§1. Holomorphic functions on algebraic set.

Any algebraic curve $A \subset \mathbf{C^2}$ in sufficiently small neighborhood U (each its point $a \in A$) is ramified covering over neighborhood of projection point a in some plane $C \subset \mathbf{C^2}$ (see. [1], chapter 2. §8. Page 175).

Observe, that on algebraic curves defining correctly not only holomorphic functions, its derivatives also. Holomorphic function and its derivatives in a point defining with the help following equality:

$$D^k f|_a = \frac{\partial^k}{\partial z^k} f(\pi^{-1}(z))|_U \,, \quad k = 0,\, 1,\, 2,\, 3,\, ...,$$

where $\pi :\, A \to \mathbf{C}$ locally biholomorphic mapping, called projection and U – neighborhood of point a, in which a restriction $\pi|_U$ - biholoorphic, $\pi^{-1}|_U$ - inverse mapping and in second member is a derivative in a point $z = \pi(a)$.

§2. The Jacoby – Hartogs series

We consider the function $f(z, w)$ holomorphic in a domain $D \times V$, $D \subset C^n$, $V \subset A$. Assume, that $\pi^{-1}(0) \in V$. Let $g(\zeta)$ - rational function on $\zeta \in C$ such that $g(0) = 0$. Then at small ρ there exist connected component Λ_ρ of set $\{\zeta :\ |g(\zeta) < \rho|\}$ such that $O \in \Lambda_\rho$, $\pi^{-1}(\Lambda_\rho) \subset V$. Since $f(z, w)$ holomorphic in the domain $D \times \pi^{-1}(\Lambda_\rho)$ then, at every fixed point $z \in D$ it can be expanded in a series Jacoby - Hartogs (see [4])

$$f(z, w) = \sum_{k=0}^{\infty} c_k(z, \pi(w)) g^k(\pi(w)) \,, \tag{1}$$

where an coefficients of series defining as follows

$$c_k(z, w) = \frac{1}{2\pi i} \int\limits_{\partial(\pi^{-1}(\Lambda_\rho)} f(z,\, \pi(\eta)) \frac{g(\pi(\eta)) - g(\pi(w))}{g^{k+1}(\pi(\eta))(\eta - w)} d\eta$$

, $k = 0, 1, 2, ...$

It follows, that $c_k(z, w)$ holomorphic functions in $D \times A$. At a fixed point $z \in D$ the series (1) converge in a lemniscates $\left\{|g(\pi(w))| < R^{(g)}(z)\right\}$, where $R^{(g)}(z)$ defining as follows

$$R^{(g)}(z) = \frac{1}{\overline{\lim\limits_{k\to\infty}} \sqrt[k]{\|c_k(z,w)\|_{\pi^{-1}(K)}}}.$$

Here $K \subset C$- arbitrary nonpolar compact, which does not hold poles g, and the limit in right-hand member of equality does not depend on choice of such compact. Notice, that the value $R^{(g)}(z)$ is a maximal radius, for which the function f is holomorphic inside of lemniscates$\left\{|g(w)| < R^{(g)}(z)\right\}$.

Lemma 1. The Jacoby – Hartogs series (1) converge uniformly inside of open set

$$G_g = \left\{(z,w) \in D \times A : \ |g(\pi(w))| < R_*^{(g)}(z) \ \right\}, \ z \in D,$$

where $R_*^{(g)}(z) = \underline{\lim}_{\xi\to z} R^{(g)}(\xi)$ - normalization from below. The function $-\ln R_*^{(g)}(z)$ is plurisubharmonic in D, $R_*^{(g)}(z) \le R^{(g)}(z), \ z \in D$ and the set $\left\{z \in D : \ R_*^{(g)}(z) < R^{(g)}(z)\right\}$ is pluripolar.

We denote by $\Re = \{g(\zeta)\}$ - countable family of all rational functions with coefficients from the set $Q + iQ$ (Q - the set of rational number) such that, each function $\Re = \{g(\zeta)\}$ has a zero only in a point $w = 0$. In order to study convergence domain, corresponding Jacoby – Hartogs series will be useful following lemma about approximation of flat sets by rational lemniscates.

Lemma 2. ([4],[6],[7]). Let Σ- close polar set from $C\backslash\{0\}$ and K - compact in $C\backslash\{\Sigma\}$. Then, there exist rational function $g \in \Re$ such that the lemniscates $\{\zeta : \ |g(z)| < 1\}$ is connected, belong to $C\backslash\{\Sigma\}$ and holding K.

§3. Some properties of pseudoconcave sets.

The properties of pseudoconcave sets has been studied in works [10-13]. Let S- pseudoconcave subset of domain $D \times V$. Assume, that S does not cross $D \times \partial V$ and

$$S_a = S \cap \{z = a\}.$$

Then:

1) the function $\ln(capS_z)$, where cap – capacity (îáîçíà÷àåò åìêîñòü (transfinite diameter) of flat set, is plurisubharmonic in D(see [13]).

2) if S_z is finite for all z from some pluripolar set $E \subset D$, then S is analytic set (see[12]).

3) if S_z is polar for all z from some nonpluripolar set $E \subset D$, then S is pluripolar set (see [10], [11]).

Defenition. The close set $S \subset D \times A$ called pseudoconcave set in the domain $D \times A$ if for any point $a \in S$ there exist some neighborhood $U \subset D \times A$ and holomorphic in $U \backslash S$ function f such that it does not converge holomorphically to the point a.

Lemma 3. Let D - a domain from $\mathbf{C^n}$ and V - a domain from irreducible algebraic curve A, such that $0 \in \pi(V)$. Let S - pseudoconcave subset of the domain $D \times V$. Assume, that S does not cross $D \times \partial V$. Then , if S_z - finite for all $z \in D$, then S- analytic subset of the domain $D \times V$.

§4. The proof of theorem.

We expand of function $f(z,w)$ in Jacoby – Hartogs series by degrees of function $g(\pi(w))$, $(g(\zeta) \in \Re\ ,\ \zeta = \pi(w),\ 0 \in \pi(V))$,

$$f(z,w) = \sum_{k=0}^{\infty} c_k(z,w) g^k(\pi(w))\,, \tag{2}$$

where $c_k(z,w) \in O(D \times V)$. It is possible, because $f(z,w)$ - holomorphic in $D \times V$ and at sufficiently small $\rho > 0$ the lemniscates $\{w :\ \ |g(w)| < \rho\}$ belong to V. According to lemma 1 the series (2) uniformly converge inside of set

$$G_g = \Big\{ (z,w)\ :\ \ |g(\pi(w))| < R_*^{(g)}(z)\ \Big\},\ z \in D$$

and consequently, its sum holomorphic in it. According to definition of family of rational functions $\Re$ the set G_g is a domain, which contain $D \times \{\pi^{-1}(0)\}$. The sum of constructed series (2) is coincident with $f(z,w)$ in neighborhood $D \times \{\pi^{-1}(0)\}$ and, thus, (2) is holomorphic continuation of $f(z,w)$ in G_g.

2. Let g_1 and $g_2$an arbitrary rational functions from family $\Re$ and let $f_1(z,w)$ and $f_2(z,w)$ are analytic continuation of function $f(z,w)$ in a domains G_{g_1} and G_{g_2}correspondingly. Since, for any point $z^0 \in D$ the function $f_1(z^0,w)$ single-valued by w and $f_1(z^0,w) = f_2(z^0,w) = f(z^0,w)$ for any $(z^0,w) \in G_{g_1} \cap G_{g_2}$, then $f(z,w)$ holomorphic in $(G_{g_1} \cup G_{g_2}) \cap \{z = z^0\}$ so, $f(z,w)$ uniquely converge in $G_{g_1} \cup\ G_{g_2}$, and it follows that the function

$f(z,w)$ uniquely converge in domain $G = \cup G_g$, where union is taken by all rational functions family $\Re$.

3. Since, at each fixed point $z^0 \in D$ the function $f(z^0, w)$ single-valued in A, then it follows that analytic continuation of function $f(z,w)$ (in $D \times A$) is uniquely. Let $\tilde{G} \subset D \times A$ an original domain for existence of function $f(z,w)$ regarding to $D \times A$. So, $\tilde{G}$ nonexpendable holomorphic in every point $(z^0, w^0) \in S = (D \times A) \backslash \tilde{G}$. From here we receive, that S - pseudoconcave subset of the domain $D \times A$.

4. Now, using lemma 2, we show, that for any point $z^0 \in D$, a set of singular point of function $f(z^0, w)$ of variable $w \in A$ coincide with layer S_{z^0} of the set S. In fact, by the terms of theorem singular set Λ of function $f(z^0, w)$ consist finite number of points, then according to lemma 2, for any compact $K \subset A \backslash \Lambda$, there exist rational function $g \in \Re$ such that, the lemniscates $\{|g(\pi(w))| < 1\}$ contains K. Consequently, the lemniscates $\left\{|g(\pi(w))| < R^{(g)}(z^0)\right\}$, so and $\left\{|g(\pi(w))| < R_*^{(g)}(z^0)\right\}$ contains (since, $R_*^{(g)}(z^0) \geq R^{(g)}(z^0)$).

5. Let Ω - an image of domain $\tilde{G} = (D \times A) \backslash S$ in maping $(z, \pi^{-1}(\zeta)) \to (z, \pi^{-1}(\frac{1}{\zeta}))$. The set $(D \times A') \backslash \Omega = L$ is also pseudoconcave. Since, S does not cross the set $D \times \{\pi^{-1}(0)\}$, then L is bounded and intersection $L \cap A'$ for any point $z^0 \in D$, consist from finite number of points, i.e. the set L is satisfying all conditions of lemma 3. Consequently, L - analytic set from here it is easy to see, that S- analytic. The proof of theorem is complete.

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