Noncommutative (generalized) sine-Gordon/massive Thirring correspondence, integrability and solitons

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Abstract

Some properties of the correspondence between the non-commutative versions of the (generalized) sine-Gordon ($\text{NCGSG}_{1,2}$) and the massive Thirring ($\text{NCGMT}_{1,2}$) models are studied. Our method relies on the master Lagrangian approach to deal with dual theories. The master Lagrangians turn out to be the NC versions of the so-called affine Toda model coupled to matter fields ($\text{NCATM}_{1,2}$), in which the Toda field g belongs to certain subgroups of $GL(3)$, and the matter fields lie in the higher grading directions of an affine Lie algebra. Depending on the form of g one arrives at two different NC versions of the $\text{NCGSG}_{1,2}/\text{NCGMT}_{1,2}$ correspondence. In the $\text{NCGSG}_{1,2}$ sectors, through consistent reduction procedures, we find NC versions of some well-known models, such as the NC sine-Gordon ($\text{NCSG}_{1,2}$) (Lechtenfeld et al. and Grisaru-Penati proposals, respectively), NC (bosonized) Bukhvostov-Lipatov ($\text{NCbBL}_{1,2}$) and NC double sine-Gordon ($\text{NCDSG}_{1,2}$) models. The $\text{NCGMT}_{1,2}$ models correspond to Moyal product extension of the generalized massive Thirring model. The $\text{NCGMT}_{1,2}$ models posses constrained versions with relevant Lax pair formulations, and other sub-models such as the NC massive Thirring ($\text{NCMT}_{1,2}$), the NC Bukhvostov-Lipatov ($\text{NCBL}_{1,2}$) and constrained versions of the last models with Lax pair formulations. We have established that, except for the well known $\text{NCMT}_{1,2}$ zero-curvature formulations, generalizations ($n_F \geq 2$, n_F =number of flavors) of the massive Thirring model allow zero-curvature formulations only for constrained versions of the models and for each one of the various constrained sub-models defined for less than n_F flavors, in the both $\text{NCGMT}_{1,2}$ and ordinary space-time descriptions (GMT), respectively. The non-commutative solitons and kinks of the $GL(3)$ $\text{NCGSG}_{1,2}$ models are investigated.

1 Introduction

Field theories in non-commutative (NC) space-times are receiving considerable attention in recent years in connection to the low-energy dynamics of D-branes in the presence of background B-field (see e.g. [1]). In particular, the NC versions of integrable systems (in two dimensions) are being considered [2]. On the other hand, conformal theories on the usual two-dimensional space-time play an important role in various aspects of modern physics, from string theory to applications in condensed matter. So, one might ask about the role played by QFTs in $(1 + 1)$-dimensional non-commutative space-time. Indeed there is reason to believe that similar applications would emerge and they deserve further investigations, since it is possible to define notions of conformal invariance, Kac-Moody and Virasoro symmetries in this context [3]. Furthermore, there is some optimism regarding the following analogy with the usual known relationship: it is believed that the integrable models, defined on two-dimensional NC *Euclidean* space, would be the NC versions of statistical models in the critical points and in the off-critical integrable directions.

The sine-Gordon type and other related integrable systems have appeared frequently in diverse areas of physics, from condensed matter to string theory, in connection to such properties as soliton solutions, integrability and duality. So, the study of their properties and the search for their solutions have greatly attracted the interest of the scientific community. In condensed matter, we can mention for example the work [4] on the nonlinear dynamics of the inhomogeneous DNA double helices chain. In topics of string theory we can mention the recent works on the magnon-type solutions on the $R \times S^n$ $(n = 2, 3)$ background geometry [5, 6].

Some non-commutative versions of the sine-Gordon model (NCSG) have been proposed in the literature [7, 8, 9, 10, 11, 12]. The relevant equations of motion have the general property of reproducing the ordinary sine-Gordon equation when the non-commutativity parameter is removed. The Grisaru-Penati version [7, 8] introduces a constraint which is non-trivial only in the non-commutative case. The constraint is required by integrability but it is satisfied by the one-soliton solutions. However, at the quantum level this model gives rise to particle production as was discovered by evaluating tree-level scattering amplitudes [8]. On the other hand, introducing an auxiliary field, Lechtenfeld et al. [12] proposed a novel NCSG model which seems to possess a factorisable and causal S-matrix.

Recently, in ordinary commutative space the so-called $sl(2)$ affine Toda model coupled to matter (Dirac) fields (ATM) has been shown to be a Master Lagrangian (ML) from which one can derive the sine-Gordon and massive Thirring models, describing the strong/weak phases of the model, respectively [13]-[16]. Besides, the ML approach was successfully applied in the non-commutative case to uncover related problems in $(2 + 1)$ dimensions regarding the duality equivalence between the Maxwell-Chern-Simons theory (MCS) and the Self-Dual (SD) model [17].

In this paper we extend some properties of the so-called $sl(3)$ generalized affine Toda model coupled to matter fields (GATM) [15] to the NC case. We define the NCGATM model by replacing the products of fields by the $\star-$products on the level of its effective action. The effective action associated to this model in ordinary space gives rise to equations of motion which can be derived from a zero-curvature equation plus some constraints. In fact, the ATM model is a constrained sub-model of an off-critical model related to the so-called conformal affine Toda model coupled to matter fields (CATM) which possesses a Lax pair formulation [18]. So, we expect the NCGATM model defined in this way does not belong to those class of NC field theories associated to a Lax pair formulation [11]. The NC $GL(2)$ case has been considered in [9], there the master Lagrangians turn out to be the NC versions of the ATM model associated to the group $GL(2)$, in which the Toda field belongs to certain representations of either $U(1) \times U(1)$ or the complexified $U(1)_C$, such that they correspond to the Lechtenfeld et al. (NCSG_1) or Grisaru-Penati (NCSG_2) proposals for the NC versions of the sine-Gordon model, respectively. Besides, the relevant NC massive Thirring ($\text{NCMT}_{1,2}$) sectors are written for two (four) types of Dirac fields corresponding to the Moyal product extension of one (two) copy(ies) of the ordinary massive Thirring model. The $\text{NCSG}_{1,2}$ models share the same one-soliton (real Toda field sector of model 2) exact solutions with their commutative counterparts, which are found without expansion in the NC parameter θ for the corresponding Toda field. Here the $GL(3)$ extension presents the above known feature regarding the appearance of two versions of the NC (generalized) sine-Gordon model ($\text{NCGSG}_{1,2}$) and the corresponding NC (generalized) massive Thirring models ($\text{NCGMT}_{1,2}$), and some new phenomena such as the appearance of the associated sub-models: three copies for each version of the NC sine-Gordon ($\text{NCSG}_{1,2}$) models, (bosonized) Bukhvostov-Lipatov models ($\text{NCbBL}_{1,2}$), double sine-Gordon models ($\text{NCDSG}_{1,2}$), and three copies for each version of the NC massive Thirring models($\text{NCMT}_{1,2}$), Bukhvostov-Lipatov models ($\text{NCBL}_{1,2}$) and the constrained $\text{NCBL}_{1,2}$ models, respectively. In addition, we have the known NC soliton solutions in the $\text{NCGSG}_{1,2}$ sectors and the appearance of a NC kink type solutions for the $\text{NCDSG}_{1,2}$ sub-models. Even though we have discussed the integrability properties of the $\text{NCGSG}_{1,2}$ models only for certain integrable directions in field space, i.e. in the $\text{NCSG}_{1,2}$ sub-models, the NC generalized massive Thirring ($\text{NCGMT}_{1,2}$) sectors present intriguing properties regarding integrability: the $\text{NCGMT}_{1,2}$ models encompass a Lax pair formulation only for a sub-model with certain eqs. of motion provided that some constraints are satisfied. Moreover, we established the integrability of certain constrained versions of the $\text{NCBL}_{1,2}$ models by providing a corresponding recipe to construct a Lax pair for each of them. The extension of the above features for the $GL(n)$ $\text{NCATM}_{1,2}$ models are straightforward.

The study of these models become interesting since the $su(n)$ ATM theories constitute excellent laboratories to test ideas about confinement [16, 19], the role of solitons in quantum field theories [13], duality transformations interchanging solitons and particles [13, 20], as well as the reduction processes of the (two-loop) Wess-Zumino-Novikov-Witten (WZNW) theory from which the ATM models are derivable [18, 15]. Moreover, the ATM type systems may also describe some low dimensional condensed matter phenomena, such as self-trapping of electrons into solitons, see e.g. [21], tunneling in the integer quantum Hall effect [22], and, in particular, polyacetylene molecule systems in connection with fermion number fractionization [23]. It has been shown that the $su(2)$ ATM model describes the low-energy spectrum of QCD_2 (*one flavor* and

N colors in the fundamental and $N = 2$ in the adjoint representations, respectively)[19]. The $sl(3)$ ATM model and its related dual sub-models GSG/GMT have been used to provide a bag model like confinement mechanism for "quarks" and it has been shown that the ATM spectrum comprises of solitons as baryons and qualitons as constituent quarks in two-dimensional QCD [24]. Moreover, the $sl(3)$ GSG model has been found to describe the low energy effective action of QCD_2 with unequal 'quark' masses, three flavors and N colors. This model has recently been used to describe the normal and exotic baryon spectrum of QCD_2 [25].

The paper is organized as follows. In the next section we present the NC extensions of the ATM model relevant to our discussions. It deals with the choice of the group representation for the Toda field g. We introduce two types of master Lagrangians ($NCATM_{1,2}$), the *first* one defined for $g \in [U(1)]^3$ with the same content of matter fields as the ordinary ATM; the *second* one defined for two copies of the $NCATM_1$ such that in this case $g, \bar{g} \in \mathcal{H} \subset SL(3, \mathbb{C})$. In section 3 the non-commutative versions ($NCGSG_{1,2}$) of the generalized sine-Gordon model (GSG) are derived from the relevant master Lagrangians through reduction procedures resembling the one performed in the ordinary GATM $\rightarrow$ GSG reduction. In section 4 we present the two NC extensions ($NCGSG_{1,2}$) of the GSG model, as well as their associated sub-models such as the $NCSG_{1,2}$, $NCbBL_{1,2}$ and $NCDSG_{1,2}$ models. In section 5 we 'decouple' on shell the theories $NCGSG_{1,2}$ and $NCGMT_{1,2}$, respectively. We discuss the conditions which must satisfy the constraints in order to have a complete decoupling, in particular for the soliton solutions. In section 6 we consider the $NCGMT_{1,2}$ models, as well as their global symmetries, associated currents and integrability properties of the constrained sub-models. In these developments the double-gauging of a U(1) symmetry in the star-localized Noether procedure to get the currents deserve a careful treatment. We discuss their associated sub-models such as the integrable $NCMT_{1,2}$, the non-integrable $NCBL_{1,2}$, and the (constrained) $NCBL_{1,2}$ models regarded as integrable sub-models. In section 7 we present the soliton and kink type solutions as a sub-set of solutions satisfying the both GSG and $NCGSG_{1,2}$ models simultaneously. Some discussions and possible directions of research to pursue in the future are presented in section 8. The Appendix A provides the usual GSG model as a reduced $sl(3)$ affine Toda model couple to matter. Some results of the zero-curvature formulation of the CATM model are provided in Appendix B, and the Lagrangian formulation of the ordinary ATM model is summarized in Appendix C.

2 The NC affine Toda models coupled to matter fields ($NCATM_{1,2}$)

In this section we present the NC versions of the so-called affine Toda model coupled to matter fields ($NCATM_{1,2}$). The case of $GL(2)$ NCATM model has been studied at the classical level in [9] and the related NC sine-Gordon/massive Thirring correspondence has been considered at the quantum level in [10]. Even though we present detailed computations for the $GL(3)$ case it can follow directly for any $GL(n)$. Two different NC extensions of the ATM model (182) are possible as long as each of them reproduce its ordinary equations of motion in the commutative limit. The commutative Toda field g in (172) belongs to the complexified abelian subgroup of $SL(3,\mathbb{C})$. The symmetry group $SL(3)$ of the ordinary ATM model (see Appendix B) when considered in the NC case is not closed under the Moyal product $\star$; then, the NC extension requires the $GL(3)$ group. In the next steps we define two versions of the non-commutative $GL(3)$ affine Toda model coupled to matter fields (NCATM$_{1,2}$). Let us define the *first* NC extension (NCATM$_1$) as

$$
\begin{aligned}
S_{NCATM_1} &\equiv S[g, W^{\pm}, F^{\pm}] \\
&= I_{WZW}[g] + \int d^2x \sum_{m=1}^{2} \{ \frac{1}{2} < \partial_- W^-_{3-m} \star [E_3\,,\, W^-_m] > - \frac{1}{2} < [E_{-3}\,,\, W^+_m] \star \partial_+ W^+_{3-m} > + \\
& < F^-_m \star \partial_+ W^+_m > + < \partial_- W^-_m \star F^+_m > + < F^-_m \star g \star F^+_m \star g^{-1} > \}, \qquad (1)
\end{aligned}
$$

where $F \star G = F \exp\Big(\frac{\theta}{2} (\overleftarrow{\partial_+} \overrightarrow{\partial_-} - \overleftarrow{\partial_-} \overrightarrow{\partial_+}) \Big) G$ and $g \in [U(1)]^3$. In fact, we have written the NC version of the ATM model presented in the eq. (182) of Appendix C. The fields $W^{\pm}_m, F^{\pm}_m$, as well as the generators $E_{\pm 3}$ of the model are defined in eqs. (166)-(171). $I_{WZW}[g]$ is a NC generalization of the WZNW action for g

$$
I_{WZW}[g] = \int d^2x \left[\partial_+ g \star \partial_- g^{-1} + \int_0^1 dy \hat{g}^{-1} \star \partial_y \hat{g} \star \left[\hat{g}^{-1} \star \partial_+ \hat{g}, \hat{g}^{-1} \star \partial_- \hat{g} \right]_\star \right], \qquad (2)
$$

where the homotopy path $\hat{g}(y)$ such that $\hat{g}(0) = \mathbf{1}$, $\hat{g}(1) = g$ ($[y, x_+] = [y, x_-] = 0$) has been defined. The WZW term in this case gives a non-vanishing contribution due to the non-commutativity. This is in contrast with the action in ordinary space, i.e. the WZW term in (182)-(183) vanishes for g belonging to an abelian subgroup of $SL(3,\mathbb{C})$. From (1) one can derive the set of equations of motion for the corresponding fields

$$
\partial_-(g^{-1} \star \partial_+ g) = \sum_{m=1}^{2} \left[F^-_m\,,\, g \star F^+_m \star g^{-1} \right]_\star \qquad (3)
$$

$$
\partial_+ F^-_m = [E_{-3}, \partial_+ W^+_{3-m}], \qquad \partial_- F^+_m = -[E_3, \partial_- W^-_{3-m}], \qquad (4)
$$

$$
\partial_+ W^+_m = -g \star F^+_m \star g^{-1}, \qquad \partial_- W^-_m = -g^{-1} \star F^-_m \star g. \qquad (5)
$$

Notice that these set of eqs. closely resemble their commutative counterparts (184)-(186) of Appendix C. Substituting the derivatives of $W^{\pm}$'s given in the eqs. (5) into the eqs. (4) one can get the equivalent set of equations

$$
\partial_+ F^-_m = -[E_{-3}\,,\, g \star F^+_{3-m} \star g^{-1}], \qquad \partial_- F^+_m = [E_3\,,\, g^{-1} \star F^-_{3-m} \star g]. \qquad (6)
$$

Notice that in the action (1) one can use simultaneously the cyclic properties of the group trace and the $\star$ product. Then, the action (1) and the equations of motion (3)-(5) have the left-right local symmetries given by

$$
g \rightarrow h_L(x_-) \star g(x_+, x_-) \star h_R(x_+), \qquad (7)
$$

$$
F^+_m \rightarrow h^{-1}_R(x_+) \star F^+_m(x_+, x_-) \star h_R(x_+), \quad W^-_m \rightarrow h^{-1}_R(x_+) \star W^-_m(x_+, x_-) \star h_R(x_+), \qquad (8)
$$

$$
F^-_m \rightarrow h_L(x_-) \star F^-_m(x_+, x_-) \star h^{-1}_L(x_-), \quad W^+_m \rightarrow h_L(x_-) \star W^+_m(x_+, x_-) \star h^{-1}_L(x_-). \qquad (9)
$$

The system of eqs. (3)-(5) is invariant under the above symmetries if the following conditions are supplied

$$
h_R(x_+) \star E_3\, h^{-1}_R(x_+) = E_3, \quad h^{-1}_L(x_-) \star E_{-3}\, h_L(x_-) = E_{-3}, \qquad (10)
$$

where $h_{L/R}(x_{\mp}) \in \mathcal{H}_0^{L/R}$, $\mathcal{H}_0^{L/R}$ being Abelian sub-groups of $GL(3)$. These symmetry transformations written in matrix form [15] are extensions of the ordinary ones to the NC case in a straightforward manner. Notice that in the ordinary space-time, in terms of the field components, the above transformations are given in the Appendix A [see eqs. (150) and (152)-(153)]; obviously, the form of the expressions given in these eqs. will change in the NC case.

Next, we define the *second* version of the $GL(3)$ NC affine Toda model coupled to matter NCATM$_2$ as

$$S_{NCATM_2} \equiv S[g, W^{\pm}, F^{\pm}] + S[\bar{g}, \mathcal{W}^{\pm}, \mathcal{F}^{\pm}], \tag{11}$$

where the independent fields g and $\bar{g}$, related to the set of matter fields $\{W^{\pm}, F^{\pm}\}$ and $\{\mathcal{W}^{\pm}, \mathcal{F}^{\pm}\}$, respectively, belong to a complexified subgroup $\mathcal{H}$ of $GL(3)$ to be specified in the subsection 4.2. As above the action $S[.\,,\,.\,,\,.]$ is defined as the Moyal extension of (182). The motivation to introduce a copy of the action functional with the set of fields $\bar{g}, \mathcal{W}^{\pm}, \mathcal{F}^{\pm}$ will be clarified below. Let us mention, in the mean time, that the second version of the NCATM$_2$ model has also been considered in [9] for the $SL(2)$ case.

The equations of motion for the NCATM$_2$ model (11) comprise the eqs. (3)-(5) written for $g \in \mathcal{H} \subset GL(3)$ and a set of analogous equations for the remaining fields $\bar{g}$, $\mathcal{F}^{\pm}$ and $\mathcal{W}^{\pm}$. Moreover, in addition to the symmetry transformations (7)-(9) one must consider similar expressions for $\bar{g}$, $\mathcal{F}^{\pm}$ and $\mathcal{W}^{\pm}$.

3 NC versions of the generalized sine-Gordon model (NCGSG$_{1,2}$)

In order to derive the NC versions of the generalized sine-Gordon model (NCGSG$_{1,2}$) we follow the master Lagrangian approach [26, 15], starting from the NCATM$_{1,2}$ models (1) and (11), respectively, as performed in the $GL(2)$ case [9]. So, let us consider first the equations of motion (3)-(5). We proceed by integrating the eqs. (4)

$$F^{-} = [E_{-3}, W^{+}_{3-m}] + f^{-}_{m}(x_{-}), \quad F^{+} = -[E_{3}, W^{-}_{3-m}] - f^{+}_{m}(x_{+}). \tag{12}$$

with the $f^{\pm}(x_{\pm})$'s being analytic functions. Next, we replace the $F^{\pm}$ of eqs. (12) and the $\partial_{\pm}W^{\pm}$ of (5), written in terms of $W^{\pm}$, into the action (1) to get

$$\begin{aligned} S'[g, W^{\pm}, f^{\pm}] \quad &= \quad I_{WZW}[g] + \int d^2x \sum_{m=1}^{2} \{ \frac{1}{2} < [E_{-3}, W^{+}_{3-m}] \star g \star f^{+}_{m} \star g^{-1} > + \\ & \frac{1}{2} < g^{-1} \star f^{-}_{m} \star g \star [E_{3}, W^{-}_{3-m}] > + < g^{-1} \star f^{-}_{m} \star g \star f^{+}_{m} > \}. \end{aligned} \tag{13}$$

As the next step, one writes the equations of motion for the $f^{\pm}(x_{\pm})$'s and solves for them; afterwards, substitutes those expressions into the intermediate action (13) getting

$$S''[g, W^{\pm}] = I_{WZW}[g] - \frac{1}{4} \int d^2x \sum_{m=1}^{2} < [E_{-3}, W^{+}_{3-m}] \star g \star [E_{3}, W^{-}_{3-m}] \star g^{-1} > . \tag{14}$$

Notice that (14) has inherited from the NCATM action the local symmetries (7)-(9). Therefore, one considers the gauge fixing

$$2i\Lambda_m^- = [E_{-3}, W_{3-m}^+], \quad 2i\Lambda_m^+ = [E_3, W_{3-m}^-], \tag{15}$$

where $\Lambda^\pm \in \hat{\mathcal{G}}_{\pm 1}$ are some constant generators in the subspaces of grade ± 1 in (180)-(181).

Then for this gauge fixing the effective action (14) becomes

$$\begin{aligned} S_{NCGSG_1}[g] &\equiv S[g] \\ &= I_{WZW}[g] + \int d^2x \sum_{m=1}^{2} [< \Lambda_m^- \star g \star \Lambda_m^+ \star g^{-1} >]. \end{aligned} \tag{16}$$

Thus, we get the equation of motion for the field g as

$$\partial_-(g^{-1} \star \partial_+ g) = \sum_{m=1}^{2} \left[\Lambda_m^-\,,\, g \star \Lambda_m^+ g^{-1}\right] \tag{17}$$

The action (16) for $g \in [U(1)]^3$ will define the first version of the non-commutative generalized sine-Gordon model (NCGSG_1). The second version requires a copy of the above action for the field $\bar{g}$

$$S_{NCGSG_2}[g] \equiv S[g] + S[\bar{g}], \tag{18}$$

where $g \in \mathcal{H} \subset SL(3)$ ($\mathcal{H}$ will be specified below).

Thus, the actions (16) and (18) are the multi-field extensions of the NC sine-Gordon models proposed earlier by Lechtenfeld et al. and Grisaru-Penati, respectively. As we will see below, these models contain as sub-models the relevant versions of the $\text{NCSG}_{1,2}$ model (in fact, each version contains three $\text{NCSG}_{1,2}$ sub-models) proposed in the literature, i.e. the Lechtenfeld et al. and Grisaru-Penati proposals for the NC extension of the sine-Gordon model, respectively. Moreover, the $\text{NCGSG}_{1,2}$ models give rise to new phenomena with interesting properties, such as the appearance of two versions of the NC Bukhvostov-Lipatov model and the NC double sine-Gordon model, respectively, as well as their NC soliton and kink type solutions.

We present below the two $\text{NCGSG}_{1,2}$ versions related to $GL(3)$, each one involving multi-field scalar fields.

4 The Toda field g parametrizations

In this section we present the two possible parametrizations of the field g, thus obtaining the two NC versions $\text{NCGSG}_{1,2}$ of the GSG model, and furthermore we obtain their relevant sub-models associated to them through consistent reductions.

4.1 First parametrization: $g \in [U(1)]^3 \subset GL(3,\mathbb{C})$

Let us write the field g in the representation

$$g = \begin{pmatrix} e_\star^{i\phi_1} & 0 & 0 \\ 0 & e_\star^{i\phi_2} & 0 \\ 0 & 0 & e_\star^{i\phi_3} \end{pmatrix} \equiv g_1 * g_2 * g_3, \text{ where} \tag{19}$$

$$g_1 = \begin{pmatrix} e_\star^{i\phi_1} & 0 & \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e_\star^{i\phi_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e_\star^{i\phi_3} \end{pmatrix} \tag{20}$$

with ϕ_i being real fields ($i = 1, 2, 3$). As we will see below, this parametrization constitutes the $GL(3,\mathbb{C})$ extension of the Lechtenfeld et al. proposal of the non-commutative version of the sine-Gordon model (NCSG$_1$) [12].

For the Λ_i's taken as

$$\begin{aligned}
\Lambda_1^+ &= \Lambda_R^1 E_{\alpha_1}^0 + \Lambda_R^2 E_\alpha^0 + \tilde{\Lambda}_R^3 E_{-\alpha_3}^1, \\
\Lambda_1^- &= \Lambda_L^3 E_{\alpha_3}^{-1} + \tilde{\Lambda}_L^1 E_{-\alpha_1}^0 + \tilde{\Lambda}_L^2 E_{-\alpha_2}^0, \\
\Lambda_2^+ &= \Lambda_R^3 E_{\alpha_3}^0 + \tilde{\Lambda}_R^1 E_{-\alpha_1}^1 + \tilde{\Lambda}_R^2 E_{-\alpha_2}^1. \\
\Lambda_2^- &= \Lambda_L^1 E_{\alpha_1}^{-1} + \Lambda_L^2 E_{\alpha_2}^{-1} + \tilde{\Lambda}_L^3 E_{-\alpha_3}^0,
\end{aligned} \tag{21}$$

the action (16) for g given in (19), upon using twice the Polyakov-Wiegmann identity

$$I_{WZW}(g_1 * g_2) = I_{WZW}(g_1) + I_{WZW}(g_2) + \int dz^2 < g_1^{-1} \star \partial_- g_1 \star \partial_+ g_2 \star g_2^{-1} >, \tag{22}$$

can be written as

$$\begin{aligned}
S_{NCSG_1}[g_1, g_2, g_3] = \quad & I_{WZW}[g_1] + I_{WZW}[g_2] + I_{WZW}[g_3] + \\
& \int d^2x \Big([\Lambda_L^3 \tilde{\Lambda}_R^3 e_\star^{i\phi_1} \star e_\star^{-i\phi_3} + \tilde{\Lambda}_L^3 \Lambda_R^3 e_\star^{i\phi_3} \star e_\star^{-i\phi_1}] + \\
& [\Lambda_L^1 \tilde{\Lambda}_R^1 e_\star^{i\phi_1} \star e_\star^{-i\phi_2} + \tilde{\Lambda}_L^1 \Lambda_R^1 e_\star^{i\phi_2} \star e_\star^{-i\phi_1}] + \\
& [\Lambda_L^2 \tilde{\Lambda}_R^2 e_\star^{i\phi_2} \star e_\star^{-i\phi_3} + \tilde{\Lambda}_L^2 \Lambda_R^2 e_\star^{i\phi_3} \star e_\star^{-i\phi_2}] \Big).
\end{aligned} \tag{23}$$

Notice that the last term in the Polyakov-Wiegmann identity (22) vanishes when written for each pair of the fields in the parametrizations (20). Then, the relevant eqs. of motion become

$$\begin{aligned}
\partial_- \Big(e_\star^{-i\phi_1} \star \partial_+ e_\star^{i\phi_1} \Big) &= [\Lambda_L^1 \tilde{\Lambda}_R^1 e_\star^{i\phi_2} \star e_\star^{-i\phi_1} - \tilde{\Lambda}_L^1 \Lambda_R^1 e_\star^{i\phi_1} \star e_\star^{-i\phi_2}] + \\
& \quad [\Lambda_L^3 \tilde{\Lambda}_R^3 e_\star^{i\phi_3} \star e_\star^{-i\phi_1} - \tilde{\Lambda}_L^3 \Lambda_R^3 e_\star^{i\phi_1} \star e_\star^{-i\phi_3}] \qquad (24) \\
\partial_- \Big(e_\star^{-i\phi_2} \star \partial_+ e_\star^{i\phi_2} \Big) &= [\Lambda_L^2 \tilde{\Lambda}_R^2 e_\star^{i\phi_3} \star e_\star^{-i\phi_2} - \tilde{\Lambda}_L^2 \Lambda_R^2 e_\star^{i\phi_2} \star e_\star^{-i\phi_3}] + \\
& \quad [\tilde{\Lambda}_L^1 \Lambda_R^1 e_\star^{i\phi_1} \star e_\star^{-i\phi_2} - \Lambda_L^1 \tilde{\Lambda}_R^1 e_\star^{i\phi_2} \star e_\star^{-i\phi_1}]. \qquad (25) \\
\partial_- \Big(e_\star^{-i\phi_3} \star \partial_+ e_\star^{i\phi_3} \Big) &= [\tilde{\Lambda}_L^3 \Lambda_R^3 e_\star^{i\phi_1} \star e_\star^{-i\phi_3} - \Lambda_L^3 \tilde{\Lambda}_R^3 e_\star^{i\phi_3} \star e_\star^{-i\phi_1}] + \\
& \quad [\tilde{\Lambda}_L^2 \Lambda_R^2 e_\star^{i\phi_2} \star e_\star^{-i\phi_3} - \Lambda_L^2 \tilde{\Lambda}_R^2 e_\star^{i\phi_3} \star e_\star^{-i\phi_2}]. \qquad (26)
\end{aligned}$$

Setting

$$\Lambda_L^j \tilde{\Lambda}_R^j = e^{i\delta_j} M_j/8, \quad j = 1,2,3; \tag{27}$$

for M_j, δ_j some constants, we define the system of eqs. (24)-(26) as the *first version* of the non-commutative generalized $GL(3,\mathbb{C})$ sine-Gordon model (NCGSG$_1$). Notice that it is defined for three real scalar fields.

In the commutative limit $\theta \to 0$ the above equations can be written as

$$\partial^2 \phi_1 \quad = \quad M_1 \sin(\phi_2 - \phi_1 + \delta_1) + M_3 \sin(\phi_3 - \phi_1 + \delta_3); \tag{28}$$

$$\partial^2 \phi_2 \quad = \quad M_2 \sin(\phi_3 - \phi_2 + \delta_2) + M_1 \sin(\phi_1 - \phi_2 - \delta_1); \tag{29}$$

$$\partial^2 \phi_3 \quad = \quad M_2 \sin(\phi_2 - \phi_3 - \delta_2) + M_3 \sin(\phi_1 - \phi_3 - \delta_3). \tag{30}$$

From the above system of equations one gets a free scalar equation of motion

$$\partial^2 \Phi \quad = \quad 0, \quad \Phi \equiv \phi_1 + \phi_2 + \phi_3. \tag{31}$$

For the particular solution $\Phi \equiv 0$ of (31) and making $M_j \to -M_j$, $\phi_1 \to -\phi_1$, one can write the first two equations (28)-(29) as

$$\partial^2 \phi_1 \quad = \quad M_1 \sin(\phi_2 + \phi_1 + \delta_1) + M_3 \sin(2\phi_1 - \phi_2 + \delta_3); \tag{32}$$

$$\partial^2 \phi_2 \quad = \quad M_2 \sin(2\phi_2 - \phi_1 - \delta_2) + M_1 \sin(\phi_1 + \phi_2 + \delta_1). \tag{33}$$

This system of eqs. is precisely the commutative generalized sine-Gordon model (GSG) [24, 25] [the form written in (32)-(33) corresponds to eqs. (161)-(162) of Appendix A].

In the following subsections we will examine certain sub-models obtained through consistent reductions of the NCGSG$_1$ system (24)-(26).

4.1.1 Non-commutative sine-Gordon model (NCSG$_1$): Lechtenfeld et al. proposal

We show that the model (24)-(26) contains as sub-models the Lechtenfeld et al. proposal for the NCSG$_1$ model. So, setting $M_2 = M_3 = 0$, $M_1 = 8M$, $\phi_3 = \delta_j = 0$ and changing $\phi_2 \to -\phi_2$ we get the system of equations [12]

$$\partial_- \left(e_\star^{-i\phi_1} \star \partial_+ e_\star^{i\phi_1} \right) \quad = \quad M [e_\star^{-i\phi_2} \star e_\star^{-i\phi_1} - e_\star^{i\phi_1} \star e_\star^{i\phi_2}] \tag{34}$$

$$\partial_- \left(e_\star^{i\phi_2} \star \partial_+ e_\star^{-i\phi_2} \right) \quad = \quad M [e_\star^{i\phi_1} \star e_\star^{i\phi_2} - e_\star^{-i\phi_2} \star e_\star^{-i\phi_1}]. \tag{35}$$

In fact, there are additional two possibilities for meaningful reductions, i.e., 1) $M_1 = M_2 = 0$, $M_3 = 8M$, $\phi_2 = \delta_j = 0$; $\phi_3 \to -\phi_3$ and 2) $M_1 = M_3 = 0$, $M_2 = 8M$, $\phi_1 = \delta_j = 0$; $\phi_3 \to -\phi_3$ respectively, providing in each case a Lechtenfeld et al. NCSG$_1$ model.

4.1.2 Non-commutative (bosonized) Bukhvostov-Lipatov model (NCbBLzx$_1$). First version

Another reduction is possible by making $M_1 = 0,\ M_2 = M_3 = -M,$ and $\phi_1 \to -\phi_1,$ in the eqs. (24)-(26) followed by the substitution $\phi_3 = \phi_1 - \phi_2$. So, one gets the set of equations

$$\partial_- \Big(e_\star^{\pm i\phi_a} \star \partial_+ e_\star^{\mp i\phi_a} \Big) = -\frac{M}{8} \Big[e_\star^{i(\phi_1-\phi_2)} \star e_\star^{\pm i\phi_a} - e_\star^{\mp i\phi_a} \star e_\star^{-i(\phi_1-\phi_2)} \Big], \qquad a = 1,2 \tag{36}$$

$$0 = \partial_- \Big[e_\star^{i\phi_1} \star \partial_+ e_\star^{-i\phi_1} + e_\star^{-i\phi_2} \star \partial_+ e_\star^{i\phi_2} + e_\star^{-i(\phi_1-\phi_2)} \star \partial_+ e_\star^{i(\phi_1-\phi_2)} \Big], \tag{37}$$

where the upper (lower) signs in (36) correspond to the index $a = 1(2)$ for the field ϕ_a. In the commutative limit the eq. (37) becomes trivial, whereas the set of equations (36) become $\partial^2\phi_1 = M\mathrm{sin}(2\phi_1 - \phi_2)$ and $\partial^2\phi_2 = M\mathrm{sin}(2\phi_2 - \phi_1)$. Defining the new fields $\psi_1 = \frac{1}{2}(\phi_1 + \phi_2)$, $\psi_2 = \frac{\sqrt{3}}{2}(\phi_1 - \phi_2)$ we arrive at the model $\partial^2\psi_1 = M\mathrm{sin}(\psi_1)\mathrm{cos}(\sqrt{3}\psi_2)$, $\partial^2\psi_2 = M\sqrt{3}\mathrm{cos}(\psi_1)\mathrm{sen}(\sqrt{3}\psi_2)$. This system of equations is precisely the bosonized form of the so-called Bukhvostov-Lipatov model [27, 28, 29, 20]. In view of these relationships we define the model (36)-(37) as the *first version* of the non-commutative bosonized Bukhvostov-Lipatov model(NCbBL$_1$).

4.1.3 Non-commutative double sine-Gordon model (NCDSG$_1$). First version

The usual double sine-Gordon model (DSG) is defined in terms of just one scalar field ϕ and the potential terms $[cos(\phi) + cos(2\phi)]$ in the action. So, we would like to reduce the above model in a consistent way in order to get a sub-model defined for just one scalar field. Let us take advantage of a particular solution of the free field equation (31). So, we consider the reduction $\phi_1 = -\phi_3 = \phi,\ \phi_2 = 0$ and substitute these relations into the equations (24)-(26). Then we obtain the next two equations

$$\partial_- \Big(e_\star^{-i\phi} \star \partial_+ e_\star^{i\phi} \Big) = M_1(e_\star^{-i\phi} - e_\star^{i\phi}) + M_3(e_\star^{-i\phi} \star e_\star^{-i\phi} - e_\star^{i\phi} \star e_\star^{i\phi}), \tag{38}$$

$$\partial_- \Big(e_\star^{i\phi} \star \partial_+ e_\star^{-i\phi} \Big) = M_1(e_\star^{i\phi} - e_\star^{-i\phi}) + M_3(e_\star^{i\phi} \star e_\star^{i\phi} - e_\star^{-i\phi} \star e_\star^{-i\phi}) \tag{39}$$

plus an equation which reduces to a trivial identity (we have imposed $M_1 = M_2$, $\delta_i = 0$).

The above two equations can be written in the equivalent form

$$\partial_- \Big(e_\star^{i\phi} \star \partial_+ e_\star^{-i\phi} - e_\star^{-i\phi} \star \partial_+ e_\star^{i\phi} \Big) = 4iM_1 \sin_\star \phi + 4iM_3 \sin_\star 2\phi \tag{40}$$

$$\partial_- \Big(e_\star^{-i\phi} \star \partial_+ e_\star^{i\phi} + e_\star^{i\phi} \star \partial_+ e_\star^{-i\phi} \Big) = 0. \tag{41}$$

The system (40)-(41) constitutes the *first* version of the non-commutative double sine-Gordon model (NCDSG$_1$) defined for just one scalar field.

The first equation (40) contains the potential terms which is the natural generalization of the ordinary double sine-Gordon potential, whereas the other one (41) has the structure of a conservation law and it can be seen as imposing an extra condition on the system. In the commutative limit, the first equation reduces to the ordinary double sine-Gordon equation (DSG), whereas the second one becomes trivial. The equations are in general complex and possess the $\mathbb{Z}_2$ symmetry of the ordinary DSG (the invariance under $\phi \to -\phi$ is easily seen in (38)-(39).

4.2 Second parametrization: $g \in \mathcal{H} \subset GL(3,\mathbb{C})$

Let us consider the parametrization

$$g = \begin{pmatrix} e_\star^{\varphi_1} \star e_\star^{\varphi_0} & 0 & 0 \\ 0 & e_\star^{-\varphi_1+\varphi_2} \star e_\star^{\varphi_0} & 0 \\ 0 & 0 & e_\star^{-\varphi_2} \star e_\star^{\varphi_0} \end{pmatrix} \equiv g_1 * g_2, \tag{42}$$

with

$$g_1 = \begin{pmatrix} e_\star^{\varphi_1} & 0 & 0 \\ 0 & e_\star^{-\varphi_1+\varphi_2} & 0 \\ 0 & 0 & e_\star^{-\varphi_2} \end{pmatrix}, \; g_2 = e_\star^{\varphi_0} \begin{pmatrix} 1 & 0 & \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{43}$$

where the fields φ_j, $j = 0, 1, 2$ are general complex fields. The additional field $\bar{g}$ is defined by substituting the fields φ_j above as $\varphi_j^\dagger$. The fields g and $\bar{g}$ are formally considered to be independent fields.

This parametrization becomes the $GL(3)$ extension of the Grisaru-Penati proposal for the non-commutative version of the sine-Gordon model (NCSG$_2$) [7, 8].

The following equations of motion can be obtained directly from the first term $S[g]$ of the action (18) for the parametrization (42)

$$\begin{aligned} \partial_-\Big(e_\star^{-\varphi_0} \star e_\star^{-\varphi_1} \star \partial_+\Big(e_\star^{\varphi_1} \;\star e_\star^{\varphi_0}\Big)\Big) &= [\Lambda_L^1 \tilde{\Lambda}_R^1 e_\star^{-\varphi_1+\varphi_2} \star e_\star^{-\varphi_1} - \tilde{\Lambda}_L^1 \Lambda_R^1 e_\star^{\varphi_1} \star e_\star^{\varphi_1-\varphi_2}] + \\ & \quad [\Lambda_L^3 \tilde{\Lambda}_R^3 e_\star^{-\varphi_2} \star e_\star^{-\varphi_1} - \tilde{\Lambda}_L^3 \Lambda_R^3 e_\star^{\varphi_1} \star e_\star^{\varphi_2}]. \qquad (44) \\ \partial_-\Big(e_\star^{-\varphi_0} \star e_\star^{\varphi_1-\varphi_2} \star \partial_+\Big(e_\star^{-\varphi_1+\varphi_2} \;\star e_\star^{\varphi_0}\Big)\Big) &= [\Lambda_L^2 \tilde{\Lambda}_R^2 e_\star^{-\varphi_2} \star e_\star^{\varphi_1-\varphi_2} - \tilde{\Lambda}_L^2 \Lambda_R^2 e_\star^{-\varphi_1+\varphi_2} \star e_\star^{\varphi_2}] + \\ & \quad [\tilde{\Lambda}_L^1 \Lambda_R^1 e_\star^{\varphi_1} \star e_\star^{\varphi_1-\varphi_2} - \Lambda_L^1 \tilde{\Lambda}_R^1 e_\star^{-\varphi_1+\varphi_2} \star e_\star^{-\varphi_1}] \qquad (45) \\ \partial_-\Big(e_\star^{-\varphi_0} \star e_\star^{\varphi_2} \star \partial_+\Big(e_\star^{-\varphi_2} \;\star e_\star^{\varphi_0}\Big)\Big) &= [\tilde{\Lambda}_L^3 \Lambda_R^3 e_\star^{\varphi_1} \star e_\star^{\varphi_2} - \Lambda_L^3 \tilde{\Lambda}_R^3 e_\star^{-\varphi_2} \star e_\star^{-\varphi_1}] + \\ & \quad [\tilde{\Lambda}_L^2 \Lambda_R^2 e_\star^{-\varphi_1+\varphi_2} \star e_\star^{\varphi_2} - \Lambda_L^2 \tilde{\Lambda}_R^2 e_\star^{-\varphi_2} \star e_\star^{\varphi_1-\varphi_2}]. \qquad (46) \end{aligned}$$

Introduce the parameters M_i, δ_i as in (27). So, we define the system of eqs. (44)-(46), supplied with the relevant eqs. of motion for the fields $\varphi_j^\dagger$ derived from the second term $S[\bar{g}]$ of the action (18), as the *second version* of the non-commutative generalized $GL(3,\mathbb{C})$ sine-Gordon model (NCGSG$_2$), where the three scalar fields φ_j are in general complex.

Next, let us examine the commutative limit. Redefining $\varphi_a \to i\,\varphi_a$ (where the new φ_a's are real), using definition (27) and taking the limit $\theta \to 0$ in the above system of equations (44)-(46) one can get

$$\partial^2 \varphi_1 = M_1 \sin(2\varphi_1 - \varphi_2 - \delta_1) + M_3 \sin(\varphi_1 + \varphi_2 - \delta_3) \tag{47}$$

$$\partial^2 \varphi_2 = M_2 \sin(2\varphi_2 - \varphi_1 - \delta_2) + M_3 \sin(\varphi_1 + \varphi_2 - \delta_3) \tag{48}$$

$$\partial^2 \varphi_0 = 0. \tag{49}$$

Thus, in (47)-(48) we recover again the equations of motion of the commutative generalized sine-Gordon model (GSG) [24, 25]. Notice that the field φ_0 decouples completely from the other fields in this limit, becoming simply a free field.

In analogy to the results of the first parametrization it is possible to get some sub-models as consistent reductions of the system (44)-(46). In the following we discuss the reductions associated to this second parametrization.

4.2.1 Non-commutative sine-Gordon model (NCSG$_2$): Grisaru-Penati proposal

A reduced single field model follows by setting $M_2 = M_3 = \delta_i = 0$, $\varphi_0 = \varphi_2 = 0$, $M_1 = -M$ and $\varphi_1 = i\varphi$ (φ, complex field). So, one gets the model

$$\partial_-(e^{\mp i\varphi}\partial_+ e^{\pm i\varphi}) = \pm M(e^{\pm 2i\varphi} - e^{\mp 2i\varphi}), \tag{50}$$

which is the Grisaru-Penati proposal for the NC extension of the sine-Gordon model (NCSG$_2$) [7, 8]. In fact, in this proposal one must consider additionally a couple of equations for $\varphi^\dagger$ obtained from the second piece in the action (18). Additional reductions, each one providing a Grisaru-Penati NCSG$_2$ model, are achieved by setting $M_1 = M_3 = \delta_i = 0$, $\varphi_0 = \varphi_1 = 0$, $M_2 = -M\ \varphi_2 = i\varphi$, and $M_1 = M_2 = \delta_i = 0$, $\varphi_0 = 0$, $M_3 = -M\ \varphi_1 = \varphi_2 = i\varphi$, respectively.

4.2.2 Non-commutative (bosonized) Bukhvostov-Lipatov model (NCbBL$_2$). Second version

A reduction leading to a two field model follows as $M_3 = 0,\ M_1 = M_2 = -M,\ \varphi_n \to i\varphi_n\ (n = 0, 1, 2)$. So, one gets the model

$$\partial_-\Big(e_\star^{-i\varphi_0} \star e_\star^{\mp i\varphi_a} \star \partial_+(e_\star^{\pm i\varphi_a} \star e_\star^{i\varphi_0})\Big) = -\frac{M}{8}\Big[e_\star^{-i(\varphi_1-\varphi_2)} \star e_\star^{\mp i\varphi_a} - e_\star^{\pm i\varphi_a} \star e_\star^{i(\varphi_1-\varphi_2)}\Big], \qquad a = 1, 2 \tag{51}$$

$$0 = \partial_-\Big[e_\star^{-i\varphi_0} \star e_\star^{-i\varphi_1} \star \partial_+(e_\star^{i\varphi_1} \star e_\star^{i\varphi_0}) + e_\star^{-i\varphi_0} \star e_\star^{i\varphi_2} \star \partial_+(e_\star^{-i\varphi_2} \star e_\star^{i\varphi_0}) + e_\star^{-i\varphi_0} \star e_\star^{i(\varphi_1-\varphi_2)} \star \partial_+(e_\star^{-i(\varphi_1-\varphi_2)} \star e_\star^{i\varphi_0})\Big], \tag{52}$$

where the upper (lower) signs in (51) correspond to the index $a = 1(2)$ of the field φ_a. In the commutative limit the eq. (52) reduces to a free scalar field equation of motion $\partial^2\varphi_0 = 0$, whereas the set of equations (51) become $\partial^2\varphi_1 = M\sin(2\varphi_1 - \varphi_2)$ and $\partial^2\varphi_2 = M\sin(2\varphi_2 - \varphi_1)$. Defining the new fields $\psi_1 = \frac{1}{2}(\varphi_1 + \varphi_2)$, $\psi_2 = \frac{\sqrt{3}}{2}(\varphi_1 - \varphi_2)$ we arrive at the model $\partial^2\psi_1 = M\sin(\psi_1)\cos(\sqrt{3}\psi_2)$, $\partial^2\psi_2 = M\sqrt{3}\cos(\psi_1)\text{sen}(\sqrt{3}\psi_2)$. As we have seen before this is just the bosonized form of the so-called Bukhvostov-Lipatov model [27, 28, 29, 20]. In view of these relationships we define the model (51)-(52) as the *second version* of the non-commutative (bosonized) Bukhvostov-Lipatov model(NCbBL$_2$).

4.2.3 Non-commutative double sine-Gordon model (NCDSG$_2$). Second version

In order to reduce the NCGSG$_2$ system of equations into another version of the NC double sine-Gordon model one takes advantage of certain properties of its commutative counterpart. In fact, the above commutative model (47)-(49) possesses the symmetry $\varphi_1 \leftrightarrow \varphi_2$; $M_1 \leftrightarrow M_2$ in the GSG sector, whereas the auxiliary

φ_0 field completely decouples in this limit. So, in the second parametrization case (42) we can impose the conditions $\varphi_1 = \varphi_2 \equiv -i\varphi,\ M_1 = M_2, \delta_i = 0$ into the system of eqs. (44)-(46) and obtain the following system of equations for complex φ

$$\partial_-\Big(e_\star^{-\varphi_0} \star e_\star^{i\varphi} \star \partial_+\Big(e_\star^{-i\varphi} \ \star e_\star^{\varphi_0}\Big)\Big) \;=\; 2iM_1 \sin_\star \varphi + 2iM_3 \sin_\star 2\varphi \tag{53}$$

$$\partial_-\Big(e_\star^{-\varphi_0} \star \partial_+ e_\star^{\varphi_0}\Big) \;=\; 0 \tag{54}$$

$$\partial_-\Big[e_\star^{-\varphi_0} \star e_\star^{i\varphi} \star \partial_+\Big(e_\star^{-i\varphi} \ \star e_\star^{\varphi_0}\Big) + e_\star^{-\varphi_0} \star e_\star^{-i\varphi} \star \partial_+\Big(e_\star^{i\varphi} \ \star e_\star^{\varphi_0}\Big)\Big] \;=\; 0. \tag{55}$$

The system (53)-(55) constitutes the *second* version of the non-commutative double sine-Gordon model (NCDSG_2) defined for two complex scalar fields.

The first equation (53) contains the potential terms generalizing the ordinary double sine-Gordon potential. The second and third ones (54)-(55) have the structure of conservation laws and can be seen as imposing extra conditions on the system. Let us examine the commutative limit $\theta \to 0$ of the NCDSG_2 system. In this limit it reduces to the usual DSG model plus a free field φ_0 equations of motion

$$\partial_-\partial_+\varphi \;=\; -2M_1 \sin\varphi - 2M_3 \sin 2\varphi \tag{56}$$

$$\partial_-\partial_+\varphi_0 \;=\; 0. \tag{57}$$

Notice that in this limit the field φ_0 decouples completely from the DSG field φ.

Some comments are in order here.

1) The NC models obtained above reproduce the usual models in the commutative limit $\theta \to 0$. So, the both versions of the $GL(3,\mathbb{C})$ non-commutative generalized sine-Gordon model ($\text{NCGSG}_{1,\,2}$) reproduce the ordinary $GL(3)$ GSG model in this limit. The both versions of the non-commutative double sine-Gordon model $\text{NCDSG}_{1,\,2}$ reproduce the usual DSG model in the ordinary space. Likewise, the both versions of the non-commutative bosonized Bukhvostov-Lipatov model $\text{NCbBL}_{1,\,2}$ lead to the usual BL model. Notice that the GSG model in ordinary space-time also contains as sub-models the variety of theories we have uncovered above, i.e. the usual SG model, Bukhvostov-Lipatov model, and the double sine-Gordon model [20, 24].

2) Regarding the integrability of the $\text{NCGSG}_{1,2}$ models they are hardly expected to possess this property since they contain as sub-models the relevant $\text{NCDSG}_{1,2}$ and $\text{NCBL}_{1,2}$ theories. The $\text{NCDSG}_{1,2}$ models are not expected to posses this property since their commutative counterpart is not integrable. The same behavior may be expected for the $\text{NCBL}_{1,2}$ models since their commutative counterpart is not classically integrable (see [20] and refs. therein), except for some restricted region in parameters space. Nevertheless, see more on this point in subsection 6.1.2 when the relevant spinor version of the (constrained) NCBL_1 model is discussed in relation to integrability.

Related to this issue, let us mention that we have not been able to write in a zero-curvature form the eq. of motion (17) of the NCGSG_1 model (17), it mainly happens due to the presence of the summation index

$m = 1, 2$ on both entries of the commutator. Actually, the eq. (17) differs from the integrable system of non-abelian affine Toda equations [30, 18].

3) The Letchenfeld et al. (34)-(35) and Grisaru-Penati (50) NC sine-Gordon models proposed in the literature appear in the context of the generalized NC sine-Gordon models as reduced sub-models of the corresponding NCGSG_1 and NCGSG_2 models, respectively. So, they are analogous to the results obtained in the commutative case in which the $GL(3)$ GSG model contains three SG sub-models as reduced models, each one associated to the positive root of the $gl(3)$ Lie algebra[24]. The group structure of the $GL(3)$ $\text{NCGSG}_{1,2}$ models allowed us to get three $\text{NCSG}_{1,2}$ sub-models, respectively, for each version, as in the commutative case.

4) In the three-field space of the $\text{NCGSG}_{1,\,2}$ models it is remarkable the appearance of three integrable directions as $\text{NCSG}_{1,\,2}$ sub-models, respectively. It suggests that there are at least three integrable directions in reduced field space of each one of the $\text{NCGSG}_{1,\,2}$ models . Examples of non-integrable reduced directions are provided by the relevant $\text{NCDSG}_{1,\,2}$ and $\text{NCBL}_{1,\,2}$ models. However, the existence of more integrable directions is suggested by the presence of certain integrable sub-models in the spinor sector of the $\text{NCGMT}_{1,2}$ models, i.e. the scalar duals of the corresponding $\text{NC(c)GMT}_{1,2}$ and $\text{NC(c)BL}_{1,2}$ spinor models, respectively (see section (6.1.1) and subsection (6.1.2)).

5) Finally, the role played by the SG model in the context of the generalized SG models is analogous to the one which happens with the correspondence between the $\lambda\phi^4$ model and the deformed linear $O(N)$-sigma model, as it was first noticed in [31]. It could be interesting to study several properties of the generalized SG models, including their non-commutative counterparts, as for example by applying and improving the quantization method described in the last reference. Let us mention that the ordinary DSG model has recently been in the center of some controversy regarding the computation of its semi-classical spectrum, see [32, 33].

5 Decoupling of $\text{NCGSG}_{1,2}$ and $\text{NCGMT}_{1,2}$ models

In the commutative case some approaches have been proposed in order to recover the GSG and GMT dual models out of the ordinary $sl(n)$ ATM model [24, 13, 14, 15, 20]. Among them, the one which proceeds by decoupling the set of equations of motion of the ATM model into the corresponding dual models [13, 15] has turned out to be more suitable in the NC case [9]. This procedure is adapted to the NC case by writing a set of mappings between the fields of the model such that the eqs. (3) and (6) when rewritten using those mappings decouple the scalar and the matter fields. So, following the procedures employed in the ordinary sl(n) case [15] and in the non-commutative $GL(2)$ ATM case [9] to the case at hand, let us consider the mappings

$$\sum_{n=1}^{2} \left[F_n^- \,,\, gF_n^+ g^{-1}\right]_\star \;=\; \sum_{n=1}^{2} \left[\Lambda_n^- \,,\, g\Lambda_n^+ g^{-1}\right]_\star, \tag{58}$$

$$\Big[E_{-3}\,,\,gF^{+}_{3-m}g^{-1}\Big]_{\star} \;=\; \Big[E_{-3}\,,\,F^{+}_{3-m}\Big]_{\star} - \frac{k_2}{2}(L^{+}_{m})^{-1}\Big[\sum_{n}\hat{J}^{-}_{n}\,,\,\hat{F}^{-}_{m}\Big]_{\star}L^{+}_{m}, \tag{59}$$

$$\Big[E_{3}\,,\,g^{-1}F^{-}_{3-m}g\Big]_{\star} \;=\; \Big[E_{3}\,,\,F^{-}_{3-m}\Big]_{\star} - \frac{k_1}{2}(L^{-}_{m})^{-1}\Big[\sum_{n}\hat{J}^{+}_{n}\,,\,\hat{F}^{+}_{m}\Big]_{\star}L^{-}_{m}, \tag{60}$$

$$F^{\pm}_{m} \;=\; \mp[E_{\pm 3}\,,\,W^{\mp}_{3-m}]_{\star}, \tag{61}$$

$$\Big[F^{\pm}_{2}\,,\,g^{\mp 1}F^{\mp}_{1}g^{\pm 1}\Big]_{\star} \;=\; 0, \tag{62}$$

where

$$\hat{J}^{\mp}_{n} \;\equiv\; \Big[\hat{F}^{\pm}_{3-n}\,,\,\hat{W}^{\mp}_{3-n}\Big]; \quad k_1, k_2 = \text{constant parameters.} \tag{63}$$

The hatted fields have the same algebraic structure as the corresponding unhatted ones except that they incorporate some parameters re-scaling the fields, those parameters will give rise to certain coupling constants between the currents of the model. Notice that the fields $\hat{J}^{\pm}_{m}$ and the constant matrices $L^{\pm}_{m}$ carry zero gradation and these will be defined below. The field g in the relations above, as defined in section 2, is assumed to belong to either $[U(1)]^3$, as in subsection 4.1, or $\mathcal{H} \subset GL(3,\mathbb{C})$, as in the second parametrization in subsection 4.2.

The relationships (58)-(61) when conveniently substituted into the ATM eqs. of motion (3) and (6) decouple them, respectively, into the NCGSG$_1$ eq. (17) and certain equations of motion incorporating only matter fields, which in matrix form become

$$\begin{aligned}\Big[E_{-3}, \partial_{+}W^{+}_{3-m}\Big]_{\star} \;&=\; +[E_{-3}, [E_{3}, W^{-}_{m}]]_{\star} - \\ &\frac{k_2}{2}\sum_{n=1}^{2}(L^{+}_{m})^{-1}[\hat{J}^{-}_{n}, [E_{-3}, \hat{W}^{+}_{3-m}]]_{\star}L^{+}_{m}\end{aligned} \tag{64}$$

$$\begin{aligned}\Big[E_{3}, \partial_{-}W^{-}_{3-m}\Big]_{\star} \;&=\; -[E_{3}, [E_{-3}, W^{+}_{m}]]_{\star} - \\ &\frac{k_1}{2}\sum_{n=1}^{2}(L^{-}_{m})^{-1}[\hat{J}^{+}_{n}, [E_{3}, \hat{W}^{-}_{3-m}]]_{\star}L^{-}_{m}\end{aligned} \tag{65}$$

We define these set of eqs. as the *first version* of the non-commutative (generalized) massive Thirring model (NCGMT$_1$).

The eqs. (62) are the constraints imposed in ref. [15] written in a compact form. These constraints, which are missing in the $GL(2)$ case, have been imposed in the non-trivial $GL(3)$ extension in order to be able to write a local Lagrangian for the off-critical and constrained ATM model out of the full set of equations of motion of the so-called conformal affine Toda model coupled to matter (CATM) [15, 18] (see the Appendices). Actually, the above 'decoupling' eqs. maintain the same form as their commutative analogs presented in eqs. (6.1)-(6.5) of the ref. [15]. We must clarify that the above 'decoupling' eqs. (58)-(60) do not completely decouple the scalar fields from the spinor-like fields due to the presence of the constraints (62). There are some instances of total decoupling, e.g. in the soliton sector of the commutative limit [20, 15]. Notice that we have not used the constraint equations (62) in order to get the eqs. (64)-(65). In order to be more specific in the discussions below we provide, in the following set of equations, the constraint eqs. (62) in terms of the component fields. Let us take the spinors as defined in (166)-(171) and the scalar field g presented in the first parametrization eq. (19), so one has

$$e_\star^{i\phi_1} \star \psi_R^1 \star e_\star^{-i\phi_2} \star \psi_L^2 = \psi_L^1 \star e_\star^{i\phi_2} \star \psi_R^2 \star e_\star^{-i\phi_3}, \tag{66}$$

$$e_\star^{i\phi_2} \star \psi_R^2 \star e_\star^{-i\phi_3} \star \widetilde{\psi}_L^3 = -\psi_L^2 \star e_\star^{i\phi_3} \star \widetilde{\psi}_R^3 \star e_\star^{-i\phi_1}; \quad e_\star^{i\phi_3} \star \widetilde{\psi}_R^3 \star e_\star^{-i\phi_1} \star \psi_L^1 = -\widetilde{\psi}_L^3 \star e_\star^{i\phi_1} \star \psi_R^1 \star e_\star^{-i\phi_2} \tag{67}$$

and

$$e_\star^{-i\phi_3} \star \widetilde{\psi}_L^2 \star e_\star^{i\phi_2} \star \widetilde{\psi}_R^1 = \widetilde{\psi}_R^2 \star e_\star^{-i\phi_2} \star \widetilde{\psi}_L^1 \star e_\star^{i\phi_1}, \tag{68}$$

$$e_\star^{-i\phi_1} \star \psi_L^3 \star e_\star^{i\phi_3} \star \widetilde{\psi}_R^2 = -\psi_R^3 \star e_\star^{-i\phi_3} \star \widetilde{\psi}_L^2 \star e_\star^{i\phi_2}; \quad e_\star^{-i\phi_2} \star \widetilde{\psi}_L^1 \star e_\star^{i\phi_1} \star \psi_R^3 = -\widetilde{\psi}_R^1 \star e_\star^{-i\phi_1} \star \psi_L^3 \star e_\star^{i\phi_3} \tag{69}$$

associated to the grades (-1) and $(+1)$ of (62), respectively.

Analogously, one can write another set of equations for the second parametrization (42) of g

$$e_\star^{\varphi_1} \star e_\star^{\varphi_0} \star \psi_R^1 \star e_\star^{-\varphi_0} \star e_\star^{\varphi_1-\varphi_2} \star \psi_L^2 = \psi_L^1 \star e_\star^{\varphi_2-\varphi_1} \star e_\star^{\varphi_0} \star \psi_R^2 \star e_\star^{-\varphi_0} \star e_\star^{\varphi_2}, \tag{70}$$

$$e_\star^{\varphi_2-\varphi_1} \star e_\star^{\varphi_0} \star \psi_R^2 \star e_\star^{-\varphi_0} \star e_\star^{\varphi_2} \star \psi_L^3 = -\psi_L^2 \star e_\star^{-\varphi_2} \star e_\star^{\varphi_0} \star \widetilde{\psi}_R^3 \star e_\star^{-\varphi_0} \star e_\star^{-\varphi_1}, \tag{71}$$

$$e_\star^{-\varphi_2} \star e_\star^{\varphi_0} \star \widetilde{\psi}_R^3 \star e_\star^{-\varphi_0} \star e_\star^{-\varphi_1} \star \psi_L^1 = -\widetilde{\psi}_L^3 \star e_\star^{\varphi_1} \star e_\star^{\varphi_0} \star \psi_R^1 \star e_\star^{-\varphi_0} \star e_\star^{\varphi_1-\varphi_2}, \tag{72}$$

and

$$e_\star^{\varphi_0} \star \widetilde{\psi}_R^2 \star e_\star^{-\varphi_0} \star e_\star^{\varphi_1-\varphi_2} \star \widetilde{\psi}_L^1 \star e_\star^{\varphi_1} = e_\star^{\varphi_2} \star \widetilde{\psi}_L^2 \star e_\star^{\varphi_2-\varphi_1} \star e_\star^{\varphi_0} \star \widetilde{\psi}_R^1 \star e_\star^{-\varphi_0}, \tag{73}$$

$$e_\star^{\varphi_0} \star \psi_R^3 \star e_\star^{-\varphi_0} \star e_\star^{\varphi_2} \star \widetilde{\psi}_L^2 \star e_\star^{\varphi_2-\varphi_1} = -e_\star^{-\varphi_1} \star \psi_L^3 \star e_\star^{-\varphi_2} \star e_\star^{\varphi_0} \star \widetilde{\psi}_R^2 \star e_\star^{-\varphi_0}, \tag{74}$$

$$e_\star^{\varphi_0} \star \widetilde{\psi}_R^1 \star e_\star^{-\varphi_0} \star e_\star^{-\varphi_1} \star \psi_L^3 \star e_\star^{-\varphi_2} = -e_\star^{\varphi_1-\varphi_2} \star \widetilde{\psi}_L^1 \star e_\star^{\varphi_1} \star e_\star^{\varphi_0} \star \psi_R^3 \star e_\star^{-\varphi_0}, \tag{75}$$

associated to the grades (-1) and $(+1)$ of (62), respectively.

Even though that the full set of the 'decoupling' equations have not been used in order to write the eqs. (64)-(65), we expect that a non-commutative version of the usual (generalized) massive Thirring model (GMT_1) [15] defined for the fields $W^{\pm}$ will emerge from these equations. In fact, we assume this point of view and study the properties of the system (64)-(65) in its own right. Nevertheless, we will recognize below certain relationships between the relevant sub-models of the both $\text{NCGSG}_{1,2}$ and $\text{NCGMT}_{1,2}$ sectors. Remarkably, these relationships will arise for certain reduced sectors obtained such that the constraints (62) become trivial, or completely decouple the spinors from the scalars in the soliton sector, which is equivalent to take the commutative limit (see below). The model (NCGMT_1) (64)-(65) is new in the literature and it is expected to correspond to the weak coupling sector of the NCGATM_1 model whose strong coupling sector is described by the first version of the non-commutative generalized sine-Gordon model (NCGSG_1) presented in subsection 4.1.

In the ordinary space the GMT equations of motion can be achieved through Hamiltonian reduction procedures, such as the Faddeev-Jackiw method, as employed in [15] for first order in time Lagrangian;

however, in the NC case, to our knowledge, there is no a similar procedure since the action of the NC GATM model involves higher order in time derivatives; actually, an infinite number of terms of increasing order in time derivatives. So, we have used the decoupling method and assumed the forms of the decoupling equations (58)-(63) to resemble the ones in the ordinary case [15], important guiding lines being the gradation structure and further, the locality of the Lagrangian in the NCGMT sector which will depend on the nature of the terms appearing in the eqs. of motion; e.g, notice the absence of terms bilinear in the spinors in the right hand side of the eqs. (64)-(65). In fact, the terms appearing in the above equations will give rise to usual kinetic and mass terms, and four-spinor coupling terms in the relevant action. The Lagrangian for the model (64)-(65) and a Lax pair formulation for a constrained version of it will be discussed below.

In order to recover the dual of the second version NCGSG_2 one must write similar decoupling expressions for the full set of fields $\{g, F^{\pm}, W^{\pm}\}$ and $\{\bar{g}, \mathcal{F}^{\pm}, \mathcal{W}^{\pm}\}$. Thus, following similar steps to the previous construction we expect to recover another version of the NC generalized massive Thirring model (NCGMT_2) defined for the fields $\{W^{\pm}, \mathcal{W}^{\pm}\}$. In the next section we propose two versions of the non-commutative (generalized) massive Thirring theories ($\text{NCGMT}_{1,2}$) by providing the relevant equations of motion and discussing their zero-curvature formulations.

6 The NC generalized massive Thirring models $\text{NCGMT}_{1,2}$

We will consider the fields $\psi^j, \widetilde{\psi}^j$ as c-number ones [9] in order to define the NC generalization of the so-called (c-number) massive Thirring model (MT) [34, 35]. In ordinary space-time these type of classical c-number multi-field massive Thirring theories have long been considered in relation to one-dimensional Dirac model of extended particles [36]. The quantization of the two-dimensional fermion model with Thirring interaction among N different massive Fermi field species has recently been performed in the functional integral approach [37].

The assumption for the fields to be c-number fields will allow the zero-curvature formulations of the $\text{NCGMT}_{1,2}$ models to be constructed resembling analogous algebraic structures present in the GATM model in the context of the affine Lie algebra $SL(3)$. This means that the c-number fields $\psi^j, \widetilde{\psi}^j$ will lie in certain higher grading directions of the principal gradation of the affine $SL(3)$ Lie algebra, as it is presented in the eqs. (168)-(171) of the Appendix B.

In ordinary space the field components of the MT model are considered to be either anti-commuting Grassmannian fields or some ordinary commuting fields (see [9] and refs. therein). Notice that the relevant (Grassmannian) GMT model would need a slightly different algebraic formulation from the one followed here for the c-number case.

6.1 NCGMT$_1$

We propose the NCGMT$_1$ action related to the eqs. of motion (64)-(65) for the fields $W_m^{\pm}$ as

$$S[W_m^{\pm}] = \int dx^2 \Big[\sum_{m=1}^{2} \{ \frac{1}{2} < [E_{-3}, W_{3-m}^{+}] \star \partial_+ W_m^{+} > -\frac{1}{2} < [E_3, W_{3-m}^{-}] \star \partial_- W_m^{-} > - < [E_{-3}, W_m^{+}] \star [E_3, W_m^{-}] > \} - \frac{1}{2} \sum_{m,n=1}^{2} < \hat{J}_m^{+} \star \hat{J}_n^{-} > \Big]. \tag{76}$$

In the last action the first two terms inside the summation provide the kinetic terms, the third one the mass terms and the last term the current-current interactions. The current-like matrices $\hat{J}_m^{\pm}$ with zero gradation appearing in the eq. (63) have the same algebraic structure as the matrix-valued currents [15]

$$J_m^{\pm} = \pm \frac{1}{4} [[E_{\mp 3}, W_m^{\pm}], W_{3-m}^{\pm}]_{\star}, \tag{77}$$

except that they are defined in terms of some hatted variables $\hat{W}_m^{\pm}$ which are constructed from the relevant unhatted ones $W_m^{\pm}$ in eqs. (168)-(171) by making the re-scalings

$$\widetilde{\psi}_L^1 \rightarrow (\frac{\lambda_1}{2})^{1/4} \widetilde{\psi}_L^1, \;\; \widetilde{\psi}_L^2 \rightarrow (\frac{\lambda_2}{2})^{1/4} \widetilde{\psi}_L^2, \;\; \psi_L^3 \rightarrow (\frac{\lambda_3}{2})^{1/4} \psi_L^3. \tag{78}$$

$$\psi_L^1 \rightarrow (\frac{\delta_1}{2})^{1/4} \psi_L^1, \;\; \psi_L^2 \rightarrow (\frac{\delta_2}{2})^{1/4} \psi_L^2, \;\; \widetilde{\psi}_L^3 \rightarrow (\frac{\delta_3}{2})^{1/4} \widetilde{\psi}_L^3. \tag{79}$$

$$\widetilde{\psi}_R^1 \rightarrow (\frac{\alpha_1}{2})^{1/4} \widetilde{\psi}_R^1, \;\; \widetilde{\psi}_R^2 \rightarrow (\frac{\alpha_2}{2})^{1/4} \widetilde{\psi}_L^2, \;\; \psi_R^3 \rightarrow (\frac{\alpha_3}{2})^{1/4} \psi_R^3. \tag{80}$$

$$\psi_R^1 \rightarrow (\frac{\beta_1}{2})^{1/4} \psi_R^1, \; \psi_R^2 \rightarrow (\frac{\beta_2}{2})^{1/4} \psi_R^2, \;\; \widetilde{\psi}_R^3 \rightarrow (\frac{\beta_3}{2})^{1/4} \widetilde{\psi}_R^3, \tag{81}$$

where the λ_j δ_j, α_j, β_j are constant parameters. These constants are introduced with the aim of recovering some coupling constants between the currents of the model.

Actually, in matrix form we have the following relationships $\hat{W}_m^{+} = L_m^{+} W_m^{+} (L_m^{+})^{-1}$ and $\hat{W}_m^{-} = L_m^{-} W_m^{-} (L_m^{-})^{-1}$. The $L_2^{\pm}$, $L_1^{\mp}$ matrices, respectively, take the following forms

$$\begin{pmatrix} \sqrt[12]{\frac{x_3}{x_1}} & 0 & 0 \\ 0 & \sqrt[12]{\frac{x_1}{x_2}} & 0 \\ 0 & 0 & \sqrt[12]{\frac{x_2}{x_3}} \end{pmatrix} \text{ and } \begin{pmatrix} \sqrt[12]{\frac{y_1}{y_3}} & 0 & 0 \\ 0 & \sqrt[12]{\frac{y_2}{y_1}} & 0 \\ 0 & 0 & \sqrt[12]{\frac{y_3}{y_2}} \end{pmatrix}, \tag{82}$$

supplied with the replacements $x \rightarrow \lambda$ for L_2^{+}, $y \rightarrow \beta$ for L_2^{-}, $x \rightarrow \alpha$ for L_1^{-}, and $y \rightarrow \delta$ for L_1^{+}.

Some relationships between these parameters will emerge below mainly arising from the consideration of current-current (generalized Thirring) type interactions among the various flavor species and integrability requirement through the zero-curvature formulation of the equations of motion.

In the following we will consider the eqs. of motion (64)-(65) in term of the field components. For future convenience let us introduce the fields $A_{R,\,L}^{i}$ as

$$A_R^1 = \sqrt[4]{\frac{\alpha_1 \beta_1}{4}} \, \psi_R^1 \star \tilde{\psi}_R^1 + \sqrt[4]{\frac{\beta_3 \alpha_3}{4}} \, \psi_R^3 \star \tilde{\psi}_R^3 \tag{83}$$

$$A_R^2 = \sqrt[4]{\frac{\alpha_2\beta_2}{4}}\, \psi_R^2 \star \tilde{\psi}_R^2 - \sqrt[4]{\frac{\alpha_1\beta_1}{4}}\, \tilde{\psi}_R^1 \star \psi_R^1 \tag{84}$$

$$A_R^3 = \sqrt[4]{\frac{\beta_3\alpha_3}{4}}\, \tilde{\psi}_R^3 \star \psi_R^3 + \sqrt[4]{\frac{\alpha_2\beta_2}{4}}\, \tilde{\psi}_R^2 \star \psi_R^2. \tag{85}$$

and

$$A_L^1 = \sqrt[4]{\frac{\delta_1\lambda_1}{4}}\, \psi_L^1 \star \tilde{\psi}_L^1 + \sqrt[4]{\frac{\delta_3\lambda_3}{4}}\, \psi_L^3 \star \tilde{\psi}_L^3 \tag{86}$$

$$A_L^2 = \sqrt[4]{\frac{\delta_2\lambda_2}{4}}\, \psi_L^2 \star \tilde{\psi}_L^2 - \sqrt[4]{\frac{\delta_1\lambda_1}{4}}\, \tilde{\psi}_L^1 \star \psi_L^1 \tag{87}$$

$$A_L^3 = \sqrt[4]{\frac{\delta_3\lambda_3}{4}}\, \tilde{\psi}_L^3 \star \psi_L^3 + \sqrt[4]{\frac{\delta_2\lambda_2}{4}}\, \tilde{\psi}_L^2 \star \psi_L^2. \tag{88}$$

In terms of these fields the currents in (76) become

$$\hat{J}_1^- = \hat{J}_2^- = -\frac{i}{2}\begin{pmatrix} A_R^1 & 0 & 0 \\ 0 & A_R^2 & 0 \\ 0 & 0 & -A_R^3 \end{pmatrix} \text{ and } \hat{J}_1^+ = \hat{J}_2^+ = -\frac{i}{2}\begin{pmatrix} A_L^1 & 0 & 0 \\ 0 & A_L^2 & 0 \\ 0 & 0 & -A_L^3 \end{pmatrix} \tag{89}$$

Therefore the action of the $NCGMT_1$ model (76) in terms of the Thirring field components become

$$\begin{aligned} S_{NCGMT_1} &= \int dx^2 \sum_{i=1}^{i=3} \Big\{ \Big[2i\tilde{\psi}_L^i \star \partial_+ \psi_L^i + 2i\tilde{\psi}_R^i \star \partial_- \psi_R^i + im_i(\tilde{\psi}_L^i \star \psi_R^i - \psi_L^i \star \tilde{\psi}_R^i) \Big] \\ &- 2(A_L^i \star A_R^i) \Big\}, \end{aligned} \tag{90}$$

Next let us write the equations of motion for the field components derived from the action above. The following three equations of motion

$$\partial_+\psi_L^3 = -\frac{1}{2}m_3\psi_R^3 - i\sqrt[4]{\frac{\delta_3\lambda_3}{4}}\,\{\psi_L^3 \star A_R^3 + A_R^1 \star \psi_L^3\} \tag{91}$$

$$\partial_+\tilde{\psi}_L^1 = -\frac{1}{2}m_1\tilde{\psi}_R^1 + i\sqrt[4]{\frac{\delta_1\lambda_1}{4}}\,\{\tilde{\psi}_L^1 \star A_R^1 - A_R^2 \star \tilde{\psi}_L^1\} \tag{92}$$

$$\partial_+\tilde{\psi}_L^2 = -\frac{1}{2}m_2\tilde{\psi}_R^2 + i\sqrt[4]{\frac{\delta_2\lambda_2}{4}}\,\{\tilde{\psi}_L^2 \star A_R^2 + A_R^3 \star \tilde{\psi}_L^2\}, \tag{93}$$

will correspond to the matrix form (64) for $m = 1$.

One can obtain the equations of motion

$$\partial_+\tilde{\psi}_L^3 = -\frac{1}{2}m_3\tilde{\psi}_R^3 + i\sqrt[4]{\frac{\delta_3\lambda_3}{4}}\,\{A_R^3 \star \tilde{\psi}_L^3 + \tilde{\psi}_L^3 \star A_R^1\} \tag{94}$$

$$\partial_+\psi_L^1 = -\frac{1}{2}m_1\psi_R^1 - i\sqrt[4]{\frac{\delta_1\lambda_1}{4}}\,\{A_R^1 \star \psi_L^1 - \psi_L^1 \star A_R^2\} \tag{95}$$

$$\partial_+\psi_L^2 = -\frac{1}{2}m_2\psi_R^2 - i\sqrt[4]{\frac{\delta_2\lambda_2}{4}}\{A_R^2 \star \psi_L^2 + \psi_L^2 \star A_R^3\}, \tag{96}$$

which in matrix form corresponds to eq. (64) for $m = 2$.

Similarly, one can obtain the equations of motion

$$\partial_- \psi_R^3 = \frac{1}{2} m_3 \psi_L^3 - i \sqrt[4]{\frac{\alpha_3\beta_3}{4}} \{\psi_R^3 \star A_L^3 + A_L^1 \star \psi_R^3\} \quad (97)$$

$$\partial_- \tilde{\psi}_R^1 = \frac{1}{2} m_1 \tilde{\psi}_L^1 + i \sqrt[4]{\frac{\alpha_1\beta_1}{4}} \{\tilde{\psi}_R^1 \star A_L^1 - A_L^2 \star \tilde{\psi}_R^1\} \quad (98)$$

$$\partial_- \tilde{\psi}_R^2 = \frac{1}{2} m_2 \tilde{\psi}_L^2 + i \sqrt[4]{\frac{\alpha_2\beta_2}{4}} \{\tilde{\psi}_R^2 \star A_L^2 + A_L^3 \star \tilde{\psi}_R^2\}, \quad (99)$$

corresponding to $m = 2$ in (65).

Finally, the equations

$$\partial_- \tilde{\psi}_R^3 = \frac{1}{2} m_3 \tilde{\psi}_L^3 + i \sqrt[4]{\frac{\alpha_3\beta_3}{4}} \{A_L^3 \star \tilde{\psi}_R^3 + \tilde{\psi}_R^3 \star A_L^1\} \quad (100)$$

$$\partial_- \psi_R^1 = \frac{1}{2} m_1 \psi_L^1 - i \sqrt[4]{\frac{\alpha_1\beta_1}{4}} \{A_L^1 \star \psi_R^1 - \psi_R^1 \star A_L^2\} \quad (101)$$

$$\partial_- \psi_R^2 = \frac{1}{2} m_2 \psi_L^2 - i \sqrt[4]{\frac{\alpha_2\beta_2}{4}} \{A_L^2 \star \psi_R^2 + \psi_R^2 \star A_L^3\}, \quad (102)$$

can be obtained from (65) in the case $m = 1$.

The set of equations of motions (91)-(102) are the $GL(3)$ extension of the equations of motion given before for the case $GL(2)$ NCMT$_1$ (see eqs. (5.11)-(5.14) of ref. [9]). In fact, the later system is contained in the $GL(3)$ extended model. For example, if one considers $\psi_L^1 = \psi_L^2 = \tilde{\psi}_L^1 = \tilde{\psi}_L^1 = 0$ in the eq. (91) then it is reproduced the equation (5.13) of reference [9] describing the single Thirring field ψ_3 provided that the parameters expression $\sqrt[4]{\frac{\delta_3\lambda_3\beta_3\alpha_3}{16}}$ corresponds to the coupling constant $\frac{\lambda}{2}$ of that reference.

The four field interaction terms in the action (90) can be re-written as a sum of Dirac type current-current terms for the various flavors ($j = 1, 2, 3$). In the constructions of the relevant currents the double-gauging of a U(1) symmetry in the star-localized Noether procedure deserves a careful treatment [38, 9]. So, one has two types of currents for each flavor [9]

$$j_k^{(1)\,\mu} = \bar{\psi}_k \gamma^\mu \star \psi_k, \quad (103)$$

$$j_k^{(2)\,\mu} = -\psi_k^T \gamma^0 \gamma^\mu \star \widetilde{\psi}_k, \quad k = 1, 2, 3.. \quad (104)$$

Notice that in the commutative limit one has $j_k^{(1)\,\mu} = j_k^{(2)\,\mu}$. In order to write as a sum of current-current interaction terms it is necessary to impose the next constraints on the $\alpha_i, \beta_i, \delta_i$, λ_i parameters

$$\frac{\delta_j \lambda_j}{\alpha_j \beta_j} = \kappa = \text{const.}; \quad j = 1, 2, 3. \quad (105)$$

Then the four-spinor interactions terms in (90), provided that (105) is taken into account, can be written as current-current interaction terms

$$\begin{aligned} -2\sum_{i=1}^{3} A_L^i A_R^i = & -g_{11}\,(j_{1\,\mu}^{(1)} \star j_1^{(1)\mu} + j_{1\,\mu}^{(2)} \star j_1^{(2)\mu}) - g_{22}\,(j_{2\,\mu}^{(1)} \star j_2^{(1)\mu} + j_{2\,\mu}^{(2)} \star j_2^{(2)\mu}) - \\ & g_{33}\,(j_{3\,\mu}^{(1} \star j_3^{(1)\mu} + j_{3\,\mu}^{(2)} \star j_3^{(2)\mu}) + g_{12}\,(j_{1\,\mu}^{(1)} \star j_2^{(2)\mu}) - \\ & g_{23}\,(j_{2\,\mu}^{(1)} \star j_3^{(1)\mu}) - g_{13}(j_{1\,\mu}^{(2)} \star j_3^{(2)\mu}), \end{aligned} \quad (106)$$

where

$$g_{jj} = \frac{1}{4}\sqrt[4]{\alpha_j\beta_j\delta_j\lambda_j}, \quad g_{jk} = \frac{1}{2}\sqrt[4]{\alpha_j\beta_j\delta_k\lambda_k}, \, (j \neq k); \quad j,k = 1,2,3. \tag{107}$$

These parameters g_{ij} define the coupling constants of the NC generalized Thirring model (NCGMT$_1$), even though that they are not mutually independent. Notice that considering the relationships (105) and (107) one has the three constraints

$$g_{ij} = 2\sqrt{g_{ii}\, g_{jj}}, \quad i \neq j. \tag{108}$$

Taking into account the constraints (108) we are left with three independent coupling parameters at our disposal, so in order to study further properties such as the integrability and the zero-curvature formulations of the model one must consider the remaining three parameters, say the independent coupling parameters g_{11}, g_{22}, g_{33}. Then, substituting in the action (90) the current-current interaction terms (106) one has

$$\begin{aligned} S_{NCGMT_1} \;\; = \;\; & \int dx^2 \Big\{ \sum_{i=1}^{i=3} \Big[2i\tilde{\psi}_L^i \star \partial_+ \psi_L^i + 2i\tilde{\psi}_R^i \star \partial_- \psi_R^i + im_i(\tilde{\psi}_L^i \star \psi_R^i - \psi_L^i \star \tilde{\psi}_R^i) \Big] \\ - \;\; & g_{11}\, (j^{(1)}_{1\,\mu} \star j_1^{(1)\mu} + j^{(2)}_{1\,\mu} \star j_1^{(2)\mu}) - g_{22}\, (j^{(1)}_{2\,\mu} \star j_2^{(1)\mu} + j^{(2)}_{2\,\mu} \star j_2^{(2)\mu}) - \\ & g_{33}\, (j^{(1}_{3\,\mu} \star j_3^{(1)\mu} + j^{(2)}_{3\,\mu} \star j_3^{(2)\mu}) + g_{12}\, (j^{(1)}_{1\,\mu} \star j_2^{(2)\mu}) - \\ & g_{23}\, (j^{(1)}_{2\,\mu} \star j_3^{(1)\mu}) - g_{13} (j^{(2)}_{1\,\mu} \star j_3^{(2)\mu}) \Big\}. \end{aligned} \tag{109}$$

We define this model as the NC (generalized) massive Thirring model NCGMT$_1$ written in terms of the component fields. Its matrix version is understood to be the action (76) once the parameters relationships (105) are taken into account.

The two types of U(1) currents $j^{(1)}_{k\,\mu}$, $j^{(2)}_{k\,\mu}$ (k=1,2,3), respectively, satisfy the conservation equations

$$\partial_+(\tilde{\psi}_L^k \star \psi_L^k) + \partial_-(\tilde{\psi}_R^k \star \psi_R^k) = 0, \quad \partial_+(\psi_L^k \star \tilde{\psi}_L^k) + \partial_-(\psi_R^k \star \tilde{\psi}_R^k) = 0, \quad k = 1,2,3. \tag{110}$$

6.1.1 (Constrained) NC(c)GMT$_1$ zero-curvature formulation

The zero-curvature condition encodes integrability even in the NC extension of integrable models (see e.g. [9] and references therein), as this condition allows, for example, the construction of infinite conserved charges for them. In order to tackle this problem it is convenient to consider the matrix form of the equations of motion of the $GL(3)$ NC Thirring model (64)-(65) and intend to write them as originating from a zero-curvature condition. So, taking into account the gradation structure of the model let us consider the following Lax pair

$$A_- \;\; = \;\; E_{-3} + a[E_{-3}, W_1^+]_\star + b[E_{-3}, W_2^+]_\star + g_1[[E_{-3}, \hat{W}_1^+], \hat{W}_2^+]_\star + g_2[[E_{-3}, \hat{W}_2^+], \hat{W}_1^+]_\star. \tag{111}$$

$$A_+ \;\; = \;\; -E_{+3} + b[E_{+3}, W_1^-]_\star + a[E_{+3}, W_2^-]_\star + \widetilde{g}_1[[E_{+3}, \hat{W}_1^-], \hat{W}_2^-]_\star + \widetilde{g}_2[[E_{+3}, \hat{W}_2^-], \hat{W}_1^-]_\star, \tag{112}$$

where $a, b, g_1, g_2, \widetilde{g}_1, \widetilde{g}_2$ are some parameters to be determined below. Notice that the potentials $A_\pm$ lie in the directions of the affine Lie algebra generators of grade $\mathcal{G}_{0,1,2,3}$ and $\mathcal{G}_{0,-1,-2,-3}$, respectively.

These matrix valued fields must be replaced into the zero-curvature equation

$$\Big[\partial_+ + A_+\,,\,\partial_- + A_-\Big]_\star = 0, \tag{113}$$

We will use the following relationships which can easily be established

$$4\hat{J}_1^- \;=\; 4\hat{J}_2^- = -[[E_{+3}, \hat{W}_1^-], \hat{W}_2^-]_\star = -[[E_{+3}, \hat{W}_2^-], \hat{W}_1^-]_\star, \tag{114}$$

$$4\hat{J}_1^+ \;=\; 4\hat{J}_2^+ = [[E_{-3}, \hat{W}_1^+], \hat{W}_2^+]_\star = [[E_{-3}, \hat{W}_2^+], \hat{W}_1^+]_\star \tag{115}$$

So, the Lax pair can be rewritten as

$$A_- \;=\; E_{-3} + a[E_{-3}, W_1^+]_\star + b[E_{-3}, W_2^+]_\star + k_1\hat{J}_1^+. \tag{116}$$

$$A_+ \;=\; -E_{+3} + b[E_{+3}, W_1^-]_\star + a[E_{+3}, W_2^-]_\star + k_2\hat{J}_1^-, \tag{117}$$

where we have introduced the new parameters $k_{1,\,2}$ such that $\widetilde{g}_1 + \widetilde{g}_2 = -\frac{k_2}{4}$, and $g_1 + g_2 = \frac{k_1}{4}$

In order to get the relevant equations of motion (64)-(65) it is useful to take into consideration the gradation structure of the various terms. So, the terms of gradation (-1) in (113), taking into account (114), become

$$\Big[E_{-3}, \partial_+ W_2^+\Big]_\star \;=\; +[E_{-3}, [E_3, W_1^-]]_\star - k_2(L_2^+)^{-1}[\hat{J}_1^-, [E_{-3}, \hat{W}_2^+]]_\star L_2^+ + \Big[F_1^+, F_2^-\Big]_\star \tag{118}$$

The equation (118) has the same structure as the equation of motion (64) (for $m = 1$) provided that we set $L_2^+ = L_1^+$, and impose the constraint

$$\Big[F_1^+, F_2^-\Big]_\star = 0. \tag{119}$$

Next, looking for the gradation $(+1)$ terms in (113) and using (115) we may get the equation

$$\Big[E_3, \partial_- W_2^-\Big]_\star \;=\; -[E_3, [E_{-3}, W_1^+]]_\star - k_1(L_2^-)^{-1}[\hat{J}_1^+, [E_3, \hat{W}_2^-]]_\star L_2^- + \Big[F_2^+, F_1^-\Big]_\star. \tag{120}$$

In a similar way, identifying $L_2^- = L_1^-$, and imposing the constraint

$$\Big[F_2^+, F_1^-\Big]_\star = 0, \tag{121}$$

one notices that the equation (120) is equal to the equation of motion (65) (for $m = 1$).

Following the process we can write for the (± 2) gradations and conclude that in order to obtain the two equations of motion in (64)-(65) for $m = 2$, it is required the same conditions $L_2^\pm = L_1^\pm$ as above, without any new constraint.

We notice that the conditions $L_2^\pm = L_1^\pm$ which are related to the equations of motion for the gradations $(\pm 1), (\pm 2)$ provide the following constraints between the initial parameters $(\alpha_i, \beta_i, \lambda_i, \delta_i)$

$$\alpha_i\beta_i = r_1; \quad \lambda_i\delta_i = r_2, \quad i = 1, 2, 3; \quad r_1,\; r_2 = \text{constants}. \tag{122}$$

In fact, these constraints are consistent with the parameters relationships (105) established above; however, eqs. (122) incorporate additional constant parameters r_1, r_2 such that $\kappa = r_2/r_1$. Additional relationships between the parameters arise by requiring that the above matrix equations derived from the zero-curvature equation to be consistent with the eqs. of motion (91)-(102). So, together with the relationships (122), it is required

$$\alpha_1\alpha_2\alpha_3 = \beta_1\beta_2\beta_3 \equiv r_1^{3/2}; \quad \lambda_1\lambda_2\lambda_3 = \delta_1\delta_2\delta_3 \equiv r_2^{3/2}, \;\; k_1 \;\;=\;\; 2^{3/4}r_1^{1/8}, \quad k_2 = 2^{3/4}r_2^{1/8} \tag{123}$$

So, the set of current-current coupling constants g_{ij} in (109), which in the last section have been assumed to be equivalent to three independent parameters, in view of the additional relationships (123) they reduce to only one independent parameter g defined by

$$g_{12} \;\;=\;\; g_{23} = g_{13} = \frac{1}{2}g; \quad g_{ii} = \frac{1}{4}g, \;\; i = 1,2,3; \;\; g \equiv (r_1 r_2)^{1/4}. \tag{124}$$

Finally, for the zero gradation term there appears the following equation

$$k_1\partial_+\hat{J}_1^+ - k_2\partial_-\hat{J}_1^- - ab[F_2^+, F_2^-] - ab[F_1^+, F_1^-] + k_1k_2[\hat{J}_1^-, \hat{J}_1^+] = 0. \tag{125}$$

We require this equation to be consistent with the full equations of motion (91)-(102) and the constraints (119) and (121). These constraints in terms of the fundamental fields become

$$\psi_R^1 * \psi_L^2 = \psi_L^1 * \psi_R^2, \;\; \psi_R^2 * \tilde{\psi}_L^3 = -\psi_L^2 * \tilde{\psi}_R^3, \;\; \tilde{\psi}_L^3 * \psi_R^1 = -\tilde{\psi}_R^3 * \psi_L^1 \tag{126}$$

and

$$\psi_R^3 * \tilde{\psi}_L^2 = -\psi_L^3 * \tilde{\psi}_R^2, \;\; \tilde{\psi}_L^1 * \psi_R^3 = -\tilde{\psi}_R^1 * \psi_L^3, \;\; \tilde{\psi}_R^2 * \tilde{\psi}_L^1 = \tilde{\psi}_L^2 * \tilde{\psi}_R^1, \tag{127}$$

respectively.

In order to establish specific relationships between the parameters a, b and r_1, r_2 let us write (125) in terms of the fundamental fields

$$i(k_1\partial_+ A_L^1 - k_2\partial_- A_R^1) \;\;=\;\; -\frac{k_1k_2}{2}(A_R^1 \star A_L^1 - A_L^1 \star A_R^1) - 2ab\{im_1(\sqrt[4]{\frac{\beta_1\lambda_1}{4}}\psi_R^1 \star \tilde{\psi}_L^1 + \sqrt[4]{\frac{\alpha_1\delta_1}{4}}\psi_L^1 \star \tilde{\psi}_R^1) + im_3(\sqrt[4]{\frac{\alpha_3\delta_3}{4}}\psi_R^3 \star \tilde{\psi}_L^3 + \sqrt[4]{\frac{\beta_3\lambda_3}{4}}\psi_L^3 \star \tilde{\psi}_R^3)\} \tag{128}$$

$$i(k_1\partial_+ A_L^2 - k_2\partial_- A_R^2)_\star \;\;=\;\; -\frac{k_1k_2}{2}(A_R^2 \star A_L^2 - A_L^2 \star A_R^2) - 2ab\{im_2(\sqrt[4]{\frac{\beta_2\lambda_2}{4}}\psi_R^2 \star \tilde{\psi}_L^2 + \sqrt[4]{\frac{\alpha_2\delta_2}{4}}\psi_L^2 \star \tilde{\psi}_R^2) - im_1(\sqrt[4]{\frac{\delta_2\alpha_2}{4}}\tilde{\psi}_R^1 \star \psi_L^1 + \sqrt[4]{\frac{\beta_1\lambda_1}{4}}\tilde{\psi}_L^1 \star \psi_R^1)\} \tag{129}$$

$$i(k_1\partial_+ A_L^3 - k_2\partial_- A_R^3)_\star \;\;=\;\; \frac{k_1k_2}{2}(A_R^3 \star A_L^3 - A_L^3 \star A_R^3) - 2ab\{im_3(\sqrt[4]{\frac{\beta_3\lambda_3}{4}}\tilde{\psi}_R^3 \star \psi_L^3 + \sqrt[4]{\frac{\delta_3\alpha_3}{4}}\tilde{\psi}_L^3 \star \psi_R^3) + im_2(\sqrt[4]{\frac{\delta_2\alpha_2}{4}}\tilde{\psi}_R^2 \star \psi_L^2 + \sqrt[4]{\frac{\beta_2\lambda_2}{4}}\tilde{\psi}_L^2 \star \psi_R^2)\}. \tag{130}$$

Substituting the fields $A^j_{R,L}$, $j = 1, 2, 3$ in the form (83)-(88) into the eqs. (128)-(130) and taking into account the set of equations of motion (91)-(102) one gets the following relationships

$$2ab = \sqrt{\frac{g}{2}}; \quad (2)^{1/4} = r_1^{1/8} + r_2^{1/8}. \tag{131}$$

Therefore, we have established a zero-curvature formulation of a constrained version of the NCGMT$_1$ model. From this point forward this constrained model will be dubbed as NC(c)GMT$_1$.

Notice that the set of equations (128)-(130) contain the relevant eq. associated to the $SL(2)$ NC massive Thirring model written for its relevant zero gradation sector analogous to (125). So, for example, if one reduces the eq. (130) to get an equation for just one field, say ψ^3, one has

$$\begin{aligned} i\Big[k_1 \sqrt[4]{\frac{r_2}{4}}\partial_+(\tilde{\psi}^3_L \star \psi^3_L) - k_2 \sqrt[4]{\frac{r_1}{4}}\partial_-(\tilde{\psi}^3_R \star \psi^3_R)\Big] \quad &= \quad -2iabm_3\Big(\sqrt[4]{\frac{\beta_3\lambda_3}{4}}\tilde{\psi}^3_R \star \psi^3_L + \sqrt[4]{\frac{\delta_3\alpha_3}{4}}\tilde{\psi}^3_L \star \psi^3_R\Big) + \\ &\frac{k_1k_2}{2}\sqrt[4]{\frac{r_1r_2}{16}}(\tilde{\psi}^3_R \star \psi^3_R \star \tilde{\psi}^3_L \star \psi^3_L - \tilde{\psi}^3_L \star \psi^3_L \star \tilde{\psi}^3_R \star \psi^3_R) \\ &. \end{aligned} \tag{132}$$

Now, taking into account $\alpha_3 = \beta_3 = \delta_3 = \lambda_3$ $[r_1 = r_2 \equiv r]$ and the identifications $\psi^3 \to i\, r^{1/16}\psi$, $r^{1/2} \to \lambda$, $[m_3 \frac{r^{1/8}}{2^{5/4}}] \to m_\psi$ we arrive at the equation $\partial_-(\tilde{\psi}_R \star \psi_R) - \partial_+(\tilde{\psi}_L \star \psi_L) = m_\psi\Big(\tilde{\psi}_R \star \psi_L + \tilde{\psi}_L \star \psi_R\Big) - i\lambda(\tilde{\psi}_R \star \psi_R \star \tilde{\psi}_L \star \psi_L - \tilde{\psi}_L \star \psi_L \star \tilde{\psi}_R \star \psi_R)$, which is the eq. (5.18) of the ref. [9] .

6.1.2 NCGMT$_1$ sub-models

In the following we discuss some reduced models associated to the action (109) and its equations of motion (91)-(102).

NC massive Thirring (NCMT$_1$) models

The reduction of the NCGMT$_1$ model equations of motion (91)-(102) to a model with just one spinor field, say the components $\psi^1_{R,L}$, $\widetilde{\psi}^1_{R,L}$ (consider the reduction $\psi^{2,3}_{R,\,L} = \widetilde{\psi}^{2,3}_{R,\,L} = 0$) reproduces the NCMT$_1$ model which has been presented in [9, 10]. Notice that in this case the constraints (126) and (127), as well as the decoupling equations (62) [or in components (66)-(69)] become trivial. Let us emphasize that the full decoupling eqs. are satisfied by a subset of soliton solutions of the field equations of the $GL(2)$ NCATM$_1$ model such that the two sectors NCSG$_1$/NCMT$_1$ completely decouple [9]. Reducing in this way it is clear the appearance of three copies of the NCMT$_1$ model associated to the spinors ψ^1, ψ^2 and ψ^3, respectively.

NC Bukhvostov-Lipatov (NCBL$_1$) model

Consider a reduced model with two fields, say $\psi^{1,2}_{R,L}$, $\widetilde{\psi}^{1,2}_{R,L}$, achieved through the reduction $\psi^3_{R,\,L} = \widetilde{\psi}^3_{R,\,L} = 0$. So, the Lagrangian (109) becomes

$$\begin{aligned} S_{NCTM} \quad = \quad &\int dx^2 \Big\{ \sum_{i=1}^{i=2} \Big[2i\tilde{\psi}^i_L \star \partial_+\psi^i_L + 2i\tilde{\psi}^i_R \star \partial_-\psi^i_R + im_i(\tilde{\psi}^i_L \star \psi^i_R - \psi^i_L \star \tilde{\psi}^i_R)\Big] \\ &-g_{11}\,(j^{(1)}_{1\,\mu} \star j^{(1)\mu}_1 + j^{(2)}_{1\,\mu} \star j^{(2)\mu}_1) - g_{22}\,(j^{(1)}_{2\,\mu} \star j^{(1)\mu}_2 + j^{(2)}_{2\,\mu} \star j^{(2)\mu}_2) \\ &+g_{12}\,(j^{(1)}_{1\,\mu} \star j^{(2)\mu}_2)\Big\}. \end{aligned} \tag{133}$$

Remember that in ordinary space there is no distinction between the type of $j_i^{(1)}$ and $j_i^{(2)}$ currents for each flavor i ; so, the model (133) when written in ordinary space-time is known in the literature as the Bukhvostov-Lipatov model (BL) [29]. It has been claimed the classical integrability of the model in two special cases $g_{12} = 0$ (2× MT model) and $g_{11} = g_{22} = 0$ (BL model) [in both cases consider $m_1 = m_2$](see [27] and refs. therein). The quantum integrability of the BL model has been discussed in [28]. In view of the above discussion we define the model (133) as the first version of the NC Bukhvostov-Lipatov model (NCBL$_1$). Actually, there are additionally two reduction processes to arrive at NCBL$_1$ models, i.e. by setting $\psi^1 = 0$ and $\psi^2 = 0$ in (109), respectively.

(Constrained) NC Bukhvostov-Lipatov (NC(c)BL$_1$) and Lax pair formulation

Let us discuss a constrained version of the model (133). In view of the developments above one can establish the zero-curvature formulation of a constrained model associated to the model (133) by setting $\psi^3_{L.R} = \widetilde{\psi}^3_{L.R} = 0$ in the matrices $W^{\pm}_{1,2}$ of the Lax pair eqs. (111)-(112), provided the constraints (126) and (127) given in the form $\psi^1_R * \psi^2_L = \psi^1_L * \psi^2_R$ and $\tilde{\psi}^2_R * \tilde{\psi}^1_L = \tilde{\psi}^2_L * \tilde{\psi}^1_R$, are considered. So, we claim that the model (133) is classically integrable provided that the above constraints are taken into account. In this way, provided that for version 2 one writes a copy of the model and their relevant constraints, one defines the (constrained) NC(c)BL$_{1,2}$ models amenable to a Lax pair formulation .

In connection to this development, let us mention that a version of the BL model for Grassmanian fields in usual space-time has also been recently shown to be associated to a Lax pair formulation provided some constraints are imposed [39].

In Fig. 1 we have outlined the various relationships. Notice that we have the two versions of NCGATM$_{1,\,2}$ and their strong/weak sectors described by the models NCGSG$_{1,\,2}$ and NCGMT$_{1,\,2}$, respectively, as well as the relevant sub-models. We have emphasized the field contents in each stage of the reductions.

Some comments are in order here.

1. The action (109) (or its matrix form (76)) defines a three species NC generalized massive Thirring model. We have tried to write its eqs. of motion (64)-(65) [or in components (91)-(102)] as deriving from a zero-curvature formulation. We have proposed a Lax pair reproducing the same equations of motion provided that the constraints (119) and (121)[or in components (126) and (127)] are imposed. This fact suggests that the NCGMT$_1$ model (76) becomes integrable only for a sub-model defined by the eqs. of motion (91)-(102) provided the constraints (119) and (121) are satisfied [40]. So, one expects that a careful introduction of the constraints trough certain Lagrange multipliers into the action will provide the Lagrangian formulation of an integrable sub-model of the NCGMT$_1$ theory.

2. Regarding the action related to the full zero-curvature equations of motion without constraints, determined by the set of eqs. (118) and (120), and the relevant eqs. in (64)-(65) written for $m = 2$, it is interesting to notice that the quadratic terms in the spinors present in the first couple of eqs. of motion (118) and (120) make it difficult to believe that one can find a local Lagrangian for the theory. Obviously, in that case we could not have a generalized massive Thirring model with a local Lagrangian involving bilinear

(kinetic and mass terms) and usual current-current terms. This fact is intimately related to the presence of the eqs. (62) [or in components (66)-(69)] in the set of decoupling eqs. (58)-(63). In the commutative case the equations of type (62) have been incorporated in order to write a local Lagrangian for the GATM model in ref. [15]. Notice that the original theory (without constraints) allows a zero-curvature formulation; in fact, its Lax pair is just the one of the so-called conformal affine Toda model coupled to matter fields [18]. However, it does not posses a local Lagrangian formulation in terms of the fields of the model; namely, the Toda and the spinor (Dirac) fields.

3. Notice that in Fig. 1 we have emphasized the duality relationship $\mathrm{NCGSG}_1 \leftrightarrow \mathrm{NCGMT}_1$ since in this case the symmetry $U(1) \times U(1) \times U(1)$ of the NCGSG_1 model is implemented in the star-localized Noether procedure to get the three $U(1)$ currents of the NCGMT_1 sector. Regarding the relationships between the sub-models of the both sectors NCGSG_1 and NCGMT_1, it is clear the appearance of the duality $\mathrm{NCSG}_1 \leftrightarrow \mathrm{NCMT}_1$ which has been discussed in the literature [9, 10]. In addition, it is expected the duality relationship $\mathrm{NCbBL}_1 \leftrightarrow \mathrm{NCBL}_1$, since in the ordinary space-time the former is the bosonized version of the later model [27, 28]. Regarding this type of duality relationships between the remaining models a more careful investigation is needed, e.g. we have not been able to describe neither the spinor model corresponding to the NCDSG_1 model, nor the scalar sectors of the (constrained) $\mathrm{NC(c)GMT}_1$ and $\mathrm{NC(c)BL}_1$ models, respectively.

6.2 NCGMT_2

As mentioned in the last paragraph of section 5 we expect that another NCGMT_2 version, with twice the number of fields of the NCGMT_1 theory, will appear when one performs a similar decoupling procedure for the extended system with $\{F^{\pm}_m, W^{\pm}_m\}$ and $\{\mathcal{F}^{\pm}_m, \mathcal{W}^{\pm}_m\}$ fields. In fact, a copy of the NCGMT_1 action (76), as well as the relevant zero-curvature equation of motion can be written for the fields $\{\mathcal{F}^{\pm}_m, \mathcal{W}^{\pm}_m\}$. Following similar steps one can construct a copy for each one of the sub-models presented above. Since it involves a direct generalization we will not present more details; however, see a corresponding construction for the $GL(2)$ case in ref. [9]. In this way one can get the NCGMT_2 model which is expected to be related to the NCGSG_2 model. Similarly to the NCGMT_1 case, one can expect that only a sub-model of NCGMT_2 will posses a zero-curvature formulation provided that a set of constraints similar to the eqs. (119) and (121), and a copy of them written for the fields $\mathcal{F}^{\pm}_m$ $(m = 1, 2)$ are considered.

7 Non-commutative solitons and kinks

It is a well known fact that the one-soliton solutions of certain models solve their NC counterparts. This feature holds for the SG model and its $\mathrm{NCSG}_{1,2}$ counterparts [9]. In the multi-field models, this feature means that the GSG model and its $\mathrm{NCGSG}_{1,2}$ extensions have a common subset of solutions, in particular the one-soliton and kink type solutions as we will see below. Of course the additional constraints, in the form of conservation laws which we have described before, e.g. the eqs. (41) and (54)-(55), respectively in the two versions of NCDSG models, must also be verified for the common subset of solutions. In fact, as we have noticed before they become trivial equations in the commutative limit.

The properties mentioned above reside on a simple observation: it is known that if $f(x_0, x_1)$ and $g(x_0, x_1)$ depend only on the combination $(x_1 - vx_0)$, then the product $f \star g$ coincides with the ordinary product $f.g$ [11, 41]. Therefore, all the $\star$ products in the NCGSG_1 system (24)-(26) reduce to the ordinary ones, so for these types of functions one has: $\text{NCGSG}_1 \rightarrow$ GSG model; the GSG model was defined in (32)-(33) [see also eqs. (161)-(162)]. In the following we record the solutions with this property, i.e, the one-soliton solutions of the NCGSG_1 model and the kink type solution of the NCDSG_1 sub-model. Actually, the same analysis can be done for the NCGSG_2 case.

7.1 Solitons and kinks

Next we write the 1-soliton and 1-kink type solutions associated to the fields $\phi_{1,2}$ of the NCGSG_1 model, which in accordance to the discussion above reduce to the GSG system of eqs. (32)-(33). We will see that these solitons are, in fact, associated to the various sine-Gordon models obtained as sub-models of the GSG theory, and the kink type solution corresponds to the double sine-Gordon sub-model [24].

1. Taking $\phi_1 = -\phi_2$ and $M_3 = M_2,\ \delta_i = 0$ in (32)-(33) one has

$$\phi_1 = 4\text{arctan}\{d\ \exp[\gamma_1(x - vt)]\}. \tag{134}$$

2. For $\phi_1 = \phi_2$ and $M_2 = M_3,\ M_1 = 0$ one has

$$\phi_1 = 4\,\text{arctan}\{d\ \exp[\gamma_2(x - vt)]\}. \tag{135}$$

Another SG model is given by setting $\phi_1 = \phi_2$ and $M_2 = M_3 = 0$ in (32)-(33) which leads to another soliton solution.

3. The kink solution is associated to the reduced double sine-Gordon model obtained by taking $\phi_1 = \phi_2 \equiv \phi$ and $M_3 = M_2,\ M_j \neq 0$. So, one has

$$\phi := 4\,\text{arctan}\,[d_K\ \sinh[\gamma_K\,(x - vt)]]\,, \tag{136}$$

which is the usual DSG kink solution [42].

The $\gamma_{1,2}, \gamma_K, d, d_K, v$ above are some constant parameters.

8 Conclusions and discussions

Some properties of the NC extensions of the GATM model and their weak-strong phases described by the $\text{NCGMT}_{1,2}$ and $\text{NCGSG}_{1,2}$ models, respectively, have been considered. The Fig. 1 summarizes the relationships we have established, as well as the field contents in each sub-model.

In the $\theta \to 0$ limit we have the following correspondences: NCGATM$_{1,2}$ $\to$ GATM; NCGSG$_{1,2}$ (the real sector of model 2) $\to$ GSG(plus a free scalar in the case of model 2); NCbBL$_{1,2}$ $\to$ bBL; NCDSG$_{1,2}$ (the real sector of model 2) $\to$ DSG(plus a free scalar in the case of model 2); NCGMT$_{1,2}$ $\to$ GMT (two copies in case of model 2); NCBL$_{1,2}$ $\to$ BL (two copies in case of model 2). In addition, the constrained versions NC(c)GMT$_{1,2}$ and NC(c)BL$_{1,2}$ give rise, in this limit, to the relevant (constrained) GMT and BL models, respectively, in ordinary space. To our knowledge, these are novel spinor integrable models.

The NCGMT$_{1,2}$ Lagrangians describe three flavor massive spinors (case 2 considers twice the number of spinors) with current-current interactions among themselves. In the process of constructing the Noether currents one recognizes the $[U(1)]^3$ symmetry in both NCGMT$_{1,2}$ models (in fact, as a subgroup of $[U(1)_C]^3$ in the model 2). We have provided the zero-curvature formulation of certain sub-models of the NCGMT$_{1,2}$. In fact, in order to write the eqs. of motion (91)-(102) as a zero-curvature equation for a suitable Lax pair one needs to impose the constraints (126)-(127), defining in this way the NC(c)GMT$_{1,2}$ models. Likewise, the (constrained) NC(c)BL$_{1,2}$ models possess certain Lax pairs.

The generalized sine-Gordon model, the usual SG model, the Bukhvostov-Lipatov model and the double sine-Gordon theory appear in the commutative limit of the both versions of the NCGSG$_{1,2}$ models. We have concluded that the NCGSG$_{1,2}$ models possess the same soliton and kink type solutions as their commutative counterparts. The appearance of the non-integrable double sine-Gordon model as a sub-model of the GSG model suggests that even the NCGSG$_{1,2}$ models are non-integrable theories for the arbitrary set of values of the parameter space, since they possess as sub-models the corresponding NCDSG$_{1,2}$ models. However, the NCGSG$_{1,2}$ models possess certain integrable directions in field space, as remarkable examples one has the NCSG$_{1,2}$ sub-models. In view of the presence of the (constrained) NC(c)GMT$_{1,2}$ and NC(c)BL$_{1,2}$ models with corresponding zero-curvature formulations, it is expected the existence of other integrable directions in the scalar sector, which we have not pursued further in the present work.

Actually, the procedures presented so far can directly be extended to the NCATM model for the affine Lie algebra $sl(n)$. Therefore one can conclude that, except for the usual MT model, a multi-flavor generalization ($n_F \geq 2$, n_F =number of flavors) of the massive Thirring model allows certain zero-curvature formulations only for its various constrained sub-models, in the both NC and ordinary space-time descriptions.

Except for the NCSG$_{1,2}$ models, which must correspond to the NCMT$_{1,2}$ models, whose Lax pair formulations have already been provided in the literature, we have not been able to find the Lax pair formulations of the NCGSG$_{1,2}$ remaining sub-models. The relevant scalar field models, and their Lax pair formulations, which must be the counterparts of the (constrained) NC(c)GMT$_{1,2}$ and NC(c)BL$_{1,2}$ models are missing; if such Lax pairs exist they are expected to contain certain nonlocal expressions of the fields of the NCGSG$_{1,2}$ models. These points deserve a careful consideration in future research.

Various aspects of the models studied above deserve attention in future research, e.g. the NC solitons and kinks of the NCGATM$_{1,2}$ models and their relations with the confinement mechanism studied in ordinary space [24], the bosonization of the NCGMT$_{1,2}$ and their sub-models, the NC zero-curvature formulation of the bosonic sector of the NC(c)GMT$_{1,2}$ and NC(c)BL$_{1,2}$ models, as discussed above. In particular, in the bosonization process of the NCGMT$_{1,2}$ models, initiated in [10] for the NCMT$_{1,2}$ case, we believe that a careful understanding of the star-localized NC Noether symmetries, as well as the classical soliton spectrum would be desirable. In view of the rich spectra and relationships present in the above models it could be interesting to apply and improve some quantization methods, such as the one proposed in [31], in order to compute the soliton and kink masses quantum corrections. Another direction of research constitutes the NC zero-curvature formulations of the NCGMT$_{1,2}$ type models defined for Grassmannian fields.

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A GSG as a reduced affine Toda model coupled to matter

We provide the algebraic construction of the $sl(3,\mathbb{C})$ conformal affine Toda model coupled to matter fields (CATM) following refs. [15, 18]. The reduction process to arrive at the classical GSG model closely follows the ref. [24]. The $sl(3,\mathbb{C})$ CATM model is a two-dimensional field theory involving four scalar fields and six Dirac spinors. The interactions among the fields are as follows: 1) in the scalars equations of motion there are the coupling of bilinears in the spinors to exponentials of the scalars. 2) Some of the equations of motion for the spinors have certain bilinear terms in the spinors themselves. That fact makes it difficult to find a local Lagrangian for the theory. Nevertheless, the model presents a lot of symmetries. It is conformally invariant, possesses local gauge symmetries as well as vector and axial conserved currents bilinear in the spinors. One of the most remarkable properties of the model is that it presents an equivalence between a U(1) vector conserved current, bilinear in the spinors, and a topological currents depending only on the first derivative of some scalars. This property allow us to implement a bag model like confinement mechanism resembling what one expects to happen in QCD. The model possesses a zero-curvature representation based on the $\hat{sl}3(C)$ affine Kac Moody algebra. It constitutes a particular example of the so-called conformal affine Toda models coupled to matter fields which has been introduced in [18]. The corresponding model associated to $\hat{sl}2(C)$ has been studied in [16] where it was shown, using bosonization techniques, that the equivalence between the currents holds true at the quantum level and so the confinement mechanism does take place in the quantum theory.

The off-critical affine Toda model coupled to matter (ATM) is defined by gauge fixing the conformal symmetry [14] and imposing certain constraints in order to write a local Lagrangian for the model [15]. These treatments of the $sl(3,\mathbb{C})$ ATM model used the symplectic and on-shell decoupling methods to unravel the classical generalized sine-Gordon (GSG) and generalized massive Thirring (GMT) dual theories describing the strong/weak coupling sectors of the ATM model [20, 15, 14]. As mentioned above the ATM model describes some scalars coupled to spinor (Dirac) fields in which the system of equations of motion has a local gauge symmetry. Conveniently gauge fixing the local symmetry by setting some spinor bilinears to constants we are able to decouple the scalar (Toda) fields from the spinors, the final result is a direct construction of the classical generalized sine-Gordon model (GSG) involving only the scalar fields. In the spinor sector we are left with a system of equations in which the Dirac fields couple to the GSG fields. Another instance in which the quantum version of the generalized sine-Gordon theory arises is in the process of bosonization of the generalized massive Thirring model (GMT), which is a multi-flavor extension of the usual massive Thirring model such that, apart from the usual current-current self-interaction for each flavor, it presents current-current interactions terms among the various U(1) flavor currents [43].

The zero-curvature condition (163) supplied with the potentials (164) gives the following equations of motion for the CATM model [18]

$$\frac{\partial^2\phi_a}{4i\,e^{\eta}} = m_1[e^{\eta-i\theta_a}\widetilde{\psi}^l_R\psi^l_L + e^{i\theta_a}\widetilde{\psi}^l_L\psi^l_R] + m_3[e^{-i\theta_3}\widetilde{\psi}^3_R\psi^3_L + e^{\eta+i\theta_3}\widetilde{\psi}^3_L\psi^3_R]; \quad a=1,2 \tag{137}$$

$$-\frac{\partial^2\widetilde{\nu}}{4} = im_1e^{2\eta-\theta_1}\widetilde{\psi}^1_R\psi^1_L + im_2e^{2\eta-\theta_2}\widetilde{\psi}^2_R\psi^2_L + im_3e^{\eta-\theta_3}\widetilde{\psi}^3_R\psi^3_L + \mathbf{m}^2e^{3\eta}, \tag{138}$$

$$-2\partial_+\psi^1_L = m_1e^{\eta+i\theta_1}\psi^1_R, \qquad -2\partial_+\psi^2_L = m_2e^{\eta+i\theta_2}\psi^2_R, \tag{139}$$

$$2\partial_-\psi^1_R = m_1e^{2\eta-i\theta_1}\psi^1_L + 2i\Big(\frac{m_2m_3}{im_1}\Big)^{1/2}e^{\eta}(-\psi^3_R\widetilde{\psi}^2_Le^{i\theta_2} - \widetilde{\psi}^2_R\psi^3_Le^{-i\theta_3}), \tag{140}$$

$$2\partial_-\psi^2_R = m_2e^{2\eta-i\theta_2}\psi^2_L + 2i\Big(\frac{m_1m_3}{im_2}\Big)^{1/2}e^{\eta}(\psi^3_R\widetilde{\psi}^1_Le^{i\theta_1} + \widetilde{\psi}^1_R\psi^3_Le^{-i\theta_3}), \tag{141}$$

$$-2\partial_+\psi^3_L = m_3e^{2\eta+i\theta_3}\psi^3_R + 2i\Big(\frac{m_1m_2}{im_3}\Big)^{1/2}e^{\eta}(-\psi^1_L\psi^2_Re^{i\theta_2} + \psi^2_L\psi^1_Re^{i\theta_1}), \tag{142}$$

$$2\partial_-\psi^3_R = m_3e^{\eta-i\theta_3}\psi^3_L, \qquad 2\partial_-\widetilde{\psi}^1_R = m_1e^{\eta+i\theta_1}\widetilde{\psi}^1_L, \tag{143}$$

$$-2\partial_+\widetilde{\psi}^1_L = m_1e^{2\eta-i\theta_1}\widetilde{\psi}^1_R + 2i\Big(\frac{m_2m_3}{im_1}\Big)^{1/2}e^{\eta}(-\psi^2_L\widetilde{\psi}^3_Re^{-i\theta_3} - \widetilde{\psi}^3_L\psi^2_Re^{i\theta_2}), \tag{144}$$

$$-2\partial_+\widetilde{\psi}^2_L = m_2e^{2\eta-i\theta_2}\widetilde{\psi}^2_R + 2i\Big(\frac{m_1m_3}{im_2}\Big)^{1/2}e^{\eta}(\psi^1_L\widetilde{\psi}^3_Re^{-i\theta_3} + \widetilde{\psi}^3_L\psi^1_Re^{i\theta_1}), \tag{145}$$

$$2\partial_-\widetilde{\psi}^2_R = m_2e^{\eta+i\theta_2}\widetilde{\psi}^2_L, \qquad -2\partial_+\widetilde{\psi}^3_L = m_3e^{\eta-i\theta_3}\widetilde{\psi}^3_R, \tag{146}$$

$$2\partial_-\widetilde{\psi}^3_R = m_3e^{2\eta+i\theta_3}\widetilde{\psi}^3_L + 2i\Big(\frac{m_1m_2}{im_3}\Big)^{1/2}e^{\eta}(\widetilde{\psi}^1_R\widetilde{\psi}^2_Le^{i\theta_2} - \widetilde{\psi}^2_R\widetilde{\psi}^1_Le^{i\theta_1}), \tag{147}$$

$$\partial^2\eta = 0, \tag{148}$$

where $\theta_1 \equiv 2\phi_1 - \phi_2$, $\theta_2 \equiv 2\phi_2 - \phi_1$, $\theta_3 \equiv \phi_1 + \phi_2$. Therefore, one has

$$\theta_3 = \theta_1 + \theta_2 \tag{149}$$

The ϕ fields are considered to be in general complex fields. In order to define the classical generalized sine-Gordon model we will consider these fields to be real.

Apart from the *conformal invariance* the above equations exhibit the $\Big(U(1)_L\Big)^2 \otimes \Big(U(1)_R\Big)^2$ *left-right* *local gauge symmetry*

$$\phi_a \rightarrow \phi_a + \xi_+^a(x_+) + \xi_-^a(x_-), \quad a = 1, 2 \tag{150}$$

$$\widetilde{\nu} \rightarrow \widetilde{\nu}\,; \qquad \eta \rightarrow \eta \tag{151}$$

$$\psi^i \rightarrow e^{i(1+\gamma_5)\Xi_+^i(x_+)+i(1-\gamma_5)\Xi_-^i(x_-)}\,\psi^i, \tag{152}$$

$$\widetilde{\psi}^i \rightarrow e^{-i(1+\gamma_5)(\Xi_+^i)(x_+)-i(1-\gamma_5)(\Xi_-^i)(x_-)}\,\widetilde{\psi}^i, \;\; i = 1, 2, 3; \tag{153}$$

$$\Xi_\pm^1 \equiv \pm\xi_\pm^2 \mp 2\xi_\pm^1, \;\; \Xi_\pm^2 \equiv \pm\xi_\pm^1 \mp 2\xi_\pm^2, \;\; \Xi_\pm^3 \equiv \Xi_\pm^1 + \Xi_\pm^2.$$

One can get global symmetries for $\xi_\pm^a = \mp\xi_\mp^a =$ constants. For a model defined by a Lagrangian these would imply the presence of two vector and two chiral conserved currents. However, it was found only half of such currents [44]. This is a consequence of the lack of a Lagrangian description for the $sl(3)^{(1)}$ CATM in terms of the B and $F^\pm$ fields (however see Appendix C for a local Lagrangian description of an off-critical and constrained sub-model). So, the vector current

$$J^\mu = \sum_{j=1}^{3} m_j \bar{\psi}^j \gamma^\mu \psi^j \tag{154}$$

and the chiral current

$$J^{5\,\mu} = \sum_{j=1}^{3} m_j \bar{\psi}^j \gamma^\mu \gamma_5 \psi^j + 2\partial_\mu(m_1\phi_1 + m_2\phi_2) \tag{155}$$

are conserved

$$\partial_\mu J^\mu = 0, \quad \partial_\mu J^{5\,\mu} = 0. \tag{156}$$

The conformal symmetry is gauge fixed by setting [14]

$$\eta = \text{const}. \tag{157}$$

The off-critical ATM model obtained in this way exhibits the vector and topological currents equivalence [18, 14]

$$\sum_{j=1}^{3} m_j \bar{\psi}^j \gamma^\mu \psi^j \equiv \epsilon^{\mu\nu}\partial_\nu(m_1\phi_1 + m_2\phi_2), \quad m_3 = m_1 + m_2, \quad m_i > 0. \tag{158}$$

In the next steps we implement the reduction process to get the GSG model through a gauge fixing of the ATM theory [24]. The local symmetries (150)-(153) can be gauge fixed through

$$i\bar{\psi}^j\psi^j = iA_j = \text{const.}; \quad \bar{\psi}^j\gamma_5\psi^j = 0. \tag{159}$$

From the gauge fixing (159) one can write the following bilinears

$$\widetilde{\psi}_R^j \psi_L^j + \widetilde{\psi}_L^j \psi_R^j = 0, \quad j = 1, 2, 3; \tag{160}$$

so, the eqs. (159) effectively comprises three gauge fixing conditions.

It can be directly verified that the gauge fixing (159) preserves the currents conservation laws (156), i.e. from the equations of motion (137)-(148) and the gauge fixing (159) together with (157) it is possible to obtain the currents conservation laws (156).

Taking into account the constraints (159) in the scalar sector, eqs. (137), we arrive at the following system of equations (set $\eta = 0$)

$$\partial^2\phi_1 = M_\psi^1 \sin(2\phi_1 - \phi_2) + M_\psi^3 \sin(\phi_1 + \phi_2), \tag{161}$$

$$\partial^2\phi_2 = M_\psi^2 \sin(2\phi_2 - \phi_1) + M_\psi^3 \sin(\phi_1 + \phi_2), \quad M_\psi^i \equiv 4A_i\, m_i, \quad i = 1, 2, 3. \tag{162}$$

The system of equations above considered for real fields $\phi_{1,\,2}$ as well as for real parameters M_ψ^i defines the *generalized sine-Gordon model* (GSG).

B The zero-curvature formulation of the $\hat{sl}(3)$ CATM model

We summarize the zero-curvature formulation of the $\hat{sl}(3)$ CATM model [18, 44]. Consider the zero-curvature condition

$$\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0. \tag{163}$$

The potentials take the form

$$A_+ = -BF^+B^{-1}, \quad A_- = -\partial_- BB^{-1} + F^-, \tag{164}$$

with

$$F^+ = F_1^+ + F_2^+, \quad F^- = F_1^- + F_2^-, \tag{165}$$

where B and $F_i^\pm$ contain the fields of the model. Let us define

$$F_m^\pm = \mp[E_{\pm 3}\,,\, W_{3-m}^\mp] \tag{166}$$

$$E_{\pm 3} = \frac{1}{6}[(2m_1 + m_2)H_1^{\pm 1} + (2m_2 + m_1)H_2^{\pm 1}], \quad m_3 = m_1 + m_2 \tag{167}$$

$$W_1^- = -\sqrt{\frac{4i}{m_3}}\psi_R^3 E_{\alpha 3}^{-1} + \sqrt{\frac{4i}{m_1}}\widetilde{\psi}_R^1 E_{-\alpha 1}^0 + \sqrt{\frac{4i}{m_2}}\widetilde{\psi}_R^2 E_{-\alpha 2}^0 \tag{168}$$

$$W_1^+ = \sqrt{\frac{4i}{m_1}}\psi_L^1 E_{\alpha 1}^0 + \sqrt{\frac{4i}{m_2}}\psi_L^2 E_{\alpha 2}^0 - \sqrt{\frac{4i}{m_3}}\widetilde{\psi}_L^3 E_{-\alpha 3}^1 \tag{169}$$

$$W_2^- = -\sqrt{\frac{4i}{m_1}}\psi_R^1 E_{\alpha 1}^{-1} - \sqrt{\frac{4i}{m_2}}\psi_R^2 E_{\alpha 2}^{-1} + \sqrt{\frac{4i}{m_3}}\widetilde{\psi}_R^3 E_{-\alpha 3}^0 \tag{170}$$

$$W_2^+ = \sqrt{\frac{4i}{m_3}}\psi_L^3 E_{\alpha 3}^0 - \sqrt{\frac{4i}{m_1}}\widetilde{\psi}_L^1 E_{-\alpha 1}^1 - \sqrt{\frac{4i}{m_2}}\widetilde{\psi}_L^2 E_{-\alpha 2}^1 \tag{171}$$

$$B = e^{i\theta_1 H_1^0 + i\theta_2 H_2^0}\, e^{\widetilde{\nu} C}\, e^{\eta Q_{ppal}} \;\equiv\; g\, e^{\widetilde{\nu} C}\, e^{\eta Q_{ppal}}. \tag{172}$$

$E^n_{\alpha_i}, H^n_1, H^n_2$ and C ($i = 1,2,3$; $n = 0, \pm 1$) are some generators of $sl(3)^{(1)}$; Q_{ppal} being the principal gradation operator. The commutation relations for an affine Lie algebra in the Chevalley basis are

$$[H^m_a, H^n_b] = mC\frac{2}{\alpha_a^2}K_{ab}\delta_{m+n,0} \tag{173}$$

$$\left[H^m_a, E^n_{\pm\alpha}\right] = \pm K_{\alpha a}E^{m+n}_{\pm\alpha} \tag{174}$$

$$\left[E^m_\alpha, E^n_{-\alpha}\right] = \sum_{a=1}^{r} l^\alpha_a H^{m+n}_a + \frac{2}{\alpha^2}mC\delta_{m+n,0} \tag{175}$$

$$\left[E^m_\alpha, E^n_\beta\right] = \varepsilon(\alpha,\beta)E^{m+n}_{\alpha+\beta}; \qquad \text{if } \alpha+\beta \text{ is a root} \tag{176}$$

$$[D, E^n_\alpha] = nE^n_\alpha, \qquad [D, H^n_a] = nH^n_a. \tag{177}$$

where $K_{\alpha a} = 2\alpha.\alpha_a/\alpha_a^2 = n^\alpha_b K_{ba}$, with n^α_a and l^α_a being the integers in the expansions $\alpha = n^\alpha_a\alpha_a$ and $\alpha/\alpha^2 = l^\alpha_a\alpha_a/\alpha_a^2$, and $\varepsilon(\alpha,\beta)$ the relevant structure constants.

Take $K_{11} = K_{22} = 2$ and $K_{12} = K_{21} = -1$ as the Cartan matrix elements of the simple Lie algebra $sl(3)$. Denoting by α_1 and α_2 the simple roots and the highest one by $\psi(= \alpha_1 + \alpha_2)$, one has $l^\psi_a = 1 (a = 1,2)$, and $K_{\psi 1} = K_{\psi 2} = 1$. Take $\varepsilon(\alpha,\beta) = -\varepsilon(-\alpha,-\beta)$, $\varepsilon_{1,2} \equiv \varepsilon(\alpha_1,\alpha_2) = 1$, $\varepsilon_{-1,3} \equiv \varepsilon(-\alpha_1,\psi) = 1$ and $\varepsilon_{-2,3} \equiv \varepsilon(-\alpha_2,\psi) = -1$.

One has $Q_{\rm ppal} \equiv \sum_{a=1}^2 \mathbf{s}_a\lambda^v_a.H + 3D$, where λ^v_a are the fundamental co-weights of $sl(3)$, and the principal gradation vector is $\mathbf{s} = (1,1,1)$ [45]. This gradation decomposes $\widehat{\mathfrak{sl}_3(\mathbb{C})}$ into the following subspaces

$$\hat{\mathcal{G}}_0 = \mathbb{C}\,H_1 \oplus \mathbb{C}\,H_2 \oplus \mathbb{C}\,C \oplus \mathbb{C}\,D = \mathbb{C}\,H_1 \oplus \mathbb{C}\,H_2 \oplus \mathbb{C}\,C \oplus \mathbb{C}\,Q_{\rm ppal}, \tag{178}$$

and

$$\hat{\mathcal{G}}_{3m} = \mathbb{C}\,H^m_1 \oplus \mathbb{C}\,H^m_2, \quad m \neq 0, \tag{179}$$

$$\hat{\mathcal{G}}_{3m+1} = \mathbb{C}\,E^m_{\alpha_1} \oplus \mathbb{C}\,E^m_{\alpha_2} \oplus \mathbb{C}\,E^{m+1}_{-\alpha_3}, \tag{180}$$

$$\hat{\mathcal{G}}_{3m+2} = \mathbb{C}\,E^{m+1}_{-\alpha_1} \oplus \mathbb{C}\,E^{m+1}_{-\alpha_2} \oplus \mathbb{C}\,E^m_{\alpha_3}. \tag{181}$$

C The off-critical and constrained $sl(3)$ ATM model

The off-critical and constrained $sl(3)$ *affine Toda model coupled to matter fields* (ATM) is defined by the action [15]

$$\begin{aligned}\frac{1}{k}I^{(3)}_{\rm ATM} = \; & I_{WZNW}[g] + \int_M d^2x\{\sum_{m=1}^2 \Big[< F^-_m\,,\, gF^+_m g^{-1} > \\ & -\frac{1}{2} < E_{-3}\,,\, [W^+_m\,,\, \partial_+ W^+_{3-m}] > + < F^-_m\,,\, \partial_+ W^+_m > \\ & +\frac{1}{2} < [W^-_m\,,\, \partial_- W^-_{3-m}]\,,\, E_3 > + < \partial_- W^-_m\,,\, F^+_m > \Big]\},\end{aligned} \tag{182}$$

where

$$I_{WZNW}[g] = \frac{1}{8}\int_M d^2x Tr(\partial_\mu g\partial^\mu g^{-1}) + \frac{1}{12}\int_D d^3x\,\epsilon^{ijk}Tr(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g), \tag{183}$$

is the Wess-Zumino-Novikov-Witten (WZNW) action for the matrix scalar field of the model. The first term inside the summation of (182) defines the form of the interactions and the remaining terms are the kinetic terms for the matrix fields associated to the spinors. The equations of motion derived from this action

$$\partial_-(g^{-1}\partial_+ g) = \sum_{m=1}^{2}\Big[F_m^-\,,\,gF_m^+g^{-1}\Big] \tag{184}$$

$$\partial_+F_m^- = [E_{-3},\partial_+W_{3-m}^+],\qquad \partial_-F_m^+ = -[E_3,\partial_-W_{3-m}^-], \tag{185}$$

$$\partial_+W_m^+ = -gF_m^+g^{-1},\qquad \partial_-W_m^- = -g^{-1}F_m^-g, \tag{186}$$

are equivalent to the above CATM equations of motion (137)-(148) provided the following constraints

$$\eta = 0 \tag{187}$$

$$\Big[F_2^{\pm}\,,\,g^{\mp 1}F_1^{\mp}g^{\pm 1}\Big] = 0, \tag{188}$$

are imposed. The first constraint defines an off-critical model, whereas the second ones allow a local Lagrangian description of the model. Let us emphasize that the constraints (188) amount to drop all the terms with spinor bilinears on the right hand side of the set of equations (140)-(142), (144)-(145) and (147), respectively. These constraints were introduced in refs. [20, 15] since they are trivially satisfied by the soliton type solutions of the full CATM model.

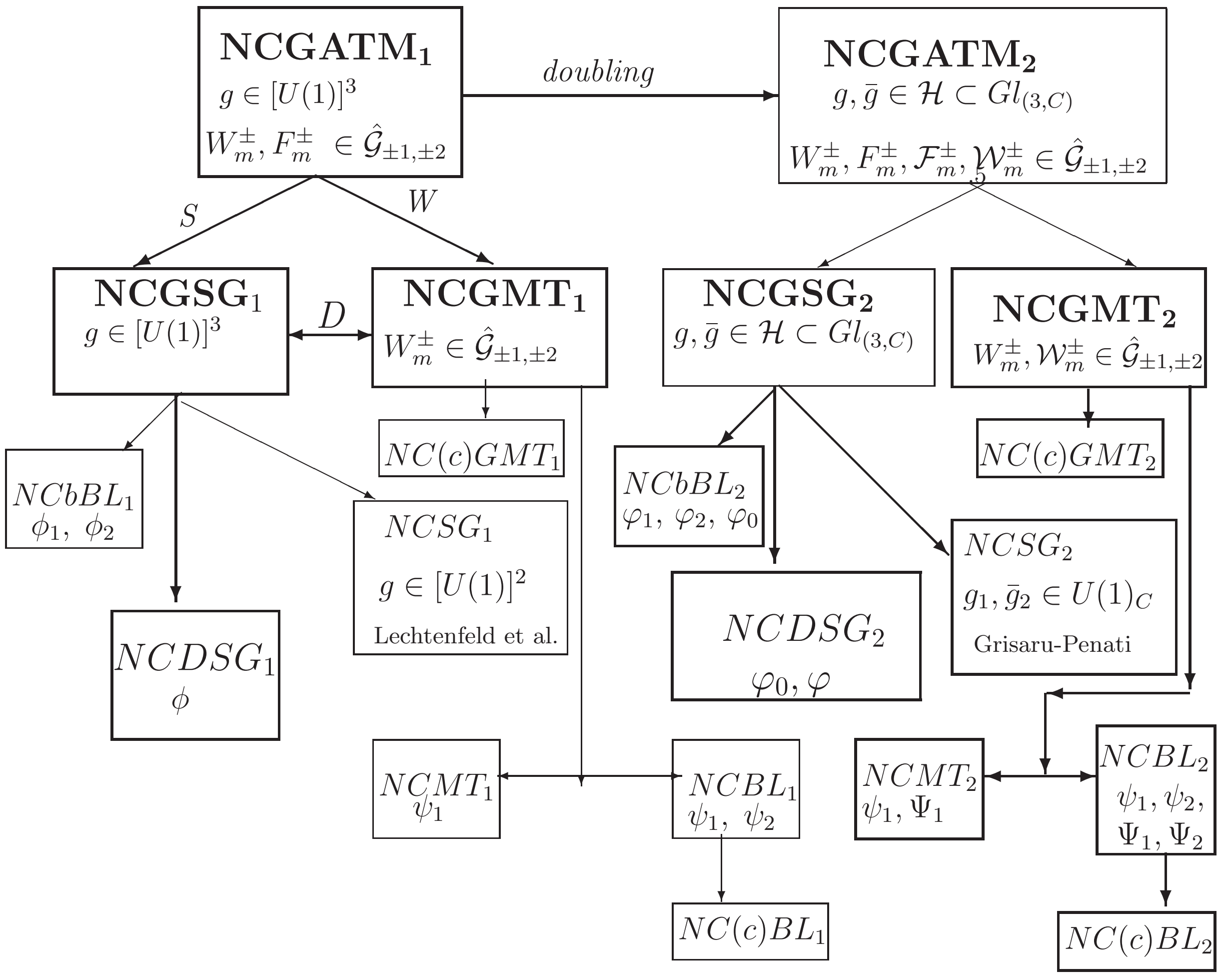

Fig. 1: NCGATM$_{1,2}$: dual sectors, sub-models and field contents.

Duality: **S**= *strong sector;* **W**= *weak sector;* **D**= *S - W duality. A Lax pair is available for NCSG$_{1,2}$/NCMT$_{1,2}$, NC(c)GMT$_{1,2}$ and NC(c)BL$_{1,2}$, respectively. Dual sectors of the models NC(c)GMT$_{1,2}$, NC(c)BL$_{1,2}$ and NCDSG$_{1,2}$ are missing in the table above and deserve future investigations.*

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