Compactons in Nonlinear Schrödinger Lattices with Strong Nonlinearity Management

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The existence of compactons in the discrete nonlinear Schrödinger equation in the presence of fast periodic time modulations of the nonlinearity is demonstrated. In the averaged DNLS equation the resulting effective inter-well tunneling depends on modulation parameters *and* on the field amplitude. This introduces nonlinear dispersion in the system and can lead to a prototypical realization of single- or multi-site stable discrete compactons in nonlinear optical waveguide and BEC arrays. These structures can dynamically arise out of Gaussian or compactly supported initial data.

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Introduction. One of the most remarkable phenomena occurring in nonlinear lattices is the existence of *discrete breathers* which arise from the interplay between discreteness, dispersion and nonlinearity [1]. These excitations are quite generic in nonlinear lattices with usual (e.g. linear) dispersion and have typical spatial profiles with exponential tails. In presence of *nonlinear dispersion* these excitations (as well as their continuous counterparts) may acquire spatial profiles with compact support and for this reason they are known as *compactons* [2]. Unlike other nonlinear excitations, compactons (having no tails) cannot interact with each other until they are in contact, this being an attractive feature for potential applications. Similarly to discrete breathers, compactons are intrinsically localized and robust excitations. The lack of exponential tails is a consequence of the nonlinear dispersive interactions which permit the vanishing of the intersite tunneling at compacton edges. The difficulty of implementing this condition in physical contexts has restricted until now investigations mainly to the mathematical side. The development of management techniques for soliton control, however, can rapidly change the situation.

Periodic management of parameters of nonlinear systems has been shown to be an effective technique for the generation of solitons with new types of properties [3]. Examples of the management technique in continuous systems are the dispersion management of solitons in optical fibers which allows to improve communication capacities [4], and the nonlinearity management of 2D and 3D Bose-Einstein condensates (BEC) or optically layered media which provides partial stabilization against collapse in the case of attractive interatomic interactions [5]. In discrete systems the diffraction management technique was used to generate spatial discrete solitons with novel properties [6, 7] which have been observed in experiments [7]. The resonant spreading and steering of discrete solitons in arrays of waveguides, induced by nonlinearity management was also investigated [8]. To date, the nonlinear management technique *for nonlinear lattices* has been considered only in the limit of weak modulations of the nonlinearity [9, 10]. The inter-well tunneling suppression has been discussed in [11] for the Bose-Hubbard chain with time periodic ramp potential and in [12] for a two-sites Bose-Hubbard model with modulated in time interactions. In both cases the tunneling suppression was uniform in the system and no apparent link with compacton formation was established. The phenomenon has also been recently observed in experiments of light propagation in waveguide arrays [13] and in BEC's in strongly driven optical lattices [14].

The aim of the present Letter is to demonstrate the existence of stable compacton excitations in the discrete nonlinear Schrödinger (DNLS) system subjected to *strong nonlinearity management* (SNLM), e.g. to fast periodic time variations of the nonlinearity. To that effect, we use an averaged DNLS Hamiltonian system to show that in the SNLM limit the inter-well tunneling can be totally suppressed for field amplitudes matching zeros of the Bessel function, introducing effective nonlinear dispersion which leads to compacton formation. We show that these compact structures not only exist in single and multi-site realizations but they generically are structurally and dynamically stable and can be generated from general classes of initial conditions. These results should enable the observation of discrete compactons in BEC and in nonlinear optical systems, both being described by the discrete NLS equation.

Theory. Consider the following lattice Hamiltonian

$$H = -\sum_n \{\kappa(u_n u_{n+1}^* + u_{n+1} u_n^*) + \frac{1}{2}(\gamma_0 + \gamma(t))|u_n|^4\}, \tag{1}$$

with the coupling constant κ quantifying the tunneling between adjacent sites (wells), γ_0 denoting the onsite constant nonlinearity and $\gamma(t)$ representing the time-dependent modulation. In the following we assume a strong management case with $\gamma(t)$ being a periodic, e.g. $\gamma(t) = \gamma(t+T)$, and rapidly varying function. As a prototypical example, we use $\gamma(t) = \frac{\gamma_1}{\varepsilon}\cos(\Omega\tau)$, with $\gamma_1 \sim O(1)$, $\varepsilon \ll 1$, $\tau = t/\varepsilon$ denoting the fast time variable and $T = 2\pi/\Omega$ the period. The dynamical system associated with (1) is the well known DNLS equation [15]

$$i\dot{u}_n + \kappa(u_{n+1} + u_{n-1}) + (\gamma_0 + \gamma(t))|u_n|^2 u_n = 0, \tag{2}$$

which serves, under suitable conditions [16], as a model for the dynamics of BEC in optical lattices subjected to SNLM (through varying the interatomic scattering length by external time-dependent magnetic fields via a Feshbach resonance), as well as for light propagation in optical waveguide arrays (here the evolution variable is the propagation distance and the SNLM consists of periodic space variations of the Kerr nonlinearity).

The existence of compacton solutions can be inferred from the fact that the averaged DNLS Hamiltonian (averaged with respect to the fast time τ), coincides with the original time independent Hamiltonian except for a rescaling of the coupling constant which depends on the Bessel function of the field amplitude. To show this, it is convenient to perform the transformation [17] $u_n(t) = v_n(t)e^{i\Gamma|v_n(t)|^2}$ with $\Gamma = \frac{1}{\epsilon}\int_0^t dt\ \gamma(\tau) = \gamma_1\Omega^{-1}\sin(\Omega\tau)$, which allows to rewrite Eq. (2) as

$$i\dot{v}_n = \Gamma v_n(|v_n|^2)_t - \kappa X - \gamma_0|v_n|^2 v_n, \tag{3}$$

with $X = v_{n+1}e^{i\Gamma\theta_+} + v_{n-1}e^{i\Gamma\theta_-}$ and $\theta_\pm = |v_{n\pm1}|^2 - |v_n|^2$. On the other hand, $(i|v_n|^2)_t = i(\dot{v}_n v_n^* + v_n\dot{v}_n^*) = i\kappa(v_n^* X - v_n X^*)$, with the star denoting the complex conjugation. Substituting this expression into Eq. (3) and averaging the resulting equation over the period T of the rapid modulation, we obtain

$$i\dot{v}_n = i\kappa|v_n|^2\langle\Gamma X\rangle - i\kappa v_n^2\langle\Gamma X^*\rangle - \kappa\langle X\rangle - \gamma_0|v_n|^2 v_n, \tag{4}$$

with $\langle\cdot\rangle \equiv \frac{1}{T}\int_0^T(\cdot)d\tau$ denoting the fast time average. The averaged terms in Eq. (4) can be calculated by means of the elementary integrals $\langle e^{\pm i\Gamma\theta_\pm}\rangle = \alpha J_0(\alpha\theta_\pm)$, $\langle\Gamma e^{\pm i\Gamma\theta_\pm}\rangle = \pm i\alpha J_1(\alpha\theta_\pm)$, with J_i being Bessel functions of order $i = 0, 1$ and $\alpha = \gamma_1/\Omega$, thus giving

$$\begin{aligned} i\dot{v}_n = -\alpha\kappa v_n[(v_{n+1}v_n^* + v_{n+1}^* v_n)J_1(\alpha\theta_+) + \\ (v_{n-1}v_n^* + v_{n-1}^* v_n)J_1(\alpha\theta_-)] - \\ \kappa[v_{n+1}J_0(\alpha\theta_+) + v_{n-1}J_0(\alpha\theta_-)] - \gamma_0|v_n|^2 v_n. \end{aligned} \tag{5}$$

Note that parameters $\gamma_1, \Omega \sim 1$, and the averaged equation is valid for times $t \leq 1/\epsilon$. This modified DNLS equation can be written as $i\dot{v}_n = \delta H_{av}/\delta v_n^*$, with averaged Hamiltonian

$$H_{av} = -\sum_n\{\kappa J_0(\alpha\theta_+)\left[v_{n+1}v_n^* + v_{n+1}^* v_n\right] + \frac{\gamma_0}{2}|v_n|^4\}. \tag{6}$$

A comparison with Eq. (1) gives the anticipated rescaling as $\kappa \to \kappa J_0(\alpha\theta_+)$; a similar rescaling was recently reported also for a quantum Bose-Hubbard dimer with time dependent onsite interaction [12].

It is worth noting that while the appearance of the Bessel function is intimately connected with harmonic modulations, the existence of compacton solutions and the lattice tunneling suppression is generic for periodic SNLM. Thus, for example, for a two-step modulation of the form $\gamma(t) = (-1)^i\gamma_1$ with $i = 0, 1$ and $\frac{i}{2} < \tau < \frac{(i+1)}{2}$, we obtain for the first term in the averaged Hamiltonian (6) $\kappa(v_{n+1}^* v_n e^{i\gamma_1\theta_+/4} + v_n^* v_{n+1}e^{-i\gamma_1\theta_+/4})\mathrm{sinc}(\gamma_1\theta_+/4)$, where $\mathrm{sinc}(x) = \sin(x)/x$, thus, in this case the suppression of tunneling exists at zeros of the sinc function. We also remark that for small $\alpha\theta_+$ the series expansion of J_0 yields the averaged Hamiltonian of the DNLS equation obtained in [9] in the limit of *weak* management.

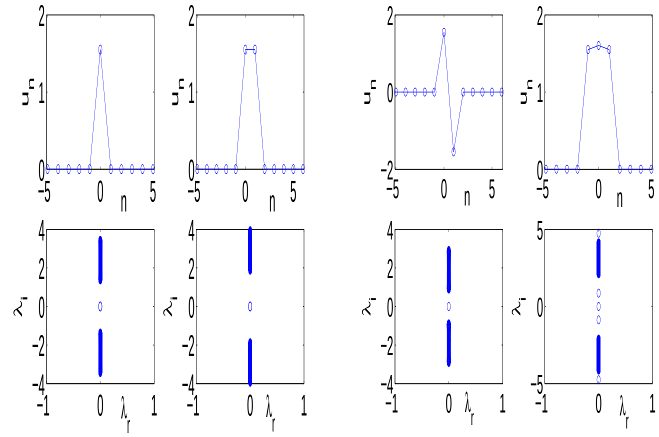

FIG. 1: Typical examples for $\kappa = 0.5$, $\alpha = 1$ of compact localized mode solutions of Eq. (7) (top panels) and of the plane (λ_r, λ_i) of their linearization eigenvalues $\lambda = \lambda_r + i\lambda_i$. 1st column: on-site, 2nd column: inter-site, in-phase, 3rd column: inter-site, out-of-phase, 4th column: three-sites. Remarkably, all solutions are *dynamically stable*.

Exact compactons and numerics. To demonstrate the existence of *exact stable* compactons in the averaged system, we seek for stationary solutions of the form $v_n = A_n e^{-i\mu t}$ for which Eq. (5) becomes

$$\begin{aligned} \mu A_n + \gamma_0 A_n^3 + \kappa(A_{n+1}J_0(\alpha\theta_+) + A_{n-1}J_0(\alpha\theta_-)) + \\ 2\alpha\kappa A_n^2[A_{n+1}J_1(\alpha\theta_+) + A_{n-1}J_1(\alpha\theta_-)] = 0. \end{aligned} \tag{7}$$

As is well known, discrete breathers can be numerically constructed with high precision using continuation procedures from the anti-continuous limit. The application of this method to Eq. (7) gives, quite surprisingly, that such modes *cannot* be continued past a critical point (of $\kappa \approx 0.32$ for $-\mu = \gamma_0 = 1$). The fact that the solutions cease to exist before reaching the limit of resonance with the linear modes ($\kappa = -\mu/2$) naturally raises the question of what type of modes may be present in the system for larger values of the coupling. In the following we show that in agreement with our theoretical prediction, the emerging excitations are genuine compactons e.g. they have vanishing tails (rather than fast double exponential decaying tails as in granular crystals [19]).

To search for compactly supported solutions one needs to consider [18] the last site of vanishing amplitude, denoted as n_0 below. In the setting of Eq. (7), this directly establishes the condition

$$J_0(\alpha A_{n_0+1}^2) = 0 \Rightarrow A_{n_0+1}^2 = 2.4048/\alpha \tag{8}$$

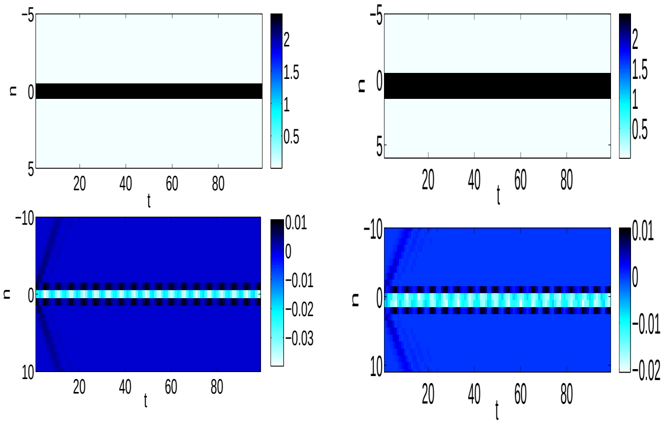

FIG. 2: Space-time evolution of a one-site (left column) and a two-sites (right column) compacton solution as obtained from direct numerical integrations of Eq. (2). Top panels in each case show the square modulus of the solution itself (large amplitude colorbar), while bottom panels (small amplitude colorbar) show the deviation from the exact solution of Eq. (7) taken as corresponding initial condition.

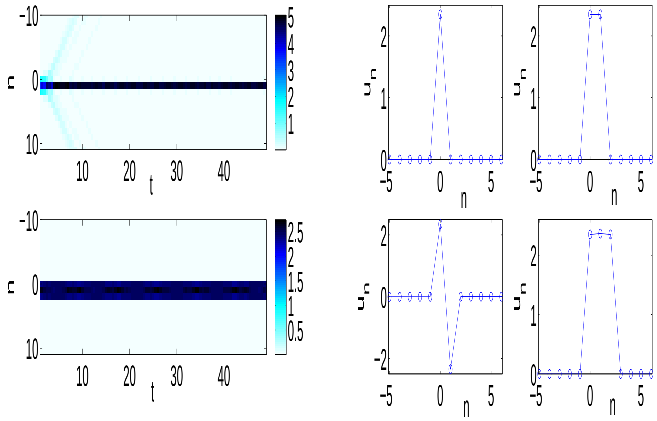

FIG. 3: Left panel: dynamical evolution for $\kappa = 1$ of a perturbed 3-sites compacton for $\epsilon = 0.1$ (top, leading to single-site evolution) and for $\epsilon = 0.025$ (bottom). Right panel: Examples of *stable* large amplitude compact modes with one-, two- (in-phase and out-of-phase) and 3-sites emerging from the second zero of the Bessel function.

which yields the solution (based on the first zero of the Bessel function) for the "boundary" of the compactly supported site. Then, for $\mu = -\gamma_0 A^2_{n_0+1}$, both the condition for compact support at $n_0 \pm 1$, and the equation for $n = n_0$ are satisfied. Hence Eq. (8) yields a single-site discrete compacton. Numerical linear stability analysis illustrates that this solution is *generically* stable (see Fig. 1). The bottom panel's eigenvalues are associated with perturbations growing as $e^{\lambda t}$. The absence of a positive real part in λ (i.e. of any λ's in the right half plane) is tantamount to linear stability. Similar results are found for two-site compactons, which are either in phase (2nd column of Fig. 1) or out-of-phase (3rd column of Fig. 1). The only thing that changes here is that in order to satisfy the equation at the non-vanishing sites, one must have $\mu = -\kappa - \gamma_0 A^2_{n_0+1}, \quad \mu = \kappa - \gamma_0 A^2_{n_0+1}$, respectively for the in-phase and out-of-phase two-site compactons (note from Fig. 1 that these solutions are both stable).

With some additional effort, one can generalize these considerations to an arbitrary number of sites. As a typical example, a three-sites compacton with amplitudes $(\ldots, 0, A_1, A_2, A_1, 0, \ldots)$ will satisfy in addition to the "no tunneling condition" $J_0(\alpha A_1^2) = 0$, the constraints:

$$\mu A_{1+i} + 2(i+1)\alpha\kappa A^2_{1+i} A_{2-i} J_1(\alpha(A^2_{2-i} - A^2_{1+i})) + \gamma_0 A^3_{1+i} + \kappa A_{2-i} J_0(\alpha(A^2_{2-i} - A^2_{1+i})) = 0,\ i = 0, 1 \quad (9)$$

which can be easily solved to yield a solution as the one shown in the 4th column of Fig. 1. We find that even such more complex solutions (which are highly unstable in DNLS [15]) are *dynamically robust* herein. This departure from the standard DNLS model can be rationalized by the fact that in the latter case the instability is mediated by the intersite tunneling/coupling [15], which for our special compacton solutions vanishes, hence endowing the solutions herein with dynamical stability.

The dynamical stability of the solutions of Fig. 1 with respect to the original DNLS model in Eq. (2) has been investigated in Fig. 2 for the one-site (left panels) and the two-site, in-phase (right panels) modes (similar findings were obtained for other modes). The top panels show the space-time contour map of the solution modulus, while the bottom panels illustrates the deviation from the initial condition. The structural stability of these compactons was ensured by adding a uniformly distributed random perturbation of small amplitude to the original solution. Both for the averaged equation (not shown here) and for the original system (see Fig. 2), the relevant perturbation stays bounded and never exceeds 2% of the solution amplitude. The waveforms remain remarkably localized in their compact shape (after a transient stage of shedding off small amplitude "radiation") and their tails never exceed an O(ϵ) correction, as theoretically expected for timescales of O($1/\epsilon$). Notice that for Eq. (2), $\gamma(t) = 1 + \frac{1}{\epsilon}\cos(t/\epsilon)$, with $\epsilon = 0.1$ was used. *However,* if one departs from the regime of validity of the averaging and from the SNLM limit, interesting deviations from the above behavior (and stability) arise. An example of this is shown in the left panels of Fig. 3. In this case, the three-site solution was initialized in Eq. (2) with $\epsilon = 0.1$ in the top panel, while $\epsilon = 0.025$ in the bottom one. In the latter, the above argued robustness of the averaged modes was observed. Yet, in the former one, the apparent lack thereof was clearly due to the use of an ϵ outside of the regime of applicability of the averaging approximation. Nevertheless, the resulting evolution has two interesting by-products. Firstly, it confirms the general preference of the system towards settling in compact modes, since the evolution asymptotes

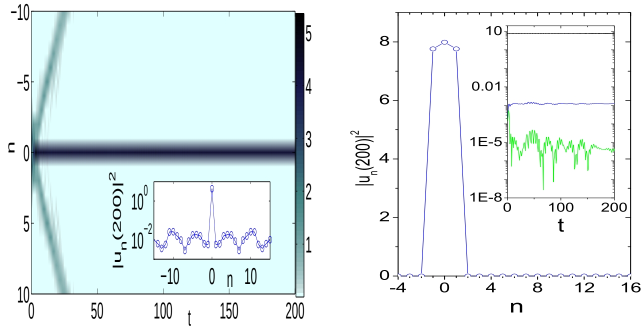

FIG. 4: Left panel: Space-time evolution of a Gaussian wavepacket $u_n(0) = 1.5e^{-0.1n^2}$ under Eq. (2) for $\kappa = 0.5$. Clearly, an essentially compact single-site excitation is produced (see the inset for $t = 200$). Right panel: Three-site compacton at time $t = 200$ generated from uniform i.c. $u_i = 2.8, i = -1, 0, 1$ for $\kappa = 0.5$. The inset shows time evolution curves (from top to bottom) of amplitudes at $0, 1, 2, 3$, respectively (the first two curves overlap in this scale).

to an essentially single-site solution. Secondly, the larger amplitude of this solution in comparison to those of Fig. 1 led us to explore the possibility of compactly supported modes associated with *higher zeros* of the Bessel function in the right panels of Fig. 3. Remarkably, such solutions again, not only exist but are stable in all the cases shown in the figure (numerical linear stability graphs are omitted). This indicates the existence of an infinite sequence of such modes, connected with the zeros of the Bessel structure of the (averaged) tunneling.

To address the robust emergence of such compact excitations, we used both few-sites uniform and even Gaussian type excitations. While in the former (experimentally realizable, see e.g. [20]) case, discrete compactons can be expected, remarkably, in either case such excitations can result. A typical example is shown for a Gaussian initial profile in the left panel of Fig. 4, which yields a single-site compact mode differing in amplitude by more than two orders of magnitude between the central site and its nearest neighbor and showing *no signs whatsoever of an exponential tail*, even in a semilog plot. In the right panel of Fig. 4 a multi-site compacton generated from uniform compactly supported data is depicted. We see that the amplitude reduces by five orders of magnitude 3-sites away from the central peak.

Let us estimate the parameters for the experimental observation of such modes, e.g. for the case of ^{7}Li condensate in a deep optical lattice. The Feshbach resonance in Li occurs at the value of external magnetic field $B = 720G$. By varying the magnetic field around this value we can easily obtain variations of the scattering length a_{s1} around the order of the background scattering length a_{s0} yielding $\gamma_1/\gamma_0 \sim 10$. In the deep optical lattice with $V_0 > 10E_R$, where V_0 is the depth of the lattice and $E_R = \hbar^2 k^2/2m$ is the recoil energy, the Gross-Pitaevskii equation can be mapped into the DNLS equation (2) [16]. Thus by changing periodically in time the magnetic field between these values with the frequency $\Omega \sim 10\omega_R$, where $\omega_R = E_R/\hbar$, we can generate matter wave compactons.

Conclusions. We predicted the existence of discrete compactons in the DNLSE with strong nonlinearity management. We found stable single and few-sites compactons of odd and even parity. They are robust and can be generated from different classes of initial conditions. Such structures may be observable in experiments on BECs in deep optical lattices with periodically varying scattering length and arrays of nonlinear optical waveguides with variable Kerr coefficient along the propagation distance.

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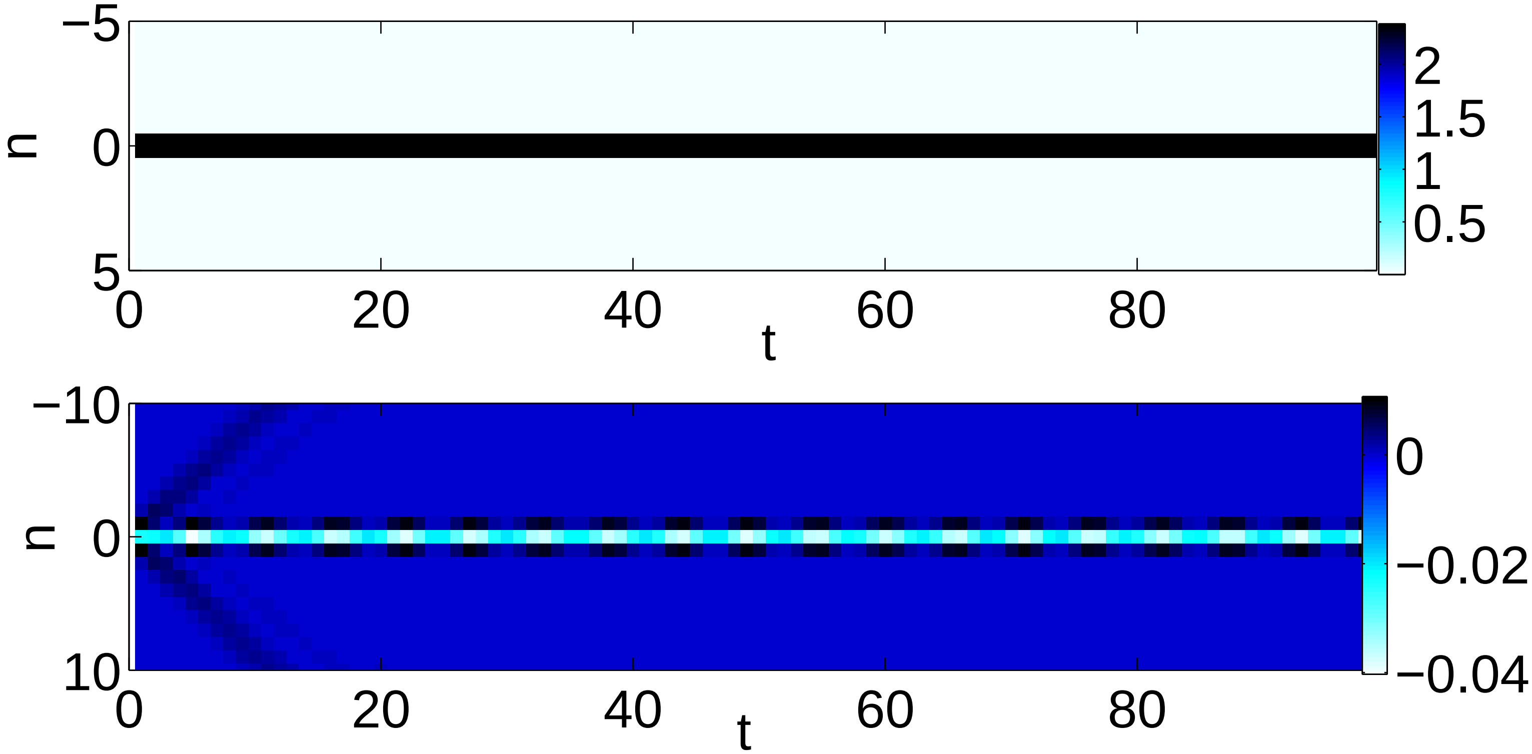

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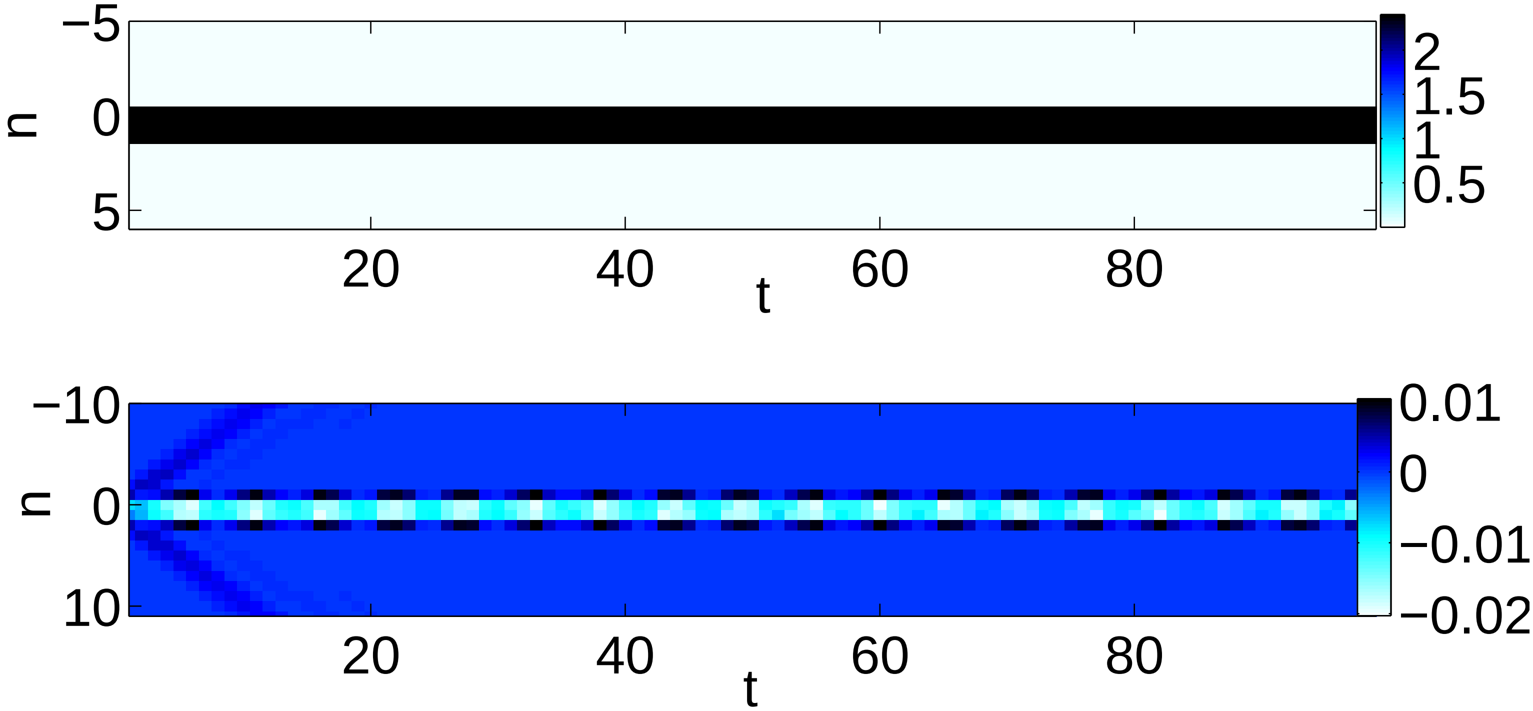

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