

Geodesic Flows and Neumann Systems on Stiefel Varieties. Geometry and Integrability*

Yuri N. Fedorov Božidar Jovanović

November 9, 2010

Abstract

We study integrable geodesic flows on Stiefel varieties $V_{n,r} = SO(n)/SO(n-r)$ given by the Euclidean, normal (standard), Manakov-type, and Einstein metrics. We also consider natural generalizations of the Neumann systems on $V_{n,r}$ with the above metrics and prove their integrability in the non-commutative sense by presenting compatible Poisson brackets on $(T^*V_{n,r})/SO(r)$. Various reductions of the latter systems are described, in particular, the generalized Neumann system on an oriented Grassmannian $G_{n,r}$ and on a sphere S^{n-1} in presence of Yang-Mills fields or a magnetic monopole field.

Apart from the known Lax pair for generalized Neumann systems, an alternative (dual) Lax pair is presented, which enables one to formulate a generalization of the Chasles theorem relating the trajectories of the systems and common linear spaces tangent to confocal quadrics. Additionally, several extensions are considered: the generalized Neumann system on the complex Stiefel variety $W_{n,r} = U(n)/U(n-r)$, the matrix analogs of the double and coupled Neumann systems.

Contents

1	Introduction	2
2	Hamiltonian Flows on Stiefel Varieties	3
3	Geodesic Flows	7
4	Integrability of Geodesic Flows	10
5	The Neumann Systems on Stiefel Varieties	13
6	Compatible Poisson Brackets and Integrability	14
7	Reduction to Grassmannians	20
8	The Dual Lax Pair and Geometric Interpretation of Integrals	22
9	Appendix 1. Magnetic Neumann Systems	29
10	Appendix 2. Rank r Double, Coupled and Neumann Systems on Complex Stiefel Manifolds	33

*AMS Subject Classification: 17B80, 53D25, 70H06, 70H33, 70H45

1 Introduction

A Stiefel variety $V_{n,r}$ is the variety of r ordered orthogonal unit vectors (e_1, \dots, e_r) in the Euclidean space \mathbb{R}^n or, equivalently, the set of $n \times r$ matrices

$$X = (e_1 \cdots e_r) \in M_{n,r}(\mathbb{R})$$

satisfying the condition

$$X^T X = \mathbf{I}_r, \tag{1.1}$$

where \mathbf{I}_r is an $r \times r$ unit matrix (e.g, see [29]). In particular, $V_{n,1}$ is a sphere S^{n-1} , while both $V_{n,n}$ and $V_{n,n-1}$ are diffeomorphic to $SO(n)$.

The integrable geodesic flows on $V_{n,2}$ and $V_{n,r}$, $2 < r < n - 1$ are constructed in [48] and [10], respectively. The geodesic flows were described in the homogeneous space representation $SO(n)/SO(n-r)$ following a general approach to the integrability of geodesic flows on homogeneous spaces [10, 11, 12].

In the first part of our paper (Sections 2, 3, 4) we study geodesic flows in the redundant coordinates X with constraints (1.1) by using a Dirac approach and the corresponding Poisson structure. This allows us to describe the flows in a quite transparent way. Our main tool is the theorem on non-commutative integrability of Hamiltonian systems (see [39, 38]) and for various examples of integrable flows, we calculate the dimension of invariant isotropic tori and give the matrix Lax representations (Section 4).

For the Manakov-type metric on $V_{n,r}$ a geometric interpretation of the motion in the form of the classical Chasles theorem for the geodesic flow on an ellipsoid is given (Section 8). In addition, the complete integrability of the geodesic flows of the $SO(n)$ -invariant Einstein metrics constructed by Jensen [27], and Arvanitoyeorgos, Dzhepko and Nikonorov [4] are proved in Sections 4 and 6.

The second part of the paper (Sections 5, 6, 7, 8) considers integrable Neumann flows on the Stiefel variety $V_{n,r} = SO(n)/SO(n-r)$ and the oriented Grassmannian variety $G_{n,r} = SO(n)/SO(r) \times SO(n-r)$, generalizing the famous Neumann system on the sphere S^{n-1} . The latter is defined as a natural mechanical system with the Hamiltonian (see [40, 35, 36]):

$$H_{neum} = \frac{1}{2}(p, p) + \frac{1}{2}(Ae, e), \quad A = \text{diag}(a_1, \dots, a_n), \tag{1.2}$$

where the cotangent bundle T^*S^{n-1} is realized as a submanifold of $\mathbb{R}^{2n}\{e, p\}$ given by the constraints $(e, e) = 1$, $(e, p) = 0$. This system, together with the Jacobi problem on the geodesic flow on an ellipsoid, provides one of the basic and most beautiful examples of application of algebraic geometric tools to integrable systems (e.g, see [24, 37, 1, 2]). It is also directly related to many other integrable models (e.g., see [36, 49, 30, 32, 22]).

The Neumann systems on $V_{n,r}$ which we consider have the kinetic energy of $SO(n)$ -invariant metrics described in Section 3 and the potential function

$$V = \frac{1}{2} \text{tr}(X^T A X) = \frac{1}{2} \sum_{i=1}^r (e_i, A e_i).$$

Two matrix Lax representations are presented. The first, a "big" one, given by Theorem 5.2 is closely related to the symmetric Clebsch–Perelomov rigid body problem [41]. For $r = 1$, it was given by Moser in [35] and for $r > 1$ and the case of the Manakov type submersion metrics by Reyman and Semenov–Tian-Shanski [44] within the framework of the R -matrix method. Note that for $r > 1$ this Lax pair does not define a Neumann system on $V_{n,r}$ uniquely and does not provide a non-commutative set of integrals, necessary for the integrability.

In contrast, the second (dual, or "small") Lax pairs, given by Theorem 8.1 are equivalent to the Neumann systems with the Euclidean and normal metrics up to an action of a finite discrete group. For the Neumann system with the Euclidean metric, the small Lax pair was first given in the unpublished manuscript [28].

In Sections 6 and 7 we give a detailed proof of the non-commutative integrability of the considered Neumann flows by using the Bolsinov completeness condition for a set of Casimir functions of the pencil of compatible Poisson brackets (see [9]). We also indicate an integrable generalization of the Neumann system on Grassmannians with a quartic potential.

In Section 8 we propose a geometric interpretation of the integrals of the Neumann systems on $V_{n,r}$ obtained from the dual Lax representation. Our geometric model generalizes the celebrated Chasles theorem adopted by Moser for the case $r = 1$ (see, e.g, Theorem 4.10 in [37]).

Magnetic Neumann flows in the rank two case (on $G_{n,2}$ and $V_{n,2}$) as well as the motion of a particle on a sphere S^{n-1} under the influence of a Yang–Mills field are presented in Appendix 1. Finally in Appendix 2, we briefly consider the rank r double and coupled Neumann flows, as well as an extension of the Neumann system onto a complex Stiefel variety $W_{n,r} = U(n)/U(n-r)$.

The geodesic flows and Neumann systems considered in this paper are written in a form appropriate for their integrable discretizations, which we describe in a separate paper [23].

2 Hamiltonian Flows on Stiefel Varieties

Stiefel varieties. As it was mentioned in Introduction, a Stiefel variety $V_{n,r}$ is the variety of r ordered orthogonal unit vectors (e_1, \dots, e_r) in the Euclidean space $(\mathbb{R}^n, (\cdot, \cdot))$, or, equivalently, the set of $n \times r$ matrices satisfying constraints (1.1). Thus $V_{n,r}$ is a smooth subvariety of dimension $N = rn - r(r+1)/2$ in the space of $n \times r$ real matrices $M_{n,r}(\mathbb{R}) = \mathbb{R}^{nr}$ and the components of X are redundant coordinates on it.

The left $SO(n)$ action on $V_{n,r}$ ($X \mapsto RX$, $R \in SO(n)$) is transitive, hence $V_{n,r}$ can be realized as a homogeneous space of the Lie group $SO(n)$ as well. If fix the orthonormal base in \mathbb{R}^n

$$E_1 = (1, 0, \dots, 0)^T, \quad E_2 = (0, 1, 0, \dots, 0)^T, \quad \dots, \quad E_n = (0, 0, 0, \dots, 1)^T \quad (2.1)$$

and take the point $X_0 = (E_1, \dots, E_r) \in V_{n,r}$, then the orthogonal transformation fixing X_0 (relative to the above basis of \mathbb{R}^n) must have the form

$$\begin{pmatrix} \mathbf{I}_r & 0 \\ 0 & B \end{pmatrix}, \quad B \in SO(n-r). \quad (2.2)$$

Since the isotropy group of X_0 is isomorphic to $SO(n-r)$, the variety $V_{n,r}$ can be identified with $SO(n)/SO(n-r)$.

The Poisson structure. The tangent bundle $TV_{n,r}$ is the set of pairs (X, \dot{X}) subject to the constraints

$$X^T X = \mathbf{I}_r, \quad X^T \dot{X} + \dot{X}^T X = 0. \quad (2.3)$$

On the other hand, the cotangent bundle $T^*V_{n,r}$ can be realized as the set of pairs of $n \times r$ matrices (X, P) that satisfy the constraints

$$X^T X = \mathbf{I}_r, \quad X^T P + P^T X = 0. \quad (2.4)$$

The latter give $r(r+1)$ independent scalar constraints

$$F_{ij} = (e_i, e_j) - \delta_{ij} = 0, \quad G_{ij} = (e_i, p_j) + (e_j, p_i) = 0, \quad 1 \leq i \leq j \leq r, \quad (2.5)$$

where p_j is the j -th column of the matrix P . This realization of $T^*V_{n,r}$ is motivated by description of the geodesic flows of the Euclidean and normal metric on $V_{n,r}$ given below, however there are other natural realizations of $T^*V_{n,r}$ (see Section 3).

The canonical symplectic structure ω on $T^*V_{n,r}$ is the restriction of the canonical 2-form in the ambient space $T^*M_{n,r}(\mathbb{R})$,

$$\omega_0 = \sum_{i=1}^n \sum_{s=1}^r dp_s^i \wedge de_s^i.$$

For our purposes it is convenient to work with the redundant variables (X, P) . The canonical Poisson structure on $T^*V_{n,r}$ can then be described by using the Dirac construction [3, 16, 35]. Namely, let $\{\cdot, \cdot\}_0$ be the canonical Poisson bracket on \mathbb{R}^{2nr}

$$\{f_1, f_2\}_0 = \sum_{i=1}^r \left(\left(\frac{\partial f_1}{\partial e_i}, \frac{\partial f_2}{\partial p_i} \right) - \left(\frac{\partial f_1}{\partial p_i}, \frac{\partial f_2}{\partial e_i} \right) \right)$$

and $C_{i,j}$ be the inverse of the matrix $\{F_i, F_j\}_0$, $i, j = 1, \dots, r(r+1)$, where, for the sake of simplicity, we denoted constraints (2.5) by $F_i = 0$, $i = 1, \dots, r(r+1)$. Then the Dirac bracket is given by

$$\{f_1, f_2\} = \{f_1, f_2\}_0 + \sum_{i,j} \{F_i, f_1\}_0 C_{i,j} \{F_j, f_2\}_0. \quad (2.6)$$

The subvariety $T^*V_{n,r}$ appears as a symplectic leaf of the Dirac bracket and the restriction of $\{f_1, f_2\}$ to $T^*V_{n,r}$ depends only on the restriction of f_1 and f_2 to $T^*V_{n,r}$.

The Hamiltonian equation

$$\dot{f} = \{f, H\}$$

can be also described by the using Lagrange multipliers. We shall write them in the matrix form:

$$\begin{aligned} \dot{X} &= \frac{\partial H}{\partial P} - X\Pi, \\ \dot{P} &= -\frac{\partial H}{\partial X} + X\Lambda + P\Pi, \end{aligned} \quad (2.7)$$

where Λ and Π are $r \times r$ symmetric matrix Lagrange multipliers uniquely determined from the condition for the trajectory $(X(t), P(t))$ to satisfy constraints (2.4).

Momentum mappings. The Lie group $SO(n)$ naturally acts on $T^*V_{n,r}$ by left multiplication:

$$R \cdot (X, P) = (RX, RP), \quad R \in SO(n). \quad (2.8)$$

Below we use the well known identification of $\Lambda^2 \mathbb{R}^n$ with a subset of $so(n)$: $x \wedge y = x \otimes y - y \otimes x = x \cdot y^T - y \cdot x^T$, $x, y \in \mathbb{R}^n$. Also, $\langle \cdot, \cdot \rangle$ is proportional to the Killing metric on $so(n)$:

$$\langle \xi_1, \xi_2 \rangle = -\frac{1}{2} \text{tr}(\xi_1 \xi_2), \quad (2.9)$$

$\xi_1, \xi_2 \in so(n)$. By the use of the scalar product (2.9) we identify $so(n)$ and $so(n)^*$.

Proposition 2.1 *The left $SO(n)$ -action (2.8) is Hamiltonian. The equivariant momentum mapping $\Phi : T^*V_{n,r} \rightarrow so(n)^* \cong so(n)$ is given by*

$$\Phi(X, P) = \sum_{i=1}^r p_i \wedge e_i, \quad (2.10)$$

or, in the matrix form $\Phi(X, P) = PX^T - XP^T$.

Proof. The left $SO(n)$ -action (2.8) on $(T^*M_{n,r}(\mathbb{R}), \{\cdot, \cdot\}_0)$ is Hamiltonian with the momentum map (2.10). The constraint functions $X^T X$ and $X^T P + P^T X$ are $SO(n)$ -invariant. Therefore

$$\{\Phi_\xi, F_i\}_0 = 0, \quad i = 1, \dots, r(r+1), \quad (2.11)$$

where $\Phi_\xi(X, P) = \langle \Phi(X, P), \xi \rangle$ is the Hamiltonian function of the action of the one-parameter subgroup $\{\exp(s\xi), s \in \mathbb{R}\}$.

In view of the definition of the Dirac bracket (2.6) and (2.11), the Hamiltonian flows of Φ_ξ on $T^*V_{n,r}$ with respect to the brackets $\{\cdot, \cdot\}_0$ and $\{\cdot, \cdot\}$ coincide. This proves the proposition. \square

Together with a left $SO(n)$ -action, we also have the natural right $SO(r)$ -action:

$$(X, P) \cdot Q = (XQ, PQ), \quad Q \in SO(r). \quad (2.12)$$

Following similar lines, one can prove

Proposition 2.2 *The right $SO(r)$ -action (2.12) is Hamiltonian. The equivariant momentum mapping $\Psi : T^*V_{n,r} \rightarrow so(r)^* \cong so(r)$ is given by*

$$\Psi(X, P) = X^T P - P^T X. \quad (2.13)$$

The momentum mappings Φ and Ψ are Poisson with respect to the (+) and (-) Lie-Poisson brackets on $so(n)$ and $so(r)$:

$$\begin{aligned} \{h_1 \circ \Phi(X, P), h_2 \circ \Phi(X, P)\} &= \{h_1(\mu), h_2(\mu)\}_{so(n)}^+, \quad \mu = \Phi(X, P), \\ \{f_1 \circ \Psi(X, P), f_2 \circ \Psi(X, P)\} &= \{f_1(\eta), f_2(\eta)\}_{so(r)}^-, \quad \eta = \Psi(X, P), \end{aligned}$$

where

$$\begin{aligned} \{h_1(\mu), h_2(\mu)\}_{so(n)}^+ &= \langle \mu, [\nabla h_1(\mu), \nabla h_2(\mu)] \rangle, \quad h_1, h_2 : so(n) \rightarrow \mathbb{R}, \\ \{f_1(\eta), f_2(\eta)\}_{so(r)}^- &= -\langle \eta, [\nabla f_1(\eta), \nabla f_2(\eta)] \rangle, \quad f_1, f_2 : so(r) \rightarrow \mathbb{R} \end{aligned}$$

and where the brackets $\langle \cdot, \cdot \rangle$ denote the scalar product (2.9) on $so(n)$ and $so(r)$, respectively.

The algebra of $SO(n)$ -invariant functions. Let $C^\infty(T^*V_{n,r})^{SO(n)}$ be the algebra of $SO(n)$ -invariant functions on $T^*V_{n,r}$. Since $SO(n)$ acts in a Hamiltonian way on $T^*V_{n,r}$, $C^\infty(T^*V_{n,r})^{SO(n)}$ is closed under the Poisson bracket.

Let $X_0 = (E_1, \dots, E_r)$. The $SO(n)$ -invariant functions, via restrictions, are in one-to-one correspondence with the $SO(n-r)$ -invariant functions on $T_{X_0}^*V_{n,r}$.

We can write $P_0 \in T_{X_0}^*V_{n,r}$ as a block matrix

$$P_0 = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, \quad (2.14)$$

where P_1 and P_2 are $r \times r$ and $(n-r) \times r$ matrixes, respectively. Then P_0 satisfies constraints (2.4) at $X = X_0$ if $P_1^T = -P_1$. Also, the $SO(n-r)$ -action on $T_{X_0}^*V_{n,r}$ is given by

$$P_1 \mapsto P_1, \quad P_2 \mapsto B \cdot P_2, \quad B \in SO(n-r). \quad (2.15)$$

Lemma 2.3 *The maximal number of functionally independent $SO(n)$ -invariant functions, i.e., the differential dimension of $C^\infty(T^*V_{n,r})^{SO(n)}$, equals*

$$\text{ddim } C^\infty(T^*V_{n,r})^{SO(n)} = \begin{cases} \dim V_{n,r} - \dim SO(n-r), & n \leq 2r \\ \dim V_{n,r} - \dim SO(n-r) + \dim SO(n-2r), & n > 2r. \end{cases} \quad (2.16)$$

Proof. The differential dimension of $C^\infty(T^*V_{n,r})^{SO(n)}$ is just the codimension of the generic $SO(n-r)$ orbit in $T_{X_0}^*V_{n,r}$. The dimension of the orbit $SO(n-r) \cdot P_0$ is

$$\dim SO(n-r) - \dim SO(n-r)_{P_0},$$

where $SO(n-r)_{P_0}$ is the isotropy group of P_0 . Since $SO(n-r)_{P_0} = \{\mathbf{I}_{n-r}\}$ for $n \leq 2r$, for a generic P_0 we get the first relation in (2.16). Further, in the case $n > 2r$, we can take $P_0 = (E_{r+1}, E_{r+2}, \dots, E_{2r})$. Then $SO(n-r)_{P_0} = SO(n-2r)$. \square

It can easily be verified that the restrictions of the $SO(n)$ -invariant functions

$$\Psi_{ij} = (e_i, p_j) - (e_j, p_i), \quad (P^T P)_{ij} = (p_i, p_j), \quad i, j = 1, \dots, r \quad (2.17)$$

to $T_{X_0}^*V_{n,r}$ define the generic orbits of the action (2.15). In particular, we get the following simple statement.

Lemma 2.4 *If a smooth function $f \in C^\infty(T^*V_{n,r})$ Poisson commutes with functions (2.17), then it Poisson commutes with all $SO(n)$ -invariant functions on $T^*V_{n,r}$.*

The Poisson bracket on $C^\infty(T^*V_{n,r})^{SO(n)}$ can be described as follows. The restriction of momentum mapping Φ to $T_{X_0}^*V_{n,r}$ establish the isomorphism

$$T_{X_0}^*V_{n,r} \cong \mathfrak{v}, \quad (2.18)$$

where \mathfrak{v} is the orthogonal complement of $so(n-r)$ in $so(n)$. Within identification (2.18), the $SO(n-r)$ -action (2.15) corresponds to the adjoint $SO(n-r)$ -action on \mathfrak{v} and the Poisson bracket on $C^\infty(T^*V_{n,r})^{SO(n)}$ corresponds to the Poisson bracket

$$\{\tilde{f}_1(\xi), \tilde{f}_2(\xi)\}_{\mathfrak{v}} = -\langle \xi, [\nabla \tilde{f}_1(\xi), \nabla \tilde{f}_2(\xi)] \rangle.$$

on the algebra $C^\infty(\mathfrak{v})^{SO(n-r)}$ of $SO(n-r)$ -adjoint invariants on \mathfrak{v} (see Thimm [48]).

Recall that the set of commuting $SO(n)$ -invariant functions \mathfrak{A} is complete if it contains maximal possible number of independent functions, that is (see [10, 12]),

$$\dim \mathfrak{A} = \dim V_{n,r} - l,$$

where $2l$ is the dimension of a generic adjoint orbit in

$$\Phi(T^*V_{n,r}) = \text{Ad}_{SO(n)} \Phi(T_{X_0}^*V_{n,r}).$$

Note that, for $n \leq 2r+1$, the generic orbit in $\Phi(T^*V_{n,r})$ is regular, while it is singular otherwise. One can prove the following relations.

Lemma 2.5

$$2l = \begin{cases} \frac{n(n-1)}{2} - \lfloor \frac{n}{2} \rfloor, & n \leq 2r+1, \\ 2r(n-r-1), & n > 2r+1. \end{cases}$$

Now consider the chain of subalgebras $so(n-r+1) \subset so(n-r+2) \subset \dots \subset so(n)$, where a matrix $\xi \in so(n-r+i)$ is included in $so(n)$ as a block matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & \xi \end{pmatrix}. \quad (2.19)$$

Let \mathfrak{A}_i be the algebra of invariants on $so(n-r+i)$ considered as a polynomials on $so(n)$ and restricted to \mathfrak{v} . Then

$$\mathfrak{A} = \mathfrak{A}_1 + \dots + \mathfrak{A}_r \quad (2.20)$$

is a complete polynomial commutative subset of $C^\infty(\mathfrak{v})^{SO(n-r)}$ (see again [10, 12]). Other complete commutative sets of $SO(n)$ -invariant functions are given in [17] and Theorem 6.4 below.

3 Geodesic Flows

$SO(n)$ -invariant metrics. An $SO(n)$ -invariant metric g on $V_{n,r}$ can be specified by a positive definite, $SO(n-r)$ -invariant scalar products g_0 at the point X_0 as follows:

$$g(R \cdot \eta_1, R \cdot \eta_2)_X = g_0(\eta_1, \eta_2), \quad \eta_1, \eta_2 \in T_{X_0}V_{n,r}, \quad X = R \cdot X_0.$$

Equivalently, an $SO(n)$ -invariant metric can be defined by using an $SO(n)$ -invariant Hamiltonian function $H(X, P)$, which is quadratic in momenta and positive definite on $T^*V_{n,r}$.

There are two natural $SO(n)$ -invariant metrics on the Stiefel variety $V_{n,r}$: the metric induced from the embedding of $V_{n,r}$ in the Euclidean space $M_{n,r}(\mathbb{R})$ and the normal metric induced from a bi-invariant metric on the Lie group $SO(n)$ (see below).

Concerning geometrical significance, one should also mention invariant Einstein metrics (see [5]) constructed in [27], [4].

The Euclidean metric. The Euclidean metric in $M_{n,r}(\mathbb{R})$ is given by the Lagrangian function

$$L_E(X, \dot{X}) = \frac{1}{2} \text{tr}(\dot{X}^T \dot{X}) = \frac{1}{2} \sum_{i=1}^r (\dot{e}_i, \dot{e}_i).$$

The Legendre transformation

$$P = \frac{\partial L_E}{\partial \dot{X}} = \dot{X} \tag{3.1}$$

yields the Hamiltonian function

$$H_E(X, P) = \frac{1}{2} \text{tr}(P^T P) = \sum_{i=1}^r \frac{1}{2} (p_i, p_i) \tag{3.2}$$

defined on the cotangent bundle $T^*M_{n,r}(\mathbb{R})$.

We shall refer to the restriction of the above metric to $V_{n,r}$ as the *Euclidean metric*, which will be denoted by ds_E^2 . The geodesic flow is described by the Euler–Lagrange equations with multipliers

$$\frac{d}{dt} \frac{\partial L_E}{\partial \dot{X}} = \frac{\partial L_E}{\partial X} + X\Lambda \iff \ddot{X} = X\Lambda, \quad \Lambda = -\dot{X}^T \dot{X},$$

where the symmetric matrix Λ is uniquely determined from the condition for the trajectory $X(t)$ to satisfy the constraints $X^T X = \mathbf{I}_r$.

Taking into account constraints (2.3) and the Legendre transformation (3.1), we see that the cotangent bundle $T^*V_{n,r}$ can be represent as a submanifold of $T^*M_{n,r}(\mathbb{R})$ given by (2.4). The corresponding Hamiltonian flow of $H_E(X, P)$ with respect to the Dirac bracket is

$$\begin{aligned} \dot{X} &= P, \\ \dot{P} &= -X P^T P. \end{aligned} \tag{3.3}$$

Submersion metrics. Let $\bar{g}(\cdot, \cdot)$ be a right-invariant metric on $SO(n)$. The subgroup $SO(n-r)$ acts freely on $SO(n)$ by isometries (right action). There is a \bar{g} -orthogonal decomposition of $T_R SO(n)$

$$SO(n-r)_R + \mathcal{H}_R = T_R SO(n), \quad R \in SO(n),$$

where $\mathcal{SO}(n-r)_R$ is the tangent space to the fibre $R \cdot \mathcal{SO}(n-r)$. By definition, the submersion metric $g(\cdot, \cdot)$ on $\mathcal{SO}(n)/\mathcal{SO}(n-r)$ is given by

$$g(\xi_1, \xi_2)_{\rho(R)} = \bar{g}(\bar{\xi}_1, \bar{\xi}_2)_R, \quad \xi_i \in T_{\rho(R)}(\mathcal{SO}(n)/\mathcal{SO}(n-r)), \quad \bar{\xi}_i \in \mathcal{H}_R, \quad \xi_i = d\rho(\bar{\xi}_i),$$

where $\rho : \mathcal{SO}(n) \rightarrow \mathcal{SO}(n)/\mathcal{SO}(n-r)$ is the canonical projection (e.g., see [5]). The vectors in $\mathcal{SO}(n-r)_R$ and \mathcal{H}_R are called vertical and horizontal respectively.

The Hamiltonian of a right invariant metric on $\mathcal{SO}(n)$ can be written in the form $h \circ \bar{\Phi}$, where h is a positive definite quadratic form on $\mathfrak{so}(n)$ and $\bar{\Phi} : T^*\mathcal{SO}(n) \rightarrow \mathfrak{so}(n)$ is the momentum mapping of the natural left $\mathcal{SO}(n)$ -action. It follows that the class of submersion metrics on $\mathcal{SO}(n)/\mathcal{SO}(n-r)$ is given by Hamiltonian functions of the form $h \circ \Phi$, where now Φ is the momentum mapping of a natural left $\mathcal{SO}(n)$ -action on $T^*(\mathcal{SO}(n)/\mathcal{SO}(n-r))$ (e.g., see [14]).

The above observation helps us to write down the Hamiltonians and the geodesic flows of the submersion metrics in the representation of the cotangent bundle $T^*V_{n,r}$ given by the constraints (2.4). The Hamiltonians are

$$H_{\mathbb{A}}(X, P) = h_{\mathbb{A}} \circ \Phi = \frac{1}{2} \langle \mathbb{A}\Phi, \Phi \rangle = -\frac{1}{4} \text{tr}(\mathbb{A}(PX^T - XP^T)(PX^T - XP^T)), \quad (3.4)$$

where $h_{\mathbb{A}}(\xi) = \frac{1}{2} \langle \mathbb{A}\xi, \xi \rangle$, $\xi \in \mathfrak{so}(n)$ and $\mathbb{A} : \mathfrak{so}(n) \rightarrow \mathfrak{so}(n)$ are positive definite operators.

Proposition 3.1 *The equations of the submersion metrics geodesic flow generated by (3.4) are*

$$\begin{aligned} \dot{X} &= \mathbb{A}(\Phi) \cdot X, \\ \dot{P} &= \mathbb{A}(\Phi) \cdot P. \end{aligned} \quad (3.5)$$

Proof. By using the chain rule $dH = dh_{\mathbb{A}} \circ d\Phi$, one gets expressions (3.5) for the derivatives of the Hamiltonian (3.4) with respect to P and $-X$. Further, it can easily be verified that then the Lagrange matrix multipliers Λ and Π in (2.7) are zero. \square

Remark 3.1 In particular, for $r = n$, the system (3.5) describes the right-invariant geodesic flow on the Lie group $\mathcal{SO}(n)$. The symmetric form of the equations differs from the symmetric representation of the rigid body equations given in [6].

The normal metric. If $\bar{g}(\cdot, \cdot)$ is a bi-invariant metric induced by the scalar product (2.9), then the submersion metric is called the *normal metric*. It is proportional to the *standard metric* induced by the negative Killing form [5]) on $\mathcal{SO}(n)/\mathcal{SO}(n-r)$. We shall denote the normal metric by ds_0^2 . Contrary to a generic submersion metric, the normal metric is also $\mathcal{SO}(n)$ -invariant. The corresponding Hamiltonian has the form

$$H_0(X, P) = \frac{1}{2} \langle \Phi, \Phi \rangle = \frac{1}{2} \text{tr}(P^T P) - \frac{1}{2} \text{tr}((X^T P)^2), \quad (3.6)$$

and its geodesic flow is given by

$$\dot{X} = \Phi \cdot X = P - XP^T X, \quad (3.7)$$

$$\dot{P} = \Phi \cdot P = -XP^T P + PX^T P. \quad (3.8)$$

Under the conditions (2.4), the relation (3.7) can be uniquely inverted and one gets

$$P = \dot{X} - \frac{1}{2} X X^T \dot{X}. \quad (3.9)$$

Relations (3.7), (3.9) give identification of $TV_{n,r}$ and $T^*V_{n,r}$ by means of the normal metric. Therefore, the Lagrangian function for the metric ds_0^2 is

$$L_0(X, \dot{X}) = \frac{1}{2} \langle \Phi_0, \Phi_0 \rangle, \quad (3.10)$$

where $\Phi_0(X, \dot{X}) = \Phi(X, P(X, \dot{X}))$:

$$\Phi_0(X, \dot{X}) = \dot{X} X^T - X \dot{X}^T - \frac{1}{2} X [X^T \dot{X} - \dot{X}^T X] X^T. \quad (3.11)$$

For a *right* $SO(n)$ -action (and therefore with opposite signs in the equations), the relation (3.11) is established in [22] by studying nonholonomic LR systems on the Lie group $SO(n)$. Note that the realization of $T^*V_{n,r}$ via (2.4) is also natural if we consider the flow given by the normal metric. Namely, defining momenta by the Legendre transformation $P = \partial L_0 / \partial \dot{X}$, from the constraints (2.3) we get the conditions (2.4).

Einstein metrics. The momentum mappings Φ and Ψ are invariant under the $SO(r)$ and $SO(n)$ actions, respectively. Therefore, the Hamiltonians of the form

$$H_\kappa(X, P) = \frac{1}{2} \langle \Phi, \Phi \rangle + \frac{\kappa}{2} \langle \Psi, \Psi \rangle = \frac{1}{2} \text{tr}(P^T P) - \left(\frac{1}{2} + \kappa \right) \text{tr}((X^T P)^2) \quad (3.12)$$

is $SO(n) \times SO(r)$ invariant. Within the class of the metrics ds_κ^2 determined by the Hamiltonian functions (3.12) there is the normal metric ($\kappa = 0$) and the Euclidean metric ($\kappa = -1/2$). Moreover, for $r = 2$ there is a unique value of κ , while for $r > 2$ there are exactly two values, such that ds_κ^2 is an Einstein metric (see [27, 4]). Following [4], we refer to these metrics as *the Jensen* metrics.

Further, in [4], new examples of the Einstein metrics are obtained within the class of metrics that we shall describe below.

Consider the Lie subalgebra

$$so(r_1) \oplus so(r_2) \oplus \cdots \oplus so(r_k) \subset so(r), \quad r_1 + r_2 + \cdots + r_k = r,$$

naturally embedded into $so(r)$. Define the Hamiltonian H_K via

$$H_K = \frac{1}{2} \langle \Phi, \Phi \rangle + K \circ \Psi, \quad (3.13)$$

where the quadratic function K is

$$K(\xi) = \frac{\kappa_0}{2} \langle \xi, \xi \rangle + \frac{\kappa_1}{2} \langle \xi_1, \xi_1 \rangle + \cdots + \frac{\kappa_k}{2} \langle \xi_k, \xi_k \rangle. \quad (3.14)$$

Here ξ_i are orthogonal projections (with respect to the $so(r)$ -Killing metric) to $so(r_i) \subset so(r)$.

In [4] Arvanitoyeorgos, Dzhepko, and Nikonorov proved that if $r_1 = r_2 = \cdots = r_k$, $\kappa_1 = \kappa_2 = \cdots = \kappa_k$, $k > 1$, $n - r > r_1 \geq 3$, than among the metrics defined by Hamiltonians (3.13) there are four Einstein metrics (two of them, with $\kappa_1 = 0$, are the Jensen metrics).

$SO(r)$ -invariant metrics. Let A be a symmetric, positive definite $n \times n$ matrix. The geodesic flows on $V_{n,r}$ with Lagrangians of the form

$$L_A(X, \dot{X}) = \frac{1}{2} \text{tr}(\dot{X}^T A \dot{X}) = \frac{1}{2} \sum_{i=1}^r (\dot{e}_i, A \dot{e}_i),$$

were studied in [7] from the point of view of the variational and optimal control problems. The Lagrangian L_A can be considered on the whole space $M_{n \times r}(\mathbb{R})$, where the Legendre transformation

$$P = \frac{\partial L_A}{\partial \dot{X}} = A\dot{X}, \quad (3.15)$$

gives the following Hamiltonian on the cotangent bundle $T^*M_{n,r}(\mathbb{R})$

$$H_A(X, P) = \frac{1}{2} \text{tr}(P^T A^{-1} P) = \frac{1}{2} \sum_{i=1}^r (p_i, A^{-1} p_i).$$

From (2.3) and (3.15) we conclude that the cotangent bundle $T^*V_{n,r}$ can be represented as a submanifold of $T^*M_{n,r}(\mathbb{R})$ given by the equations

$$X^T X = \mathbf{I}_r, \quad X^T A^{-1} P + P^T A^{-1} X = 0. \quad (3.16)$$

Again, one defines the Dirac bracket with respect to the constraints (3.16).

Let ds_A^2 be the metric defined by the Lagrangian L_A and $\{\cdot, \cdot\}_A$ be the new Dirac bracket. Then the geodesic flow of ds_A^2 can be described by the Hamilton equations

$$\dot{f} = \{f, H_A(X, P)\}_A$$

restricted to the symplectic leaf (3.16). The corresponding matrix form of the flow is

$$\begin{aligned} \dot{X} &= A^{-1} P, \\ \dot{P} &= X \Lambda, \end{aligned} \quad (3.17)$$

where Λ is a symmetric $r \times r$ matrix uniquely determined from the condition

$$\Lambda X^T A^{-1} X + X^T A^{-1} X \Lambda + 2P^T A^{-2} P = 0.$$

Note that for $r = 1$, the metric ds_A^2 is a standard metric on the ellipsoid $(x, A^{-1}x) = 1$, while for r it is a right-invariant Manakov rigid body metric on $SO(n)$ (see [35]). Furthermore, ds_A^2 is right $SO(r)$ -invariant and, via submersion, induces a metric on the oriented Grassmannian variety $G_{n,r}$ (see Section 7). It would be interesting to prove integrability of the corresponding geodesic flows.

4 Integrability of Geodesic Flows

The normal metric. As shown in [10, 11], the geodesic flows of the normal metrics ds_0^2 on the homogeneous spaces G/H of compact Lie groups G are completely integrable in the non-commutative sense. The proof is based on the following geometrical observation. Let (M, G, Φ) be a Hamiltonian G -space with an equivariant momentum mapping $\Phi : M \rightarrow \mathfrak{g}^*$, where G is a compact group. Consider the following two natural sets of functions on M : the functions obtained by pulling-back the algebra $C^\infty(\mathfrak{g}^*)$ by the moment map and the set of G -invariant functions $C^\infty(M)^G$. They are closed under the Poisson bracket and according to the Noether theorem $\{\Phi^*(C^\infty(\mathfrak{g}^*)), C^\infty(M)^G\} = 0$ [26]. Moreover, $\Phi^*(C^\infty(\mathfrak{g}^*)) + C^\infty(M)^G$ is a complete set of functions [11]. That is, any Hamiltonian system with those integrals is non-commutatively integrable [39, 38]. In particular, consider the case when (M, G, Φ) is a cotangent bundle $T^*(G/H)$ with a natural G action. Since the Hamiltonian H_0 of the normal metric is of the form $h \circ \Phi$, where h is an invariant quadratic polynomial on \mathfrak{g}^* , the function H_0 Poisson commutes with all G -invariant functions (the Noether theorem), as well as with all the functions in $\Phi^*(C^\infty(\mathfrak{g}^*))$ (the mapping $h \mapsto h \circ \Phi$ is a morphism of Poisson structures). Therefore, the flow of H_0 is non-commutatively integrable.

Let $2l$ be the dimension of a generic orbit in $\Phi(T^*V_{n,r})$ (see Lemma 2.5).

Theorem 4.1 [10, 11] *The geodesic flow of the normal metric (3.7), (3.8) is completely integrable in the non-commutative sense. The complete algebra of first integrals is*

$$\Phi^*(C^\infty(\mathfrak{so}(n)) + C^\infty(T^*V_{n,r})^{SO(n)}).$$

The generic motions of the system are quasi-periodic over the isotropic tori of dimension

$$\delta_0 = 2 \dim V_{n,r} - \text{ddim } C^\infty(T^*V_{n,r})^{SO(n)} - 2l.$$

Recall that the number of functionally independent $SO(n)$ -invariant functions is given by (2.16). Thus we have

$$\delta_0 = \begin{cases} r, & n \geq 2r \\ \dim V_{n,r} + \dim SO(n-r) - \dim SO(n) + \text{rank } SO(n), & n < 2r \end{cases}$$

It is interesting that for $n \geq 2r$ the dimension of the invariant tori coincides with the dimension of invariant tori of geodesic flows of normal metrics on the corresponding Grassmannian manifolds $G_{n,r}$.

Manakov metrics. The above construction has a natural generalization to a class of geodesic flows of submersion metrics given by the Hamiltonians (3.4), such that the corresponding Euler equations on $\mathfrak{so}(n)$:

$$\dot{f} = \{f, h_{\mathbb{A}}\}_{\mathfrak{so}(n)}^+ \iff \dot{\xi} = [\nabla h_{\mathbb{A}}(\xi), \xi] = [\mathbb{A}\xi, \xi], \quad \xi \in \mathfrak{so}(n) \quad (4.1)$$

are completely integrable. For example, choose the Manakov operator

$$\mathbb{A}(E_i \wedge E_j) = \frac{b_i - b_j}{a_i - a_j} E_i \wedge E_j, \quad 1 \leq i < j \leq n, \quad \text{i.e., } \mathbb{A}(\xi) = \text{ad}_A^{-1} \text{ad}_B \xi, \quad (4.2)$$

where all the eigenvalues of $A = \text{diag}(a_1, \dots, a_n)$ and $B = \text{diag}(b_1, \dots, b_n)$ are distinct and \mathbb{A} is positive definite. Then the Euler equations (4.1) are completely integrable ([33, 20]). Moreover, for generic A , among the integrals $\text{tr}(\xi + \lambda A)^k$ one can always find a complete set of commuting integrals h_1, \dots, h_l on a generic adjoint orbit in $\Phi(T^*V_{n,r})$ (see Brailov [14] and Bolsinov [9]). Therefore, according to Theorem 2.2 in [11], we have

Theorem 4.2 *The geodesic flow of the submersion metric (3.5) with the metric given by the Manakov operator (4.2) is completely integrable in the non-commutative sense with a complete set of polynomial integrals (2.17) and*

$$\text{tr} \left(\lambda A + \sum_{i=1}^r p_i \wedge e_i \right)^k, \quad k = 1, \dots, n, \quad \lambda \in \mathbb{R}. \quad (4.3)$$

The Generic motion of the system is quasi-periodic over the isotropic tori of dimension

$$\delta = 2 \dim V_{n,r} - \text{ddim } C^\infty(T^*V_{n,r})^{SO(n)} - l.$$

As in the case of geodesic flows of normal metrics, for $n \geq 2r$ the dimension of generic invariant tori is simply $\delta = r(n - r)$.

Remark 4.1 The non-commutative integrability implies the usual commutative integrability, at least by means of smooth commuting integrals [11]. For the case of the geodesic flows considered above, the commuting integrals can be taken to be the polynomials (2.20) and (4.3).

Like in the case of the right-invariant metric on $SO(n)$, the Manakov metric on $V_{n,r}$ possess a matrix Lax pair.

Theorem 4.3 *Equations (3.5) with the metric given by the Manakov operator (4.2) imply the matrix equation with a spectral parameter λ*

$$\begin{aligned} \frac{d}{dt}\mathcal{L}_{man}(\lambda) &= [\mathcal{A}_{man}(\lambda), \mathcal{L}_{man}(\lambda)], \\ \mathcal{L}_{man}(\lambda) &= \Phi + \lambda A, \quad \mathcal{A}_{man}(\lambda) = \mathbb{A}(\Phi) + \lambda B. \end{aligned}$$

The dual Lax pair for the Manakov flows. Consider the Manakov operator (4.2) for $B = A^2$. Then

$$\mathbb{A}(\xi) = A\xi + \xi A, \quad (4.4)$$

and equations (3.5) become

$$\begin{aligned} \dot{X} &= A(PX^T - XP^T)X + (PX^T - XP^T)AX, \\ \dot{P} &= A(PX^T - XP^T)P + (PX^T - XP^T)AP. \end{aligned} \quad (4.5)$$

Theorem 4.4 *Up to the action of a discrete group \mathbb{Z}_2^n generated by reflections with respect to the coordinate hyperplanes in \mathbb{R}^n ,*

$$\begin{aligned} (X, P) &\longmapsto (S_i X, S_i P), \quad i = 1, \dots, n, \\ S_i(x_1, \dots, x_n) &= (y_1, \dots, y_n), \quad y_j = x_j, \quad j \neq i, \quad y_i = -x_i, \end{aligned} \quad (4.6)$$

the geodesic flow (4.5) is equivalent to the matrix equation

$$\frac{d}{dt}\mathcal{L}_{man}^*(\lambda) = [\mathcal{L}_{man}^*(\lambda), \mathcal{A}_{man}^*(\lambda)] \quad (4.7)$$

with a spectral parameter λ and $2r \times 2r$ matrices

$$\mathcal{L}_{man}^*(\lambda) = \begin{pmatrix} -X^T(\lambda\mathbf{I}_n - A)^{-1}P & -X^T(\lambda\mathbf{I}_n - A)^{-1}X \\ P^T(\lambda\mathbf{I}_n - A)^{-1}P & P^T(\lambda\mathbf{I}_n - A)^{-1}X \end{pmatrix}, \quad (4.8)$$

$$\mathcal{A}_{man}^*(\lambda) = \begin{pmatrix} X^T(A + \lambda\mathbf{I}_n)^{-1}P & X^T(A + \lambda\mathbf{I}_n)^{-1}X \\ -P^T(A + \lambda\mathbf{I}_n)^{-1}P & -P^T(A + \lambda\mathbf{I}_n)^{-1}X \end{pmatrix}. \quad (4.9)$$

Note that after imposing the condition $X^T P = 0$, equations (4.5) formally coincide with the equations describing rank r solutions of the Manakov system on $so(n)$ (see [21]).

The Euclidean and Jensen's metrics. Since the Hamiltonian (3.12) is $SO(n) \times SO(r)$ -invariant, we can apply the general construction used in Theorem 4.1 with respect to the $SO(n) \times SO(r)$ -action (see [11]). Let $C^\infty(T^*V_{n,r})^{SO(n) \times SO(r)}$ be the algebra of $SO(n) \times SO(r)$ -invariant functions on $T^*V_{n,r}$.

Theorem 4.5 *The geodesic flows of Jensen's metrics ds_κ^2 with the Hamiltonian functions (3.12) are completely integrable in the non-commutative sense. The complete algebra of first integrals is*

$$\Phi^*(C^\infty(so(n))) + \Psi^*(C^\infty(so(r))) + C^\infty(T^*V_{n,r})^{SO(n) \times SO(r)}. \quad (4.10)$$

In particular, the geodesic flow (3.3) of the Euclidean metric is completely integrable.

The complete commutative set of polynomials \mathfrak{F} within $C^\infty(T^*V_{n,r})^{SO(n) \times SO(r)}$ as well as the integrability of the geodesic flows with Hamiltonians (3.13) will be given below (see (6.20) in Section 6, Theorem 6.4 and Corollary 6.5).

Remark 4.2 Both the geodesic flows of the Euclidean and the normal metric share the isotropic foliation defined by integrals (4.10), but do not share the isotropic foliation defined in Theorem 4.1. Namely, the straightforward calculations show that functions (p_i, p_j) in (2.17) do not Poisson commute with $\langle \Psi, \Psi \rangle$, and, therefore, the algebra of $SO(n)$ -invariant functions is not conserved along the geodesic flow of the Euclidean metric (see Lemma 2.4).

Here we note the following characterization of $SO(n) \times SO(r)$ -invariant metrics.

Proposition 4.6 *If the metric ds^2 on $V_{n,r}$ is $SO(n) \times SO(r)$ -invariant, then, up to multiplication by a constant, it coincides with ds_κ^2 for some κ .*

Proof. The statement follows from the fact that the restriction of the Hamiltonian function to $T_{X_0}^* V_{n,r}$ is a quadratic form invariant with respect to the transformations (2.15) and

$$P_1 \longmapsto Q^{-1}P_1Q, \quad P_2 \longmapsto P_2Q, \quad Q \in SO(r),$$

where P_1 and P_2 are defined in (2.14). □

5 The Neumann Systems on Stiefel Varieties

The celebrated Neumann system on a sphere S^{n-1} is defined as a natural mechanical system with the quadratic Hamiltonian (1.2). By analogy, we define a *Neumann on the Stiefel variety* $V_{n,r}$ as a natural mechanical system with an $SO(n)$ -invariant kinetic energy and the quadratic potential function

$$V = \frac{1}{2} \operatorname{tr}(X^T A X) = \frac{1}{2} \sum_{i=1}^r (e_i, A e_i). \quad (5.1)$$

Note that the above potential is constant for r .

While on the sphere S^{n-1} an $SO(n)$ -invariant kinetic energy is unique (up to multiplication by a constant factor), on the variety $V_{n,r}$ with $r > 1$ there are many different $SO(n)$ -invariant metrics. Following Sections 3 and 4, we consider the kinetic energy determined by the metrics ds_κ^2 (see eq. (3.12)). Thus, the Hamiltonian has the form

$$H_{neum,\kappa}(X, P) = \frac{1}{2} \operatorname{tr}(P^T P) - \left(\frac{1}{2} + \kappa \right) \operatorname{tr}((X^T P)^2) + \frac{1}{2} \operatorname{tr}(X^T A X). \quad (5.2)$$

Proposition 5.1 *The Neumann system with Hamiltonian (5.2) is given by*

$$\begin{aligned} \dot{X} &= P - (1 + 2\kappa) X P^T X, \\ \dot{P} &= -AX - X P^T P + (1 + 2\kappa) P X^T P + X X^T A X. \end{aligned} \quad (5.3)$$

Proof. It is a direct calculation. We have

$$\begin{aligned} \dot{X} &= \frac{\partial H}{\partial P} - X \Pi = P - (1 + 2\kappa) X P^T X - X \Pi, \\ \dot{P} &= -\frac{\partial H}{\partial X} + X \Lambda + P \Pi = -AX + (1 + 2\kappa) P X^T P + X \Lambda + P \Pi. \end{aligned}$$

Differentiating the constraints (2.4) with respect to time gives

$$\dot{X}^T X + X^T \dot{X} = 0, \quad \dot{X}^T P + X^T \dot{P} + \dot{P}^T X + P^T \dot{X} = 0.$$

The first equation implies that the Lagrange multiplier Π equals zero, while the second one yields $\Lambda = X^T AX - P^T P$. \square

Note that Hamiltonians (5.2) are right $SO(r)$ -invariant, so the momentum mapping (2.13) is conserved by the flows (5.3) for any parameter κ . In particular, for $\kappa = 0$ we get the *Neumann system with the normal metric* given by

$$\begin{aligned}\dot{X} &= P - XP^T X, \\ \dot{P} &= -AX + PX^T P + X\Lambda = -AX + PX^T P - XP^T P + XX^T AX,\end{aligned}\tag{5.4}$$

while for $\kappa = -1/2$ we get the *Neumann system with the Euclidean metric* with the corresponding Hamilton equations

$$\begin{aligned}\dot{X} &= P, \\ \dot{P} &= -AX + X\Lambda = -AX - XP^T P + XX^T AX.\end{aligned}\tag{5.5}$$

The Lax pair. Although for different κ the flows (5.3) do not coincide, the derivatives of the momentum Φ and of the symmetric matrix XX^T are the same:

$$\frac{d}{dt}\Phi = [XX^T, A], \quad \frac{d}{dt}(XX^T) = [\Phi, XX^T].\tag{5.6}$$

As a result, the following theorem holds.

Theorem 5.2 *Equations (5.3), in particular (5.4) and (5.5), imply the same $n \times n$ matrix Lax representation with a spectral parameter λ :*

$$\frac{d}{dt}\mathcal{L}_{neum}(\lambda) = [\mathcal{A}_{neum}(\lambda), \mathcal{L}_{neum}(\lambda)]\tag{5.7}$$

$$\mathcal{L}_{neum}(\lambda) = \lambda\Phi + XX^T - \lambda^2 A, \quad \mathcal{A}_{neum}(\lambda) = \Phi - \lambda A.\tag{5.8}$$

The proof is immediate. The coefficients of the spectral curve

$$\Gamma \subset \mathbb{C}^2\{\lambda, \nu\} : \det(\mathcal{L}_{neum}(\lambda) - \mu\mathbf{I}_n) = 0\tag{5.9}$$

give us the commuting integrals of both systems, which can be expressed in the form

$$\mathfrak{F} = \{\text{tr}(\lambda(PX^T - XP^T) + XX^T - \lambda^2 A)^k \mid k = 1, \dots, n, \lambda \in \mathbb{R}\}.\tag{5.10}$$

The Lax representation (5.7) is closely related to the Clebsch–Perelomov rigid body system [41]. For $r = 1$ it was given by Moser in [35] and for $r > 1$ in [44], without giving explicitly the equations of motion. (As was mentioned above, the Lax pair does not define the system itself.) The book [44] also describes the Neumann flows on Grassmannians $G_{n,r}$, as well as the magnetic Neumann flow on $G_{n,2}$. We shall consider these systems together with the magnetic Neumann flows on $V_{n,2}$ in detail in Section 7 and Appendix 2, respectively.

Alternative (or dual) Lax pairs, which are does equivalent to equations (5.5), (5.4) (up to the action of a finite discrete group) are given below in Section 8.

6 Compatible Poisson Brackets and Integrability

The Neumann systems on $V_{n,r}$ admitting Lax pairs with the Lax matrix (5.8) can be obtained as appropriate reductions of integrable n -dimensional tops having symmetric inertia

tensors and moving in a force field with a quadratic potential (the Bogoyavlenski generalization of the Clebsch–Perelomov system [8]), see Appendix 1 and [44]. However, these are reductions on *singular* coadjoint orbits in the corresponding Lie coalgebras, hence the integrability of the Neumann systems does not follow directly from that of the n -dimensional tops.

We now prove a non-commutative integrability of systems (5.3) by using the Bolsinov condition for a set of Casimir functions of the pencil of compatible Poisson brackets to be complete [9].

Theorem 6.1 *Let all the eigenvalues of A be distinct. Then the Neumann systems (5.3), in particular (5.4) and (5.5), are completely integrable in the non-commutative sense with the non-commutative set of integrals given by (5.10) and by the components of the $SO(r)$ -momentum mapping (2.13). The generic trajectory $(X(t), P(t))$ corresponding to the maximal rank of the momentum Ψ is quasi-periodic over isotropic tori of dimension*

$$\frac{1}{2} \left(2r(n-r) + \frac{r(r-1)}{2} - \left[\frac{r}{2} \right] \right) + \left[\frac{r}{2} \right]. \quad (6.1)$$

Proof. First, we give the interpretation of the integrals (5.10) from the bi-Hamiltonian point of view. Consider the Lie algebra $gl(n)$ of $n \times n$ real matrices equipped with the scalar product $\langle X, Y \rangle = -\frac{1}{2} \text{tr}(XY)$ and the orthogonal decomposition $gl(n) = so(n) + Sym(n)$ onto skew-symmetric and symmetric matrices:

$$[so(n), Sym(n)] \subset Sym(n), \quad [Sym(n), Sym(n)] \subset so(n). \quad (6.2)$$

The scalar product $\langle \cdot, \cdot \rangle$ is positive definite on $so(n)$ and negative definite on $Sym(n)$.

Let us identify $gl(n)^*$ and $gl(n)$ by means of $\langle \cdot, \cdot \rangle$. On $gl(n)$ we have a pair of compatible Poisson brackets given by the following Poisson tensors

$$\begin{aligned} \Lambda_1(\xi + \eta, \zeta + \theta)|_x &= \langle x, [\xi, \zeta] + [\xi, \theta] + [\eta, \zeta] \rangle, \\ \Lambda_2(\xi + \eta, \zeta + \theta)|_x &= \langle x - A, [\xi + \eta, \zeta + \theta] \rangle, \end{aligned} \quad (6.3)$$

where $x \in gl(n)$, $\xi, \zeta \in so(n)$, $\eta, \theta \in Sym(n)$ (see [43, 9]). The tensor Λ_1 corresponds to the canonical Lie–Poisson brackets in the dual to the semi-direct product $so(n) \oplus_{\text{ad}} Sym(n)$.

Consider the Poisson pencil

$$\Lambda_{\lambda_1, \lambda_2} = \lambda_1 \Lambda_1 + \lambda_2 \Lambda_2, \quad \Pi = \{ \Lambda_{\lambda_1, \lambda_2} \mid \lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1^2 + \lambda_2^2 \neq 0 \}.$$

For $\lambda_2 \neq 0$, the bracket $\Lambda_{\lambda_1, \lambda_2}$ is isomorphic to the canonical Lie–Poisson bracket on $gl(n)$ and its Casimir functions have the form

$$f(x) = \text{tr} \left(\sqrt{\frac{\lambda_2}{\lambda_1 + \lambda_2}} h + v - \frac{\lambda_2}{\lambda_1 + \lambda_2} A \right)^k, \quad k = 1, \dots, n,$$

where $x = h + v$, $h \in so(n)$, $v \in Sym(n)$ (see [43, 9]). Let

$$\mathcal{F} = \{ \text{tr}(\lambda h + v - \lambda^2 A)^k \mid k = 1, 2, \dots, n, \lambda \in \mathbb{R} \}. \quad (6.4)$$

be the union of all the Casimir functions of the brackets with $\lambda_1 + \lambda_2 \neq 0$, $\lambda_2 \neq 0$. Then \mathcal{F} is a commutative set with respect to all the brackets in Π and, if the eigenvalues of A are distinct, \mathcal{F} is a complete commutative set on a generic symplectic leaf in $(gl(n), \Lambda_1)$ (see Theorem 1.5 in [9]). The mapping

$$\Theta = \Phi + XX^T = \sum_{i=1}^r p_i \wedge e_i + \sum_{i=1}^r e_i \otimes e_i$$

defines the Poisson mapping between $(T^*V_{n,r}, \{\cdot, \cdot\})$ and $(gl(n), \Lambda_1)$ which is invariant with respect to the right $SO(r)$ -action (2.12) and the transformation $(X, P) \mapsto (-X, -P)$. Indeed, Θ is the composition of the following two Poisson mapping:

$$\begin{aligned}\Theta_1(X, P) &= \Phi(X, P) + X, \quad (X, P) \in T^*V_{n,r}, \\ \Theta_2(\xi, Y) &= \xi + YY^T, \quad \xi \in so(n), Y \in M_{n,r}(\mathbb{R}).\end{aligned}$$

The mapping Θ_1 realizes $T^*V_{n,r}$ as a coadjoint orbit in the dual space of the semi-direct product $so(n) \oplus_\rho M_{n,r}(\mathbb{R})$ (e.g., see equation (29.11), page 225, [26]; here ρ denotes the usual multiplication of matrices) and Θ_2 is a Poisson mapping between $(so(n) \oplus_\rho M_{n,r}(\mathbb{R}))^*$ and $(so(n) \oplus_{\text{ad}} Sym(n))^*$ (see Lemma 7.1 of [44]).

We have also that the algebra of integrals (5.10) is the pull-back of (6.4):

$$\mathfrak{F} = \Theta^* \mathcal{F}.$$

However, the image $\Theta(T^*V_{n,r})$ is the union of *singular* symplectic leaves in $(gl(n), \Lambda_1)$. Namely, the generic symplectic leaf in $(gl(n), \Lambda_1)$ has the dimension $n^2 - n$, while the dimension of generic leaf in $\Theta(T^*V_{n,r})$ is

$$2l = 2r(n - r) + \frac{r(r - 1)}{2} - \left\lfloor \frac{r}{2} \right\rfloor < n^2 - n \quad (6.5)$$

(see Lemma 6.2 below). Nevertheless, it can be proved that the set of the functions (6.4) is complete on a generic orbit laying in $\Theta(T^*V_{n,r})$ as well (see Lemma 6.3 below). That is, among the integrals \mathcal{F} there is at least l polynomials p_1, \dots, p_l independent on the symplectic leaves in $\Theta(T^*V_{n,r})$.

The rest of the proof follows the idea of [11, 51]. Namely, since $SO(r)$ acts on $T^*V_{n,r}$ freely and preserves the Poisson bracket $\{\cdot, \cdot\}$, the quotient space $(T^*V_{n,r})/SO(r)$ carries natural induced Poisson bracket $\{\cdot, \cdot\}'$. Let $\sigma : T^*V_{n,r} \rightarrow (T^*V_{n,r})/SO(r)$ be the canonical projection. Then, by definition,

$$\{f, g\}'(\sigma(X, P)) = \{F, G\}(X, P), \quad F = f \circ \sigma, \quad G = g \circ \sigma. \quad (6.6)$$

The Casimir functions $j_1, \dots, j_{\lfloor r/2 \rfloor}$ of the brackets $\{\cdot, \cdot\}'$ can be obtained from the $SO(r)$ -invariant functions $J_k = \text{tr}(\Psi^{2k})$ via $j_k \circ \sigma = J_k$.

The mapping Θ induces \mathbb{Z}_2 -Poisson covering

$$\theta : ((T^*V_{n,r})/SO(r), \{\cdot, \cdot\}') \rightarrow (\Theta(T^*V_{n,r}), \Lambda_1), \quad \theta \circ \sigma = \Theta.$$

Hence the functions $p_1 \circ \theta, \dots, p_l \circ \theta$ are independent on a generic symplectic leaf and, together with the Casimir functions $j_1, \dots, j_{\lfloor r/2 \rfloor}$, form a complete commutative set of functions within $(T^*V_{n,r})/SO(r)$. In other words, the functions

$$p_1 \circ \theta, \dots, p_l \circ \theta, \quad J_1, \dots, J_{\lfloor \frac{r}{2} \rfloor} \quad (6.7)$$

form a complete commutative set of functions in the algebra of $SO(r)$ -invariant functions on $T^*V_{n,r}$. Note that the independency of the functions $J_1, \dots, J_{\lfloor r/2 \rfloor}$ at (X, P) is equivalent to the regularity of $\text{Ad}_{SO(r)}$ -orbit of $\Psi(X, P)$. Therefore, (6.1) holds for the invariant manifolds where the rank of $\Psi(X, P)$ is maximal.

Then, according to Theorem 1 in [13], the functions (6.7) together with $\Psi^*(C^\infty(so(r)))$ form a complete non-commutative set of functions in $T^*V_{n,r}$. Therefore the integrals $\Theta^* \mathcal{F} + \Psi^*(C^\infty(so(r)))$ of the Neumann system form a complete non-commutative set. Moreover, the functions (6.7) commute with all the integrals and therefore their Hamiltonian flows generate generic leaves of the isotropic foliation given by $\Theta^* \mathcal{F} + \Psi^*(C^\infty(so(r)))$. Hence, the dimension of the generic isotropic tori is given by (6.1). \square

Remark 6.1 The Hamiltonian of the *Neumann system with the normal metric* has the form $H_{neum_0} = H \circ \Theta$, where

$$H = \frac{1}{2} \langle h, h \rangle - \langle A, v \rangle,$$

defines the completely integrable (by means of the integrals (6.4)) Hamiltonian flow

$$\dot{h} = [v, A], \quad \dot{v} = [h, v] \quad (6.8)$$

on $(so(n) \oplus_{\text{ad}} Sym(n))^*$. This system belongs to the class of Clebsch–Perelomov–Bogoyavlenski rigid body systems ([41, 8], see also Section 7). Note that the representation of the Neumann system on the sphere as a system on a adjoint orbit is given by Ratiu [42].

Lemma 6.2 *The dimension of a generic symplectic leaf in $(\Theta(T^*V_{n,r}), \Lambda_1)$ is given by formula (6.5).*

Proof. Without loss of generality, choose a point $X_0 = (E_1, \dots, E_r) \in V_{n,r}$ and a generic point $(X_0, P_0) \in T_{X_0}^*V_{n,r}$. Denote

$$h = \Phi(X_0, P_0), \quad v = X_0 X_0^T$$

Then h is a generic $so(n)$ -matrix of the form

$$h = \begin{pmatrix} h_1 & h_2 \\ -h_2^T & 0 \end{pmatrix}, \quad h_1 \in so(r), \quad h_2 \in M_{r, n-r}(\mathbb{R}) \quad \text{and} \quad v = \begin{pmatrix} \mathbf{I}_r & 0 \\ 0 & 0 \end{pmatrix}.$$

From the definition (6.3) of the tensors Λ_1, Λ_2 we get $\xi + \eta \in \ker \Lambda_1(h + v)$, $\xi \in so(n)$, $\eta \in Sym(n)$ if and only if

$$[\xi, v] = 0, \quad (6.9)$$

$$[\xi, h] + [\eta, v] = 0. \quad (6.10)$$

The first equation gives the condition for ξ to belongs to the subalgebra $so(r) \oplus so(n-r)$. Denote

$$\xi = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 & \eta_3 \\ \eta_3^T & \eta_2 \end{pmatrix},$$

where $\xi_1 \in so(r)$, $\xi_2 \in so(n-r)$, $\eta_1 \in Sym(r)$, $\eta_2 \in Sym(n-r)$, $\eta_3 \in M_{r, n-r}(\mathbb{R})$. Since

$$[\eta, v] = \begin{pmatrix} 0 & -\eta_3 \\ \eta_3^T & 0 \end{pmatrix},$$

from (6.2) and (6.10) we find that $[\xi_1, h_1] = 0$, ξ_2, η_1, η_2 are arbitrary, and η_3 is uniquely determined from the equation

$$\left[\begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}, \begin{pmatrix} 0 & h_2 \\ -h_2^T & 0 \end{pmatrix} \right] + \begin{pmatrix} 0 & -\eta_3 \\ \eta_3^T & 0 \end{pmatrix} = 0.$$

For generic h_1 , the solutions of $[h_1, \xi_1] = 0$ form a maximal commutative subalgebra of $so(r)$. That is, the dimension of the space of the solution of (6.9), (6.10) is

$$\dim \ker \Lambda_1(h + v) = \text{rank } so(r) + \dim so(n-r) + \dim Sym(r) + \dim Sym(n-r).$$

Finally, the dimension of a generic symplectic leaf in $(\Theta(T^*V_{n,r}), \Lambda_1)$ is equal to $n^2 - \dim \ker \Lambda_1(h + v) = 2l$. \square

Lemma 6.3 *If all eigenvalues of A are distinct, then the set of the functions (6.4) is a complete commutative set on a generic symplectic leaf in $(\Theta(T^*V_{n,r}), \Lambda_1)$*

Proof. We keep the notation from the proof of Lemma 6.2. The proof presented here is a modification of that of Theorem 1.6 in [9]. According to Theorem 1.1 in [9], the set of the functions (6.4) is complete on the symplectic leaf containing the point $x = h + v$ if and only if

- (A1) All the brackets $\Lambda_{\lambda_1, \lambda_2}$ non-proportional to Λ_1 have the maximal rank $n^2 - n$ at x .
(A2) The kernel of the bracket $\Lambda_{1, -1}$ at x , restricted to the linear space $\ker \Lambda_1$, has dimension n ,

$$\dim\{(\xi + \eta \in \ker \Lambda_1(x) \mid \Lambda_{1, -1}(\xi + \eta, \ker \Lambda_1(x))|_x = 0\}. \quad (6.11)$$

Here all the objects are assumed to be complexified. Since the conditions (A1) and (A2) are both generic, it is sufficient to find $x_1 \in \Theta(T^*V_{n,r})$ for which (A1) holds and $x_2 \in \Theta(T^*V_{n,r})$ which satisfies (A2). Then the set of $x \in \Theta(T^*V_{n,r})$ satisfying both conditions will be open and dense everywhere in the induced topology on $\Theta(T^*V_{n,r})$.

If $\lambda_1 + \lambda_2 \neq 0$, $\lambda_2 \neq 0$, then the bracket $\Lambda_{\lambda_1, \lambda_2}$ is isomorphic to the canonical Lie–Poisson bracket on $gl(n, \mathbb{C})$. Thus the brackets $\Lambda_{\lambda_1, \lambda_2}$ ($\lambda_1 + \lambda_2 \neq 0$, $\lambda_2 \neq 0$) have the maximal rank $n^2 - n$ at $x = h + v$ if and only if the complex line

$$\mathcal{L} = \{h + v - \lambda A \mid \lambda \in \mathbb{C}\}$$

intersects the set of singular points of $gl(n, \mathbb{C})$ only at $x = h + v$. This condition is obviously satisfied if all the eigenvalues of A are distinct.

To prove (A1) we have to find the (complex) dimension of $\ker \Lambda_{1, -1}$. From the definition (6.3) we get

$$\Lambda_{1, -1}(\xi + \eta, \zeta + \theta)|_{h+v} = -\langle h, [\eta, \theta] \rangle + \langle A, [\xi, \theta] + [\eta, \zeta] \rangle \quad (6.12)$$

and $\xi + \eta \in \ker \Lambda_{1, -1}(h + v)$, $\xi \in so(n, \mathbb{C})$, $\eta \in Sym(n, \mathbb{C})$ if and only if

$$[\eta, A] = 0, \quad (6.13)$$

$$[\xi, A] - [\eta, h] = 0. \quad (6.14)$$

The solutions of (6.13) are all diagonal matrices. For the given diagonal matrix η , the matrix ξ is uniquely determined from (6.14). Therefore, $\dim \ker \Lambda_{1, -1}(h + v)$.

It remains to check the condition (A2). Take h of the form

$$h = \begin{pmatrix} h_1 & 0 \\ 0 & 0 \end{pmatrix},$$

h_1 being a generic element of $so(r)$. From the proof of Lemma 6.2 we have

$$\ker \Lambda_1(h + v) = so(r, \mathbb{C})_{h_1} \oplus Sym(r, \mathbb{C}) \oplus gl(n - r, \mathbb{C}), \quad (6.15)$$

where $so(r, \mathbb{C})_{h_1} = \{\xi_1 \in so(r, \mathbb{C}) \mid [\xi_1, h_1] = 0\}$ is a Cartan subalgebra of $so(r, \mathbb{C})$.

From (6.12), (6.15) we conclude that $\xi + \eta$ belongs to $\ker \Lambda_{1, -1}(x)|_{\ker \Lambda_1(x)}$ if and only if

$$[A_1, \xi_1] - [h_1, \eta_1] = 0, \quad (6.16)$$

$$[A_1, \eta_1] \in so(r, \mathbb{C})^\perp, \quad (6.17)$$

$$\xi_2 = 0, \quad (6.18)$$

$$[A_2, \eta_2] = 0, \quad (6.19)$$

where η_1 and η_2 are defined in the proof of Lemma 6.2 and

$$A_1 = \text{diag}(a_1, \dots, a_r), \quad A_2 = \text{diag}(a_{r+1}, \dots, a_n).$$

Equations (6.16), (6.17) form a closed system within $gl(r, \mathbb{C})$, and from the proof of Theorem 1.6 [9], the dimension of solution of (6.16), (6.17) is equal to r . On the other side, the solution of (6.19) are all diagonal matrices in $gl(n-r, \mathbb{C})$. Hence (6.11) holds. \square

Singular matrices A and the Einstein metrics. Now suppose that not all the eigenvalues of A are distinct:

$$a_1 = \dots = a_{k_1}, \quad a_{k_1+1} = \dots = a_{k_1+k_2}, \quad \dots, \quad a_{n+1-k_r} = \dots = a_n, \quad k_1 + k_2 + \dots + k_r.$$

Then we have a non-trivial isotropy algebra

$$so(n)_A = \{\xi \in so(n) \mid [\xi, A] = 0\} = so(k_1) \oplus so(k_2) \oplus \dots \oplus so(k_r).$$

Let \mathcal{G} be the set of linear functions on $so(n)_A$. The set $\mathcal{F} + \mathcal{G}$ is a complete non-commutative set of function on $(gl(n), \Lambda_1)$ (Theorem 1.5 [9]). By modifying the proof of Lemma 6.2 and Bolsinov's Theorem 1.5 [9], one can prove that the set of functions $\mathcal{F} + \mathcal{G}$ is complete on $\Theta(T^*V_{n,r})$ as well, implying non-commutative integrability of the Neumann systems (5.3). The complete verification is out the scope of this paper.

Let $SO(n)_A = SO(k_1) \times SO(k_2) \times \dots \times SO(k_r) \subset SO(n)$ be the adjoint isotropy group of A . The momentum mapping of the left $SO(n)_A$ -action is given by

$$\Phi_A = \text{pr}_{so(n)_A} \Phi = \text{pr}_{so(n)_A} (PX^T - XP^T)$$

and $\Theta^*\mathcal{G}$ are exactly Noether integrals arising from the $SO(n)_A$ -symmetry of the Neumann flows.

In particular, when $A = 0$, we get the integrals of the geodesic flow of the metric ds_{κ}^2 in the form $\Phi^*(C^\infty(so(n))) + \Psi^*(C^\infty(so(r))) + \mathfrak{F}$, where now

$$\mathfrak{F} = \{\text{tr}(\lambda(PX^T - XP^T) + XX^T)^k \mid k = 1, \dots, n, \lambda \in \mathbb{R}\}. \quad (6.20)$$

We shall mention the following important corollary of the above construction.

Let $\mathbb{B} : so(r) \rightarrow so(r)$ be positive definite and $h_{\mathbb{B}} = \frac{1}{2}\langle \xi, \mathbb{B}\xi \rangle$. Suppose that the Euler equations

$$\dot{f} = \{f, h_{\mathbb{B}}\}_{so(r)}^- \iff \dot{\xi} = [\xi, \nabla h_{\mathbb{B}}(\xi)] = [\xi, \mathbb{B}\xi], \quad \xi \in so(r) \quad (6.21)$$

are completely integrable with a complete commutative set of functions \mathcal{B} .

Theorem 6.4 (i) $\mathfrak{B} + \mathfrak{F}$ is a complete commutative set of $SO(n)$ -invariant functions on $T^*V_{n,r}$, where \mathfrak{F} is given by (6.20) and $\mathfrak{B} = \Psi^*(\mathcal{B})$.

(ii) The geodesic flow of the $SO(n)$ -invariant metric $ds_{\mathbb{B}}^2$ on $V_{n,r}$ defined by the Hamiltonian function

$$H_{\mathbb{B}} = \frac{1}{2} \text{tr}(P^T P) - \frac{1}{4} \text{tr}((X^T P - P^T X)\mathbb{B}(X^T P - P^T X))$$

is completely integrable in the non-commutative sense. The complete set of first integrals is $\Phi^*(C^\infty(so(n))) + \mathfrak{F} + \mathfrak{B}$, and the generic trajectories of the system are quasi-periodic over the isotropic tori of dimension

$$\text{ddim}(\mathfrak{F} + \mathfrak{B}) = \dim V_{n,r} - l.$$

Here $2l$ is the dimension of a generic adjoint orbit in $\Phi(T^*V_{n,r})$ (see Lemma 2.5).

For example, if the matrix $B = \text{diag}(B_1, \dots, B_r)$ has distinct eigenvalues, we can take $\mathbb{B}(\xi) = B\xi + \xi B$ (the Manakov operator on $so(r)$) and the commutative set

$$\mathfrak{B} = \{ \text{tr}(\Psi + \mu B)^k, \mu \in \mathbb{R}, k = 1, 2, \dots, r \}. \quad (6.22)$$

Now, by using the chain method for the construction of complete commuting sets of functions on Lie algebras developed by Mikityuk [34], one can easily prove that the Hamiltonian function (3.14) defines completely integrable system on $so(r)$. Therefore we get

Corollary 6.5 *The $SO(n)$ -invariant geodesic flows determined by Hamiltonian functions (3.13) are completely integrable. In particular, the geodesic flow of the Einstein metrics constructed in [4] are completely integrable.*

Note that the set of functions given in Theorem 6.4 differs from those described in Theorem 2.6. Another proof of integrability of geodesic flows of the Einstein metrics, based on using singular Manakov flows, is recently obtained in [17].

Commutative integrability of the Neumann flows. We turn back to the Neumann flows. According to Theorem 6.1, systems (5.3) are integrable in the noncommutative sense by means of the integrals $\Psi^*(C^\infty(so(r)) + \mathfrak{F})$, where \mathfrak{F} is given by (5.10). However, the Neumann flows (5.3) are integrable in the commutative (Liouville) sense as well: instead of $\Psi^*(C^\infty(so(r)))$ one should take, for example, the commutative set (6.22). Moreover, it follows:

Corollary 6.6 *Let all the eigenvalues of A be distinct. Suppose that the Euler equations (6.21) are completely integrable with a complete commutative set of functions \mathfrak{B} . Then the Neumann system with the kinetic energy given by the $SO(n)$ -invariant metric $ds_{\mathbb{B}}^2$*

$$H_{neum, \mathbb{B}} = \frac{1}{2} \text{tr}(P^T P) - \frac{1}{4} \text{tr}((X^T P - P^T X)\mathbb{B}(X^T P - P^T X)) + \frac{1}{2} \text{tr}(X^T A X)$$

is completely integrable. The complete commutative set of first integrals is $\mathfrak{F} + \mathfrak{B}$, where \mathfrak{F} is given by (5.10) and $\mathfrak{B} = \Psi^(\mathfrak{B})$.*

In particular, the Neumann systems with the kinetic energy determined by $SO(n)$ -invariant Einstein metrics constructed in [4] are completely integrable.

7 Reduction to Grassmannians

By definition, the points of the oriented Grassmannian variety $G_{n,r}$ are r -dimensional oriented planes passing through the origin in the Euclidean space \mathbb{R}^n . The usual action of the group $SO(n)$ on \mathbb{R}^n yields a transitive action on the set of all r -dimensional planes, i.e., on $G_{n,r}$. The isotropy group of the r -plane spanned by the vectors E_1, \dots, E_r (relative to the base (2.1)) has the form

$$\begin{pmatrix} SO(r) & 0 \\ 0 & SO(n-r) \end{pmatrix} \cong SO(r) \times SO(n-r).$$

It follows that $G_{n,r} \cong SO(n)/(SO(r) \times SO(n-r))$.

The oriented Grassmannian can also be seen as a quotient space of the Stiefel manifold by the right $SO(r)$ -action described in Section 1. The quotient mapping $V_{n,r} \rightarrow G_{n,r}$ is

$$X = (e_1, \dots, e_r) \mapsto e_1 \wedge \dots \wedge e_r.$$

The symplectic leaves in $(T^*V_{n,r})/SO(r), \{\cdot, \cdot\}'$ with $\{\cdot, \cdot\}'$ given by (6.6) are the Marsden–Weinstein symplectic reduced spaces of $T^*V_{n,r}$. In particular, the reduced space that corresponds to zero value of the momentum mapping

$$\Psi^{-1}(0)/SO(r),$$

is symplectomorphic to the cotangent bundle $T^*G_{n,r}$ equipped with a canonical symplectic structure. Note that (X, P) belongs to $\Psi^{-1}(0)$ if and only if $X^T P = 0$.

The last condition also implies that, although the Neumann systems (5.4) and (5.5) are different on the whole $T^*V_{n,r}$, they coincide on $\Psi^{-1}(0)$, hence their reductions onto the cotangent bundle $T^*G_{n,r}$ are the same.

The reduced flow can be written in the alternative Euler–Lagrange form. Namely, we have

$$\frac{d}{dt}(e_1 \wedge \cdots \wedge e_r) = \sum_{i=1}^r e_1 \wedge \cdots \wedge \dot{e}_i \wedge \cdots \wedge e_r$$

and, in view of the matrix equation (5.5), the reduced system is

$$\begin{aligned} \frac{d^2}{dt^2}(e_1 \wedge \cdots \wedge e_r) &= - \sum_{i=1}^r e_1 \wedge \cdots \wedge A e_i \wedge \cdots \wedge e_r \\ &+ 2 \sum_{1 \leq i < j \leq r} e_1 \wedge \cdots \wedge \dot{e}_i \wedge \cdots \wedge \dot{e}_j \wedge \cdots \wedge e_r + \lambda(e_1 \wedge \cdots \wedge e_r), \quad (7.1) \\ \lambda = \text{tr } \Lambda &= \text{tr}(X^T A X - \dot{X}^T \dot{X}) = \sum_{i=1}^r ((e_i, A e_i) - (\dot{e}_i, \dot{e}_i)), \end{aligned}$$

where

$$(e_i, e_j) = \delta_{ij}, \quad (\dot{e}_i, e_j) = 0, \quad i, j = 1, \dots, r. \quad (7.2)$$

Note that (7.1) is $SO(r)$ -invariant. We refer to (7.1) as a *Neumann system* on the oriented Grassmannian variety $G_{n,r}$.

Theorem 7.1 *Suppose that all the eigenvalues of A are distinct. Then the Neumann system on $T^*G_{n,r}$ is completely integrable in the Liouville sense by means of the integrals induced from the $SO(r)$ -invariant functions (5.10).*

Proof. Since $T^*G_{n,r} = \Psi^{-1}(0)/SO(r)$ is a singular symplectic leaf in $T^*V_{n,r}/SO(r)$, the statement of the theorem does not follow directly from Theorem 6.1.

We keep the notation from the proofs of Theorem 6.1, Lemma 6.2 and Lemma 6.3. Let $h = \Psi(X_0, P_0)$, $v = X_0 X_0^T$, where $(X_0, P_0) \in \Psi^{-1}(0) \cap T_{X_0}^* V_{n,r}$ is in a generic position. Then $h_1 = 0$. From the proof of Lemma 6.2 we get

$$\dim \ker \Lambda_1(h + v) = \dim(gl(r) \oplus gl(n - r)).$$

and the mappings Θ and θ map $\Psi^{-1}(0)$ and $\Psi^{-1}(0)/SO(r)$ to the single symplectic leaf in $(gl(n), \Lambda_1)$. Now, by modifying the proof of Lemma 6.2 one can prove conditions (A1) and (A2) for $x = h + v$ with $h_1 = 0$ as well. Therefore, among functions (6.4) one can find exactly

$$r(n - r) = \frac{1}{2}(\dim gl(n) - \dim gl(r) - \dim gl(n - r)) = \dim G_{n,r}$$

independent functions $p_1, \dots, p_{r(n-r)}$. Thus the functions $p_1 \circ \theta, \dots, p_{r(n-r)} \circ \theta$ provide a complete commutative set of functions on $\Psi^{-1}(0)/SO(r)$. \square

The special case $r = 1$. The Stiefel variety $V_{n,n-1}$ is diffeomorphic to $SO(n) = V_{n,n}$: to e_1, \dots, e_{n-1} one can associate the unique unit vector e_n such that e_1, \dots, e_n is the orthonormal base with the same orientation as E_1, \dots, E_n :

$$X = (e_1, \dots, e_{n-1}) \mapsto \mathcal{X} = (e_1, \dots, e_{n-1}, e_n) \in SO(n). \quad (7.3)$$

Similarly, the oriented Grassmannian variety $G_{n,n-1}$ is diffeomorphic to $G_{n,1} = S^{n-1}$ via the mapping

$$e_1 \wedge \dots \wedge e_{n-1} \mapsto e_n. \quad (7.4)$$

It is natural to expect that in this case the Neumann system (7.1) gives rise to the classical Neumann system on the sphere S^{n-1} . Indeed, due to conditions (7.2), for $r = 1$ the second term in the right-hand side of equations (7.1) vanishes. Then, under identification (7.4), these equations give rise to

$$\ddot{e}_n = (A - \text{tr } A \mathbf{I})e_n + \lambda e_n = Ae_n - ((e_n, Ae_n) + (\dot{e}_n, \dot{e}_n))e_n, \quad (7.5)$$

which describes the motion on the sphere $S^{n-1} = \{ \langle e_n, e_n \rangle = 1 \}$ with the potential $-\frac{1}{2} \langle Ae_n, e_n \rangle$.

The 4-th degree potential. By using Cartan models of symmetric spaces, a class of new integrable potential systems on such spaces was obtained by Saksida [45]. As noticed in [46], in the case of the sphere S^{n-1} , such a system is a generalization of the Neumann system in presence of a potential of degree 4. The latter system is separable in the spherical elliptic coordinates and was found previously in [50].

In addition, it can be proved that the construction of [45] on the Grassmannian varieties gives rise to the Hamiltonian

$$H(X, P) = \frac{1}{2} \text{tr}(P^T P) + \text{tr}(X^T A^2 X) - \text{tr}(X^T A X X^T A X), \quad (7.6)$$

which for $r = 1$, takes the well known form $H = \frac{1}{2}(p, p) + \sum_{i=1}^n a_i^2 e_i^2 - (\sum_{i=1}^n a_i e_i^2)^2$ [50, 46]. For $r > 1$, the Hamiltonian flow on $T^*V_{n,r}$ determined by the Hamiltonian function (7.6) is integrable after its restriction to the invariant manifold $\Psi^{-1}(0) \subset T^*V_{n,r}$ and the reduction to $T^*G_{n,r}$. This system will be discussed elsewhere.

8 The Dual Lax Pair and Geometric Interpretation of Integrals

As mentioned in Section 5, like the classical Neumann system on the sphere S^{n-1} , the Neumann systems on $V(r, n)$ also admit dual Lax representations.

Theorem 8.1 *Up to the action of a discrete group \mathbb{Z}_2^n generated by reflections (4.6), the Neumann flows (5.4) and (5.5) are equivalent to the following $2r \times 2r$ matrix Lax pair with a rational spectral parameter λ*

$$\frac{d}{dt} \mathcal{L}_{neum}^*(\lambda) = [\mathcal{L}_{neum}^*(\lambda), \mathcal{A}_{neum}^*(\lambda)], \quad (8.1)$$

$$\mathcal{L}_{neum}^*(\lambda) = \begin{pmatrix} -X^T(\lambda \mathbf{I}_n - A)^{-1} P & -X^T(\lambda \mathbf{I}_n - A)^{-1} X \\ \mathbf{I}_r + P^T(\lambda \mathbf{I}_n - A)^{-1} P & P^T(\lambda \mathbf{I}_n - A)^{-1} X \end{pmatrix}, \quad (8.2)$$

where for system (5.4), respectively (5.5), one should put

$$\mathcal{A}_{neum}^*(\lambda) = \begin{pmatrix} X^T P & \mathbf{I}_r \\ \Lambda - \lambda \mathbf{I}_r & -P^T X \end{pmatrix}, \quad \text{respectively,} \quad \mathcal{A}_{neum}^*(\lambda) = \begin{pmatrix} 0 & \mathbf{I}_r \\ \Lambda - \lambda \mathbf{I}_r & 0 \end{pmatrix}, \quad (8.3)$$

where $\Lambda = X^T A X - P^T P$.

The statement is checked straightforwardly by using constraints (2.4) and the matrix identities

$$A(\lambda \mathbf{I}_n - A)^{-1} = (\lambda \mathbf{I}_n - A)^{-1} A = \lambda(\lambda \mathbf{I}_n - A)^{-1} - \mathbf{I}_n.$$

The dual Lax pair (8.3) for the generalized Neumann system (5.5) was first given in unpublished manuscript [28]. For $r = 1$ it gives the known 2×2 Lax pair for the Neumann system indicated in several publications (see, e.g., [47] and references therein).

Remark 8.1 For r the considered Neumann flows become geodesic flows of bi-invariant metrics on $SO(n)$. Then both Lax representations ($n \times n$ and $2n \times 2n$) give the integrals for the geodesic flow of a bi-invariant metric, but also they give the integrals for the right-invariant Manakov geodesic flows on $SO(n)$ described by equations (3.5), (4.2).

The spectral curve and integrals. Let

$$a(\lambda) = (\lambda - a_1) \cdots (\lambda - a_n).$$

The spectral curve of $\mathcal{L}_{neum}^*(\lambda)$ can be written in form

$$|a(\lambda)\mathcal{L}_{neum}^*(\lambda) - w\mathbf{I}_n| \equiv w^{2r} + w^{2r-2}a(\lambda)\mathcal{I}_2(\lambda) + \cdots + w^2 a^{2r-3}(\lambda)\mathcal{I}_{2r-2}(\lambda) + a^{2r-1}(\lambda)\mathcal{I}_{2r}(\lambda) = 0,$$

where $\mathcal{I}_{2l}(\lambda)$ are invariant polynomials in the components of the wedge products $e_{j_1} \wedge \cdots \wedge e_{j_l}$ and

$$\begin{aligned} & e_1 \wedge \cdots \wedge e_r, \\ & e_1 \wedge \cdots \wedge e_r \wedge p_i, \quad i = 1, \dots, r, \\ & \dots\dots\dots \\ & e_1 \wedge \cdots \wedge e_r \wedge p_1 \wedge \cdots \wedge p_r. \end{aligned} \quad (8.4)$$

Note that, due to the symplectic block structure of $\mathcal{L}_{neum}^*(\lambda)$, the coefficients at odd powers of w in the spectral curve are zero.

In the case $2r \leq n$ the polynomials can be written in form

$$\begin{aligned}
\mathcal{I}_2(\lambda) &= \sum_{i=1}^n \frac{a(\lambda)}{\lambda - a_i} ((e_1^i)^2 + \cdots + (e_r^i)^2) + \sum_{1 \leq i < j \leq n} \frac{a(\lambda)}{(\lambda - a_i)(\lambda - a_j)} \Phi_{ij}^2, \\
&\dots \dots \\
\mathcal{I}_{2l}(\lambda) &= \sum_{I_l} \frac{a(\lambda)}{(\lambda - a_{i_1}) \cdots (\lambda - a_{i_l})} \sum_{J_l = \{j_1, \dots, j_l\}} (e_{j_1} \wedge \cdots \wedge e_{j_l})_{I_l}^2 \\
&\quad + \sum_{I_{l+1}} \frac{a(\lambda)}{(\lambda - a_{i_1}) \cdots (\lambda - a_{i_{l+1}})} \sum_{J_l, 1 \leq i \leq r} (e_{j_1} \wedge \cdots \wedge e_{j_l} \wedge p_i)_{I_{l+1}}^2 + \cdots, \\
&\dots \dots \\
\mathcal{I}_{2r}(\lambda) &= \sum_{I_r} \frac{a(\lambda)}{(\lambda - a_{i_1}) \cdots (\lambda - a_{i_r})} (e_1 \wedge \cdots \wedge e_r)_{I_r}^2 \\
&\quad + \sum_{I_{r+1}} \frac{a(\lambda)}{(\lambda - a_{i_1}) \cdots (\lambda - a_{i_{r+1}})} \sum_{i=1}^r (e_1 \wedge \cdots \wedge e_r \wedge p_i)_{I_{r+1}}^2 \\
&\quad + \sum_{I_{r+2}} \frac{a(\lambda)}{(\lambda - a_{i_1}) \cdots (\lambda - a_{i_{r+2}})} \sum_{1 \leq i < j \leq r} (e_1 \wedge \cdots \wedge e_r \wedge p_i \wedge p_j)_{I_{r+2}}^2 + \cdots \\
&\quad + \sum_{I_{2r}} \frac{a(\lambda)}{(\lambda - a_{i_1}) \cdots (\lambda - a_{i_{2r}})} |\Phi|_{I_{2r}}^{I_{2r}}, \tag{8.5}
\end{aligned}$$

where $I_k = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ is the multi-index with distinct indices $1 \leq i_1 < \cdots < i_k \leq n$ and $|\Phi|_{I_k}^{I_k}$ is the $k \times k$ diagonal minor of the momentum matrix Φ corresponding to the multi-index I_k . Note that, in view of definition of Φ ,

$$|\Phi|_{I_{2r}}^{I_{2r}} = (e_1 \wedge \cdots \wedge e_r \wedge p_1 \wedge \cdots \wedge p_r)_{I_{2r}}^2.$$

In the case $2r > n$ the polynomials $\mathcal{I}_{2l}(\lambda)$ have the same form with the only difference: the terms with the wedge products of e_i, p_j of order $> n$ are absent.

It follows that in both cases $\mathcal{I}_{2l}(\lambda)$ are polynomials in λ of degree $n - l$ and that the leading coefficients of $\mathcal{I}_2(\lambda), \dots, \mathcal{I}_{2r}(\lambda)$ produce trivial constants on $V(r, n)$. Hence, as a simple counting shows, the number of nontrivial integrals on $T^*V(r, n)$ provided by the Lax matrix $\mathcal{L}_{neum}^*(\lambda)$ in (8.3) equals

$$N = (n - 1) + (n - 2) + \cdots + (n - r) = r(n - r) + r(r - 1)/2,$$

which coincides with the dimension of the Stiefel variety.

Note that, although the Lax matrix $\mathcal{L}_{neum}^*(\lambda)$ is not invariant under the right $SO(r)$ -action, the spectral curve and therefore all the integrals $\mathcal{I}_{2l}(\lambda)$ are $SO(r)$ -invariant.

Since the number N is bigger than half of dimension (6.5) of a generic symplectic leaf within $(T^*V_{n,r})/SO(r)$, some of the integrals are dependent. Like the "big" Lax matrix $\mathcal{L}_{neum}(\lambda)$ in (5.7), the dual Lax matrix $\mathcal{L}_{neum}^*(\lambda)$ does not produce explicitly the momenta integrals Ψ_{ij} .

Geometric interpretation of the integrals $\mathcal{I}_{2r}(\lambda)$. The components of forms (8.4) that appear in the last invariant polynomial $\mathcal{I}_{2r}(\lambda)$ have a transparent geometric interpretation: they are Plücker coordinates of the $2r$ -dimensional linear subspace ($2r$ -plane)

$$\bar{\Sigma} = \bar{\Sigma}(X, P) \subset \mathbb{R}^{n+r}(x_1, \dots, x_n, y_1, \dots, y_r)$$

spanned by the columns of the $2r \times (n+r)$ matrix

$$\mathcal{V} = \mathcal{V}(X, P) = \begin{pmatrix} e_1 & \cdots & e_r & p_1 & \cdots & p_r \\ 0 & \cdots & 0 & & & \\ \vdots & & \vdots & & \mathbf{I}_r & \\ 0 & \cdots & 0 & & & \end{pmatrix}, \quad (8.6)$$

\mathbf{I}_r being the identity $r \times r$ matrix. Indeed (see, e.g., [25]), for any k, m ($k < m$), the Plücker coordinates of a k -plane π in $\mathbb{R}^m(x_1, \dots, x_m)$ spanned by independent vectors $v_1, \dots, v_k \in \mathbb{R}^m$ are the coefficients G_I of the polynomial

$$v_1 \wedge \cdots \wedge v_k = \sum_I G_I dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ is the multi-index with $1 \leq i_1 < \cdots < i_k \leq n$.

Then the Plücker coordinates of $\bar{\Sigma}$ are given by all $2r \times 2r$ minors of \mathcal{V} . In particular, the $2r \times 2r$ minors that completely contain \mathbf{I}_r give the Plücker coordinates of the r -plane $\text{span}(e_1, \dots, e_r) \subset \mathbb{R}^n$.

Now consider the following family of confocal cones in $\mathbb{R}^{n+r}(x_1, \dots, x_n, y_1, \dots, y_r)$

$$\bar{Q}(\lambda) = \left\{ \frac{x_1^2}{\lambda - a_1} + \cdots + \frac{x_n^2}{\lambda - a_n} + y_1^2 + \cdots + y_r^2 = 0 \right\}, \quad \lambda \in \mathbb{R}. \quad (8.7)$$

The following theorem is a first variant of a generalization of the remarkable Chasles theorem describing a geometric relation between the geodesic flow on an ellipsoid and common tangent lines of confocal quadrics ([15, 35, 30]).

Theorem 8.2 *Let the $2r$ -plane $\bar{\Sigma}(t) \subset \mathbb{R}^{n+r}$ be associated to a generic solution $(X(t), P(t))$ of the Neumann systems (5.4) and (5.5) on $V_{n,r}$ as described above. Then $\bar{\Sigma}(t)$ is tangent simultaneously to $n-r$ fixed confocal cones $\bar{Q}(c_1), \dots, \bar{Q}(c_{n-r})$, where c_1, \dots, c_{n-r} are the roots of the invariant polynomial $\mathcal{I}_{2r}(\lambda)$.*

One can show that for real solutions $(X(t), P(t))$ all these cones are real.

In the particular case $r = 1$, one can also consider the section of $\bar{\Sigma}$ and of the family $\bar{Q}(\lambda)$ by the subspace $\{y_1 = 1\} \cong \mathbb{R}^n$, which give respectively an affine line $l(t) = p(t) + \text{span}\{e(t)\}$ and the family of confocal quadrics

$$Q(\lambda) = \left\{ \frac{x_1^2}{a_1 - \lambda} + \cdots + \frac{x_n^2}{a_n - \lambda} = 1 \right\}.$$

Then, due the above theorem, $l(t)$ is tangent to $n-1$ fixed quadrics $Q(c_1), \dots, Q(c_{n-r})$, and we recover the following variant of the Chasles theorem given for the Neumann system¹

Proposition 8.3 (Moser, [35])

(i) *Let $(e(t), p(t))$ be a solution of the system on T^*S^{n-1} with the Hamiltonian*

$$\mathcal{H} = \sum_{i=1}^n \alpha_i F_i, \quad F_i = e_i^2 + \sum_{j \neq i} \frac{(e_i p_j - e_j p_i)^2}{a_i - a_j},$$

α_i being arbitrary constants. Then the associated line $l(t)$ is tangent to $n-1$ fixed confocal quadrics of the family $Q(\lambda)$.

¹It can be formulated in two different ways (see Theorem 12 in [20] given for the Clebsch–Perelomov systems and Theorem 4.10 in [37]).

- (ii) If $(e(t), p(t))$ is a solution of the system on T^*S^{n-1} with $\mathcal{H} = \sum_{i=1}^n F_i/a_i$ restricted to $\mathcal{H} = 0$, the corresponding line $l(t)$ is tangent to the ellipsoid $Q(0)$, on which the contact point $l \cap Q(0)$ traces a geodesic.

The proof of Theorem 8.2 is based upon the following property described in [20].

Proposition 8.4 *Let G_I , $I = \{i_1, \dots, i_k\}$ be the Plücker coordinates of a k -plane passing through the origin in \mathbb{R}^m . The set of all such k -planes that are tangent to a nondegenerate cone $\{\langle x, Bx \rangle = 0\}$, $B = \text{diag}(b_1, \dots, b_m)$ is the intersection of the (non-oriented) Grassmannian $G(m, k) \subset \wedge^k \mathbb{R}^m$ with the quadric*

$$\left\{ \sum_I |B|_I^I G_I^2 = 0 \right\} \subset \wedge^k \mathbb{R}^m, \quad |B|_I^I = b_{i_1} \cdots b_{i_k}. \quad (8.8)$$

Proof of Theorem 8.2. Let us now set $m + r$ and consider the family of cones (8.7). Let, as above, $\bar{\Sigma}$ be a $2r$ -plane in \mathbb{R}^{n+r} spanned by the columns of \mathcal{V} and associated to a point X, P on $T^*V_{n,r}$, and let

$$B = \text{diag} \left(\frac{1}{\lambda - a_1}, \dots, \frac{1}{\lambda - a_n}, 1, \dots, 1 \right).$$

Then, in view of Proposition 8.4 and the structure of \mathcal{V} , for a fixed $\lambda = \lambda^*$ the set of the $2r$ -planes that are tangent to $\bar{Q}(\lambda^*)$ is described by the following quadratic equation in terms of the Plücker coordinates of $\bar{\Sigma}$

$$\begin{aligned} & \sum_{I_r} \frac{(e_1 \wedge \cdots \wedge e_r)_{I_r}^2}{(\lambda^* - a_{i_1}) \cdots (\lambda^* - a_{i_r})} + \sum_{I_{r+1}} \frac{1}{(\lambda^* - a_{i_1}) \cdots (\lambda^* - a_{i_{r+1}})} \sum_{i=1}^n (e_1 \wedge \cdots \wedge e_r \wedge p_i)_{I_{r+1}}^2 \\ & + \sum_{I_{r+2}} \frac{1}{(\lambda^* - a_{i_1}) \cdots (\lambda^* - a_{i_{r+2}})} \sum_{1 \leq i < j \leq n} (e_1 \wedge \cdots \wedge e_r \wedge p_i \wedge p_j)_{I_{r+2}}^2 + \cdots \\ & + \sum_{I_{2r}} \frac{1}{(\lambda^* - a_{i_1}) \cdots (\lambda^* - a_{i_{2r}})} (e_1 \wedge \cdots \wedge e_r \wedge p_1 \wedge \cdots \wedge p_r)_{I_{2r}}^2 = 0. \end{aligned}$$

Due to (8.5), this coincides with the equation $\mathcal{I}_{2r}(\lambda^*) = 0$ up to multiplication by $a(\lambda^*)$.

Since $\mathcal{I}_{2r}(\lambda)$ is also an invariant polynomial of degree $n - r$ in λ , for a fixed plane $\bar{\Sigma}$ there exist precisely $n - r$ fixed cones of the family $\bar{Q}(\lambda)$ tangent to $\bar{\Sigma}$. This establishes the theorem. \square

Restriction to \mathbb{R}^n . By analogy with the case $r = 1$, one can consider the restriction of family (8.7) to the linear subspace $\{y_1 = \cdots = y_r = 1\}$:

$$Q_r(\lambda) = \left\{ \frac{x_1^2}{a_1 - \lambda} + \cdots + \frac{x_n^2}{a_n - \lambda} = r \right\} = \mathbf{i}^{-1} (\bar{Q}(\lambda) \cap \{y_1 = 1, \dots, y_r = 1\}), \quad (8.9)$$

where $\mathbf{i} : \mathbb{R}^n \rightarrow \mathbb{R}^{n+r}$ is the natural inclusion

$$\mathbf{i}(x_1, \dots, x_n) = (x_1, \dots, x_n, 1, \dots, 1).$$

This gives a family of confocal quadrics in \mathbb{R}^n .

Further, the section of $\bar{\Sigma}$ by the subspace $\{y_1 = \cdots = y_r = 1\}$ defines an affine r -plane $\Sigma(t) = \mathbf{i}^{-1} (\bar{\Sigma} \cap \{y_1 = \cdots = y_r = 1\})$ in $\mathbb{R}^n(x_1, \dots, x_n)$, which is spanned by the orthogonal vectors e_1, \dots, e_r and passes through the point $p_1 + \cdots + p_r$. As a result, to a generic

solution $(X(t), P(t))$ of the Neumann system on $V_{n,r}$ one can uniquely associate the moving r -plane

$$\Sigma(t) = p_1(t) + \cdots + p_r(t) + \text{span}\{e_1(t), \dots, e_r(t)\}.$$

In contrast to the case $r = 1$, due to dimensional reasons, for $r > 1$ the r -plane $\Sigma(t)$ is not necessarily tangent to the quadrics $Q_r(c_1), \dots, Q_r(c_{n-r})$. More precisely, since

$$d\mathbf{i}(T_{(x_1, \dots, x_n)}Q(\lambda)) = T_{\mathbf{i}(x_1, \dots, x_n)}\bar{Q}(\lambda) \cap \{y_1 = 1, \dots, y_r = 1\},$$

the tangency of $\bar{\Sigma}(t)$ and $\bar{Q}(c_i)$, for a fixed t , either implies the tangency of the corresponding affine r -plane $\Sigma(t)$ and the quadric $Q_r(c_i)$, or $\Sigma(t)$ does not intersect $Q_r(c_i)$. As a result, one cannot formulate a natural generalization of the Chasles theorem in \mathbb{R}^n that involves this r -plane.

Another feature of the case $r > 1$ is that, although the first integrals given by the polynomial $\mathcal{I}_{2r}(\lambda)$ are invariant with respect to the right $SO(r)$ -action on (X, P) , the $2r$ -plane $\bar{\Sigma}$ and r -plane Σ do not have this property. Thus, a generic polynomial $\mathcal{I}_{2r}(\lambda)$ corresponds to a whole family of $2r$ -planes (r -planes, respectively) that are tangent to the same set of confocal cones and is obtained as the orbit of $\bar{\Sigma}$ (Σ , respectively) under the right $SO(r)$ -action.

Then, it natural to replace $\bar{\Sigma}$ by the moving cylinder $\bar{\Delta}(t)$, the union of $2r$ -planes $\bar{\Sigma}(X(t)B, P(t)B)$ spanned by the columns of the $2r \times (n+r)$ matrices

$$\mathcal{V}(X(t)B, P(t)B), \quad B \in SO(r),$$

where $\mathcal{V}(X, P)$ is given by (8.6). The cylinder $\bar{\Delta}(t)$ is $SO(r)$ -invariant and, due to the construction, is tangent simultaneously to $n-r$ fixed confocal cones $\bar{Q}(c_1), \dots, \bar{Q}(c_{n-r})$.

Next, the section of $\bar{\Delta}(t)$ by the subspace $\{y_1 = \cdots = y_r = 1\}$ defines the moving $(2r-1)$ -dimensional cylinder

$$\Delta(t) = \left\{ \sum_{i,j} B_{i,j} p_i(t) \mid B \in SO(r) \right\} + \text{span}\{e_1(t), \dots, e_r(t)\},$$

which is now an appropriate object for the second generalization of the Chasles theorem:

Theorem 8.5 *Let the $(2r-1)$ -dimensional cylinder $\Delta(t) \subset \mathbb{R}^n$ be associated to a generic solution $(X(t), P(t))$ of the Neumann systems (5.4) or (5.5) on $V_{n,r}$ as described above. Then $\Delta(t)$ is tangent simultaneously to $n-r$ fixed confocal quadrics $Q_r(c_1), \dots, Q_r(c_{n-r})$ of the confocal family (8.7), where c_1, \dots, c_{n-r} are the roots of the invariant polynomial $\mathcal{I}_{2r}(\lambda)$.*

Proof. First, note that the plane $\bar{\Sigma}(X(t)B, P(t)B)$ can be obtained from $\bar{\Sigma}(X(t), P(t))$ by rotating it in the coordinates y_1, \dots, y_r by the matrix B^{-1} . That is, the cylinder $\bar{\Delta}(t)$ can be regarded as the orbit of $\bar{\Sigma}(X(t), P(t))$ with respect to the $SO(r)$ -action in the coordinates y_1, \dots, y_r . This property is related to the $SO(r)$ -symmetry of the cones $\bar{Q}(\lambda)$ in y_1, \dots, y_r .

Indeed, the $2r \times (n+r)$ matrices

$$\mathcal{V}(XB, PB) \quad \text{and} \quad \mathcal{V}(XB, PB) \begin{pmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} e_1 & \cdots & e_r & p_1 & \cdots & p_r \\ 0 & \cdots & 0 & & & \\ \vdots & & \vdots & & & \\ 0 & \cdots & 0 & & B^{-1} & \end{pmatrix}$$

define the same $2r$ -plane $\bar{\Sigma}(X(t)B, P(t)B)$, whereas the second matrix is obtained from $\mathcal{V}(X, P)$ by left multiplication by the block matrix $\text{diag}(\mathbf{I}_{n-r}, B^{-1})$.

Now let $l(t)$ be the line along which the plane $\bar{\Sigma}(X(t), P(t))$ is tangent to the cone $\bar{Q}(c_i)$,

$$l = l(t) = \text{span}\{v(t) = (v_1, v_2, \dots, v_{n+r})^T\} \subset \bar{\Sigma}(X(t), P(t)) \subset \mathbb{R}^{n+r}$$

and $l_B(t) = \text{span}\{v_B(t)\}$ be the tangency line of the rotated plane $\bar{\Sigma}(X(t)B, P(t)B)$ and $\bar{Q}(c_i)$. Due to the above observation, $v_B(t) = \text{diag}(\mathbf{I}_n, B^{-1})v(t)$.

One can always find $B^* \in SO(r)$ depending on i and t , such that the last r coordinates of $v_{B^*}(t)$ are equal. Then the r -plane

$$\Sigma_{B^*}(t) = i^{-1} (\bar{\Sigma}(X(t)B^*, P(t)B^*) \cap \{y_1 = \dots = y_r = 1\}) = \sum_{i,j} B_{i,j}^* p_i(t) + \text{span}\{e_1, \dots, e_r\}$$

is tangent to the quadric $Q_r(c_i)$ at the point $i^{-1}(l_{B^*}(t) \cap \{y_1 = \dots = y_r = 1\})$. Since for any i and t , $\Sigma_{B^*}(t)$ is a subspace of the cylinder $\Delta(t)$, we arrive at the statement of the theorem. \square

Chasles Theorem for Manakov Flows. Similar statement holds for the geodesic flows of submersion metrics defined by Manakov operators (see also Theorem 12 in [20]).

Suppose $2r < n$. The dual Lax pair for the flow with Manakov operator (4.4) given in Theorem 4.4 gives the set of commuting integrals $\mathcal{J}_{2l}(\lambda)$, the coefficients with term w^{2r-2l} in the expression $|a(\lambda)\mathcal{L}_{man}^*(\lambda) - w\mathbf{I}_n|$. In particular,

$$\mathcal{J}_{2r}(\lambda) = \sum_{I_{2r}} \frac{a(\lambda)}{(\lambda - a_{i_1}) \cdots (\lambda - a_{i_{2r}})} |\Phi|_{I_{2r}}$$

is a polynomial of degree $n - 2r$ in λ .

Consider the following family of confocal cones in \mathbb{R}^n :

$$Q_0(\lambda) = \left\{ \frac{x_1^2}{a_1 - \lambda} + \dots + \frac{x_n^2}{a_n - \lambda} = 0 \right\}, \quad \lambda \in \mathbb{R}.$$

Repeating the arguments of Theorem 8.2, and using the fact that the Manakov geodesic flows with different choices of the matrix B in (4.2) are quasi-periodic motions over the same isotropic toric foliation of $T^*V_{n,r}$, we get:

Theorem 8.6 *Let the $2r$ -plane*

$$\Sigma(t) = \text{span}\{e_1(t), \dots, e_r(t), p_1(t), \dots, p_r(t)\} \subset \mathbb{R}^n$$

be associated to a generic solution $(X(t), P(t))$ of the geodesic flow (3.5) given by the Manakov operator (4.2). Then $\Sigma(t)$ is tangent simultaneously to $n - 2r$ fixed confocal cones $Q_0(c_1), \dots, Q_0(c_{n-2r})$, where c_1, \dots, c_{n-2r} are the roots of the invariant polynomial $\mathcal{J}_{2r}(\lambda)$.

Remark 8.2 Since all objects in theorems 8.5 and 8.6 are right $SO(r)$ -invariant they are also valid for the Neumann system on the oriented Grassmannian variety $G_{n,r}$ as well as for the geodesic flows on $G_{n,r}$ obtained by submersion from the Manakov flows.

The case of the Poisson sphere. As an illustrative example, consider the case $r = 1$ and Manakov operator (4.2) with $B = -A^{-1}$. Then the submersion metric on S^{n-1} takes the following form (see [14])

$$ds^2 = \frac{1}{\langle A^{-1}e, e \rangle} \sum_{i=1}^n a_i de_i^2. \quad (8.10)$$

In the elliptic coordinates the metric (8.10) is of the Stäckel type, and its geodesic flow

$$\begin{aligned}\dot{e} &= (A^{-1}e, e)A^{-1}p - (A^{-1}e, p)A^{-1}e, \\ \dot{p} &= (A^{-1}e, p)A^{-1}p - (A^{-1}p, p)A^{-1}e\end{aligned}$$

is completely integrable [14]. For $n = 3$, the metric (8.10) is proportional to the metric on the Poisson sphere S^2 , i.e., to the metric obtained after $SO(2)$ reduction of the free rigid body motion around a fixed point with the inertia tensor $I = A^{-1}$.

From Theorem 8.6, we get

Corollary 8.7 *Let $e(t)$ be a geodesic line of the Poisson sphere metric (8.10). Then the moving 2-plane*

$$\Sigma(t) = \text{span}\{e(t), p(t)\} = \text{span}\{e(t), Ae(t)\}$$

is tangent to $n - 2$ fixed confocal quadrics of the family $Q_0(\lambda)$.

9 Appendix 1. Magnetic Neumann Systems

The Neumann system on S^{n-1} in presence of the Yang–Mills fields. We now go back to the reduction of the Neumann system to Grassmannians described in Section 7 and consider in detail the case $r = 1$.

Apart from the complete reduction of the Neumann system onto $\Psi^{-1}(0)/SO(n-1) \cong T^*S^{n-1}$, it is also convenient to describe partially reduced flows for non-zero values of the momentum Ψ , that is, the flows on

$$(T^*V_{n,n-1})/SO(n-1) \cong so(n) \times S^{n-1}, \quad (9.1)$$

as well as the reduced flows on the symplectic leaves

$$\Psi^{-1}(\eta)/SO(n-1)_\eta \cong \Psi^{-1}(\mathcal{O}_\eta)/SO(n-1) \subset (T^*V_{n,n-1})/SO(n-1),$$

where \mathcal{O}_η is the adjoint orbit of η . It is known that for $\eta \neq 0$, the quotients $\Psi^{-1}(\mathcal{O}_\eta)/SO(n-1)$ are diffeomorphic to \mathcal{O}_η -bundles over (co)tangent bundle of the sphere S^{n-1} and that the reduced systems are natural mechanical systems with the influence of the Yang–Mills fields (see, e.g., [26], Chapter III, which provides a detailed geometrical analysis of such systems).

We shall derive the reduced equations and Yang–Mills fields directly from the equations of motion. Namely, for a matrix $\xi \in so(n)$ and an unit vector $e_n \in S^{n-1}$, define projections

$$\text{pr}_{e_n}(\xi) = \xi e_n \otimes e_n + e_n \otimes e_n \xi, \quad \text{pr}_{e_n^\perp}(\xi) = \xi - \text{pr}_{e_n}(\xi)$$

with respect to the orthogonal decomposition

$$so(n) = \{e_n \wedge \mathbb{R}^n\} \oplus \{e_n \wedge \mathbb{R}^n\}^\perp. \quad (9.2)$$

Note that $\{e_n \wedge \mathbb{R}^n\}^\perp \cong so(n-1)$ and $\{e_n \wedge \mathbb{R}^n\}$ can be naturally identified with the tangent space $T_{e_n}S^{n-1}$. Therefore the reduced phase space $T^*V_{n,n-1}/SO(n-1)$ is also represented as a $so(n-1)$ -bundle over TS^{n-1} that we shall denote by $so(n-1) \times_s TS^{n-1}$. There is the natural inclusion

$$so(n-1) \times_s TS^{n-1} \subset so(n) \times TS^{n-1} : (\xi, e_n, \dot{e}_n) \in so(n-1) \times_s TS^{n-1} \Leftrightarrow \text{pr}_{e_n}(\xi) = 0.$$

Proposition 9.1 (i) *The reduced Neumann system on the quotient variety (9.1) has the form*

$$\dot{\Omega} = [A, e_n \otimes e_n], \quad \dot{e}_n = \Omega e_n, \quad \Omega \in so(n). \quad (9.3)$$

(ii) The second derivative of the vector e_n is

$$\ddot{e}_n = Ae_n - ((e_n, Ae_n) + (\dot{e}_n, \dot{e}_n))e_n + F_\Omega \dot{e}_n, \quad (9.4)$$

where $F_\Omega = \text{pr}_{e_n^\perp}(\Omega) = \Omega - \Phi(e_n, \dot{e}_n)$ and $\Phi(e_n, \dot{e}_n) = \dot{e}_n \wedge e_n$ is the standard $SO(n)$ -momentum mapping on TS^{n-1} .

If we restrict the flow to the invariant submanifold $\Psi^{-1}(\mathcal{O}_\eta)/SO(r)$, then $(F_\Omega, e_n, \dot{e}_n)$ ranges over the subbundle $\mathcal{O}_\eta \times_s TS^{n-1} \subset so(n-1) \times_s TS^{n-1}$ obtained by replacing fibers $so(n-1)$ by the adjoint orbits $\mathcal{O}_\eta \subset so(n-1)$. In particular, for $\eta = 0$ we recover (7.5). For $\eta \neq 0$, the additional term $F_\Omega \dot{e}_n$ can be interpreted as the influence of the Yang–Mills field with the internal symmetry group $SO(n-1)$ and charge type \mathcal{O}_η ([26]). In the special case $n = 3, r = 2$, after identification of the Lie algebras $(so(3), [\cdot, \cdot])$ and (\mathbb{R}^3, \times) , equation (9.4) takes the form

$$\ddot{e}_3 = Ae_3 - ((e_3, Ae_3) + (\dot{e}_3, \dot{e}_3))e_3 + \epsilon e_3 \times \dot{e}_3,$$

which describes the motion of the particle with the charge ϵ on S^2 in the magnetic monopole field. Here $\eta = \epsilon E_1 \wedge E_2$ and $F_\Omega \dot{e}_3 = \epsilon e_3 \times \dot{e}_3$ represents the Lorentz force of the magnetic monopole.

Since we already proved the completeness of the commuting integrals (5.10) on a generic symplectic leaf within (9.1), we arrive at the following statement.

Theorem 9.2 *The Neumann system perturbed by the Yang–Mills field, i.e., the restriction of (9.3) to $\mathcal{O}_\eta \times_s TS^{n-1}$, is completely integrable for a generic value $\eta \in so(n-1)$.*

Proof of Proposition 9.1. Let us identify $V_{n,n-1}$ and $SO(n)$ via (7.3) and consider $SO(n)$ as the configuration space of the rigid body moving around a fixed point. Then the vectors e_1, \dots, e_n are fixed in the body, the matrix $\mathcal{X} = (e_1, \dots, e_n)$ maps the fixed frame to the frame attached to the body and

$$M = \mathcal{P}\mathcal{X}^T - \mathcal{X}\mathcal{P}^T$$

plays the role of the angular momentum of the body *in the space frame*. Here we denoted the $n \times n$ momentum matrix by \mathcal{P} .

From the identity $e_n \otimes e_n = \mathbf{I}_n - \sum_{i=1}^{n-1} e_i \otimes e_i$ we see that, after identification (7.3), the Neumann system with the normal metric on $V_{n,n-1}$ corresponds to the motion of the rigid body with the Hamiltonian

$$H_{CP} = \frac{1}{2}\langle M, M \rangle + \frac{1}{2}(\text{tr } A - (e_n, Ae_n)). \quad (9.5)$$

This is a special, symmetric case of the Clebsch–Perelomov rigid body problem: the inertia operator of the body is the identity on the Lie algebra $so(n)$ (see [41]). Therefore the angular velocity in *the space frame* $\Omega = \mathcal{X} \cdot \mathcal{X}^{-1}$ and the angular velocity in *the body frame* $\omega = \mathcal{X}^{-1}\dot{\mathcal{X}}$ are equal to the angular momentum in the space frame M and to the angular momentum in the body frame $m = \mathcal{X}^{-1}M\mathcal{X}$ respectively.

The Hamiltonian (9.5) is invariant with respect to rotations in $\mathbb{R}^{n-1} = \text{span}(e_1, \dots, e_{n-1})$, i.e., it is right $SO(n-1)$ -invariant. In the right trivialization, the motion of the body is described by the Euler–Poincaré equations

$$\dot{\Omega} = [A, e_n \otimes e_n] \quad (9.6)$$

together with the Poisson equations

$$\dot{e}_i = \Omega e_i, \quad i = 1, \dots, n. \quad (9.7)$$

Whence it is clear that the reduced flow is given by (9.6) and the last Poisson equation

$$\dot{e}_n = \Omega e_n. \quad (9.8)$$

To prove the second assertion of Proposition 9.1, we rewrite equations (9.3) with respect to the orthogonal decomposition (9.2). Then, from (9.8), we have $\dot{e}_n = \text{pr}_{e_n}(\Omega)e_n$ and

$$\text{pr}_{e_n}(\Omega) = \dot{e}_n \wedge e_n = \Phi(e_n, \dot{e}_n). \quad (9.9)$$

In view of (9.6) and (9.9), the time derivation of (9.8) reads

$$\begin{aligned} \ddot{e}_n &= \dot{\Omega}e_n + \Omega\dot{e}_n = \dot{\Omega}e_n + \text{pr}_{e_n}(\Omega)\dot{e}_n + \text{pr}_{e_n^\perp}(\Omega)\dot{e}_n \\ &= (Ae_n \otimes e_n - e_n \otimes e_n A)e_n + (\dot{e}_n \wedge e_n)\dot{e}_n + \text{pr}_{e_n^\perp}(\Omega)\dot{e}_n \\ &= Ae_n - ((e_n, Ae_n) + (\dot{e}_n, \dot{e}_n))e_n + F_\Omega \dot{e}_n, \end{aligned}$$

which concludes the proof. \square

Symmetric Clebsch–Perelomov–Bogoyavlenski rigid body systems. A similar reduction can be made for the Neumann systems on $V_{n,r}$ and $G_{n,r}$ by considering the motion of a symmetric rigid body with the Hamiltonian

$$H_{CPB} = \frac{1}{2}\langle M, M \rangle + \frac{1}{2} \sum_{i=1}^r (e_i, Ae_i),$$

which corresponds to a special (symmetric) case of the Bogoyavlenski generalization of the Clebsch–Perelomov system (see [8]). Namely, in the space frame the motion is described by the Euler–Poincaré equations

$$\dot{\Omega} = [e_1 \otimes e_1 + \cdots + e_r \otimes e_r, A] \quad (9.10)$$

together with the Poisson equations (9.7). After substitutions

$$h = \Omega \quad \text{and} \quad v = e_1 \otimes e_1 + \cdots + e_r \otimes e_r$$

they take the closed form (6.8).

The system is right $SO(r) \times SO(n-r)$ -invariant with the momentum mapping

$$\Psi = \Psi_{so(r)} + \Psi_{so(n-r)}, \quad \Psi_{so(r)} = \text{pr}_{so(r)}(\mathcal{X}^{-1}\Omega\mathcal{X}), \quad \Psi_{so(n-r)} = \text{pr}_{so(n-r)}(\mathcal{X}^{-1}\Omega\mathcal{X}),$$

where

$$so(r) = \text{span}\{E_i \wedge E_j, 1 \leq i < j \leq r\}, \quad so(n-r) = \text{span}\{E_i \wedge E_j, r+1 \leq i < j \leq n\}.$$

The reductions of (9.10), (9.7) to $\Psi_{so(n-r)}^{-1}(\mathcal{O}_{\eta_{so(n-r)}})/SO(n-r)$ and to

$$\Psi^{-1}(\mathcal{O}_{\eta_{so(r)}} \times \mathcal{O}_{\eta_{so(n-r)}})/SO(r) \times SO(n-r)$$

lead to the Neumann systems on the Stiefel variety $V_{n,r}$ and, respectively, to the oriented Grassmannian variety $G_{n,r}$ under the influence of the Yang–Mills fields. In particular, if $\eta_{so(r)} = 0$ ($\eta_{so(r)} = \eta_{so(n-r)} = 0$), we get the Neumann system with the *normal metric* on $V_{n,r}$ (respectively, the Neumann system on $G_{n,r}$).

Magnetic Neumann flows on $V_{n,2}$ and $G_{n,2}$. The adjoint orbits in $so(2)$ are points, so a symplectic reduced space $\Psi^{-1}((\eta_{so(2)}, 0))/SO(2) \times SO(n-2)$ is diffeomorphic to $T^*G_{n,2}$. For $\eta_{so(2)} = \epsilon E_1 \wedge E_2 \neq 0$, it represents the magnetic cotangent bundle $T^*G_{n,2}$: the canonical symplectic structure of $T^*G_{n,2}$ is "twisted" by adding the magnetic form, which is exactly Kirillov–Konstant symplectic form on $G_{n,2}$ multiplied by ϵ (for more details see, e.g., [31, 13]).

As above, for us it is convenient to write the equations in the Euler–Lagrange form. Due to the presence of the magnetic form, the tangent bundle momentum mapping of the left $SO(n)$ -action is modified by adding the term $\epsilon e_1 \wedge e_2$ (see [19, 13])

$$\Phi_\epsilon = \dot{e}_1 \wedge e_1 + \dot{e}_2 \wedge e_2 + \epsilon e_1 \wedge e_2 = [e_1 \wedge e_2, \dot{e}_1 \wedge e_2 + e_1 \wedge \dot{e}_2] + \epsilon e_1 \wedge e_2, \quad (9.11)$$

whereas the right-hand side of the Neumann system (7.1) is modified by adding the term $\epsilon \Phi_0$ (see [13])

$$\begin{aligned} \frac{d^2}{dt^2} (e_1 \wedge e_2) &= -Ae_1 \wedge e_2 - e_1 \wedge Ae_2 \\ &+ ((e_1, Ae_1) - (\dot{e}_1, \dot{e}_1) + (e_2, Ae_2) - (\dot{e}_2, \dot{e}_2)) e_1 \wedge e_2 \\ &+ 2\dot{e}_1 \wedge \dot{e}_2 + \epsilon [e_1 \wedge e_2, \dot{e}_1 \wedge e_2 + e_1 \wedge \dot{e}_2]. \end{aligned} \quad (9.12)$$

Here $e_1, e_2, \dot{e}_1, \dot{e}_2$ satisfy the conditions (7.2). (Clearly, a similar symplectic reduction with a magnetic term can also be applied on the Stiefel variety $V_{n,n-2}$ (see [44]).)

To construct the magnetic Neumann flows on $T^*V_{n,2}$, consider closed 2-form

$$\omega_{mag} = de_1 \wedge de_2 = \sum_{i=1}^n de_1^i \wedge de_2^j$$

restricted to $V_{n,2}$. Let $\pi : T^*V_{n,2} \rightarrow V_{n,2}$ be the canonical projection and define the symplectic form

$$\omega_\epsilon = \omega + \epsilon \pi^* \omega_{mag}, \quad (9.13)$$

where ω is the canonical form on $T^*V_{n,2}$ (see Section 2).

The following two propositions can be verified by straightforward computations.

Proposition 9.3 *The left $SO(n)$ -action on $(T^*V_{n,2}, \omega_\epsilon)$ is Hamiltonian with the momentum mapping given by*

$$\Phi_\epsilon = \Phi + \epsilon e_1 \wedge e_2 = p_1 \wedge e_1 + p_2 \wedge e_2 + \epsilon e_1 \wedge e_2. \quad (9.14)$$

Proposition 9.4 *The Hamiltonian equations defined by the Hamiltonians of the Neumann systems with the Euclidean metric and the Normal metric with respect to the symplectic structure (9.13) read*

$$\begin{aligned} \dot{e}_1 &= p_1, \\ \dot{e}_2 &= p_2, \\ \dot{p}_1 &= -Ae_1 + ((e_1, Ae_1) - (p_1, p_1) - \epsilon(e_1, p_2)) e_1 + ((e_1, Ae_2) - (p_1, p_2)) e_2 + \epsilon p_2, \\ \dot{p}_2 &= -Ae_2 + ((e_1, Ae_2) - (p_1, p_2)) e_1 + ((e_2, Ae_2) - (p_2, p_2) + \epsilon(e_2, p_1)) e_2 - \epsilon p_1 \end{aligned} \quad (9.15)$$

and, respectively,

$$\begin{aligned} \dot{e}_1 &= \Phi_0 e_1 = p_1 - (e_1, p_2) e_2, \\ \dot{e}_2 &= \Phi_0 e_2 = p_2 - (e_2, p_1) e_1, \\ \dot{p}_1 &= \Phi_0 p_1 - Ae_1 + ((e_1, Ae_1) - \epsilon(e_1, \Phi_0 e_2)) e_1 + (e_1, Ae_2) e_2 + \epsilon \Phi_0 e_2, \\ \dot{p}_2 &= \Phi_0 p_2 - Ae_2 + (e_2, Ae_1) e_1 + ((e_2, Ae_2) + \epsilon(e_2, \Phi_0 e_1)) e_2 - \epsilon \Phi_0 e_1. \end{aligned} \quad (9.16)$$

Equations (9.15), (9.16) are right $SO(2)$ -invariant and have integral $\Psi_{12} = (e_1, p_2) - (e_2, p_1)$. The magnetic Neumann system (9.12) can be also seen as a reduction of the system (9.15), or (9.16) with respect to the right $SO(2)$ -action.

Furthermore, for the systems (9.12), (9.15), (9.16) the relation (5.6) still holds in the form

$$\frac{d}{dt}\Phi_\epsilon = [e_1 \otimes e_1 + e_2 \otimes e_2, A], \quad \frac{d}{dt}(e_1 \otimes e_1 + e_2 \otimes e_2) = [\Phi_\epsilon, e_1 \otimes e_1 + e_2 \otimes e_2],$$

which implies the Lax representation (5.7), where instead of the momentum mapping Φ one should use Φ_ϵ in (9.14).

Theorem 9.5 *The magnetic Neumann systems (9.12), (9.15) and (9.16) are completely integrable in the commutative sense with respect to the twisted symplectic structures described above.*

The proof is a simple modification of those of Theorems 6.1 and 7.1.

10 Appendix 2. Rank r Double, Coupled and Neumann Systems on Complex Stiefel Manifolds

In this section we briefly consider several natural generalizations of the Neumann flows on Stiefel varieties. We present their equations of motion and Lax representations, however a complete verification of the integrability is out of the scope of this paper.

Rank r double Neumann system. In [47] Suris introduced the double Neumann system describing the motion of 2 points $x, y \in \mathbb{R}^n$ which interact via the bilinear potential $(x, Ay)/2$ under the constraint $(x, y) = 1$.

We consider *rank r double Neumann system* defined by the Lagrangian function

$$L(X, Y, \dot{X}, \dot{Y}) = \text{tr}(\dot{X}^T \dot{Y}) - \text{tr}(X^T AY), \quad (10.1)$$

where the $n \times r$ matrices $X, Y \in M_{n,r}(\mathbb{R})$ are subject to the constraints

$$X^T Y = \mathbf{I}_r. \quad (10.2)$$

The corresponding Euler–Lagrange equations with $r \times r$ matrix multipliers read:²

$$\ddot{X} = -AX + X\Lambda^T, \quad \ddot{Y} = -AY + Y\Lambda, \quad (10.3)$$

where

$$\Lambda = X^T AY - \dot{X}^T \dot{Y}. \quad (10.4)$$

The rank r double Neumann system is an extension of the Neumann system on $V_{n,r}$ with the Euclidean metric: if $(X(t), Y(t))$ is a solution of the system (10.3) with the initial conditions $X = Y$, $\dot{X} = \dot{Y}$, then $(X(t), P(t)) = (X(t), \dot{X}(t))$ is a solution of (5.5). The Lax representation (8.1) extends as follows.

Theorem 10.1 *The equations (10.3) imply the following $2r \times 2r$ matrix Lax pair with the parameter λ*

$$\frac{d}{dt}\mathcal{L}(\lambda) = [\mathcal{L}(\lambda), \mathcal{A}(\lambda)], \quad (10.5)$$

$$\mathcal{L}(\lambda) = \begin{pmatrix} -X^T(\lambda\mathbf{I}_n - A)^{-1}\dot{Y} & -X^T(\mathbf{I}_n - \lambda A)^{-1}Y \\ \mathbf{I}_r + \dot{X}^T(\mathbf{I}_n - \lambda A)^{-1}\dot{Y} & \dot{X}^T(\lambda\mathbf{I}_n - A)^{-1}Y \end{pmatrix}, \quad \mathcal{A}(\lambda) = \begin{pmatrix} 0 & \mathbf{I}_r \\ \Lambda - \lambda\mathbf{I}_r & 0 \end{pmatrix},$$

Λ being given by (10.4).

²For simplicity here we only give the Lagrangian description.

Apart from the integrals provided by the Lax matrix $L(\lambda)$, the equations also possess the matrix integral $X^T \dot{Y} - Y^T \dot{X}$ associated to the $GL(n, \mathbb{R})$ -symmetry

$$(X, Y, \dot{X}, \dot{Y}) \longmapsto (XR^T, YR^{-1}, \dot{X}R^T, \dot{Y}R^{-1}), \quad R \in GL(n, \mathbb{R}).$$

Coupled Neumann system on $V_{n,r}$. This systems generalizes the motion of 2 points x, y on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ that interact via the bilinear potential $(x, Ay)/2$. The latter system was introduced in [44] (see also [47]).

Namely, let matrices $X, Y \in M_{n,r}(\mathbb{R})$ define two points on $V_{n,r}$ and let $P, Q \in M_{n,r}(\mathbb{R})$ represent their momenta such that

$$X^T P + P^T X = 0, \quad Y^T Q + Q^T Y = 0.$$

Assume that the evolution of X, Y is described by the Hamiltonian

$$H = T_\kappa + \text{tr}(X^T A Y), \quad (10.6)$$

$$T_\kappa = \frac{1}{2} \text{tr}(P^T P) - \left(\frac{1}{2} + \kappa\right) \text{tr}((X^T P)^2) + \frac{1}{2} \text{tr}(Q^T Q) - \left(\frac{1}{2} + \kappa\right) \text{tr}((Y^T Q)^2),$$

where T_κ is the kinetic energy of the points defined by an $SO(n)$ -invariant metric on $V_{n,r}$, which depends on the parameter κ . As above (see Section 3), for $\kappa = -1/2$ we have the Euclidean metric and for $\kappa = 0$ the normal one. (Note that in the case $r = 1$ we have $X^T P = Y^T Q = 0$, and all the above metrics coincide.)

The Hamilton equations with multipliers have the form

$$\begin{aligned} \dot{X} &= P - (1 + 2\kappa)X P^T X, \\ \dot{P} &= -AY + (1 + 2\kappa)P X^T P + X\Lambda, \\ \dot{Y} &= Q - (1 + 2\kappa)Y Q^T Y, \\ \dot{Q} &= -AX + (1 + 2\kappa)Q Y^T Q + Q\Pi, \end{aligned} \quad (10.7)$$

with

$$\Lambda = \frac{1}{2}(X^T A Y + Y^T A X) - P^T P, \quad \Pi = \frac{1}{2}(X^T A Y + Y^T A X) - Q^T Q. \quad (10.8)$$

Borrowing the terminology of [47], we call (10.7) the *r-coupled Neumann systems* on $V_{n,r}$. They are invariant with respect to the right diagonal $SO(r)$ -action on the product $T^*V_{n,r} \times T^*V_{n,r}$, and the corresponding matrix momentum $X^T P - P^T X + Y^T Q - Q^T Y$ is preserved along their flows.

The book [44] presented a "big" Lax pair of the *r-coupled systems* with the $so(n, n)$ -matrices depending on parameter ν

$$\begin{aligned} \dot{L}(\nu) &= [L(\nu), A(\nu)], \\ L(\nu) &= \begin{pmatrix} \mathbf{0} & A \\ A & \mathbf{0} \end{pmatrix} \nu + \begin{pmatrix} P X^T - X P^T & \mathbf{0} \\ \mathbf{0} & Q Y^T - Y Q^T \end{pmatrix} + \begin{pmatrix} \mathbf{0} & X Y^T \\ Y X^T & \mathbf{0} \end{pmatrix} \nu^{-1}, \\ A(\nu) &= L_+(\lambda) = \begin{pmatrix} \mathbf{0} & A \\ A & \mathbf{0} \end{pmatrix} \nu + \begin{pmatrix} P X^T - X P^T & \mathbf{0} \\ \mathbf{0} & Q Y^T - Y Q^T \end{pmatrix}, \end{aligned} \quad (10.9)$$

$\mathbf{0}$ being zero $n \times n$ block. Curiously, this Lax pair holds only for the case of the normal metric ($\kappa = 0$), and not for the Euclidean one, as one might expect and as happens in the case of the Neumann flows on $T^*V_{n,r}$.

Below we also present the dual "small" Lax representation. Introduce $r \times r$ matrix

$$F_\lambda(X, Y) = X^T (A^2 - \lambda^2)^{-1} Y.$$

Theorem 10.2 *The coupled Neumann system (10.7) with $\kappa = 0$ admits the following Lax pair with the spectral parameter λ*

$$\frac{d}{dt}\mathcal{L}(\lambda) = [\mathcal{L}(\lambda), \mathcal{A}(\lambda)], \quad (10.10)$$

$\mathcal{L}(\lambda), \mathcal{A}(\lambda)$ being $4r \times 4r$ matrices

$$\mathcal{L}(\lambda) = \begin{pmatrix} \lambda F_\lambda(P, X) & \lambda F_\lambda(P, P) & F_\lambda(AP, Y) & -\mathbf{I}_r + F_\lambda(AP, Q) \\ -\lambda F_\lambda(X, X) & -\lambda F_\lambda(X, P) & -F_\lambda(AX, Y) & -F_\lambda(AX, Q) \\ F_\lambda(AQ, X) & -\mathbf{I}_r + F_\lambda(AQ, P) & \lambda F_\lambda(Q, Y) & \lambda F_\lambda(Q, Q) \\ -F_\lambda(AY, X) & -F_\lambda(AY, P) & -\lambda F_\lambda(Y, Y) & -\lambda F_\lambda(Y, Q) \end{pmatrix},$$

$$\mathcal{A}(\lambda) = \begin{pmatrix} X^T P & \Lambda & 0 & -\lambda \mathbf{I}_r \\ \mathbf{I}_r & X^T P & 0 & 0 \\ 0 & -\lambda \mathbf{I}_r & Y^T Q & \Pi \\ 0 & 0 & \mathbf{I}_r & Y^T Q \end{pmatrix},$$

and Λ, Π are given by (10.8).

Again, the proof is straightforward. For $r = 1$ the Lax pair (10.10) was given by Suris in [47].

It is still not clear whether the r -coupled Neumann system with the Euclidean metric admits a Lax representation. However, we can consider the following perturbation of the Hamiltonian (10.6):

$$H_\kappa = T_\kappa + \text{tr}(X^T AY) - 2\kappa \text{tr}(X^T P Y^T Q).$$

Then the corresponding flows imply the Lax representation with matrices (10) for any κ , while the Lax representation (10.11) holds with $X^T P$ and $Y^T Q$ on the diagonal of $\mathcal{A}(\lambda)$ replaced by $(1 + 2\kappa)X^T P + 2\kappa Y^T Q$ and $2\kappa X^T P + (1 + 2\kappa)Y^T Q$, respectively. In particular, by taking $\kappa = -1/2$, we get the Lax representation of an r -coupled Neumann system with the Euclidean metric and with an additional interacting term $\text{tr}(X^T P Y^T Q)$.

Complex Stiefel manifolds. The Neumann systems and geodesic flows on $V_{n,r}$ can be extended to the complex Stiefel varieties $W_{n,r}$ as well. Recall that $W_{n,r}$ is the space of r ordered orthogonal vectors (z_1, \dots, z_n) in \mathbb{C}^n endowed with the standard Hermitian metric, or equivalently, the set of $n \times r$ matrices $Z \in M_{n,r}(\mathbb{C})$ satisfying

$$\bar{Z}^T Z = \mathbf{I}_r \quad (10.11)$$

(see, e.g., [29]). The variety $W_{n,r}$ can also be identified with the homogeneous space of the unitary group: $W_{n,r} \cong U(n)/U(n-r)$. The real Stiefel variety $V_{n,r}$ is thus a submanifold of $W_{n,r}$ given by the condition $Z = \bar{Z}$.

While the idea of integrable geodesic flows on $W_{n,r}$ follows from the general construction given for compact homogeneous spaces [10, 11, 12], to our knowledge, potential systems on $W_{n,r}$ for $r > 1$ were not studied yet.

We shall consider the Neumann system with the metric induced by the *Hermitian metric* and defined by the Lagrangian function

$$L(Z, \bar{Z}, \dot{Z}, \dot{\bar{Z}}) = \frac{1}{2} \text{tr}(\dot{Z}^T \dot{\bar{Z}}) - \frac{1}{2} \text{tr}(Z^T A \bar{Z}). \quad (10.12)$$

As above, the matrix A is a real diagonal $n \times n$ matrix.

The Euler–Lagrange equations with multipliers read:

$$\ddot{Z} = -AZ + Z\Lambda, \quad \ddot{\bar{Z}} = -A\bar{Z} + \bar{Z}\bar{\Lambda}, \quad (10.13)$$

where

$$\Lambda = \bar{Z}^T AZ - \dot{\bar{Z}}^T \dot{Z} = \bar{\Lambda}^T. \quad (10.14)$$

Then the Neumann system on $V_{n,r}$ with the Euclidean metric (5.5) can be regarded as a subsystem of (10.13): if $Z(t)$ is its solution on the complex Stiefel variety $W_{n,r}$ with the initial conditions satisfying $Z = \bar{Z}$, $\dot{Z} = \dot{\bar{Z}}$, then $(X(t), P(t)) = (Z(t), \dot{Z}(t))$ is a solution of the Neumann system (5.5) on $V_{n,r}$, and vice versa.

We also have

Theorem 10.3 *The Neumann system on the complex Stiefel manifold (10.13) imply the $2r \times 2r$ matrix representation with the spectral parameter λ*

$$\begin{aligned} \frac{d}{dt} \mathcal{L}(\lambda) &= [\mathcal{L}(\lambda), \mathcal{A}(\lambda)], \quad (10.15) \\ \mathcal{L}(\lambda) &= \begin{pmatrix} -Z^T(\lambda \mathbf{I}_n - A)^{-1} \dot{\bar{Z}} & -Z^T(\mathbf{I}_n - \lambda A)^{-1} \bar{Z} \\ \mathbf{I}_r + \dot{Z}^T(\mathbf{I}_n - \lambda A)^{-1} \dot{\bar{Z}} & \dot{Z}^T(\lambda \mathbf{I}_n - A)^{-1} \bar{Z} \end{pmatrix}, \quad \mathcal{A}(\lambda) = \begin{pmatrix} 0 & \mathbf{I}_r \\ \bar{\Lambda} - \lambda \mathbf{I}_r & 0 \end{pmatrix}, \end{aligned}$$

with Λ given by (10.14).

The system is invariant with respect to a right $U(r)$ -action and the symmetry with respect to the left action of $U(1)^n$ defined by

$$Z \mapsto \text{diag}(\rho_1, \dots, \rho_n) \cdot Z, \quad \rho_i \in U(1), \quad i = 1, \dots, n. \quad (10.16)$$

The symmetries imply the conservation of $u(n)$ and $u(1)^n$ momentum maps

$$\psi_0 = Z^T \dot{\bar{Z}} - \dot{Z}^T \bar{Z} \quad \text{and} \quad \psi_j = \sum_{i=1}^r \dot{z}_i^j \bar{z}_i^j - z_i^j \dot{\bar{z}}_i^j, \quad j = 1, \dots, n,$$

where $Z = (z_1, \dots, z_r)$.

Under the condition $\rho_1 = \dots = \rho_n$, (10.16) defines the left $U(1)$ -action, which is free. Then we have a well defined reduced Neumann flow on the quotient space (*complex projective Stiefel variety*) $PW_{n,r} = W_{n,r}/U(1)$.

On the other hand, the right $U(r)$ -symmetry enables one to reduce the Neumann system to the complex Grassmann variety $G_{\mathbb{C}}(n, r) \cong U(n)/U(r) \times U(n-r)$ of r -dimensional complex planes in \mathbb{C}^n , or, in general, to $U(n)$ -adjoint orbits $\mathcal{O} = U(n)/U(k_1) \times \dots \times U(k_l) \times U(n-r)$, where $k_1 + k_2 + \dots + k_l = r$.

Acknowledgments

We are grateful to Yuri G. Nikonorov on useful discussions. The research of B. J. was supported by the Serbian Ministry of Science, Project 174020, Geometry and Topology of Manifolds, Classical Mechanics and Integrable Dynamical Systems. The research of Yu. F. was supported by the MICINN-FEDER grant MTM2009-06973 and CUR-DIUE grant 2009SGR859.

References

- [1] *M. R. Adams, J. Harnad, E. Previato*, Isospectral Hamiltonian flows in finite and infinite dimensions. I. Generalized Moser systems and moment maps into loop algebras. *Comm. Math. Phys.* **117** (1988), no. 3, 451–500.
- [2] *M. R. Adams, J. Harnad, J. Hurtubise*, Isospectral Hamiltonian flows in finite and infinite dimensions. II. Integration of flows. *Comm. Math. Phys.* **134** (1990), no. 3, 555–585.
- [3] *V. I. Arnold, V. V. Kozlov, A. I. Neishtadt*, Mathematical aspects of classical and celestial mechanics. *Itogi Nauki i Tekhniki. Sovr. Probl. Mat. Fundamental'nye Napravleniya*, Vol. **3**, VINITI, Moscow 1985 (Russian). English transl.: *Encyclopaedia of Math. Sciences*, Vol. **3**, Springer-Verlag, Berlin 1989.
- [4] *A. Arvanitoyeorgos, V.V. Dzhepko, Yu. G. Nikonorov*, Invariant Einstein metrics on some homogeneous spaces of classical Lie groups, *Canadian Journal of Mathematics* **61** (2009), no. 6, 1201–1213, arXiv: math/0612504 [math.DG].
- [5] *A. Besse*, *Einstein Manifolds*, Springer, A Series of Modern Surveys in Mathematics, 1987.
- [6] *A. M. Bloch, P. E. Crouch, J. E. Marsden, T. S. Ratiu*, The symmetric representation of the rigid body equations and their discretization, *Nonlinearity*, **15**(2002), 1309–1341.
- [7] *A. M. Bloch, P. E. Crouch, A. K. Sanyal*, A variational problem on Stiefel manifolds, *Nonlinearity*, **19** (2006), 2247–2276.
- [8] *O. I. Bogoyavlenski*, New integrable problem of classical mechanics. *Comm. Math. Phys.* **94** (1984), 255–269
- [9] *A. V. Bolsinov*, Compatible Poisson brackets on Lie algebras and the completeness of families of functions in involution, *Izv. Acad. Nauk SSSR, Ser. matem.* **55** (1991), No.1, 68–92 (Russian); English translation: *Math. USSR-Izv.*, **38** (1992), No.1, 69–90.
- [10] *A. V. Bolsinov, B. Jovanović*, Integrable geodesic flows on homogeneous spaces. *Matem. Sbornik* **192** (2001) no. 7, 21–40 (Russian); English translation: *Sb. Mat.* **192** (2001), no. 7–8, 951–968.
- [11] *A. V. Bolsinov, B. Jovanović*, Non-commutative integrability, moment map and geodesic flows. *Annals of Global Analysis and Geometry* **23** (2003), no. 4, 305–322, arXiv: math-ph/0109031.
- [12] *A. V. Bolsinov, B. Jovanović*, Complete involutive algebras of functions on cotangent bundles of homogeneous spaces, *Mathematische Zeitschrift* **246** (2004), no. 1–2, 213–236.
- [13] *A. V. Bolsinov, B. Jovanović*, Magnetic Flows on Homogeneous Spaces, *Com. Mat. Helv.*, **83** (2008), no. 3, 679700, arXiv: math-ph/0609005.
- [14] *A. V. Brailov*, Construction of complete integrable geodesic flows on compact symmetric spaces. *Izv. Acad. Nauk SSSR, Ser. matem.* **50** (1986), no.2, 661–674, (Russian); English translation: *Math. USSR-Izv.* **50** (1986), No.4, 19–31.
- [15] *M. Chasles*, Les lignes géodésiques et les lignes de courbure des surfaces du second degré. *Journ. de Math.* **11** (1846), 5–20.
- [16] *P. A. Dirac*, On generalized Hamiltonian dynamics. *Can. J. Math.* **2** (1950), no.2, 129–148.
- [17] *V. Dragović, B. Gajić, B. Jovanović*, Singular Manakov Flows and Geodesic Flows on Homogeneous Spaces, *Transform. Groups* **14** (2009), no. 3, 513–530, arXiv:0901.2444 [math-ph]
- [18] *B. A. Dubrovin, I. M. Krichever, S. P. Novikov*, Integrable Systems I, *Itogi Nauki i Tekhniki. Sovr.Probl.Mat. Fund.Naprav.* Vol. **4**, VINITI, Moscow 1985 (Russian). English transl.: *Encyclopaedia of Math.Sciences*, Vol. **4**, 173–280, Springer-Verlag, Berlin 1989.

- [19] *D. I. Efimov*, The magnetic geodesic flows on a homogeneous symplectic manifold. *Siberian Mathematical Journal* **46** (2005), no.1, 83-93.
- [20] *Yu. N. Fedorov*, Integrable systems, Lax representation and confocal quadrics, *Amer. Math. Soc. Transl. (2)* Vol. **168**, 173-199 (1995).
- [21] *Yu. N. Fedorov*, Integrable flows and Backlund transformations on extended Stiefel varieties with application to the Euler top on the Lie group $SO(3)$, *J.Non. Math. Phys.* **12** (2005), Suppl. 2, 77-94.
- [22] *Yu. N. Fedorov, B. Jovanović*, Nonholonomic LR systems as Generalized Chaplygin systems with an Invariant Measure and Geodesic Flows on Homogeneous Spaces, *J. Nonlinear Sci.* **14** (2004), no. 4, 341–381. , arXiv: math-ph/0307016
- [23] *Yu. N. Fedorov, B. Jovanović*, Continuous and discrete Neumann systems on Stiefel varieties as matrix generalizations of the Jacobi–Mumford systems, in preparation.
- [24] *H. Flaschka*, Towards an algebro-geometric interpretation of the Neumann system. *Tohoku Math. J. (2)* **36** (1984), no. 3, 407–426.
- [25] *F. Griffiths, J. Harris*, *Principles of Algebraic Geometry*. Wiley Interscience, New York 1978.
- [26] *V. Guillemin, S. Sternberg*, *Symplectic techniques in physics*. Cambridge University press, 1984.
- [27] *G. Jensen*, Einstein metrics on principal fiber bundles, *J. Diff. Geom.* **8** (1973), 599-614.
- [28] *S. Kapustin*, The Neumann system on Stiefel varieties. Preprint 1992 (Russian).
- [29] *S. Kobayashi, K. Nomizu*, *Foundation of Differential Geometry*, Volume II. John Willey & Sons, New York, 1969, 468 p.
- [30] *H. Knörrer*, Geodesics on quadrics and a mechanical problem of C.Neumann. *J. Reine Angew. Math.* **334** (1982), 69–78.
- [31] *M. Kummer*, On the construction of the reduced phase space of a Hamiltonian system with symmetry. *Indiana Univ. Math. J.* **30** (1981), 281-291.
- [32] *V. B. Kuznetsov*, Isomorphism of an n -dimensional Neumann system and an n -site Gaudin magnet. (Russian) *Funktsional. Anal. i Prilozhen.* **26** (1992), no. 4, 88–90 (Russian); translation in *Funct. Anal. Appl.* **26** (1992), no. 4, 302–304.
- [33] *S. V. Manakov*, Note on the integrability of the Euler equations of n -dimensional rigid body dynamics, *Funkt. Anal. Prilozh.* **10** (1976), no. 4, 93-94 (Russian).
- [34] *I. V. Mikityuk*, Integrability of the Euler equations associated with filtrations of semisimple Lie algebras. *Matem. Sbornik* **125(167)** No.4 (1984) (Russian); English translation: *Math. USSR Sbornik* **53** (1986), No.2, 541-549.
- [35] *J. Moser*, Geometry of quadric and spectral theory. In: *Chern Symposium 1979*, Berlin–Heidelberg–New York, 147–188, 1980.
- [36] *J. Moser*, Various aspects of integrable Hamiltonian systems. *Dynamical systems (C.I.M.E. Summer School, Bressanone, 1978)*, pp. 233–289, *Progr. Math.*, 8, Birkhuser, Boston, Mass., 1980.
- [37] *D. Mumford*, *Tata lectures on theta*, Birkhäuser, Boston 1984.
- [38] *A. S. Mishchenko, A. T. Fomenko*, Generalized Liouville method of integration of Hamiltonian systems, *Funkt. Anal. Prilozh.* **12** (1978), No.2, 46-56 (Russian); English translation: *Funct. Anal. Appl.* **12** (1978), 113-121.

- [39] *N. N. Nekhoroshev*, Action-angle variables and their generalization, Tr. Mosk. Mat. O.-va. **26** (1972), 181-198, (Russian); English translation: Trans. Mosc. Math. Soc. **26** (1972), 180-198.
- [40] *C. Neumann*, De probleme quodam mechanico, quod ad primam integralium ultra-ellipticorum classem revocatum. J. Reine Angew. Math. **56** (1859).
- [41] *A. M. Perelomov*, Some remarks on the integrability of the equations of motion of a rigid body in an ideal fluid, Funkt. Anal. Prilozh **15** (1981), no. 2, 83-85 (Russian); English translation: Funct. Anal. Appl. **15** (1981), 144-146.
- [42] *T. Ratiu*, The C. Neumann problem as a completely integrable system on an adjoint orbit, Trans. Amer. Math. Soc. **264** (1981), no.2, 321-329.
- [43] *A. G. Reyman*, Integrable Hamiltonian systems connected with graded Lie algebras, Zap. Nauchn. Semin. LOMI AN SSSR **95** (1980), 3-54 (Russian); English translation: J. Sov. Math. **19** (1982), 1507-1545.
- [44] *A. G. Reyman, M. A. Semonov-Tian-Shanski*, Group theoretical methods in the theory of finite dimensional integrable systems. In. Dynamical systems VII Itogi Nauki i Tekhniki. Sovr.Probl.Mat. Fund.Naprav. Vol.16, VINITI, Moscow 1987 (Russian). English transl.: Encyclopaedia of Math.Sciences, Vol. **16**, Springer 1994.
- [45] *P. Saksida*, Nahm's Equations and Generalizations of the Neumann System, Proc. London Math. Soc. **78** (1999), no. 3, 701-720.
- [46] *P. Saksida*, Integrable anharmonic oscillators on spheres and hyperbolic spaces. Nonlinearity **14** (2001), 977-994.
- [47] *Yu. B. Suris*, The problem of integrable discretization: Hamiltonian approach. Progress in Mathematics, **219**. Birkhuser Verlag, Basel, 2003.
- [48] *A. Thimm*, Integrable geodesic flows on homogeneous spaces *Ergod. Th. & Dynam. Sys.* **1** (1981), 495-517.
- [49] *A. P. Veselov*, Finite zone potentials and integrable systems on a sphere with quadratic potential. Funkt. Anal. Prilozh. **14** (1980), no. 1, 48-50 (Russian); English translation: Funct. Anal. Appl. **14** 37-39 (1980).
- [50] *S. Wojciechowski*, Integrable one-particle potentials related to the Neumann system and the Jacobi problem of geodesic motion on an ellipsoid, Phys. Lett. A **107** (1985), 107-111.
- [51] *N. T. Zung*, Torus actions and integrable systems. In: Bolsinov A.V., Fomenko A.T., Oshepkov A.A. (eds.) Topological methods in the theory of integrable systems. Cambridge Scientific Publ., 2006, 289-328; arXiv: math.DS/0407455.

Yuri N. Fedorov
 Department de Matemàtica I
 Universitat Politècnica de Catalunya
 Barcelona, E-08028 Spain
 e-mail: Yuri.Fedorov@upc.es

Božidar Jovanović
 Mathematical Institute SANU
 Serbian Academy of Sciences and Arts
 Kneza Mihaila 36, 11000, Belgrad, Serbia
 e-mail: bozaj@mi.sanu.ac.yu