Bak–Sneppen type models and rank-driven processes

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We analyse a surprising connection between Bak–Sneppen type models and much simpler Markov processes, which we call rank-driven processes (RDPs).

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INTRODUCTION

In [1], Bak and Sneppen introduced a very fruitful and simple model of evolution that has proved surprisingly hard to analyse. Despite the nearly 900 citations to that paper in the literature [2], only a small number of rigorous results have been obtained, such as those of Meester and Znamenski [3] on the non-triviality of the steady-state distribution. We hope that the approach of this letter will stimulate new analytical results in this area.

The classical Bak–Sneppen model (BS) involves a ring of N sites. Initially, we associate with the k-th site ($k \in \{1, \dots, N\}$) a fitness value $x_k \in [0, 1]$ chosen from the uniform distribution $U[0, 1]$. To perform an update of the process, we choose the smallest of all the x_k, x_{kmin} say, and replace x_{kmin} and its two nearest neighbours $x_{kmin\pm1}$ (indices calculated modulo N) by new independent $U[0, 1]$ random numbers. If the algorithm is iterated via simulation, and if N is large, the marginal distribution of the fitness at any particular site can be seen numerically to evolve to a $U[s^*, 1]$ distribution, where simulations suggest that $s^* \approx 0.667$.

Several variations on this model have been considered in the literature. One is the *anisotropic* Bak–Sneppen (aBS) model, in which, in addition to the smallest fitness, only its *right-hand* nearest neighbour is replaced. The phenomena are qualitatively the same as in the original BS model: the aBS model also gives rise (according to large-N simulations) to a threshold value s^*, which in this case is approximately 0.724 [4]. The aBS model is the main focus of this letter. The reason for choosing aBS over BS is that the calculations are easier while similar arguments can be used in the two cases.

Another variant on the BS model is the 'mean field' version of [5, 6], in which one replaces the smallest fitness and $K - 1$ randomly chosen ones; a generalization of the $K = 2$ version of that model is studied in [7].

RANK DRIVEN PROCESSES

As in BS, consider a set of N sites, each site k populated by a real number x_k in $[0, 1]$; now we do not specify any topology for the sites. A *rank driven process* (RDP) is a discrete-time Markov process on the 'N-simplex'

$$\Delta_N = \{(x_{(1)}, \dots, x_{(N)}) : 0 \le x_{(1)} \le \cdots \le x_{(N)} \le 1\};$$

$x_{(1)}, \dots, x_{(N)}$ are the (increasing) *order statistics* of $x_1, \dots, x_N$. The RDP evolves according to the following Markovian rule. At each step, K of the x_k-values are selected according to rank by sampling (without replacement, according to some specified probability distribution) from $\{1, 2, \dots, N\}$; that is, the sample from $\{1, 2, \dots, N\}$ specifies the $x_{(k)}$ that are chosen. The chosen x_k-values are replaced by new (independent) random numbers, e.g. from $U[0, 1]$ (as we will do below), or from some other probability distribution.

An example of a RDP is one that evolves by picking the smallest value with probability one and replacing it by a $U[0, 1]$ value. One that always picks the smallest value together with some other value selected with equal probability $1/(N-1)$ for each possibility, is also an RDP, and is the mean-field version of aBS considered by de Boer *et al.* [5] and Labzowsky and Pis'mak [6].

We are most interested in the case where one replaces the smallest value and the k-th ranked value, $k \ge 2$, with probability $f_N(k)$. Thus $f_N(k) \ge 0$, $k \ge 2$ and $\sum_{k=2}^{N} f_N(k) = 1$. Here we discuss only the case where the replacement fitness values are drawn from the $U[0, 1]$ distribution. We will call such an RDP a *class A RDP with distribution f_N*. Thus the model of [5] is a class A RDP with $f_N(k) = 1/(N - 1)$.

In [7], by considering the random walk associated with a class A RDP, we show that a crucial quantity is

$$\alpha = \lim_{n\to\infty} \lim_{N\to\infty} \sum_{k=2}^{n} f_N(k), \qquad (1)$$

assuming that the N-limit exists. Here $\alpha \in [0,1]$ measures the "atomicity" of f_N as $N \to \infty$. For example, for the mean field aBS of [5] one has $\alpha = 0$, while if we always replace the smallest and the second smallest elements, $\alpha = 1$. The main result of [7] is that the threshold in the limiting ($N \to \infty$) stationary distribution of x values in a class A RDP is given by

$$s^* = \frac{1+\alpha}{2}. \qquad (2)$$

We give a sketch of the argument for (2) in the next section.

A second result of [7] shows that the limiting marginal distribution at stationarity is $U[s^*, 1]$, where s^* is given by (2), provided that the selection distribution f_N is 'eventually uniform' in the sense that

$$f_N(k) \approx \frac{1-\alpha}{N} \qquad (3)$$

for k sufficiently large. This condition is satisfied with $\alpha = 0$ for the mean field aBS of [5], showing that the limiting distribution is indeed $U[1/2, 1]$ as indicated by [5].

THRESHOLDS IN CLASS A RDPS

We give a brief indication of the origin of the threshold formula (2); see [7] for details. Consider the s-counting process $N_t(s)$ defined to be the number of x_k-values in the interval $[0, s]$ after t iterations of the RDP. Then $N_t(s)$ is a Markov chain on the finite state-space $\{0, 1, \ldots, N\}$. The threshold s^* relates to the limiting ($t \to \infty$ then $N \to \infty$) marginal distribution of an arbitrary x_k. The probability that a randomly chosen x_k-value is less than s is $E[N_t(s)]/N$ (where E denotes expected value). Thus a natural way to define a threshold s^* is

$$s^* = \sup\{s \geq 0 : \lim_{N\to\infty} \lim_{t\to\infty} N^{-1} E[N_t(s)] = 0\};$$

the t-limit exists by Markov chain limit theory and it can be shown that the N-limit exists too, so that s^* is well defined [7].

To evaluate s^*, we compute the mean drift of $N_t(s)$:

$$\begin{aligned} &E[N_{t+1}(s) - N_t(s) \mid N_t(s) = n] \\ &\quad = 2s - (1 + F_N(n))\mathbf{1}\{n > 0\}, \end{aligned} \qquad (4)$$

where $F_N(n) = \sum_{k=2}^{n} f_N(k)$ and an empty sum is 0. Heuristically, for large N and large n, $F_N(n) \approx \alpha$ by (1) so that this drift is approximately $2s - 1 - \alpha$, and setting this equal to zero gives (2). One expects that the drift being zero indicates the threshold behaviour, because a positive (negative) drift would mean $N_t(s)$ increases (decreases). In this argument there are several limits involved (n, N, t all going to ∞) that need to be handled with care.

We outline the half of the argument showing that $s^* \geq (1+\alpha)/2$. We need to show that $\lim_{N\to\infty} \lim_{t\to\infty} N^{-1} E[N_t(s)] = 0$ for $s < (1+\alpha)/2$. Fix $s < (1+\alpha)/2$ for the remainder of this section. We have from (4) and (1) that for some $\varepsilon > 0$, some $A < \infty$, and some $N_0 < \infty$,

$$E[N_{t+1}(s) - N_t(s) \mid N_t(s) = n] \leq -\varepsilon,$$

for all $n \geq A$ and all $N \geq N_0$. That is, $N_t(s)$ satisfies a Foster–Lyapunov condition [8] uniformly in N. Thus $N_t(s)$ is an ergodic Markov process, and a uniform integrability argument shows that $\lim_{t\to\infty} E[N_t(s)]$ exists, and is bounded uniformly in N. This is a key step in showing $\lim_{N\to\infty} \lim_{t\to\infty} E[N_t(s)] < \infty$, which is a stronger result than the one required. We refer to [7] for the details.

RDPS AND ABS

Surprisingly, Bak–Sneppen type models, which are defined in terms of nearest neighbours of the smallest element, are closely connected to RDPs. *Assume* that for aBS, we can define a probability distribution $f_N(\cdot)$ on $\{2, \ldots N\}$, $f_N(k)$ being the equilibrium probability that the right nearest neighbour of the smallest element is k-th ranked. In other words, if we let $P(k, M)$ be the number of times the k-th ranked element, $k \geq 2$, is the right neighbour of the smallest element in M iterations of the aBS algorithm, we put

$$f_N(k) = \lim_{M\to\infty} \frac{1}{M} P(k, M). \qquad (5)$$

Heuristically, we expect that given suitable ergodicity properties for the aBS Markov process on the uncountable state space $[0,1]^N$, this limit will exist with probability one.

If f_N given by (5) is well-defined, one can consider the class A RDP with the same distribution f_N. We claim that this RDP shares many properties with the aBS process. We first describe simulation evidence to this effect. At the end of this section we discuss further progress that is required to obtain a theoretical understanding of the link between the two processes.

Numerically, we have convincing evidence that f_N given by (5) is well-defined and monotone decreasing, and that the limit $\lim_{N\to\infty} f_N(k)$ is well-defined for all k, so that α defined by (1) exists. Figure 1 shows approximations to $f_N(k)$ for small values of k for different values of N. We see that for a given N, $f_N(k)$ decays rapidly for small k before settling down to a uniform value. In fact, it appears that there is a constant C such that $f_N(k) = C/N$ for large enough k. Thus the numerical evidence supports the eventual uniformity condition (3). Hence $\alpha = 1 - C$. Numerical results give $\alpha \approx 0.445$ and hence $s^* \approx 0.723$, in close agreement with the simulations of [4]. Note that $f_N(2) \approx 0.209$ for all the values of N in Figure 1.

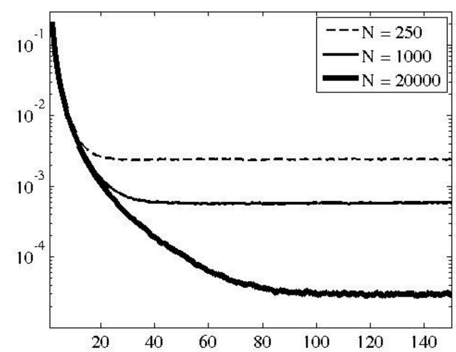

FIG. 1. Plot of $f_N(k)$, $k \in \{2, \ldots, 150\}$ for $N = 250, 1000, 20000$.

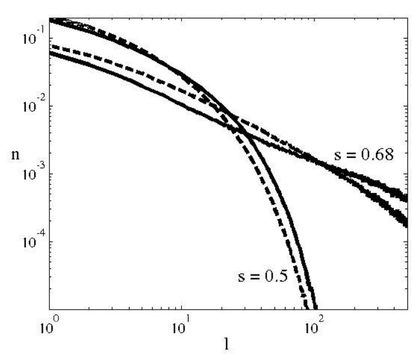

FIG. 2. Size distribution $n(l)$ of s avalanches in aBS (solid line) and RDP (dashed) for $s = 0.5,\ 0.68$ and $N = 1000$.

Following [1] we define the length of an s avalanche to be l if the number of consecutive steps for which the smallest fitness value stays below s is l. We compute the distribution $n(l)$ of avalanche lengths for aBS and our RDP. Representative distributions are given in Figure 2. As s approaches s^* we find that $n(l)$ shows the power law behaviour characteristic of self-organized criticality, although there is a small but clear difference in the exponents of the two processes.

The numerical evidence, beyond the coincidence of the thresholds, strongly suggests that the RDP with f_N given by (5) is closely related to aBS. For example, as noted above, the two processes share the $U[s^*, 1]$ limiting distribution. The exact relationship of the two processes remains to be characterized rigorously. If one wished to define a Markov process on Δ_N whose stationary distribution coincided with the projection onto Δ_N of the stationary distribution of aBS, a natural candidate would be a RDP with state-dependent selection distribution: instead of a single $f_N(\cdot)$ one would have a family $f_N(\,\cdot\,; x)$ selection distributions conditioned on the state $x \in \Delta_N$. Thus, assuming it exists, one would take $f_N(\,\cdot\,; x)$ to be the stationary distribution for aBS of the right-neighbour of the smallest element *conditional* on the projection of the current state onto Δ_N being x. The fact that the numerical evidence described above suggests that one can proceed not with a state-dependent RDP based on $f_N(\,\cdot\,; x)$ but with the simpler class A RDP based on $f_N(\,\cdot\,)$ (which is an average of the $f_N(\,\cdot\,; x)$) seems to point to some important underlying property of aBS itself. Two possible explanations would be:

(a) $f_N(\,\cdot\,) = f_N(\,\cdot\,; x)$ for all x, i.e., at stationarity there is some independence between the order statistics and the permutation that maps sites to ranks; or

(b) $f_N(\,\cdot\,; x)$ satisfies (uniformly in x) the same asymptotic conditions as $f_N(\,\cdot\,)$ that are central to the limit behaviour, namely (1) and (3).

The stronger fact (a) would suggest that the stationary distribution of the class A RDP coincides with the projection of the stationary distribution of aBS onto Δ_N, so that the two processes share the same detailed equilibrium properties. The weaker fact (b) would suffice to explain why the two processes share the same threshold and characteristic $U[s^*, 1]$ limit distribution. Finally, we remark that the distributions $f_N(\,\cdot\,; x)$ seem to be very difficult to evaluate numerically for aBS.

CONCLUSIONS

The distribution $f_N(k)$ and the quantity α of (1) capture the build-up of correlations in Bak–Sneppen type algorithms, the threshold behaviour of which can be analysed exactly by considering the $N_t(s)$ Markov process on a countable state space.

The class of RDPs that we have introduced is of interest in its own right. Numerical evidence suggests that by choosing as parameter for the RDP an appropriate statistic ($f_N(\,\cdot\,)$) of aBS, one can replicate the asymptotic behaviour of aBS by the RDP, for which one can prove rigorous results more easily. The remaining analytical challenge is to clarify the relationship between aBS and the class A RDP. This involves at least two main parts: (i) proving the existence of the distributions f_N and of the limit as $N \to \infty$; and (ii) determining the property of aBS that allows us to use $f_N(\,\cdot\,)$ instead of the conditional version $f_N(\,\cdot\,; x)$. If one can make precise the connection between aBS and the RDP, one should be able to transfer rigorous results for RDPs [7] to aBS.

We note that in the case of the classical BS process the same argument apply, though now f_N is a function of two variables, $f_N(k, \ell)$, $k \in \{2, \ldots N-1\}$, $\ell \in \{k+1, \ldots, N\}$.

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