On rational solutions of multicomponent and matrix KP hierarchies

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Abstract

We derive some rational solutions for the multicomponent and matrix KP hierarchies generalising an approach by Wilson. Connections with the multicomponent version of the KP/CM correspondence are discussed.

1 Introduction

The *KP hierarchy* is an integrable hierarchy of partial differential equations generated by a pseudo-differential operator of the form

$$L = D + u_1 D^{-1} + u_2 D^{-2} + \dots \tag{1}$$

Here the $(u_i)_{i\geq 1}$ are elements of a differential algebra $\mathcal{A}$ of smooth functions of a variable x (with $D = \partial/\partial x$) and a further infinite family of variables $\boldsymbol{t} = (t_i)_{i\geq 1}$. The evolution of L is determined by the following system of Lax-type equations:

$$\partial_k L = [B_k, L] \qquad (k \geq 1) \tag{2}$$

where $\partial_k = \partial/\partial t_k$ and $B_k = (L^k)_+$. All these equations commute, and the variable x may be identified with t_1. This hierarchy of equations is naturally viewed as a dynamical system defined on an infinite-dimensional Grassmann manifold, as discovered by Sato [9]. In [10] Segal and Wilson developed a very general framework for building solutions to the KP hierarchy out of the points of a certain Grassmannian $\mathrm{Gr}(H)$ of closed subspaces in a Hilbert space H.

The *multicomponent KP hierarchy* (mcKP) is a generalisation of the KP hierarchy obtained by replacing $\mathcal{A}$ with the differential algebra of $r \times r$ matrices whose entries belong to an algebra of smooth functions of a variable x and r further families of variables

$$\boldsymbol{t}^{(1)} = (t_i^{(1)})_{i\geq 1} \quad \dots \quad \boldsymbol{t}^{(r)} = (t_i^{(r)})_{i\geq 1}$$

which we will collectively denote by $\bar{\boldsymbol{t}}$. The operator (1) now becomes

$$L = D + U_1 D^{-1} + U_2 D^{-2} + \dots \tag{3}$$

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where $(U_i)_{i\geq 1}$ are $r \times r$ matrices. To define the evolution of L we introduce a set of r matrix pseudo-differential operators R_α of the form[1]

$$R_\alpha = E_\alpha + R_{1\alpha} D^{-1} + R_{2\alpha} D^{-2} + \dots \tag{4}$$

where E_α is the matrix with 1 in the entry (α, α) and zero elsewhere. These operators are required to satisfy the equations $[L, R_\alpha] = 0$, $[R_\alpha, R_\beta] = 0$ and $\sum_\alpha R_\alpha = I_r$ (it can be shown that such operators do exist). The evolution equations for the mcKP hierarchy are then

$$\partial_{k\alpha} L = [B_{k\alpha}, L] \qquad (k \geq 1,\, \alpha = 1 \dots r) \tag{5}$$

where $\partial_{k\alpha} = \partial / \partial t_k^{(\alpha)}$ and $B_{k\alpha} = (L^k R_\alpha)_+$. The variable x may be identified with $\sum_\gamma t_1^{(\gamma)}$; if we define also for each $k \geq 2$ the new variables $t_k = \sum_\gamma t_k^{(\gamma)}$ then the corresponding flows determine a sub-hierarchy of equations that we call the *matrix KP hierarchy*.

In [12] George Wilson obtained a complete classification of the rational solutions to the KP hierarchy. More precisely, he proved that the coefficients of an operator L satisfying (1) are proper (i.e., vanishing at infinity) rational functions of x exactly when L comes from a point of a certain sub-Grassmannian of $\mathrm{Gr}(H)$, which he called the *adelic Grassmannian*. This space turns out to be in one-to-one correspondence with the phase space of the rational Calogero-Moser system [13]; this provides a geometric explanation for the phenomenon, first noticed by Krichever in [6], that the poles of a rational solution to the KP equation evolve as a system of point particles described by the Calogero-Moser Hamiltonian. On the other hand, the adelic Grassmannian may also be seen as the moduli space of isomorphism classes of right ideals in the Weyl algebra [2], thereby linking the subject to the emerging field of noncommutative algebraic geometry. This web of connections is sometimes referred to as the "KP/CM correspondence" (see also [1] for a wider perspective on that matter).

The purpose of this paper is to establish some rationality results, obtained in the author's PhD Thesis [11], for the solutions of multicomponent and matrix KP hierarchies. Our main motivation is to understand how the above-mentioned results generalise to the multicomponent setting. The paper is organised as follows. In section 2 we briefly recall the mapping between points in the r-component Segal-Wilson Grassmannian $\mathrm{Gr}(r)$ and solutions of multicomponent KP. In section 3 we analyse the multicomponent rational Grassmannian $\mathrm{Gr}^{\mathrm{rat}}(r)$ and display an explicit formula for the Baker and tau functions associated to these points. In section 4 we prove two rationality results: in Theorem 2 we consider mcKP solutions coming from certain (very special) points in $\mathrm{Gr}^{\mathrm{rat}}(r)$, whereas in Theorem 3 we restrict to the matrix KP hierarchy but consider a much larger subset of $\mathrm{Gr}^{\mathrm{rat}}(r)$. Finally in section 5 we briefly comment on the relevance of these results for the multicomponent version of the KP/CM correspondence.

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2 Preliminaries

We start by briefly recalling the definition of the r-component Segal-Wilson Grassmannian [10, 8]. Consider the Hilbert space $H^{(r)} = L^2(S^1, \mathbb{C}^r)$; its elements can be thought of as functions $\mathbb{C} \to \mathbb{C}^r$

[1] Greek indices will henceforth run from 1 to r.

by embedding S^1 in the complex plane as the circle γ_R with centre 0 and radius $R \in \mathbb{R}^+$. We have the splitting $H^{(r)} = H_+^{(r)} \oplus H_-^{(r)}$ in the two subspaces consisting of functions with only positive (resp. negative) Fourier coefficients, with associated orthogonal projections $\pi_\pm$. The *Segal-Wilson Grassmannian* of $H^{(r)}$, denoted by $\mathrm{Gr}(r)$, is the set of all closed linear subspaces $W \subseteq H^{(r)}$ such that $\pi_+|_W$ is a Fredholm operator of index zero and $\pi_-|_W$ is a compact operator.

If we take the elements of $H^{(r)}$ to be row vectors for definiteness, the loop group $\mathrm{LGL}(r, \mathbb{C})$ naturally acts on $H^{(r)}$ (hence on $\mathrm{Gr}(r)$) by matrix multiplication from the right. We define $\Gamma_+(r)$ as the subgroup consisting of diagonal matrices of the form[2]

$$\mathrm{diag}(g_1, \ldots, g_r) \quad \text{with } g_\alpha \in \Gamma_+ \tag{6}$$

where Γ_+ is the group of analytic functions $g \colon S^1 \to \mathbb{C}^*$ that extend to holomorphic functions on the disc $\{\, z \in \mathbb{C}P^1 \mid |z| \leq R \,\}$ and such that $g(0) = 1$ (cfr. [10]). It follows that for each g_α there exists a holomorphic function f_α such that $g_\alpha = \mathrm{e}^{f_\alpha}$ (with $f_\alpha(0) = 0$), and by letting $f_\alpha = \sum_{i \geq 1} t_i^{(\alpha)} z^i$ we have

$$g_\alpha(z) = \exp \sum_{i \geq 1} t_i^{(\alpha)} z^i \tag{7}$$

Thus a generic matrix $g \in \Gamma_+(r)$ may be written in the form

$$g = \exp \mathrm{diag}(\sum_{i \geq 1} t_i^{(1)} z^i, \ldots, \sum_{i \geq 1} t_i^{(r)} z^i) \tag{8}$$

and is totally described by the family of coefficients $\bar{\boldsymbol{t}} = (\boldsymbol{t}^{(1)}, \ldots, \boldsymbol{t}^{(r)})$.

We now recall the mapping between points of $\mathrm{Gr}(r)$ and solutions to the multicomponent KP hierarchy. In what follows we will say that a matrix-valued function $\psi(z)$ belongs to a subspace $W \in \mathrm{Gr}(r)$ if and only if each row of ψ, seen as an element of $H^{(r)}$, belongs to W.

For every $W \in \mathrm{Gr}(r)$ we define

$$\Gamma_+(r)^W := \{\, g \in \Gamma_+(r) \mid W g^{-1} \text{ is transverse} \,\} \tag{9}$$

where "transverse" means that the orthogonal projection $W g^{-1} \to H_+^{(r)}$ is an isomorphism. For any $g \in \Gamma_+(r)^W$ the *reduced Baker function* associated to W and g is the matrix-valued function ψ whose row ψ_α is the inverse image of $e_\alpha \in H_+^{(r)}$ (the α-th element of the canonical basis of $\mathbb{C}^r$) by $\pi_+|_{W g^{-1}}$. It follows straightforwardly that

$$\tilde{\psi}_W(g, z) = I_r + \sum_{i \geq 1} W_i(g) z^{-i} \tag{10}$$

for some matrices $(W_i)_{i \geq 1}$. Now, since each row of the matrix $\tilde{\psi}_W$ belongs to the subspace $W g^{-1}$, each row of the product matrix $\tilde{\psi}_W g$ will belong to W; the *Baker function* associated to W is the map ψ_W which sends $g \in \Gamma_+(r)^W$ to this matrix:

$$\psi_W(g, z) = \left(I_r + \sum_{i \geq 1} W_i(g) z^{-i} \right) g(z) \tag{11}$$

[2]More generally, we could consider the subgroup determined by a maximal torus of type $\underline{r}$ (where $\underline{r}$ is any partition of r) in $\mathrm{GL}(r, \mathbb{C})$; this gives rise to the so-called *Heisenberg flow of type* $\underline{r}$. However, as shown e.g. in [4], these flows are simply the pullback on $\mathrm{Gr}(r)$ of multiple lower-dimensional mcKP flows.

Notice that for every $\eta \in \mathrm{LGL}(r,\mathbb{C})$ one has $\tilde{\psi}_{W\eta}(g,z) = \tilde{\psi}_W(g\eta^{-1},z)$, so that

$$\psi_{W\eta}(g,z) = \psi_W(g\eta^{-1},z)\cdot\eta \tag{12}$$

We now recall that expressions such as (11) are in one-to-one correspondence with zeroth-order pseudo-differential operators of the form

$$K_W = I_r + \sum_{i\geq 1} W_i(\bar{\boldsymbol{t}})D^{-i} \tag{13}$$

where we consider g as a function of the coefficients $\bar{\boldsymbol{t}}$ defined by (8). From K_W we can define a first order pseudo-differential operator L_W via the following prescription ("dressing"):

$$L_W = K_W D K_W^{-1} \tag{14}$$

The following result was proved in [3]:

Theorem 1. *For any point $W \in \mathrm{Gr}(r)$ the operator L_W defined by* (14) *satisfies the multicomponent KP equation.*

We remark that the correspondence (14) is not one-to-one: if $K' = KC$ where $C = I_r + \sum_{j\geq 1} C_j D^{-j}$ for some family $(C_j)_{j\geq 1}$ of constant diagonal matrices then $L = L'$. At the level of $\mathrm{Gr}(r)$ this "gauge freedom" is expressed by the action of the group $\Gamma_-(r)$ consisting of diagonal $r\times r$ matrices of the form $\mathrm{diag}(h_1,\ldots,h_r)$, where each h_α belongs to the group (denoted Γ_- in [10]) of analytic functions $h\colon S^1 \to \mathbb{C}^*$ that extend to holomorphic functions on the disc $\{\, z \in \mathbb{C}P^1 \mid |z| \geq R \,\}$ and such that $h(\infty) = 1$.

Another way to describe the Baker function associated to a point of $\mathrm{Gr}(r)$ relies on the so-called *tau function*. Here we do not need to enter into the details of its definition (see again [3]); the key fact is that to each subspace $W \in \mathrm{Gr}(r)$ we can associate certain holomorphic functions on $\Gamma_+(r)$, denoted τ_W and $\tau_{W\alpha\beta}$ for each pair of indices $\alpha \neq \beta$, determined up to constant factors, such that the following equality ("Sato's formula") holds:

$$\tilde{\psi}_W(g,z)_{\alpha\beta} = \begin{cases} \dfrac{\tau_W(gq_{z\alpha})}{\tau_W(g)} & \text{if } \alpha = \beta \\[2ex] z^{-1}\dfrac{\tau_{W\alpha\beta}(gq_{z\beta})}{\tau_W(g)} & \text{if } \alpha \neq \beta \end{cases} \tag{15}$$

where for each $z \in \mathbb{C}$ and $\alpha = 1,\ldots,r$ we define $q_{z\alpha}$ to be the element of $\Gamma_+(r)$ that has $q_z(\zeta) := 1 - \frac{\zeta}{z}$ at the (α,α) entry and 1 elsewhere on the diagonal.

3 The multicomponent rational Grassmannian

3.1 Definition

For the sake of brevity we will denote by $\mathcal{R}$ the space of rational functions on the complex projective line $\mathbb{C}P^1$, by $\mathcal{P}$ the subspace of polynomials and by $\mathcal{R}_-$ the subspace of proper (i.e., vanishing at infinity) rational functions. We consider the space $\mathcal{R}^r$ of r-tuples of rational functions with the direct sum decomposition

$$\mathcal{R}^r = \mathcal{P}^r \oplus \mathcal{R}^r_- \tag{16}$$

and associated canonical projection maps $\pi_+ : \mathcal{R}^r \to \mathcal{P}^r$ and $\pi_- : \mathcal{R}^r \to \mathcal{R}^r_-$.

We define the Grassmannian $\mathbb{G}\mathrm{r}^{\mathrm{rat}}(r)$ as the set of closed linear subspaces $W \subseteq \mathcal{R}^r$ for which there exist polynomials $p, q \in \mathcal{P}$ such that

$$p\mathcal{P}^r \subseteq W \subseteq q^{-1}\mathcal{P}^r \tag{17}$$

The *virtual dimension* of W, denoted $\operatorname{vdim} W$, is the index of the (Fredholm) operator $p_+ := \pi_+|_W$; one has

$$\operatorname{vdim} W = \dim W' - r \deg p \tag{18}$$

where $W' := W/p\mathcal{P}^r$ is finite-dimensional. We denote by $\mathrm{Gr}^{\mathrm{rat}}(r)$ the subset of $\mathbb{G}\mathrm{r}^{\mathrm{rat}}(r)$ consisting of subspaces of virtual dimension zero.

This space can be embedded in the r-component Segal-Wilson Grassmannian $\mathrm{Gr}(r)$ by the following procedure: given $W \in \mathrm{Gr}^{\mathrm{rat}}(r)$, we choose the radius $R \in \mathbb{R}^+$ involved in the definition of $\mathrm{Gr}(r)$ such that every root of the polynomial q appearing in (17) is contained in the open disc $|z| < R$; then the restrictions $f|_{\gamma_R}$ for all $f \in W$ determine a linear subspace whose L^2-closure belongs to $\mathrm{Gr}(r)$. This embedding automatically defines a topology on $\mathrm{Gr}^{\mathrm{rat}}(r)$ and its subspaces by restriction.

Lemma 1. *A subspace $W \in \mathbb{G}\mathrm{r}^{\mathrm{rat}}(r)$ has virtual dimension zero if and only if the codimension of the inclusion $W \subseteq q^{-1}\mathcal{P}^r$ coincides with $r \deg q$.*

Proof. Condition (17) may be rewritten as $qp\mathcal{P}^r \subseteq qW \subseteq \mathcal{P}^r$, so that the codimension of W in $q^{-1}\mathcal{P}^r$ is the same as the codimension of qW in $\mathcal{P}^r$, namely $\dim \mathcal{P}^r/qW$. Taking the quotient of both those spaces by the common subspace $qp\mathcal{P}^r$ we get an isomorphic linear space which is the quotient of two finite-dimensional spaces:

$$\frac{\mathcal{P}^r}{qW} \cong \frac{\mathcal{P}^r/qp\mathcal{P}^r}{qW/qp\mathcal{P}^r}$$

Moreover, $qW/qp\mathcal{P}^r \cong W/p\mathcal{P}^r = W'$, so that

$$\operatorname{codim}_{q^{-1}\mathcal{P}^r} W = r(\deg q + \deg p) - \dim W'$$

By using (18) we finally get

$$\operatorname{codim}_{q^{-1}\mathcal{P}^r} W = r \deg q - \operatorname{vdim} W \tag{19}$$

from which the lemma follows. □

Let's introduce a more algebraic description for $\mathbb{G}\mathrm{r}^{\mathrm{rat}}(r)$ in the same vein as [12]. For every $k = 1, \ldots, r$, $s \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ we define the linear functional $\mathrm{ev}_{k,s,\lambda}$ on $\mathcal{P}^r$ by

$$\langle \mathrm{ev}_{k,s,\lambda}, (p_1, \ldots, p_r) \rangle = p_k^{(s)}(\lambda)$$

These functionals are easily seen to be linearly independent; we denote by $\mathscr{C}^{(r)}$ the linear space they generate, and think of it as a space of "differential conditions" we can impose on r-tuples of polynomials. We also set

$$\mathscr{C}^{(r)}_\lambda := \operatorname{span}\{\mathrm{ev}_{k,s,\lambda}\}_{1 \le k \le r,\, s \in \mathbb{N}}$$

and

$$\mathscr{C}^{(r)}_{t,\lambda} := \operatorname{span}\{\operatorname{ev}_{k,s,\lambda}\}_{1\le k\le r,\, 0\le s<t}$$

with the convention that $\mathscr{C}^{(r)}_{0,\lambda} = \{0\}$.

Given $c \in \mathscr{C}^{(r)}$, the finite set of points $\lambda \in \mathbb{C}$ such that the projection of c on $\mathscr{C}^{(r)}_{\lambda}$ is nonzero will be called the *support* of c. For every linear subspace $C \subseteq \mathscr{C}^{(r)}$, its annihilator

$$V_C := \{\, (p_1,\dots,p_r) \in \mathcal{P}^r \mid \langle c, (p_1,\dots,p_r)\rangle = 0 \text{ for all } c \in C \,\}$$

is a linear subspace in $\mathcal{P}^r$.

Lemma 2. *A subspace $W \subseteq \mathcal{R}^r$ belongs to $\mathbb{G}\mathrm{r}^{\mathrm{rat}}(r)$ if and only if there exist a finite-dimensional subspace $C \subseteq \mathscr{C}^{(r)}$ and a polynomial q such that $W = q^{-1}V_C$; moreover $W \in \mathrm{Gr}^{\mathrm{rat}}(r)$ if and only if $r \deg q = \dim C$.*

Proof. Let's suppose that $W = q^{-1}V_C$ for some finite-dimensional subspace C in $\mathscr{C}^{(r)}$ and let $\{\lambda_1,\dots,\lambda_m\}$ be the support of C. For every $i = 1,\dots,m$ let t_i be the maximum value of s for the functionals $\operatorname{ev}_{k,s,\lambda_i}$ involved in the elements of C. By letting $p := \prod_{i=1}^m (z-\lambda_i)^{t_i+1}$ it follows that $p\mathcal{P}^r \subseteq V_C$, hence a fortiori $pq\mathcal{P}^r \subseteq V_C \subseteq \mathcal{P}^r$ and dividing by q we see that $W \in \mathbb{G}\mathrm{r}^{\mathrm{rat}}(r)$. Now, if $r\deg q = \dim C$ then $\operatorname{codim}_{\mathcal{P}^r} V_C = \dim C = r \deg q$ and lemma 1 implies that $W \in \mathrm{Gr}^{\mathrm{rat}}(r)$.

For the converse, let $W \in \mathbb{G}\mathrm{r}^{\mathrm{rat}}(r)$. Then there exist $p, q \in \mathcal{P}$ such that $qp\mathcal{P}^r \subseteq qW \subseteq \mathcal{P}^r$; this means in particular that the linear space qW is obtained by imposing a certain (finite) number of linearly independent conditions in the dual space of $\mathcal{P}^r/qp\mathcal{P}^r$. The latter can be identified with $\mathbb{C}^r \otimes U$, where U is the linear space of polynomials with degree less than $\deg p + \deg q$, so that qW is determined by a finite-dimensional subspace in $(\mathbb{C}^r \otimes U)^*$. On the other hand, this space is generated by the elements of $\mathscr{C}^{(r)}$ (e.g. using the functionals $\frac{1}{s!}\operatorname{ev}_{k,s,0}$ that extract the s-th coefficient of the k-th polynomial). Thus, there exists a linear subspace $C \subseteq \mathscr{C}^{(r)}$ of finite dimension such that $V_C = qW$. If moreover $\operatorname{vdim} W = 0$, then by (18) the linear space $W' \cong qW/qp\mathcal{P}$ has dimension $r\deg p$, so that it must be defined by $r \deg p$ linearly independent conditions. It follows that the subspace C has dimension $r \deg q$. $\square$

In the sequel, the subspace $q^{-1}V_C$ singled out by this lemma will be denoted simply by $(C,q)^*$.

Lemma 3. *Two subspaces $W_1 = (C,q_1)^*$ and $W_2 = (C,q_2)^*$ determined by the same conditions space C in $\mathrm{Gr}^{\mathrm{rat}}(r)$ lie in the same $\Gamma_-(r)$-orbit.*

Actually, the matrix $\eta := \frac{q_1}{q_2} I_r$ belongs to $\Gamma_-(r)$ (notice that q_1 and q_2 are both of degree $r \dim C$) and is such that $W_1\eta = W_2$.

We say that a finite-dimensional subspace $C \subseteq \mathscr{C}^{(r)}$ is *homogeneous* if it admits a basis consisting of "1-point conditions", i.e. differential conditions each one involving a single point:

$$C = \bigoplus_{\lambda\in\mathbb{C}} C_\lambda \quad \text{where} \quad C_\lambda := C \cap \mathscr{C}^{(r)}_\lambda \tag{20}$$

We denote by $\mathrm{Gr}^{\mathrm{hom}}(r)$ the set of subspaces $(C,q)^* \in \mathrm{Gr}^{\mathrm{rat}}(r)$ such that C is homogeneous, $\dim C_\lambda = rn_\lambda$ for some natural numbers $n_\lambda \in \mathbb{N}$ and $q = q_C$, where

$$q_C := \prod_{\lambda\in\mathbb{C}} (z-\lambda)^{n_\lambda} \tag{21}$$

3.2 The Baker and tau functions

We would now like to determine the Baker function of a point $(C,q)^* \in \mathrm{Gr}^{\mathrm{rat}}(r)$. By Lemma 3 it is enough to consider the case $q = z^d$. Thus, we suppose that $\dim C = rd$ and take the subspace $W = (C, z^d)^*$; we claim that ψ_W must have the form

$$\psi_W(g,z) = \left(I_r + \sum_{j=1}^{d} W_j(g) z^{-j} \right) g(z) \tag{22}$$

Indeed, by standard arguments (cfr. [12, Sect. 4]), on the one hand we have that each row of the matrix-valued function $z^d \psi_W(g,z)$ (for every fixed g) belongs to the L^2-closure of V_C in $H_+^{(r)}$ (for some value of the radius R), and on the other hand that each functional $\mathrm{ev}_{k,s,\lambda}$ extends uniquely to a continuous functional on $H_+^{(r)}$; hence $(z^d \psi_W)_\alpha \in V_C$ for every α. Now,

$$(z^d \psi_W)_{\alpha\beta} = \left(z^d \delta_{\alpha\beta} + \sum_{j \geq 1} W_{j\alpha\beta}(g) z^{d-j} \right) g_\beta(z)$$

and if we require every matrix element to be a polynomial, we see that every W_j with $j > d$ must be the zero matrix.

To determine the matrices $W_1, \dots, W_d$, let $(c_1, \dots, c_{rd})$ be a basis for C; for each α we take the α-th row of $z^d \psi_W$ and impose the equalities $\langle c_i, (z^d \psi_W)_\alpha \rangle = 0$ (with $i = 1, \dots, rd$). This yields the following linear system of equations:

$$\langle c_i, \left(z^d \delta_{\alpha\beta} + \sum_{j=1}^{d} W_{j\alpha\beta}(g) z^{d-j} \right) g_\beta(z) \rangle = 0 \tag{23}$$

In other words, we have a family of r linear systems, each of which involves rd equations, for a total of $r^2 d$ scalar equations. The unknowns are of course the r^2 entries of the d matrices $\{W_1, \dots, W_d\}$; the coefficients of these unknowns involve, as in the scalar case, the g_β's and their derivatives evaluated at the points in the support of C.

The tau functions associated to $W = (C, z^d)^* \in \mathrm{Gr}^{\mathrm{rat}}(r)$ are readily obtained by imitating the calculations in [3]; they are built out of the following family of $(rd+1) \times (rd+1)$ matrices (indexed by $\alpha, \beta = 1, \dots, r$):

$$M_{\alpha\beta} := \begin{pmatrix} \langle c_1, g_1 \rangle & \dots & \langle c_{rd}, g_1 \rangle & 0 \\ \vdots & & \vdots & z^{-d} \\ \langle c_1, g_r \rangle & \dots & \langle c_{rd}, g_r \rangle & 0 \\ \langle c_1, z g_1 \rangle & \dots & \langle c_{rd}, z g_1 \rangle & 0 \\ \vdots & & \vdots & z^{1-d} \\ \langle c_1, z g_r \rangle & \dots & \langle c_{rd}, z g_r \rangle & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \langle c_1, z^{d-1} g_1 \rangle & \dots & \langle c_{rd}, z^{d-1} g_1 \rangle & 0 \\ \vdots & & \vdots & z^{-1} \\ \langle c_1, z^{d-1} g_r \rangle & \dots & \langle c_{rd}, z^{d-1} g_r \rangle & 0 \\ \langle c_1, z^d g_\alpha \rangle & \dots & \langle c_{rd}, z^d g_\alpha \rangle & \delta_{\alpha\beta} \end{pmatrix} \tag{24}$$

where in the last column the only nonzero element is on the β-th row of each block. Notice that in an expression like $\langle c_i, g_\gamma\rangle$, g_γ must be interpreted as the row vector having g_γ in its γ-th entry and zero elsewhere.

In terms of these matrices, the "diagonal" tau function τ_W is simply the cofactor of the element $\delta_{\alpha\beta} = 1$ in the lower right corner, or equivalently the determinant of the $rd \times rd$ minor obtained by deleting the last column and the last row:

$$\tau_W(g) = \det(\langle c_i, z^{j-1} g_\gamma\rangle)_{\substack{i=1\dots rd \\ j=1\dots d,\, \gamma=1\dots r}} \tag{25}$$

This is just the matrix of coefficients of the linear system (23), so that the system has a solution exactly when $g \in \Gamma_+(r)^W$, as expected. The "off-diagonal" tau function $\tau_{W\alpha\beta}$ (with $\alpha \neq \beta$) is the cofactor of the element z^{-1} in the last column:

$$\tau_{W\alpha\beta}(g) = (-1)^{r-\beta} \det(\langle c_i, z^{j-1}(g_\gamma + \delta_{jd}\delta_{\gamma\beta}(z g_\alpha - g_\gamma))\rangle)_{\substack{i=1\dots rd \\ j=1\dots d,\, \gamma=1\dots r}} \tag{26}$$

Observe that an expression such as $z^{j-1} g_\gamma$ may equivalently be read as $\partial_{1\gamma}^{j-1} g_\gamma$; this will be useful in what follows.

4 Rational solutions

We can now prove a rationality result for the solutions of the mcKP hierarchy coming from subspaces in $\mathrm{Gr}^{\mathrm{hom}}(r)$ defined by a set of condition whose support is a single point.

Theorem 2. *Let $W \in \mathrm{Gr}^{\mathrm{hom}}(r)$ and suppose that $W = (C, q_C)^*$ with C supported on a single point $\lambda \in \mathbb{C}$. Then:*

1. *Each diagonal entry of the reduced Baker function $\tilde{\psi}_W$ is a rational function of the times $t_1^{(1)}, \dots, t_1^{(r)}$ that tends to 1 as $t_1^{(\alpha)} \to \infty$ for any α;*
2. *The tau function τ_W is a polynomial in $t_1^{(1)}, \dots, t_1^{(r)}$ with constant leading coefficient.*

Proof. The two statements are equivalent by virtue of the diagonal part of Sato's formula (15), so it suffices to prove one of them; we choose the first. By hypothesis we have $W = (C, (z-\lambda)^d)^*$ with $C \subseteq \mathscr{C}_\lambda^{(r)}$ of dimension rd; let $(c_1, \dots, c_{rd})$ be a basis for it. Consider the subspace $U := (C, z^d)^* \in \mathrm{Gr}^{\mathrm{rat}}(r)$; its tau function is given by (25). To compute the diagonal elements of the corresponding Baker function we use Sato's formula for U:

$$\tilde{\psi}_{U\alpha\alpha} = \frac{\tau_U(g q_{\zeta\alpha})}{\tau_U(g)} \tag{27}$$

(here we consider ζ as a parameter and z as a variable). Since we are only interested in the times with subscript 1 we will work in the stationary setting, i.e. we put $t_k^{(\alpha)} = 0$ for every $k \geq 2$, $\alpha = 1 \dots m$. Each condition c_i is supported at λ, hence we can define a family of polynomials $\{\phi_{i\gamma}\}_{i=1\dots rd,\, \gamma=1\dots r}$ (with $\phi_{i\gamma}$ only depending on $t_1^{(\gamma)}$) by the equation

$$\langle c_i, g_\gamma\rangle = g_\gamma(\lambda)\phi_{i\gamma} \tag{28}$$

To apply (27) we need to know the determinant of the matrices $\partial_{1\gamma}^{j-1}\langle c_i, g_\gamma\rangle$ and $\partial_{1\gamma}^{j-1}\langle c_i, g_\gamma(1-\delta_{\alpha\gamma}\frac{z}{\zeta})\rangle$. As for the first, by using (28) its generic element can be written in the form

$$\partial_{1\gamma}^{j-1}\langle c_i, g_\gamma\rangle = \partial_{1\gamma}^{j-1}(g_\gamma(\lambda)\phi_{i\gamma}) = g_\gamma(\lambda)(\partial_{1\gamma}+\lambda)^{j-1}\phi_{i\gamma} \tag{29}$$

For the second matrix we have

$$\partial_{1\gamma}^{j-1}\langle c_i, g_\gamma(1-\delta_{\alpha\gamma}\frac{z}{\zeta})\rangle = \partial_{1\gamma}^{j-1}\langle c_i, g_\gamma\rangle - \partial_{1\gamma}^{j-1}\langle c_i, \delta_{\alpha\gamma} g_\gamma \frac{z}{\zeta}\rangle \tag{30}$$

The first term is exactly (29), whereas the second is

$$\delta_{\alpha\gamma}\partial_{1\gamma}^{j}\langle c_i, g_\gamma\rangle\zeta^{-1} = \delta_{\alpha\gamma}\partial_{1\gamma}^{j}(g_\gamma(\lambda)\phi_{i\gamma})\zeta^{-1} = \delta_{\alpha\gamma} g_\gamma(\lambda)(\partial_{1\gamma}+\lambda)^{j}\phi_{i\gamma}\zeta^{-1}$$

By putting all together, equation (30) becomes

$$g_\gamma(\lambda)(\partial_{1\gamma}+\lambda)^{j-1}\big(\phi_{i\gamma} - \delta_{\alpha\gamma}(\partial_{1\gamma}\phi_{i\gamma}+\lambda\phi_{i\gamma})\zeta^{-1}\big)$$

that we can rewrite as

$$g_\gamma(\lambda)(1-\delta_{\alpha\gamma}\frac{\lambda}{\zeta})(\partial_{1\gamma}+\lambda)^{j-1}\big(\phi_{i\gamma} - \delta_{\alpha\gamma}\frac{1}{\zeta-\lambda}\partial_{1\gamma}\phi_{i\gamma}\big) \tag{31}$$

But $(1-\delta_{\alpha\gamma}\frac{\lambda}{\zeta}) = q_{\zeta\alpha}(\lambda)_\gamma$, so plugging (29) and (31) into (27) we obtain

$$\tilde\psi_{U\alpha\alpha}(g,\zeta) = (q_\zeta(\lambda))^d \frac{\det\Big((\partial_{1\gamma}+\lambda)^{j-1}(\phi_{i\gamma} - \delta_{\alpha\gamma}\frac{1}{\zeta-\lambda}\partial_{1\gamma}\phi_{i\gamma})\Big)}{\det\big((\partial_{1\gamma}+\lambda)^{j-1}\phi_{i\gamma}\big)} \tag{32}$$

The factor $(q_\zeta(\lambda))^d$ disappears when we go back from ψ_U to ψ_W; we are left with the ratio of two determinants of matrices with polynomial entries in the times $t_1^{(\gamma)}$, which is clearly a rational function. Moreover, if we expand the numerator of (32) by linearity over the sum, we see that the term obtained by always choosing $\phi_{i\gamma}$ exactly reproduces the polynomial at the denominator, and all the other terms involve a polynomial which has degree strictly lower than τ in some $t_1^{(\gamma)}$ (since we replace $\phi_{i\gamma}$ with one of its derivatives); this proves that $\tilde\psi_{W\alpha\alpha}\to 1$ as all the $t_1^{(\alpha)}$ tend to infinity. □

We now consider the evolution on $\mathrm{Gr}(r)$ described by the flows of the matrix KP hierarchy. Recall from the Introduction that these are given by the vector fields $\partial/\partial t_k$, where $t_k = \sum_\gamma t_k^{(\gamma)}$. Equivalently, we can take the matrix $g\in\Gamma_+(r)$ to be of the form

$$g = \mathrm{diag}(\mathrm{e}^{\xi(r^{-1}\boldsymbol{t},z)},\ldots,\mathrm{e}^{\xi(r^{-1}\boldsymbol{t},z)}) \tag{33}$$

where $\boldsymbol{t} = (t_k)_{k\geq 1}$ (and $t_1 = x$). Let's define $\tilde g := \mathrm{e}^{\xi(r^{-1}\boldsymbol{t},z)}$ and $h := \mathrm{e}^{\xi(\boldsymbol{t},z)}$ (so that $g = \tilde g I_r$ and $\tilde g^r = h$); then the Baker and tau functions for the matrix KP hierarchy are naturally expressed in terms of h only, since that single function completely controls the flows of the hierarchy.

Theorem 3. *Let $W\in\mathrm{Gr}^{\mathrm{hom}}(r)$, then:*

1. *$\tilde\psi_W(h,z)$ is a matrix-valued rational function of x that tends to I_r as $x\to\infty$;*

2. $\tau_W(h)$ and $\tau_{W\alpha\beta}(h)$ are polynomial functions of x with constant leading coefficients.

Proof. Let $W = (C, q_C)^*$ with C homogeneous and let $(\lambda_1, \dots, \lambda_d)$ be its support with each point counted according to its multiplicity (so that the λ_i are not necessarily distinct); finally for each $i = 1 \dots d$ let $(c_{ij})_{j=1\dots r}$ be a set of r linearly independent conditions at λ_i. In this way we get a basis $(c_{11}, \dots, c_{dr})$ for C made of 1-point conditions. Now consider the subspace $U := (C, z^d)^* \in \mathrm{Gr}^{\mathrm{rat}}(r)$; it is related to W by the following element of $\Gamma_-(r)$:

$$\eta = \prod_{i=1}^{d} q_z(\lambda_i)^{-1} I_r = \exp\left(\sum_{k\geq 0}\sum_{i=1}^{d} \frac{\lambda_i^k}{k} z^{-k}\right) I_r \tag{34}$$

This corresponds to multiplying the tau function by

$$\hat{\eta} = \prod_{\alpha=1}^{r} \exp\left(-\sum_{k\geq 0}\sum_{i=1}^{d} \lambda_i^k t_k^{(\alpha)}\right) = \prod_{\alpha=1}^{r}\prod_{i=1}^{d} g_\alpha(\lambda_i)^{-1} = \prod_{i=1}^{d} h(\lambda_i)^{-1} \tag{35}$$

since $g_\alpha = \tilde{g}$ for every α and $\tilde{g}^r = h$.

Let's define the family of polynomials $\{\phi_{ij\gamma}\}$ (for $i = 1 \dots d$, $j = 1 \dots r$, $\gamma = 1 \dots r$) by the equation

$$\langle c_{ij}, g_\gamma \rangle = \tilde{g}(\lambda_i)\phi_{ij\gamma} \tag{36}$$

Again, this works precisely because each c_{ij} is a 1-point condition; notice that, although $g_\gamma = \tilde{g}$ for every γ, the polynomials ϕ still depend on γ since it is the index of the only nonzero entry of the row vector on which c_{ij} acts.

We can now easily compute the tau functions associated to U by retracing the same steps as in the proof of theorem 2, but now using the polynomials defined by (36). Then $\tau_U(g)$ is the determinant of the matrix whose generic element is

$$\tilde{g}(\lambda_i)(\partial_{1\gamma} + \lambda_i)^{k-1}\phi_{ij\gamma}$$

The term $\tilde{g}(\lambda_i)$ does not depend on the row indices (k, γ) so that we can factor it out from the determinant and get

$$\tau_U = \prod_{i=1}^{d} (\tilde{g}(\lambda_i))^r \det\left((\partial_{1\gamma} + \lambda_i)^{k-1}\phi_{ij\gamma}\right) \tag{37}$$

But $\prod_i (\tilde{g}(\lambda_i))^r = \prod_i h(\lambda_i)$ is exactly the inverse of (35), so that

$$\tau_W = \det\left((\partial_{1\gamma} + \lambda_i)^{k-1}\phi_{ij\gamma}\right) \tag{38}$$

is the determinant of a matrix with polynomial entries in $t_1^{(\alpha)} = \frac{x}{r}$, hence a polynomial in x and the coefficient of the top degree term involves only the constants λ_i. By Sato's formula, this implies that $\tilde{\psi}_{W\alpha\alpha} \to 1$ as $x \to \infty$.

Now take $\alpha, \beta \in \{1, \dots, m\}$, $\alpha \neq \beta$ and consider the off-diagonal tau function $\tau_{U\alpha\beta}(g)$; it is given by the determinant of a matrix $M_{\alpha\beta}(g)$ which coincides with the one involved in the definition of $\tau_U(g)$ except for the row corresponding to $k = d-1$, $\gamma = \beta$ which is replaced by the row $\langle c_{ij}, \partial_{1\alpha}^d g_\alpha \rangle$. But since $g_\alpha = g_\beta = \tilde{g}$ we can again collect out of the determinant the same factor as before, so that $\tau_{W\alpha\beta}(g) = \det \Phi_{ij,k\gamma}$ with

$$\Phi_{ij,k\gamma} := \begin{cases} (\partial_{1\gamma} + \lambda_i)^{k-1}\phi_{ij\gamma} & \text{if } k \neq d \text{ or } \gamma \neq \beta \\ (\partial_{1\alpha} + \lambda_i)^{d}\phi_{ij\alpha} & \text{if } k = d \text{ and } \gamma = \beta \end{cases} \tag{39}$$

This is also a polynomial in x; moreover we can write

$$(\partial_{1\alpha}+\lambda_i)^d\phi_{ij\alpha}=(\partial_{1\alpha}+\lambda_i)^{d-1}(\lambda_i\phi_{ij\alpha}+\partial_{1\alpha}\phi_{ij\alpha}) \tag{40}$$

But now $M_{\alpha\beta}$ has also a row (for $k=d-1$, $\gamma=\alpha$) whose generic element reads $(\lambda_i+\partial_{1\alpha})^{d-1}\phi_{ij\alpha}$, and we can subtract this row multiplied by λ_1 (say) to the row (40) without altering the determinant, so that

$$\Phi_{ij,k\gamma}:=\begin{cases}(\partial_{1\gamma}+\lambda_i)^{k-1}\phi_{ij\gamma} & \text{if } k\neq d \text{ or } \gamma\neq\beta\\ (\partial_{1\alpha}+\lambda_i)^{d-1}((\lambda_i-\lambda_1)\phi_{ij\alpha}+\partial_{1\alpha}\phi_{ij\alpha}) & \text{if } k=d \text{ and } \gamma=\beta\end{cases}$$

This means that $\tau_{W\alpha\beta}$ is the determinant of a matrix whose generic entry is equal or of degree strictly lower than the corresponding one on τ_W; it follows that the degree of $\tau_{W\alpha\beta}$ is strictly lower than τ_W, and this (again by Sato's formula) implies that the off-diagonal components of $\tilde{\psi}_W$ tend to zero as $x\to\infty$. □

To see how some of the new solutions look like, take for example the homogeneous subspace with 1-point support $W=(C,\frac{1}{z-\lambda})^*$ with C generated by the r conditions

$$c_k=\mathrm{ev}_{k,1,\lambda}+a_{k1}\,\mathrm{ev}_{1,0,\lambda}+\dots+a_{kr}\,\mathrm{ev}_{r,0,\lambda}\qquad(k=1\dots r) \tag{41}$$

determined by the $r\times r$ matrix $A=(a_{ij})$. These are, in a sense, the most general conditions involving only the functionals $\mathrm{ev}_{k,s,\lambda}$ with $s\leq 1$. For the sake of brevity, let's set $\boldsymbol{t}_\lambda:=r^{-1}(x+2t_2\lambda+3t_3\lambda^2+\dots)$; then

$$\tau_W(\boldsymbol{t})=\det(\boldsymbol{t}_\lambda I_r+A^t)\qquad \tau_{W\alpha\beta}(\boldsymbol{t})=-\operatorname{cof}_{\beta,\alpha}(\boldsymbol{t}_\lambda I_r+A^t)$$

$$\tilde{\psi}(\boldsymbol{t},z)=I_r-(\boldsymbol{t}_\lambda I_r+A^t)^{-1}\frac{1}{z-\lambda}$$

where $\operatorname{cof}_{\beta,\alpha}$ stands for the (β,α)-cofactor of a matrix and A^t for the transpose of A.

More generally, we could take the element of $\mathrm{Gr}^{\mathrm{hom}}(r)$ determined by the support $\{\lambda_1,\dots,\lambda_n\}$ and, for each $i=1\dots n$, a matrix A_i that specifies a set of conditions of the form (41) to be imposed at the point λ_i. The tau function of such a subspace is the determinant of the following block matrix:

$$\begin{pmatrix} Y_1 & \dots & Y_n\\ \lambda_1Y_1+I_r & \dots & \lambda_nY_n+I_r\\ \vdots & \ddots & \vdots\\ \lambda_1^{n-1}Y_1+(n-1)\lambda_1^{n-2}I_r & \dots & \lambda_n^{n-1}Y_n+(n-1)\lambda_n^{n-2}I_r\end{pmatrix}$$

where $Y_i:=\boldsymbol{t}_{\lambda_i}I_r+{A_i}^t$. The off-diagonal tau functions $\tau_{W\alpha\beta}$ are obtained in the usual way (i.e., replacing the β-th line of the bottom blocks with the α-th line of the blocks $\lambda_i^nY_i+n\lambda_i^{n-1}I_r$), and the matrix Baker function is then given by Sato's formula. When $r=1$, these subspaces are exactly the ones described in [13, Sect. 3].

5 Relationship with the multicomponent KP/CM correspondence

Recall from [13] that a crucial ingredient of the scalar KP/CM correspondence is the bijective map $\beta\colon\mathcal{C}\to\mathrm{Gr}^{\mathrm{ad}}$ between the phase space of the Calogero-Moser system $\mathcal{C}$ and the adelic Grassmannian

$\mathrm{Gr}^{\mathrm{ad}}$. In the multicomponent case, an analogous mapping should be defined between the phase space of the Gibbons-Hermsen system [5] (also known as "spin Calogero-Moser") and some space of solutions for the matrix KP hierarchy, as suggested by a well-known calculation [7].

In Wilson's unpublished notes [14], a map that fulfils this rôle is conjectured. In order to describe (part of) the definition of this map let's recall (see e.g. [13, Sect. 8]) that the completed phase space of the n-particle, r-component Gibbons-Hermsen system is the (smooth, irreducible) affine algebraic variety defined by the following symplectic reduction:

$$\mathcal{C}_{n,r} = \{\, (X,Y,v,w) \mid [X,Y] + vw = -I \,\} / \mathrm{GL}(n,\mathbb{C})$$

where $(X,Y,v,w) \in \mathrm{End}(\mathbb{C}^n) \oplus \mathrm{End}(\mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r) \oplus \mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^n)$ and $\mathrm{GL}(n,\mathbb{C})$ acts as a change of basis in $\mathbb{C}^n$. Now denote by $\mathcal{C}''_{n,r}$ the subspace of $\mathcal{C}_{n,r}$ consisting of equivalence classes of quadruples for which Y is diagonalisable with distinct eigenvalues; from each of these classes we can select a representative such that $Y = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$, X is a Moser matrix associated to Y (i.e. $X_{ij} = 1/(\lambda_i - \lambda_j)$ and $X_{ii} = \alpha_i$ is any complex number) and each pair (v_i, w_i) (where we denote by v_i the i-th row of v and by w_i the i-th column of w) belongs to the algebraic variety $\{\, (\xi,\eta) \in \mathbb{C}^r \times \mathbb{C}^r \mid \xi\cdot\eta = -1 \,\}/\mathbb{C}^*$, where $\lambda \in \mathbb{C}^*$ acts as

$$\lambda.(\xi,\eta) = (\lambda\xi, \lambda^{-1}\eta) \tag{42}$$

Notice in particular that none of the v_i's and w_i's can be the zero vector, by virtue of the normalisation condition $v_i w_i = -1$.

According to [14], a point $[X,Y,v,w] \in \mathcal{C}''_{n,r}$ corresponds to the subspace $W \in \mathrm{Gr}^{\mathrm{rat}}(r)$ defined by the following prescriptions:

1. functions in W are regular except for (at most) a simple pole in each λ_i and a pole of any order at infinity;
2. if $f = \sum_{k\geq -1} f^{(i)}_k (z-\lambda_i)^k$ is the Laurent expansion of $f \in W$ in λ_i then:
   (a) $f^{(i)}_{-1}$ is a scalar multiple of v_i, and
   (b) $(f^{(i)}_0 + \alpha_i f^{(i)}_{-1}) \cdot w_i = 0$.

Our purpose is now to show how these prescriptions translate in the language of the previous sections.

So suppose that $W \in \mathrm{Gr}^{\mathrm{rat}}(r)$ satisfies conditions (1–2); we must find a finite-dimensional, homogeneous space of conditions C such that $W = (C, q_C)^*$ where q_C is given by (21). Condition 1 clearly implies that C has support $\{\lambda_1,\ldots,\lambda_n\}$ and $q_C = \prod_{i=1}^n (z-\lambda_i)$. Thus, we only need to find, for each $i = 1\ldots n$, a subspace $C_i \subseteq \mathscr{C}^{(r)}_{\lambda_i}$ of dimension r whose elements satisfy the conditions (2a–2b). Let's define the n polynomials

$$q_i := \frac{q_C}{z-\lambda_i} = \prod_{j\neq i} (z-\lambda_j)$$

for $i = 1\ldots n$; then by a direct computation we have that

$$\mathrm{ev}_{k,0,\lambda_i}(q_C f) = q_i(\lambda_i) f^{(i)}_{-1,k}$$

and similarly

$$\mathrm{ev}_{k,1,\lambda_i}(q_C f) = \sum_{\ell\neq i}\prod_{j\neq i,\ell}(\lambda_i-\lambda_j)f^{(i)}_{-1,k} + q_i(\lambda_i)f^{(i)}_{0,k}$$

whence (putting $\delta_i := \sum_{\ell\neq i}(\lambda_i-\lambda_\ell)^{-1}$)

$$q_i(\lambda_i)f^{(i)}_{0,k} = (\mathrm{ev}_{k,1,\lambda_i} - \delta_i\,\mathrm{ev}_{k,0,\lambda_i})(q_C f)$$

Using these expressions, (2a) reads

$$\mathrm{ev}_{k,0,\lambda_i}(q_C f) = q_i(\lambda_i)a_i v_{ik} \quad \text{for all } k = 1\dots r$$

for some $a_i \in \mathbb{C}$, but only $r-1$ of these conditions are independent because of the $\mathbb{C}^*$-action (42) on v_i. As we already noticed, at least one entry of v_i is nonzero, say $v_{i1}\neq 0$. Then we can use the $\mathbb{C}^*$-action to normalise $v_{i1}=1$, so that $\mathrm{ev}_{1,0,\lambda_i}(q_C f) = q_i(\lambda_i)a_i$ and the remaining $r-1$ conditions can be written as

$$\mathrm{ev}_{k,0,\lambda_i}(q_C f) - v_{ik}\,\mathrm{ev}_{1,0,\lambda_i}(q_C f) = 0 \quad \text{for all } k = 2\dots r$$

The further condition (2b) directly translates as

$$\sum_{k=1}^{r} w_{ki}\left(\mathrm{ev}_{k,1,\lambda_i}(q_C f) + (\alpha_i-\delta_i)\,\mathrm{ev}_{k,0,\lambda_i}(q_C f)\right) = 0$$

where w_{1i} is fixed by the equation $v_i w_i = -1$. We conclude that, in the $(r-1)$-dimensional affine cell where $v_{i1}\neq 0$, we can take as C_i the subspace

$$C_i = \left\langle \mathrm{ev}_{2,0,\lambda_i} - v_{i2}\,\mathrm{ev}_{1,0,\lambda_i},\dots,\mathrm{ev}_{r,0,\lambda_i} - v_{ir}\,\mathrm{ev}_{1,0,\lambda_i}, \sum_{k=1}^{r} w_{ki}(\mathrm{ev}_{k,1,\lambda_i} + (\alpha_i-\delta_i)\,\mathrm{ev}_{k,0,\lambda_i})\right\rangle$$

Analogous descriptions are available in the other cells, where $v_{ik}=0$ for every $k<k^*$ and $v_{ik^*}\neq 0$. Repeating this argument for every $i = 1\dots n$, we obtain a homogeneous subspace $C = C_1\oplus\dots\oplus C_n$ in $\mathscr{C}^{(r)}$ such that $W = (C, q_C)^*$, as we wanted.

To see a concrete example in the simplest possible case ($n=r=2$), consider the space associated to a point

$$\left[\begin{pmatrix}\alpha_1 & \frac{v_1w_2}{\lambda_1-\lambda_2}\\ \frac{v_2w_1}{\lambda_2-\lambda_1} & \alpha_2\end{pmatrix}, \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}, \begin{pmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{pmatrix}, \begin{pmatrix}w_{11} & w_{12}\\ w_{21} & w_{22}\end{pmatrix}\right] \in \mathcal{C}''_{2,2}$$

Suppose further that $v_{11}\neq 0$ and $v_{21}\neq 0$; the corresponding subspace $W\in \mathrm{Gr}^{\mathrm{hom}}(2)$ is then given by $(z-\lambda_1)^{-1}(z-\lambda_2)^{-1}V_C$, where $C\subseteq \mathscr{C}^{(2)}$ is generated by the four conditions

$$\begin{aligned}
c_{11} &= \mathrm{ev}_{2,0,\lambda_1} - v_{12}\,\mathrm{ev}_{1,0,\lambda_1}\\
c_{12} &= w_{11}\,\mathrm{ev}_{1,1,\lambda_1} + w_{21}\,\mathrm{ev}_{2,1,\lambda_1} + w_{11}(\alpha_1+\delta)\,\mathrm{ev}_{1,0,\lambda_1} + w_{21}(\alpha_1+\delta)\,\mathrm{ev}_{2,0,\lambda_1}\\
c_{21} &= \mathrm{ev}_{2,0,\lambda_2} - v_{22}\,\mathrm{ev}_{1,0,\lambda_2}\\
c_{22} &= w_{12}\,\mathrm{ev}_{1,1,\lambda_2} + w_{22}\,\mathrm{ev}_{2,1,\lambda_2} + w_{12}(\alpha_2-\delta)\,\mathrm{ev}_{1,0,\lambda_2} + w_{22}(\alpha_2-\delta)\,\mathrm{ev}_{2,0,\lambda_2}
\end{aligned}$$

where $\delta := \delta_2 = -\delta_1 = (\lambda_2-\lambda_1)^{-1}$, $w_{11} = -1 - v_{12}w_{21}$ and $w_{12} = -1 - v_{22}w_{22}$. After some computation we find

$$\tau_W(\boldsymbol{t}) = (\boldsymbol{t}_{\lambda_1}+\alpha_1)(\boldsymbol{t}_{\lambda_2}+\alpha_2) + \delta^2(v_{22}w_{21}+w_{11})(v_{12}w_{22}+w_{12})$$

$$\tau_{W12}(\boldsymbol{t}) = -w_{12}v_{22}(\boldsymbol{t}_{\lambda_1}+\alpha_1) - w_{11}v_{12}(\boldsymbol{t}_{\lambda_2}+\alpha_2) + \delta\big(2v_{12}v_{22}(w_{12}w_{21}-w_{11}w_{22}) + v_{22}w_{12} - v_{12}w_{11}\big)$$

$$\tau_{W21}(\boldsymbol{t}) = -w_{22}(\boldsymbol{t}_{\lambda_1}+\alpha_1) - w_{21}(\boldsymbol{t}_{\lambda_2}+\alpha_2) + \delta\big(2w_{21}w_{22}(v_{22}-v_{12}) + w_{21} - w_{22}\big)$$

We remark that τ_W coincides with the determinant of the matrix X under the evolution described by the Gibbons-Hermsen flows $H_k = \operatorname{tr} Y^k$ (taking t_k as the time associated to H_k):

$$\tau_W(\boldsymbol{t}) = \det \begin{pmatrix} \alpha_1 + \boldsymbol{t}_{\lambda_1} & \frac{\tilde{g}(\lambda_1)}{\tilde{g}(\lambda_2)} \frac{v_1 w_2}{\lambda_1 - \lambda_2} \\ \frac{\tilde{g}(\lambda_2)}{\tilde{g}(\lambda_1)} \frac{v_2 w_1}{\lambda_2 - \lambda_1} & \alpha_2 + \boldsymbol{t}_{\lambda_2} \end{pmatrix}$$

This result fits nicely with the general formulae derived in [14].

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