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Development of instability of dark solitons generated by a flow of Bose-Einstein condensate past a concave corner

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The stability of dark solitons generated by a supersonic flow of a Bose-Einstein condensate past a concave corner (or a wedge) is studied. It is shown that solitons in the dispersive shock wave generated at the initial moment of time demonstrate a snake instability during their evolution to stationary curved solitons. Time of decay of soliton to vortices agrees very well with analytical estimates of the instability growth rate.

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Flow of quantum liquids has attracted much attention since discovery of superfluidity. At first it was discussed mainly in relation with the existence of the critical flow velocity past obstacles (or boundaries of capillary tubes) above which superfluidity is lost. According to Landau criterium it occurs at the velocity corresponding to the threshold of generation of linear excitations in the fluid (for instance, rotons in HeII). However, it was found that the experimental value of the critical velocity in HeII was much less than that predicted by Landau and this contradiction between the theory and the experiment was explained by Feynman who indicated that the threshold for generation of nonlinear excitations, namely vortices, is much less than the Landau value and agrees, at least semi-quantitatively, with the experimental value. Later these predictions were confirmed by analytical and numerical study of a superfluid flow modeled by the Gross-Pitaevskii equation for a weakly interacting Bose gas [1, 2]. As was found in these studies, a 2D flow past a disk-shaped obstacle with the radius not much greater than the healing length looses its superfluidity at velocity equal to $v_{cr} = \sqrt{2/11}c_s \cong 0.43c_s$, c_s being the sound velocity, i.e. the velocity of Bogoliubov excitations in the long wavelength limit. Here v_{cr} equals to the threshold velocity above which vortices are generated by the flow at the boundary of the obstacle. It was noticed also in the numerical simulations that a stationary wave pattern is generated by a supersonic flow past an obstacle and this pattern consists of two different regions separated by the Mach cone. These theoretical findings got new interest after experimental realization of the flow of Bose-Einstein condensate (BEC) of dilute atomic gas past obstacles [3]. In theoretical works motivated by these experiments it was shown that the region outside the Mach cone is occupied by the so called "ship waves" formed due to interference of Bogoliubov waves radiated by the obstacle by means of Cherenkov effect [3–5], whereas in the region inside the Mach cone a fan of oblique dark solitons is located [6, 7] and these oblique solitons are effectively stable if the flow velocity is large enough [8].

However, if we consider flow of BEC past an extended obstacle, the wave pattern can change drastically. In particular, the ship wave not far from the obstacle's boundary becomes nonlinear and transforms into a stationary dispersive shock wave, i.e., in some approximation, into a lattice of curved dark solitons [9]. In vicinity of the boundary of the obstacle both the local sound velocity and the direction of the flow are changed compared with their values in the incident flow and, hence, the question of stability of dark solitons in the dispersive shocks should be reconsidered. Their instability was noticed in numerical simulations of the highly supersonic flow past a slender wedge [9] and it became crucially important for moderately supersonic flows past not so slender wedges when the instability destroyed very fast the dispersive shock pattern [10]. The aim of this paper is to give theoretical estimates of the instability growth rate in such flows and to confirm the theory by numerical simulations.

First, we notice that if the obstacle is large and not too slender, then transition to a stationary state formed by the "curved solitons" is accompanied by propagation of bending waves along such solitons; these waves include unstable modes which evolution can lead eventually to decay of oblique solitons to vortex-antivortex pairs. Let at the initial moment of time $t = 0$ the condensate be confined in the region outside the trap potential having a form of a corner with one its wall located along negative x axis ($-\infty < x < 0$) and another one along the line $y = \tan\alpha \cdot x$, α being the angle of the slope ($0 < \alpha < \pi/2$). The flow with velocity M (equal to the Mach number in our standard non-dimensional units with the flow velocity measured in units of the sound velocity) along the x axis is switched on at $t = 0$. Evidently, in vicinity of the sloping boundary the wave appears which is equivalent to one generated by a piston moving with velocity $M \sin\alpha$; hence, a non-stationary dispersive shock wave is generated [9–11] with wave crests parallel to the sloping boundary. If $M \sin\alpha < \sqrt{n_0}$, n_0 being the density of incident BEC, then the waves

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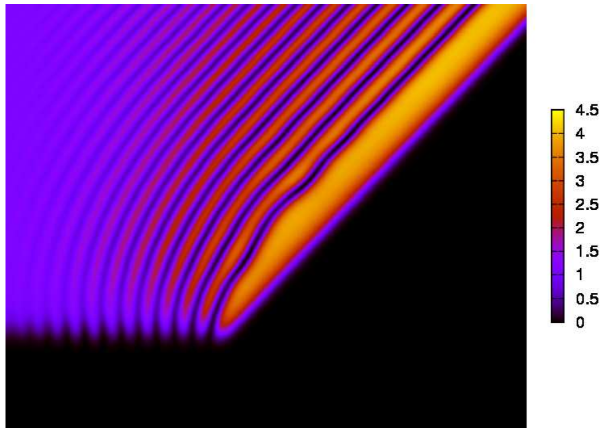

FIG. 1: Density plot of the wave pattern generated by the flow of BEC with the Mach number $M = 2.5$ past a corner with the angle $\alpha = 52^\circ$ at the moment of evolution $t = 8.22$. The oblique solitons of the dispersive shock are modulated by the wave of bending. These wave packets include the unstable modes which lead eventually to decay of oblique solitons to vortices.

closest to the boundary can be considered as oblique solitons [7]. However, these oblique solitons are cut at their edge located about the line $y = -\cot\alpha \cdot x$. These their edges are not attached to the obstacles and, hence, cannot become effectively stable due to transition from their absolute instability to a convective one at large enough flow velocity (see [8]). Therefore these oblique solitons must change their form transforming to "curved solitons" studied analytically in [9] for $M \gg 1$, $\alpha \ll 1$. Hence, bending waves start their propagation along oblique solitons and we suppose that the component of the wave packet with maximal value of the instability growth rate Γ leads to decay of oblique solitons to vortices observed numerically in [10].

To perform quantitative comparison of analytical estimates with results of numerical simulations, we define a numerical time $T(\alpha)$ of the instability growth as follows. Direct numerical solution of the GP equation

$$i\psi_t + \tfrac{1}{2}(\psi_{xx} + \psi_{yy}) - |\psi|^2\psi = V(x,y)\psi \tag{1}$$

with the corner-like shape of the trap potential $V(x,y)$ shows that solitons bend in the shock with time in such a way that, for instance, the deepest one is centered around the curve $y' = Y(x',t)$, where x' is the axis along the sloping side of the corner and y' is the axis normal to x', that is (x',y') axes are obtained by counter-clockwise rotation of (x,y) axes to the angle α and $Y(x',t)$ has a meaning of the distance of the soliton's trough line from the boundary at the moment t. A typical shape of this wave pattern is shown in Fig. 1 and one can see that a wave packet propagates along the soliton (x' axis). Our calculations show that the packet's velocity is equal to $M\cos\alpha$, that is this packet is built from non-propagating unstable modes and its motion is caused by convection due to flow of BEC along the boundary. In Fig. 2 the forms of wave packets corresponding to different values of α at such moments of time that their amplitudes reach a certain value chosen the same one for all α. As we see, the forms of these wave packets are very similar to each other, too, and we choose just these moments of time as definitions of numerical values of the instability growth time $T(\alpha)$.

To estimate the analytical growth rate values, we notice that the problem of formation of dispersive shock wave by a piston moving with a constant velocity is equivalent to well-known problem of evolution of a step-like pulse [12, 13] and its solution was discussed in [9, 11]. In particular, it was found that the condensate's density between the corner's boundary (piston) and the soliton edge of the dispersive shock wave is equal to

$$n = \left(\sqrt{n_0} + \tfrac{1}{2}M\sin\alpha\right)^2, \tag{2}$$

where n_0 is the density of BEC in the incident flow. The velocity of the deepest soliton at the edge of the shock is equal to

$$V = \sqrt{n_0} + \tfrac{1}{2}M\sin\alpha. \tag{3}$$

These formulae determine the parameters of the soliton which stability must be studied. Its analytical form is well known and can be expressed as

$$\begin{gathered}\psi_s = \Phi_s(\zeta)e^{-int}, \quad \Psi_s(\zeta) = k\tanh(k\zeta) - iV = \sqrt{n_s(\zeta)}\,e^{i\varphi_s(\zeta)}, \quad \frac{\partial\varphi_s}{\partial\zeta} = -V\left(1 - \frac{n}{n_s}\right) \\ k = \sqrt{n - V^2}, \quad \zeta = x + Vt.\end{gathered} \tag{4}$$

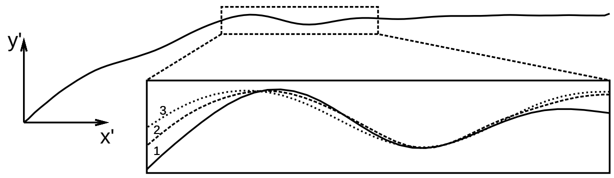

FIG. 2: Line of trough of the deepest dark soliton generated by the flow of BEC with $M = 2.5$ at the moment of time $t = 8.22$ past a corner with the angle $\alpha = 52°$. In the insert the wave packets with the same amplitude are shown for the angles (1) $\alpha = 52°$, (2) $\alpha = 37°$, (3) $\alpha = 32°$. The corresponding moments of time are chosen as times $T(\alpha)$ of the instability growth.

Then the perturbed wave function reads

$$\psi = (\Phi_s + \phi)e^{-int} = (\Phi_s + \phi' + i\phi'')e^{-int} = \sqrt{n_s + \delta n}\, e^{i(\varphi_s + \delta\varphi) - int}, \tag{5}$$

where the real variables ϕ' and ϕ'' are related with perturbations of the density δn and the phase $\delta\phi$ by the formulae

$$\phi' = \frac{\cos\varphi_s}{2\sqrt{n_s}}\delta n + \sqrt{n_s}\,\sin\varphi_s \cdot \delta\varphi, \quad \phi'' = \frac{\sin\varphi_s}{2\sqrt{n_s}}\delta n - \sqrt{n_s}\,\cos\varphi_s \cdot \delta\varphi. \tag{6}$$

Substitution of (5) into (1) and linearization of the resulting equation with respect to ϕ gives

$$i\phi_t + n\phi + \tfrac{1}{2}(\phi_{xx} + \phi_{yy}) - (2|\Phi_s|^2\phi + \Phi_s^2\phi^*) = 0, \tag{7}$$

where asterisk denotes complex conjugation. We are interested in propagation of a harmonic disturbance of the soliton along y axis. As usual in linear problems, such a disturbance can be represented in a complex form, δn, $\delta\varphi \propto \exp(\Gamma t + ipy)$ or $\phi = (\phi' + i\phi'')\exp(\Gamma t + ipy)$, $\phi^* = (\phi' - i\phi'')\exp(\Gamma t + ipy)$, where we keep previous notation for the amplitudes $\phi' = \phi'(\zeta)$, $\phi'' = \phi''(\zeta)$. For this form of the disturbance, Eq. (7) reduces to

$$i\Gamma\phi + iV\phi_\zeta + n\phi + \tfrac{1}{2}\phi_{\zeta\zeta} - \tfrac{1}{2}p^2\phi - (2|\Phi_s|^2\phi + \Phi_s^2\phi^*) = 0. \tag{8}$$

Substitution of (4) and separation of real and imaginary parts yields the system

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} \phi' \\ \phi'' \end{pmatrix} = \Gamma \begin{pmatrix} \phi' \\ \phi'' \end{pmatrix} \tag{9}$$

where

$$\begin{aligned} L_{11} &= -V\partial_\zeta - 2kV\tanh(k\zeta), \quad L_{12} = -\tfrac{1}{2}\partial_{\zeta\zeta} - n + n_s + 2V^2 + \tfrac{1}{2}p^2, \\ L_{21} &= \tfrac{1}{2}\partial_{\zeta\zeta} + n - 3n_s + 2V^2 - \tfrac{1}{2}p^2, \quad L_{22} = -V\partial_\zeta + 2kV\tanh(k\zeta). \end{aligned} \tag{10}$$

This eigenvalue problem was studied in [14] and we used it for numerical calculation of the instability growth rate $\Gamma(p)$ with the background density n and the soliton's velocity V given by (2) and (3), respectively, for $M = 2.5$ and several values of the angle α. These values of α as well as corresponding values of the background density and solitons velocities are given in the first three columns of Table 1. The instability growth rate depends on the wave number p and its maximal value Γ_{max} for each α is indicated in the fourth column of Table 1. As we see, it decreases with growth of α; for convenience we have normalized its values dividing them by the value at $\alpha = 52°$:

$$\gamma_{theor}(\alpha) = \frac{\Gamma_{max}(\alpha)}{\Gamma_{max}(52°)}. \tag{11}$$

In the last two columns we compare these theoretical predictions with the corresponding numerical characteristics defined as

$$\gamma_{num}(\alpha) = \frac{T(52°)}{T(\alpha)} \tag{12}$$

where $T(\alpha)$ is the defined above instability growth time. One can see very good agreement which confirms our approach to the problem of instability of oblique solitons.

Table I. Analytical and numerical estimates of instability growth rate of oblique solitons (Mach number $M = 2.5$).

α	n	V	Γ_{max}	$T(\alpha)$	$\gamma_{num}(\alpha)$	$\gamma_{theor}(\alpha)$
52°	4.16	0.07	1.09	8.22	1	1
47°	3.82	0.13	0.99	9.1	1.09	1.11
42°	3.48	0.19	0.89	10.34	1.22	1.26
37°	3.12	0.26	0.79	11.41	1.38	1.39
32°	2.80	0.35	0.68	13.03	1.60	1.59

Thus, our results show that the size of the obstacle is crucially important for realization of transition to effectively stable dark soliton states. In numerical experiment of Ref. [7] and theoretical estimates of Ref. [8] the size of the obstacle did not play any essential role. However, if a characteristic size l of the obstacle is such that the time l/v_{conv} (where v_{conv} is the velocity of convection) is greater than the inverse instability growth rate Γ_{max}^{-1}, then the unstable wave packet has enough time for evolution into its nonlinear stage with formation of vortex-antivortex pairs. Hence, if we wish to observe effectively stable dark solitons in the flow of BEC past a concave corner or a wedge, then the parameters of the flow must satisfy the condition

$$\frac{l}{M\cos\alpha} \ll \frac{1}{\Gamma_{max}}. \tag{13}$$

This condition was fulfilled in most of numerical experiments of Ref. [9] but was not fulfilled in numerical experiments of Ref. [10] what has lead to fast decay of oblique solitons into vortices. Our estimates show that the location of the "seed" disturbance where the first vortices are formed is determined by the distance $M\cos\alpha/\Gamma_{max}$ along the inclined wall of the corner.

It is worth noticing also that importance of existence of such a period in evolution of dark solitons that the unstable modes have enough time for their development into vortices manifested itself also in evolution of dark ring solitons [15, 16]: the instability develops effectively near the turning point of the ring soliton evolution when its parameters do not practically change and the mode with maximal corresponding value of Γ dominates in the evolution of the soliton's shape.x

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