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Properties of Harmonic Meromorphic Functions which are Starlike of Complex Order with Respect to Conjugate Points

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Abstract

Let $\sum_{\mathcal{H}}$ denote the class of functions f which are harmonic, meromorphic, orientation preserving and univalent in $\widetilde{U} = \{z : |z| > 1\}$. This paper defines and investigates a class of meromorphic univalent harmonic functions that are starlike of complex order with respect to conjugate points. The authors obtain convolution and convex combination properties.

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1 Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathcal{D} \subseteq \mathcal{C}$ if both u and v are real harmonic in $\mathcal{D}$. Hengartner and Schober [1], among other things, investigated the family $\sum_{\mathcal{H}}$ of functions $f = h + \overline{g}$ which are harmonic, meromorphic, orientation preserving and univalent in $\widetilde{U} = \{z : |z| > 1\}$ where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}, \quad g(z) = \sum_{n=1}^{\infty} b_n z^{-n}, \quad a_n \geq 0,\ b_n \geq 0,\ z \in \widetilde{U}. \tag{1}$$

Motivated by the results of [1], Jahangiri and Silverman [3] and Jahangiri [2] studied the classes of functions in $\sum_{\mathcal{H}}$ which are starlike or convex in $\widetilde{U}$. In particular, they investigated starlike and convex functions in the class $\sum_{\bar{\mathcal{H}}}$ consisting of functions $f = h + \overline{g}$ where h and g are of the form

$$h(z) = z + \sum_{n=1}^{\infty} |a_n| z^{-n}, \quad g(z) = -\sum_{n=1}^{\infty} |b_n| z^{-n}, \quad z \in \widetilde{U}. \tag{2}$$

In [5], Nasr and Aouf introduced the class of starlike functions of complex order b. Denote $\mathcal{S}^*(b)$ to be the class consisting of functions which are analytic and starlike of complex order b(b is a non-zero complex number) and satisfying the following condition

$$Re \left\{ 1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0, \; z \in U = \{z : |z| < 1\}.$$

In [4], Janteng and Abdul Halim were motivated to form a new subclass of $\sum_{\mathcal{H}}$ based on Nasr and Aouf's class as follows.

Definition 1.1 *Let $f \in \sum_{\mathcal{H}}$. Then $f \in \sum_{\mathcal{H}} S_c^{\star}(b, \alpha)$ is said to be harmonic meromorphic starlike of complex order with respect to conjugate points, if and only if, for $0 \leq \alpha < 1$, b non-zero complex number with $|b| \leq 1$, $z' = \frac{\partial}{\partial \theta}(z = re^{i\theta})$, $f'(z) = \frac{\partial}{\partial \theta}(f(z) = f(re^{i\theta}))$, $1 < r < \infty$ and $0 \leq \theta \leq 2\pi$,*

$$Re \left\{ 1 + \frac{1}{b} \left(\frac{2zf'(z)}{z'(f(z) + \overline{f(\overline{z})})} - 1 \right) \right\} \geq \alpha, \; |z| = r > 1.$$

If f takes the form (2) then we denote it as $\sum_{\bar{\mathcal{H}}} S_c^{\star}(b, \alpha)$.

2 Main Results

The following theorem proved by Janteng and Abdul Halim in [4] will be used throughout in this paper.

Theorem 2.1 *Let $f = h + \overline{g}$ be given by (1). If*

$$\sum_{n=1}^{\infty} \left(\frac{n + 1 - |b| + \alpha|b|}{(1-\alpha)|b|} \right) |a_n| + \sum_{n=1}^{\infty} \left(\frac{n - 1 + |b| - \alpha|b|}{(1-\alpha)|b|} \right) |b_n| \leq 1, \tag{3}$$

where $0 \leq \alpha < 1$ and b a non-zero complex number with $|b| \leq 1$ then f is harmonic, orientation preserving, univalent in $\widetilde{U}$ and $f \in \sum_{\mathcal{H}} S_c^{\star}(b, \alpha)$. Condition (3) is also necessary if $f \in \sum_{\bar{\mathcal{H}}} S_c^{\star}(b, \alpha)$.

Next, we show that the class $\sum_{\bar{\mathcal{H}}} S_c^{\star}(b,\alpha)$ is invariant under convolution and convex combinations of its members.

For harmonic functions $f(z)=z+\sum_{n=1}^{\infty}|a_n|z^{-n}-\sum_{n=1}^{\infty}|b_n|\bar{z}^{\,-n}$ and $F(z)=z+\sum_{n=1}^{\infty}|A_n|z^{-n}-\sum_{n=1}^{\infty}|B_n|\bar{z}^{\,-n}$, we define the convolution of f and F as

$$(f\star F)(z)=z+\sum_{n=1}^{\infty}|a_nA_n|z^{-n}-\sum_{n=1}^{\infty}|b_nB_n|\bar{z}^{\,-n}. \tag{4}$$

In the next theorem, we examine the convolution properties of the class $\sum_{\bar{\mathcal{H}}} S_c^{\star}(b,\alpha)$.

Theorem 2.2 *For $0\le\beta\le\alpha<1$, let $f\in\sum_{\bar{\mathcal{H}}} S_c^{\star}(b,\alpha)$ and $F\in\sum_{\bar{\mathcal{H}}} S_c^{\star}(b,\beta)$. Then $(f\star F)\in\sum_{\bar{\mathcal{H}}} S_c^{\star}(b,\alpha)\subset\sum_{\bar{\mathcal{H}}} S_c^{\star}(b,\beta)$.*

Proof. Write $f(z)=z+\sum_{n=1}^{\infty}|a_n|z^{-n}-\sum_{n=1}^{\infty}|b_n|\bar{z}^{\,-n}$ and $F(z)=z+\sum_{n=1}^{\infty}|A_n|z^{-n}-\sum_{n=1}^{\infty}|B_n|\bar{z}^{\,-n}$. Then the convolution of f and F is given by (4).

Note that $|A_n|\le 1$ and $|B_n|\le 1$ since $F\in\sum_{\bar{\mathcal{H}}} S_c^{\star}(b,\beta)$. Then we have

$$\begin{aligned}&\sum_{n=1}^{\infty}[n-(|b|-\alpha|b|-1)]|a_n||A_n|+\sum_{n=1}^{\infty}[n+(|b|-\alpha|b|-1)]|b_n||B_n|\\ &\le\sum_{n=1}^{\infty}[n-(|b|-\alpha|b|-1)]|a_n|+\sum_{n=1}^{\infty}[n+(|b|-\alpha|b|-1)]|b_n|.\end{aligned}$$

Therefore, $(f\star F)\in\sum_{\bar{\mathcal{H}}} S_c^{\star}(b,\alpha)\subset\sum_{\bar{\mathcal{H}}} S_c^{\star}(b,\beta)$ since the right hand side of the above inequality is bounded by $(1-\alpha)|b|$ while $(1-\alpha)|b|\le(1-\beta)|b|$.

Now, we determine the convex combination properties of the members of $\sum_{\bar{\mathcal{H}}} S_c^{\star}(b,\alpha)$.

Theorem 2.3 *The class $\sum_{\bar{\mathcal{H}}} S_c^{\star}(b,\alpha)$ is closed under convex combination.*

Proof. For $i=1,2,3,\ldots$, suppose that $f_i\in\sum_{\bar{\mathcal{H}}} S_c^{\star}(b,\alpha)$ where f_i is given by

$$f_i(z)=z+\sum_{n=1}^{\infty}|a_{n,i}|z^{-n}-\sum_{n=1}^{\infty}|b_{n,i}|\bar{z}^{\,-n}.$$

For $\sum_{i=1}^{\infty}c_i=1,\ 0\le c_i\le 1$, the convex combinations of f_i may be written as

$$\begin{aligned}\sum_{i=1}^{\infty}c_if_i(z) &= c_1z+\sum_{n=1}^{\infty}c_1|a_{n,1}|z^{-n}-\sum_{n=1}^{\infty}c_1|b_{n,1}|\bar{z}^{\,-n}+c_2z+\sum_{n=1}^{\infty}c_2|a_{n,2}|z^{-n}-\sum_{n=1}^{\infty}c_2|b_{n,2}|\bar{z}^{\,-n}\ldots\\ &= z\sum_{i=1}^{\infty}c_i+\sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}c_i|a_{n,i}|\right)z^{-n}-\sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}c_i|b_{n,i}|\right)\bar{z}^{\,-n}\\ &= z+\sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}c_i|a_{n,i}|\right)z^{-n}-\sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}c_i|b_{n,i}|\right)\bar{z}^{\,-n}.\end{aligned}$$

Next, consider

$$\sum_{n=1}^{\infty}\left([n-(|b|-\alpha|b|-1)]\left|\sum_{i=1}^{\infty}c_i|a_{n,i}|\right|\right)$$

$$+\sum_{n=1}^{\infty}\left([n+(|b|-\alpha|b|-1)]\left|\sum_{i=1}^{\infty}c_i|b_{n,i}|\right|\right)$$

$$\begin{aligned}
= \quad & c_1\sum_{n=1}^{\infty}[n-(|b|-\alpha|b|-1)]|a_{n,1}|+...\\
& +c_m\sum_{n=2}^{\infty}[n-(|b|-\alpha|b|-1)]|a_{n,m}|+...\\
& +c_1\sum_{n=1}^{\infty}[n+(|b|-\alpha|b|-1)]|b_{n,1}|+...\\
& +c_m\sum_{n=1}^{\infty}[n+(|b|-\alpha|b|-1)]|b_{n,m}|+...\\
= \quad & \sum_{i=1}^{\infty}c_i\Big\{\sum_{n=1}^{\infty}[n-(|b|-\alpha|b|-1)]|a_{n,i}|\\
& +\sum_{n=1}^{\infty}[n+(|b|-\alpha|b|-1)]|b_{n,i}|\Big\}.
\end{aligned}$$

Now, $f_i \in \sum_{\bar{\mathcal{H}}} S_c^{\star}(b,\alpha)$, therefore from Theorem 2.1, we have

$$\sum_{n=1}^{\infty}[n-(|b|-\alpha|b|-1)]|a_{n,i}|+\sum_{n=1}^{\infty}[n+(|b|-\alpha|b|-1)]|b_{n,i}| \le (1-\alpha)|b|\ .$$

Hence

$\sum_{n=1}^{\infty}\left([n-(|b|-\alpha|b|-1)]\left|\sum_{i=1}^{\infty}c_i|a_{n,i}|\right|\right)$

$+\sum_{n=1}^{\infty}\left([n+(|b|-\alpha|b|-1)]\left|\sum_{i=1}^{\infty}c_i|b_{n,i}|\right|\right)$

$\le (1-\alpha)|b|\sum_{i=1}^{\infty}c_i$

$=(1-\alpha)|b|.$

By using Theorem 2.1 again, we have $\sum_{i=1}^{\infty}c_i f_i \in \sum_{\bar{\mathcal{H}}} S_c^{\star}(b,\alpha)$.

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