Appl.Math.Lett 8,N4,(1995),87-90.

EXAMPLES OF NONUNIQUENESS FOR AN INVERSE PROBLEM OF GEOPHYSICS*

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Abstract. Two different velocity profiles and a source term are constructed such that the surface data are the same for all times and are not identically zero. The governing equation is $c^{-2}(x)u_{tt} - \Delta u = f(x,t)$ in $D \times [0,\infty)$, $u = 0$ for $t < 0$, $u_N = 0$ on ∂D, $D \subset \mathbb{R}^n_+ := \{x : x_n > 0\}$, $f(x,t) \not\equiv 0$. The data are the values $u(x,t)$, $\forall x \in S$, $\forall t > 0$. Here S is the part of ∂D which lies on the plane $x_n = 0$, $D = \{x : a_j \le x_j \le b_j, 1 \le j \le n, a_n = 0\}$.

I. Introduction.

Let $D \subset \mathbb{R}^n_+ := \{x : x \in \mathbb{R}^n, x_n \ge 0\}$ be a bounded domain, part S of the boundary Γ of D is on the plane $x_n = 0$, $f(x,t)$ is a source of the wavefield, $c(x) > 0$ is a velocity profile. The wavefield, e.g., the acoustic pressure, solves the problem:

$$c^{-2}(x)u_{tt} - \Delta u = f(x,t) \quad \text{in} \quad D \times [0,\infty), \quad f(x,t) \not\equiv 0, \tag{1}$$

$$u_N = 0 \quad \text{on} \quad \Gamma \tag{2}$$

$$u = u_t = 0 \quad \text{at} \quad t = 0. \tag{3}$$

Here N is the unit outer normal to Γ, u_N is the normal derivative of u on Γ. If $c^2(x)$ is known, then the direct problem (1)-(3) is uniquely solvable. The inverse problem (IP) we are interested in is the following one:

(IP) *Given the data* $u(x,t)$ $\quad \forall x \in S$, $\forall t > 0$, *can one recover* $c^2(x)$ *uniquely?*

The basic result of this paper is: *the answer to the above question is no.*

An analytical construction is presented of two constant velocities $c_j > 0$, $c_1 \ne c_2$, which can be chosen arbitrary, and a source, which is constructed after $c_j > 0$ are chosen, such that the solutions to problems (1)-(3) with $c^2(x) = c_j^2$ produce the same surface data on S for all times:

$$u_1(x,t) = u_2(x,t) \quad \forall x \in S, \quad \forall t > 0. \tag{4}$$

The domain D we use is a box: $D = \{x : a_j \le x_j \le b_j, 1 \le j \le n\}$.

This construction is given in section II. At the end of section II the data on S are suggested, which allow one to uniquely determine $c^2(x)$.

*This work was done under the auspices of DOE.

II. Example of nonuniqueness of the solution to IP.

Our construction is valid for any $n \geq 2$. For simplicity we take $n = 2$, $D = \{x : 0 \leq x_1 \leq \pi, 0 \leq x_2 \leq \pi\}$. Let $c^2(x) = c^2 = \text{const} > 0$. The solution to (1)-(3) with $c^2(x) = c^2 = \text{const}$ can be found analytically

$$u(x,t) = \sum_{m=0}^{\infty} u_m(t)\phi_m(x), \quad m = (m_1, m_2) \tag{5}$$

where

$$\begin{gathered}\phi_m(x) = \gamma_{m_1 m_2} \cos(m_1 x_1)\cos(m_2 x_2), \\ \int_D \phi_m^2(x)dx = 1, \quad \Delta\phi_m + \lambda_m \phi_m = 0, \\ \phi_{mN} = 0 \quad \text{on} \quad \Gamma, \quad \lambda_m := m_1^2 + m_2^2, \\ \gamma_{00} = \frac{1}{\pi}, \quad \gamma_{m_1 0} = \gamma_{0 m_2} = \frac{\sqrt{2}}{\pi}, \\ \gamma_{m_1 m_2} = 2/\pi \quad \text{if} \quad m_1 > 0 \quad \text{and} \quad m_2 > 0,\end{gathered} \tag{6}$$

$$u_m(t) := u_m(t,c) = \frac{c}{\sqrt{\lambda_m}} \int_0^t \sin[c\sqrt{\lambda_m}(t-\tau)] f_m(\tau)d\tau, \quad f_m(t) := \int_D f(x,t)\phi_m(x)dx \tag{6'}$$

The data are

$$u(x_1, 0, t) = \sum_{m=0}^{\infty} u_m(t,c)\gamma_{m_1 m_2}\cos(m_1 x_1). \tag{7}$$

For these data to be the same for $c = c_1$ and $c = c_2$, it is necessary and sufficient that

$$\sum_{m_2=0}^{\infty} \gamma_{m_1 m_2} u_m(t, c_1) = \sum_{m_2=0}^{\infty} \gamma_{m_1 m_2} u_m(t, c_2), \quad \forall t > 0, \quad \forall m_1. \tag{8}$$

Taking Laplace transform of (8) and using (6′) one gets an equation, equivalent to (8),

$$\sum_{m_2=0}^{\infty} \gamma_{m_1 m_2} \overline{f}_m(p) \left[\frac{c_1^2}{p^2 + c_1^2\lambda_m} - \frac{c_2^2}{p^2 + c_2^2\lambda_m}\right] = 0, \quad \forall p > 0, \quad \forall m_1. \tag{9}$$

Take $c_1 \neq c_2$, $c_1, c_2 > 0$, arbitrary and find $\overline{f}_m(p)$ for which (9) holds. This can be done by infinitely many ways. Since (9) is equivalent to (8), the desired example of nonuniqueness of the solution to IP is constructed.

Let us give a specific choice: $c_1 = 1$, $c_2 = 2$, $\overline{f}_{m_1 m_2} = 0$ for $m_1 \neq 0$, $m_2 \neq 1$ or $m_2 \neq 2$, $\overline{f}_{02}(p) = \frac{1}{p+1}$, $\overline{f}_{01}(p) = -\frac{p^2+1}{(p+1)(p^2+16)}$. Then (9) holds. Therefore, if

$$f(x,t) = \frac{\sqrt{2}}{\pi}\left[f_{01}(t)\cos(x_2) + f_{02}(t)\cos(2x_2)\right], \quad c_1 = 1, \quad c_2 = 2 \tag{10}$$

then the data $u_1(x,t) = u_2(x,t) \quad \forall x \in S$, $\forall t > 0$. In (10) the values of the coefficients are

$$f_{01}(t) = -\frac{2}{17}\exp(-t) - \frac{15}{17}\left[\cos(4t) - \frac{1}{4}\sin(4t)\right], \quad f_{02}(t) = \exp(-t). \tag{11}$$

Remark 1. The above example brings out the question: What data on S are sufficient for the unique identifiability of $c^2(x)$? The answer to this question one can find in [1] and [2].

In particular, if one takes $f(x,t) = \delta(t)\delta(x-y)$, and allows x and y run through S, then the data $u(x,y,t) \quad \forall x, y \in S$, $\forall t > 0$, determine $c^2(x)$ uniquely. In fact, the low frequency surface data $\tilde{u}(x,y,k)$, $\forall x, y \in S \quad \forall k \in (0, k_0)$, where $k_0 > 0$ is an arbitrary small fixed number, determine $c^2(x)$ uniquely under mild assumptions on D and $c^2(x)$. By $\tilde{u}(x,y,k)$ is meant the Fourier transform of $u(x,y,t)$ with respect to t.

Remark 2. One can check that the non-uniqueness example with constant velocities is not possible to construct as was done above if the sources are concentrated on S, that is, if $f(x_1, x_2, t) = \delta(x_2) f_1(x_1, t)$.

Acknowledgement. The author thanks Prof. S. Seatzu who drew his attention to the IP discussed above.

References

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