Solving second order ordinary differential equations by extending the PS method

L.G.S. Duarte[*], L.A. da Mota[†] and J.E.F.Skea[‡]

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Abstract

An extension of the ideas of the Prelle-Singer procedure to second order differential equations is proposed. As in the original PS procedure, this version of our method deals with differential equations of the form $y'' = M(x, y, y')/N(x, y, y')$, where M and N are polynomials with coefficients in the field of complex numbers C. The key to our approach is to focus not on the final solution but on the first-order invariants of the equation. Our method is an attempt to address algorithmically the solution of SOODEs whose first integrals are elementary functions of x, y and y'.

[*]Universidade do Estado do Rio de Janeiro, Instituto de Física, Departamento de Física Teórica, R. São Francisco Xavier, 524, Maracanã, CEP 20550–013, Rio de Janeiro, RJ, Brazil. E-mail: lduarte@dft.if.uerj.br

[†]idem. E-mail: damota@dft.if.uerj.br

[‡]idem. Email: jimsk@dft.if.uerj.br

1 Introduction

The fundamental position of differential equations (DEs) in scientific progress has, over the last three centuries, led to a vigorous search for methods to solve them. The overwhelming majority of these methods are based on classification of the DE into types for which a method of solution is known, which has resulted in a gamut of methods that deal with specific classes of DEs. This scene changed somewhat at the end of the 19th century when Sophus Lie developed a general method to solve (or at least reduce the order of) ordinary differential equations (ODEs) given their symmetry transformations [1, 2, 3]. Lie's method is very powerful and highly general, but first requires that we find the symmetries of the differential equation, which may not be easy to do. Search methods have been developed [4, 5] to extract the symmetries of a given ODE, however these methods are heuristic and cannot guarantee that, if symmetries exist, they will be found.

On the other hand in 1983 Prelle and Singer (PS) presented a deductive method for solving first order ODEs (FOODE) that presents a solution in terms of elementary functions if such a solution exists [6]. The attractiveness of the PS method lies not only in its basis on a totally different theoretical point of view but, also in the fact that, if the given FOODE has a solution in terms of elementary functions, the method guarantees that this solution will be found (though, in principle it can admittedly take an infinite amount of time to do so). The original PS method was built around a system of two autonomous FOODEs of the form $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ with P and Q in $C[x, y]$ or, equivalently, the form $y' = R(x, y)$, with $R(x, y)$ a rational function of its arguments. Here we propose a generalization that allows us to apply the techniques developed by Prelle and Singer to second order differential equations (SOODEs). The key idea is to focus not on the final solution of the equation, but rather its invariants.

This paper is organized as follows: in section 2, the reader is introduced to the PS procedure; section 3 addresses our approach extending the ideas of the PS procedure to the case of SOODEs and discusses how generally applicable the method is to such equations. Section 4 is dedicated to some examples solved via our procedure and, finally, conclusions are presented in section 5.

2 The Prelle-Singer Procedure

Despite its usefulness in solving FOODEs, the Prelle-Singer procedure is not very well known outside mathematical circles, and so we present a brief overview of the main ideas of the procedure.

Consider the class of FOODEs which can be written as

$$y' = \frac{dy}{dx} = \frac{M(x, y)}{N(x, y)} \tag{1}$$

where $M(x, y)$ and $N(x, y)$ are polynomials with coefficients in the complex field C.

In [6], Prelle and Singer proved that, if an elementary first integral of (1) exists, it is possible to find an integrating factor R with $R^n \in C$ for some (possible non-integer) n, such that

$$\frac{\partial(RN)}{\partial x} + \frac{\partial(RM)}{\partial y} = 0. \tag{2}$$

The ODE can then be solved by quadrature. From (2) we see that

$$N\,\frac{\partial R}{\partial x} + R\,\frac{\partial N}{\partial x} + M\,\frac{\partial R}{\partial y} + R\,\frac{\partial M}{\partial y} = 0. \tag{3}$$

Thus

$$\frac{D[R]}{R} = -\left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right), \tag{4}$$

where

$$D \equiv N\frac{\partial}{\partial x} + M\frac{\partial}{\partial y}. \tag{5}$$

Now let $R = \prod_i f_i^{n_i}$ where f_i are irreducible polynomials and n_i are non-zero integers. From (5), we have

$$\begin{aligned} \frac{D[R]}{R} &= \frac{D[\prod_i f_i^{n_i}]}{\prod_i f_k^{n_k}} = \frac{\sum_i f_i^{n_i-1} n_i D[f_i] \prod_{j\neq i} f_j^{n_j}}{\prod_k f_k^{n_k}} \\ &= \sum_i \frac{f_i^{n_i-1} n_i D[f_i]}{f_i^{n_i}} = \sum_i \frac{n_i D[f_i]}{f_i}. \end{aligned} \tag{6}$$

From (4), plus the fact that M and N are polynomials, we conclude that $D[R]/R$ is a polynomial. Therefore, from (6), we see that $f_i | D[f_i]$.

We now have a criterion for choosing the possible f_i (build all the possible divisors of $D[f_i]$) and, by using (4) and (6), we have

$$\sum_i \frac{n_i D[f_i]}{f_i} = -\left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right). \tag{7}$$

If we manage to solve (7) and thereby find n_i, we know the integrating factor for the FOODE and the problem is reduced to a quadrature. Risch's algorithm [7] can then be applied to this quadrature to determine whether a solution exists in terms of elementary functions.

3 Extending the Prelle-Singer Procedure

In the previous section, the main ideas and concepts used in the Prelle-Singer procedure were introduced. Here we present an extension of these ideas applicable to SOODEs. The main idea is to focus on the first order invariants of the ODE rather than on the solutions.

3.1 Introduction

Consider the SOODE

$$y'' = \frac{d^2y}{dx^2} = \frac{M(x,y,y')}{N(x,y,y')}, \tag{8}$$

where $M(x,y,y')$ and $N(x,y,y')$ are polynomials with coefficients in C. We assume that (8) has a solution in terms of elementary functions, in which case there are two independent elementary functions of x y and y' which are constant on all solutions of (8), namely the first order invariants

$$I_i(x,y,y') = C_i \qquad i = 1,2. \tag{9}$$

Without loss of generalization we consider one of these and, dropping the index on I_i we have

$$\mathrm{d}I = \frac{\partial I}{\partial x}\,\mathrm{d}x + \frac{\partial I}{\partial y}\,\mathrm{d}y + \frac{\partial I}{\partial y'}\,\mathrm{d}y' = 0. \tag{10}$$

Now, introducing the notation $\frac{\partial I}{\partial u} \equiv I_u$, we have

$$I_x + I_y y' + I_{y'} y'' = 0, \tag{11}$$

and so

$$y'' = -\frac{I_x + I_y y'}{I_{y'}}, \tag{12}$$

which is (8) in terms of the differential invariant I. Rewriting (8) as

$$\frac{M}{N}\,\mathrm{d}x - \mathrm{d}y' = 0 \tag{13}$$

and observing that

$$y'\,\mathrm{d}x = \mathrm{d}y, \tag{14}$$

we can add the identically null term $S(x, y, y')y'\,\mathrm{d}x - S(x, y, y')\,\mathrm{d}y$ to (13) and obtain the 1-form

$$\left(\frac{M}{N} + Sy'\right)\,\mathrm{d}x - S\,\mathrm{d}y - \mathrm{d}y' = 0. \tag{15}$$

Notice that the 1-form (16) must be proportional to the 1-form (10). So, since the 1-form (10) is exact, we can multiply (16) by the integrating factor $R(x, y, y')$ to obtain

$$\mathrm{d}I = R(\phi + Sy')\,\mathrm{d}x - RS\,\mathrm{d}y - R\,\mathrm{d}y' = 0, \tag{16}$$

where $\phi \equiv M/N$.

Comparing equations (10) and (16),

$$\begin{aligned} I_x &= R(\phi + Sy'), \\ I_y &= -RS, \\ I_{y'} &= -R. \end{aligned} \tag{17}$$

Now equations (17) must satisfy the compatibility conditions $I_{xy} = I_{yx}, I_{xy'} = I_{y'x}$ and $I_{yy'} = I_{y'y}$. This implies that

$$D[S] = -\phi_y + S\phi_{y'} + S^2, \tag{18}$$

$$D[R] = -R(S + \phi_{y'}), \tag{19}$$

$$R_y = R_{y'} S + S_{y'} R, \tag{20}$$

where the differential operator D is defined as

$$D \equiv \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \phi\frac{\partial}{\partial y'}. \tag{21}$$

Combining (18) and (19) we obtain

$$D[RS] = -R\phi_y. \tag{22}$$

So if the product of S and the integrating factor R is a rational function of x, y and y', then $D[RS]$ is too. Since ϕ is rational (and so, therefore, is ϕ_y), equation (22) tells us that R is rational. Using (19) and similar arguments we conclude that S must be a rational function of x, y and y'.

In summary, from (17) it follows that the supposition that RS is rational can be equated to the existence of a first order invariant whose derivatives in relation to x, y and y' are rational functions. With this in mind we restate the original supposition in the form of a conjecture.

3.2 The Conjecture

We first state a result proved in [6]: **Theorem:** *Let K be a differential field of functions in $n+1$ variables and L an elementary extension of K. Let f be in K and assume there exists a nonconstant g in L such that g is constant on all solutions of $y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)})$. Then there exist $w_1, \ldots, w_m$ algebraic over K and constants $c_i, \ldots, c_m$ such that*

$$w_0(x, y, y', y'', .., y^{(n-1)}) + \sum_i c_i \log(w_i(x, y, y', y'', \ldots, y^{(n-1)})) \tag{23}$$

is a constant on all solutions of $y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)})$.

This result shows that for the particular case of SOODEs whose solutions are elementary, there are two independent first order invariants of the form

$$w_0(x, y, y') + \sum_i c_i \log w_i(x, y, y'). \tag{24}$$

Our conjecture is that if these two first order invariants exist it is always possible to find a function of them (which will, therefore, itself be a first order invariant) of the form

$$z_0(x, y, y') + \sum_i c_i \log[z_i(x, y, y')], \tag{25}$$

where z_i are *rational* functions of x, y and y'.

Conjecture: *Let K be a differential field of functions in three variables and L an elementary extension of K. Let f be in K and assume there exist two independent non-constant $\{g_1, g_2\}$ in L such that g_i are constant on all solutions of $y'' = f(x, y, y')$. Then there exists at least one constant of the form*

$$z_0(x, y, y') + \sum_i c_i \log(z_i(x, y, y')) \tag{26}$$

where the z_i are in K.

By the previous reasoning it can be seen that (26) implies that the product RS is a rational function of x, y and y'.

If this conjecture holds, then our extension of the PS method applies to all SOODEs of the form (8). Though we have not been able to prove our conjecture, extensive trials while developing this procedure has not revealed any counter example. Even if the conjecture is false, our experience with real test cases has shown that the method is, at least, applicable to the vast majority of SOODEs of the form (8).

3.3 Finding R and S

Our conjecture implies that, if the SOODE to be solved has an elementary general solution, then S is a rational function which we may write as

$$S = \frac{S_n}{S_d} = \frac{\sum_{i,j,k} a_{ijk} x^i y^j {y'}^k}{\sum_{i,j,k} b_{ijk} x^i y^j {y'}^k}. \tag{27}$$

We can also see that (18) does not involve R. So, given a degree bound on the polynomials S_n and S_d, we may find a set of solutions to this equation which are then candidates to solve the system of equations (18)–(20).

From (19) we have

$$\frac{D[R]}{R} = -(S + \phi_{y'}) = -\frac{S_n}{S_d} - \left(\frac{M}{N}\right)_{y'} = -\frac{S_n N^2 + S_d(N M_{y'} - M N_{y'})}{S_d N^2} \tag{28}$$

which can be rewritten as

$$\frac{\mathcal{D}[R]}{R} = -S_n N^2 + S_d(N M_{y'} - M N_{y'}), \tag{29}$$

where the differential operator $\mathcal{D}$ is defined as

$$\mathcal{D} \equiv (S_d N^2) D. \tag{30}$$

We keep in mind that

- S_n, S_d, N and M are polynomials in x, y and y';
- $\mathcal{D}$ is a linear differential operator whose coefficients of $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial y'}$ are polynomials in x, y and $, y'$;
- R is a rational function of x, y and y', which we may write as

$$R = \frac{R_n}{R_d} = \frac{\sum_{i,j,k} c_{ijk} x^i y^j {y'}^k}{\sum_{i,j,k} d_{ijk} x^i y^j {y'}^k}. \tag{31}$$

If we have a theoretical limit on the degrees of R_m and R_d (a *degree bound*), we may use a procedure analogous to that described in section 2 to obtain candidates for the integrating factor R. We simply construct all polynomials in x, y and y' up to the degree bound.

3.4 Reduction of the SOODE

Once R and S have been determined using equations (17) we have all the partial first derivatives of the first order differential invariant, $I(x, y, y')$, which is constant on the solutions. This invariant can then be obtained as

$$\begin{aligned} I(x,y,y') = &\int R\,(\phi + S y')\,dx\ - \\ &\int \Big[RS + \frac{\partial}{\partial y}\int R\,(\phi + S y')\,dx\Big] dy\ - \\ \int \left[R + \frac{\partial}{\partial y'}\left(\int R\,(\phi + S y')\,dx - \int \Big[RS + \frac{\partial}{\partial y}\int R\,(\phi + S y')\,dx\Big] dy\right)\right] dy'. \end{aligned} \tag{32}$$

The equation $I(x, y, y') = C_1$ can then be solved to obtain a FOODE for y': the *reduced* ODE

$$y' = \varphi(x, y, C1). \tag{33}$$

To obtain the general solution of the original ODE, we can apply the Prelle-Singer method in its original form to this reduced ODE. Thus, if our conjecture is correct, the method proposed here (for SOODEs of the form (8)) is as algorithmic as the original PS method for FOODEs. We note that the original PS method fails to be what is strictly an algorithm because no theoretical degree bound is yet known for the candidate polynomials which enter in the prospective solution, and so the procedure has no effective terminating condition for the case when an elementary solution does not exists. In practice, a terminating condition is put in by hand (it is found that polynomials of degree higher than 4 lead to computations which are overly complex for the average desktop computer). However, should such a degree bound be established, and our conjecture shown to be true, then the method proposed here would be an algorithm for deciding whether elementary solutions of SOODEs of the form (8) exist.

4 Examples

In this section we present examples of physically motivated SOODEs that are solved by our procedure[1]. As a simple illustrative example, we begin with the classical harmonic oscillator and then consider some nonlinear SOODEs which arise from astrophysics and general relativity.

Example 1: The Simple Harmonic Oscillator
In its simplest form, the equation for the simple harmonic oscillator is

$$y'' = -y. \tag{34}$$

For this ODE equations (18), (19) and (20) are

$$\begin{aligned} S_x + y'S_y - yS_{y'} &= 1 + S^2, && (35) \\ R_x + y'R_y - yR_{y'} &= -RS, && (36) \\ R_y - R_{y'}S - S_{y'}R &= 0. && (37) \end{aligned}$$

One possible solution to these equations is

$$S = \frac{y}{y'}, \quad R = y'. \tag{38}$$

From this, and using (32), we get the reduced ODE

$$C1 = y^2 + y'^2, \tag{39}$$

which, of course, represents the energy conservation for the oscillator.

This example is very simple and leads to a form of ϕ which is independent of x and y'. And, as with all linear ODEs, alternative and more straightforward solution methods exist. The other examples illustrate the solution method at work for non-linear SOODEs which can be placed in the form (8).

[1]We present only the reduction of the SOODEs since the integration of the resulting FOODE can be achieved by various methods, including the PS method itself.

Example 2: An Exact Solution in General Relativity

A rich source of non-linear DEs in physics are the highly non-linear equations of General Relativity. In general, Einstein's equations are, of course, partial DEs, but there exist classes of equations where the symmetry imposed reduces these equations to ODEs in one independent variable. One such class is that of static, spherically symmetric solutions for stellar models, which depend only on the radial variable, r. The metric for a general statically spherically spacetime has two free functions, $\lambda(r)$ and $\mu(r)$ say. On imposing the condition that the fluid is a perfect fluid, Einstein's equations reduce to two coupled ODEs for $\lambda(r)$ and $\mu(r)$. Specifying one of these functions reduces the problem to solving an ODE (of first or second order) for the other.

Following this procedure, Buchdahl [8] obtained an exact solution for a relativistic fluid sphere by considering the so-called isotropic metric

$$\dot{s}^2 = (1-f)^2(1+f)^{-2}\dot{t}^2 - (1+f)^4[\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2]$$

with $f = f(r)$. The field equations for $f(r)$ reduce to

$$f f'' - 3f'^2 - r^{-1} f f' = 0.$$

Changing notation with $y(x) = f(r)$, equations (18), (19) and (20) assume the form

$$\begin{aligned} S_x + y' S_y + \frac{y'\,(3\,y'\,x + y)}{xy} S_{y'} &= -\frac{y'}{xy} + \frac{y'\,(3\,y'\,x + y)}{xy^2} + \\ &\quad \left(\frac{3\,y'\,x + y}{xy} + 3\,\frac{y'}{y}\right) S + S^2, \quad (40) \\ R_x + y' R_y + \frac{y'\,(3\,y'\,x + y)}{xy} R_{y'} &= -R\left(S + \frac{3\,y'\,x + y}{xy} + 3\,\frac{y'}{y}\right), \quad (41) \\ R_y - R_{y'} S - S_{y'} R &= 0. \quad (42) \end{aligned}$$

One solution of those equations is

$$S = -3\,\frac{y'}{y}, \quad R = \frac{1}{xy^3} \quad (43)$$

By using (32) we obtain the reduced FOODE:

$$C_1 = y'/(y^3 x) \quad (44)$$

which is separable and easily integrated to obtain the general solution

$$y(x)^2 = \left(-C_1\,x^2 + C_2\right)^{-1}. \quad (45)$$

Example 3: A Static Gaseous General-Relativistic Fluid Sphere

In a later paper [9], Buchdahl approaches the problem of the general relativistic fluid sphere using a different coordinate system from the previous example. For ease in comparison of the originals,

Substituting the $\xi(r)$ of the original Writing $y(x)$ instead of the $\xi(r)$ in the original, arrives at the equation

$$y'' = \frac{x^2 y'^2 + y^2 - 1}{x^2 y}, \quad (46)$$

For this SOODE, eqs (18, 19 and 20) become:

$$
\begin{aligned}
S_x + y' S_y + \frac{x^2 y'^2 + y^2 - 1}{x^2 y} S_{y'} &= -2\, x^{-2} + \frac{x^2 y'^2 + y^2 - 1}{y^2 x^2} \\
&\quad +2\, \frac{y'\, S}{y} + S^2 \qquad (47)\\
R_x + y' R_y + \frac{x^2 y'^2 + y^2 - 1}{x^2 y} R_{y'} &= -R \left(S + 2\, \frac{y'}{y} \right) \qquad (48)\\
R_y - R_{y'} S - S_{y'} R &= 0 \qquad (49)
\end{aligned}
$$

One solution to those equations is:

$$
S = \frac{-x^2 y'^2 - x y y' + 1}{x y^2 + x^2 y y'}, \;\; R = \frac{y + x y'}{x y^2}. \qquad (50)
$$

From this, using eq. (32), we get the reduced FOODE:

$$
C_1 = \frac{2\, x y y' + y^2 + x^2 y'^2 - 1}{2 x^2 y^2}. \qquad (51)
$$

which can be solved to:

$$
y(x)^2 = \frac{\tan(\sqrt{2}\sqrt{C_1}\, (C_2 + x))^2}{\left(2\, C_1 + 2\, \tan(\sqrt{2}\sqrt{C_1}\, (C_2 + x))^2 C_1\right) x^2} \qquad (52)
$$

This example has an extra feature: It is not solved by other solvers we have tried (mainly the Maple solver, in the version 5, that we believe to be the best). So, apart from the (already) very interesting fact that our approach is an algorithmic attempt to solve SOODEs, we have also this present fact, i.e., some SOODEs are solved via our method and "escape" from other very powerful solvers.

5 Conclusion

In this paper, we presented an approach that is an extension of the ideas developed by Prelle-Singer [6] to tackle FOODEs. We believe it to be the first technique to address algorithmically the solution of SOODEs with elementary first integrals.

Here, we dealt with a restrict class of SOODEs (namely, the ones of the form (8)). However, we can use our method in solving SOODEs where $\phi(x, y, y')$ depends on elementary functions of x, y, y', following the developments for the Prelle-Singer approach for FOODEs [10, 11]. We are presently working on those ideas.

The generality of our approach is based on a conjecture (see section (3.2)) that we have already proved for many special cases. Even if the conjecture is proven false, our approach is a powerful tool in dealing with SOODEs since we have extensively tested it with many equations, both from mathematics and physical origin. In fact, since all the examples we have encounter have been solved by our approach, we are preparing a computational package implementing the Prelle-Singer procedure (and our present extension) to be submitted to Computer Physics Communications.

References

[1] Stephani, H. *Differential equations: their solution using symmetries*, ed. M.A.H. MacCallum, Cambridge University Press, New York and London (1989).

[2] G.W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Applied Mathematical Sciences **81**, Springer-Verlag, (1989).

[3] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, (1986).

[4] E.S. Cheb-Terrab, L.G.S. Duarte and L.A.C.P. da Mota, *Computer Algebra Solving of First Order ODEs Using Symmetry Methods. Comput.Phys.Commun.*, **101**, (1997), 254.

[5] E.S. Cheb-Terrab, L.G.S. Duarte and L.A.C.P. da Mota, *Computer Algebra Solving of Second Order ODEs Using Symmetry Methods. Comput.Phys.Commun.*, **108**, (1998), 90.

[6] M Prelle and M Singer, Elementary first integral of differential equations. *Trans. Amer. Math. Soc.*, **279** 215 (1983).

[7] R H Risch, *The Problem of Integration in Finite Terms Trans. Amer. Math. Soc.*, **139** 167–189 (1969).

[8] H.A.Buchdahl (1964) *A Relativistic Fluid Sphere Resembling the Emden Polytrope of Index 5 Ap. J.* **140** 1512–1516.

[9] H.A.Buchdahl (1967) *General Relativistic Fluid Spheres III. A Static Gaseous Model Ap. J.* **147** 310–316.

[10] Y K Man, Computing Closed Form Solutions of First Order ODEs Using the Prelle-Singer Procedure. *J. Symbolic Computation*, **16** 423–443 (1993)

[11] Y K Man, First Integrals of Autonomous Systems of Differential Equations and the Prelle-Singer Procedure. *J. Phys. A*, **27** L329–332 (1994)