Indifference price with general semimartingales

Sara Biagini* Marco Frittelli § Matheus Grasselli ‡

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Abstract

For utility functions u finite valued on $\mathbb{R}$, we prove a duality formula for utility maximization with random endowment in general semimartingale incomplete markets. The main novelty of the paper is that possibly non locally bounded semimartingale price processes are allowed. Following Biagini and Frittelli [BF06], the analysis is based on the duality between the Orlicz spaces $(L^{\widehat{u}}, (L^{\widehat{u}})^*)$ naturally associated to the utility function. This formulation enables several key properties of the indifference price $\pi(B)$ of a claim B satisfying conditions weaker than those assumed in literature. In particular, the indifference price functional π turns out to be, apart from a sign, a convex risk measure on the Orlicz space $L^{\widehat{u}}$.

Key words: Indifference price - utility maximization – non locally bounded semimartingale – random endowment - incomplete market – Orlicz space – convex duality - convex risk measure
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1 Introduction

The main purpose of this paper is to study the indifference pricing framework in markets where the underlying traded assets are described by general semimartingales which are *not assumed to be locally bounded*. Following Hodges and Neuberger [HN89], we define the *(seller) indifference price* $\pi(B)$ of a claim B as the implicit solution of the equation

$$\sup_{H\in\mathcal{H}^W} E\left[u\left(x+\int_0^T H_t dS_t\right)\right] = \sup_{H\in\mathcal{H}^W} E\left[u\left(x+\pi(B)+\int_0^T H_t dS_t - B\right)\right], \tag{1}$$

where $x \in \mathbb{R}$ is the constant initial endowment, $T < \infty$ is a fixed time horizon while S is an $\mathbb{R}^d-$valued càdlàg semimartingale defined on a filtered stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ that satisfies the usual assumptions. The $\mathbb{R}^d-$valued portfolio process H belongs to an appropriate class $\mathcal{H}^W$ of admissible integrands defined in Section 2.1 through a random variable W that controls the losses incurred in trading. B is an $\mathcal{F}_T-$measurable random variable corresponding to a financial liability at time T and satisfies the integrability conditions discussed in Section 3.1.

*Università di Pisa, Italy. Email: sara.biagini@ec.unipi.it
§Università di Milano, Italy. Email: marco.frittelli@mat.unimi.it
‡McMaster University, Canada. Email: grasselli@math.mcmaster.ca

Throughout the paper, the utility function u is assumed to be an increasing and concave function $u : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{x\to-\infty} u(x) = -\infty$.

Neither *strict monotonicity nor strict concavity are required*, but we exclude that u is constant on $\mathbb{R}$.

In principle, a general way to compute the indifference price in (1) is to solve the two utility maximization problems, in the sense of finding the optimizers in the class of admissible integrands. Such optimizers then correspond to the optimal trading strategies that an investor should follow with or without the claim B, therefore providing a corresponding notion of *indifference hedging* for the claim. However, it is generally possible to employ duality arguments to obtain the optimal *values* for utility maximization problems under broader assumptions than those necessary to find their optimizer. Since these values are all that is necessary for calculating the indifference price itself, the main goal here is the pursuit of such duality results rather than a full analysis of the indifference hedging problem which is deferred to future work (even though some partial results in this direction are provided in Proposition 3.18).

The key to establish such duality above is to choose convenient dual spaces as the ambient for the domains of optimization. Our approach is to use the Orlicz space $L^{\widehat{u}}$ - and its dual space $L^{\widehat{\Phi}}$ - that arises naturally from the choice of the utility function u and was previously used in [BF06] for the special case of $B = 0$, as explained in Section 2.

We then use this general framework for the case of a random endowment B in Section 3 and prove in Theorem 3.15 a duality result of the type

$$\sup_{H\in\mathcal{H}^W} E\left[u\left(x + \int_0^T H_t dS_t - B\right)\right] \tag{2}$$

$$= \min_{\lambda>0,\ Q\in\mathcal{M}^W} \left\{\lambda x - \lambda Q(B) + E\left[\Phi\left(\lambda\frac{dQ^r}{dP}\right)\right] + \lambda\|Q^s\|\right\}. \tag{3}$$

where W is a loss control and in the dual problem (3), $\Phi : \mathbb{R}_+ \to \mathbb{R}$ is the convex conjugate of the utility function u, defined by

$$\Phi(y) := \sup_{x\in\mathbb{R}} \{u(x) - xy\}, \tag{4}$$

while $\mathcal{M}^W$ is the appropriate set of linear pricing functionals Q, which admit the decomposition

$$Q = Q^r + Q^s$$

into regular and singular parts. The penalty term in the right-hand side of (3) is split into the expectation $E\left[\Phi\left(\lambda\frac{dQ^r}{dP}\right)\right]$, associated only with the regular part of Q, and the norm $\|Q^s\|$, associated only with its singular part.

¿From the previous results [BF06] in the case $B = 0$, we expected the presence of the singular part $\|Q^s\|$, due to the fact that we allow possibly unbounded semimartingales. As shown in the Examples in Section 3.6.1 and discussed in Section 3.5, when also the claim B is present and is not sufficiently integrable, in the above duality an additional singular term appears from $Q(B) = E_{Q^r}[B] + Q^s(B)$.

The above duality result (2)-(3) holds under the assumptions that B belongs to the set $\mathcal{A}_u$ of *admissible claims* (see definition 3.2). Even though we admit price processes represented by general semimartingale, the above assumptions on B are weaker than those assumed in the literature for the locally bounded case - see the discussion in Sections 3.1 and 5. This is a nice consequence of the selection of the Orlicz space duality.

Regarding the primal utility maximization problem with random endowment, in Theorem 3.15 we also prove the existence of the optimal solution f_B in a slightly enlarged set than $\{\int_0^T H_t dS_t \mid H \in \mathcal{H}^W\}$. As it happens in the literature for $B = 0$ this optimal solution exists under additional assumptions on the utility function u (or similar growth conditions on its conjugate), which are introduced in Section 3.4.

Since the most well–studied utility function in the class considered in this paper is the exponential utility, we specialize the duality result for this case in Section 3.6, thereby obtaining a generalization of the results in Bellini and Frittelli [BeF02], the "Six Authors paper" [6Au02] and Becherer [Be03]. Some interesting examples of exponential utility optimization with random endowment are presented, where the singular part shows up. These examples are simple, one period market models, but surprising since they display a quite different behavior from the locally bounded case, which is thoroughly interpreted.

While the notion of the indifference price was introduced in 1989 by Hodges and Neuberger [HN89], the analysis of its dual representation in terms of (local) martingale measures was performed in the late '90. It started with Frittelli [F00] and was considerably expanded by [6Au02] and, in a dynamic context, by El Karoui and Rouge [EkR00]. An extensive survey of the recent literature on this topic can be found in [C08], Volume on Indifference Pricing.

Armed with the duality result of Theorem 3.15, the indifference price of a claim B is addressed in Section 4. The classical approach of Convex Analysis - basically the Fenchel-Moreau Theorem - was first applied in Frittelli and Rosazza [FR02] to deduce the dual representation of convex risk measures on L^p spaces. Based on the duality results proven in [F00], in [FR02] it is also shown that, for the exponential utility function, the indifference price of a bounded claim defines - except for the sign - a convex risk measure. In recent years this connection has been deeply investigated by many authors (see Barrieu and N. El Karoui [BK05] and the references therein).

In Section 4 of this paper these results are further extended thanks to the Orlicz space duality framework. This enables us to establish the properties of the indifference price π summarized in Proposition 4.4, including the expected convexity, monotonicity, translation invariance and volume asymptotics. More interestingly, in (65) we provide a new and fairly explicit representation for the indifference price, which is obtained applying recent results from the theory of convex risk measures developed in Biagini and Frittelli [BF07]. In fact, in Proposition 4.4 it is also shown that the map π, as a convex monotone functional on the Orlicz space $L^{\widehat{u}}$, is continuous and subdifferential on the interior of its proper domain $\mathcal{B}$, which is considerably large as it coincides with $-\text{int}(\text{Dom}(I_u))$, i.e. the opposite of the interior of the proper domain of the integral functional $I_u(f) = E[u(f)]$ in $L^{\widehat{u}}$. The minus sign is only due to the fact that $\pi(B)$ is the seller indifference price. In Corollary 4.6 we show that when B and the loss control W are "very nice" (i.e., they are in the special subspace $M^{\widehat{u}}$ of $L^{\widehat{u}}$), the indifference price π has also the Fatou property.

The regularity of the map π itself allows then for a very nice, short proof of some bounds on the indifference price $\pi(B)$ of a fixed claim B as a consequence of the Max Formula in Convex Analysis.

Section 5 concludes the paper with a comparison with the existing literature on utility maximization in incomplete semimartingale markets with random endowment (the reader is deferred to [BF06] and the literature therein for the case of no random endowment) and utility functions finite valued on $\mathbb{R}$ (see Hugonnier and Kramkov [HK04] and the literature therein for utility functions finite valued on $\mathbb{R}_+$).

2 The set up for utility maximization

In this section we recall the set up of [BF06] for the utility maximization problem in an Orlicz space framework with zero random endowment, corresponding to the left–hand side of (1). Similar arguments can then be used in the next section for the optimization problem in the presence of a random endowment as in (2). In particular, the class of admissible integrands as well as the relevant Orlicz spaces and dual variables are the same for both problems.

2.1 Admissible integrands, suitability and compatibility

Given a non–negative random variable $W \in \mathcal{F}_T$, the domain of optimization for the primal problem (2) is the following set of W*–admissible strategies*:

$$\mathcal{H}^W := \left\{ H \in L(S) \mid \exists c > 0 \text{ such that } \int_0^t H_s dS_s \geq -cW, \forall t \in [0,T] \right\}, \tag{5}$$

where $L(S)$ denotes the class of predictable, S-integrable processes. In other words, the random variable W controls the losses in trading. This extension of the classic notion of admissibility, which requires $W = 1$, was already used in Schachermayer ([S94] Section 4.1) in the context of the fundamental theorem of asset pricing, as well as in Delbaen and Schachermayer [DS99].

In order to build a reasonable utility maximization, W should satisfy two conditions that are mathematically useful and economically meaningful. The first condition depends only on the vector process of traded assets S and guarantees that the set of W–admissible strategies is rich enough for trading purposes:

Definition 2.1. We say that a random variable $W \geq 1$ is *suitable* for the process S if for each $i = 1, \dots, d$, there exists a process $H^i \in L(S^i)$ such that

$$P(\{\omega \mid \exists t \geq 0 \text{ such that } H^i_t(\omega) = 0\}) = 0 \tag{6}$$

and

$$\left| \int_0^t H^i_s dS^i_s \right| \leq W, \quad \forall t \in [0,T]. \tag{7}$$

The class of suitable random variables is denoted by $\mathbb{S}$.

The second condition depends only on the utility function and measures to what extent the investor accepts the risk of a large loss:

Definition 2.2. We say that a positive random variable W is *strongly compatible* with the utility function u if

$$E[u(-\alpha W)] > -\infty \text{ for all } \alpha > 0 \tag{8}$$

and that it is *compatible* with u if

$$E[u(-\alpha W)] > -\infty \text{ for some } \alpha > 0. \tag{9}$$

Given a suitable and compatible random variable W, the first step to apply duality arguments to problem (2) is to rewrite it in terms of an optimization over random variables, as opposed to an optimization over stochastic processes. To this end, we define the set of terminal values obtained from W–admissible trading strategies as

$$K^W = \left\{ \int_0^T H_t dS_t \mid H \in \mathcal{H}^W \right\}, \tag{10}$$

and consider the modified primal problem

$$\sup_{k \in K^W} E[u(x+k)]. \tag{11}$$

The next step is to identify a good dual system and invoke some duality principle. Classically, the system (L^∞, ba) has been successfully used when dealing with locally bounded traded assets. In order to accommodate more general markets and inspired by the compatibility conditions above, in the next section we argue instead for the use of an appropriate Orlicz spaces duality, naturally induced by the utility function.

Remark 2.3. When S is locally bounded, $W = 1$ is automatically suitable and compatible (see [BF05], Proposition 1), and we recover the familiar set of trading strategies. Therefore, the locally bounded setup is a special case of our more general framework.

Remark 2.4. The conditions of suitability and compatibility on W put integrability restrictions on the jumps of the semimartingale S. For a toy example that illustrates the various situations, see [BF06, Example 4].

Remark 2.5. It is not difficult (see for instance Biagini [B04], where the utility maximization for possibly non locally bounded semimartingales was addressed with a new class of strategies) to build a different set up, where the definitions of admissibility, suitability and compatibility are formulated in terms of stochastic processes, instead of random variables, leading to an *adapted* control of the losses from trading.

A real, adapted and nonnegative process Y could be defined to be suitable to S if for each $i = 1, \ldots, d$, there exists a process $H^i \in L(S^i)$ satisfying (6) and $|H^i dS^i| \leq Y$, and to be compatible with u if

$$E[u(-\alpha Y_T^*)] > -\infty \text{ for some } \alpha > 0,$$

where $Y_t^* = \sup_{s \leq t} |Y_s|$ is the maximal process of Y. The admissible integrands become then

$$\mathcal{H}^Y := \{H \in L(S) \mid \exists c > 0 \text{ such that } HdS \geq -cY\}.$$

It is then easy to check that if a process Y satisfies the two requirements above, then the random variable $W := Y_T^*$ is suitable and compatible, in the sense of the Definitions 2.1, 2.2, and that

$\mathcal{H}^Y \subset \mathcal{H}^W$. This shows that this set up with processes does not achieve more generality than that one with random variables and for this reason we continue to use the framework described in (5) and in Definitions 2.1 and 2.2.

Remark 2.6. Alternatively, the same definition of suitability as in the previous remark could be used, but the process Y could be defined to be compatible with u if it satisfies the following less stringent condition:

$$E[u(-\alpha_t Y_t)] > -\infty \text{ for some } \alpha_t > 0, \text{ for all } t \in [0, T]. \tag{12}$$

The problem with this definition is that in general (12) does not guarantee the existence of a uniform bound (in the form of a single random variable) on the stochastic integrals $\int H_t dS_t$ satisfying the integrability condition required for the Ansel and Stricker Lemma [AS94]. To the best of our knowledge, without this latter result one cannot show that the regular elements of the dual variables are sigma martingale measures, a key property that justifies the interpretation of the dual variables as pricing measures (see the subsequent Section 2.3 or [DS98], [B04, Prop. 6], [BF05, Prop. 6] and [BF06, Prop. 19]).

2.2 The Orlicz space framework

This new framework for utility maximization was first introduced by Biagini [B08] and then considerably expanded in [BF06], upon which this section is mostly based. The key observation is that the function $\widehat{u} : \mathbb{R} \to [0, +\infty)$ defined as

$$\widehat{u}(x) = -u(-|x|) + u(0),$$

is a Young function (a reference book is [RR91]). Thus, its corresponding Orlicz space

$$L^{\widehat{u}}(\Omega, \mathcal{F}, P) = \{f \in L^0(\Omega, \mathcal{F}, P) \mid E[\widehat{u}(\alpha f)] < +\infty \text{ for some } \alpha > 0\},$$

is a Banach space (and a Banach lattice) when equipped with the Luxemburg norm

$$N_{\widehat{u}}(f) = \inf \left\{ c > 0 \mid E\left[\widehat{u}\left(\frac{f}{c}\right)\right] \leq 1 \right\}. \tag{13}$$

Since the probability space $(\Omega, \mathcal{F}, P)$ is fixed throughout the paper, set $L^p := L^p(\Omega, \mathcal{F}, P)$, $p \in [0, +\infty]$, and $L^{\widehat{u}} := L^{\widehat{u}}(\Omega, \mathcal{F}, P)$. Under our assumptions on the utility u, it is not difficult to see that $L^\infty \subseteq L^{\widehat{u}} \subseteq L^1$. Next consider the subspace of "very integrable" elements in $L^{\widehat{u}}$

$$M^{\widehat{u}} := \{f \in L^{\widehat{u}} \mid E[\widehat{u}(\alpha f)] < +\infty \text{ for all } \alpha > 0\}.$$

Due to the fact that $\widehat{u}$ is continuous and finite on $\mathbb{R}$, $M^{\widehat{u}}$ contains L^∞ and moreover it coincides with the closure of L^∞ with respect to the Luxemburg norm. However, the inclusion $M^{\widehat{u}} \subset L^{\widehat{u}}$ is in general strict, since bounded random variables are not necessarily dense in $L^{\widehat{u}}$ (see [RR91, Prop. III.4.3 and Cor. III.4.4]). This will play a central role in our work.

As observed in [B08] and [BF06], the Young function $\widehat{u}$ carries information about the utility on large losses, in the sense that for $\alpha > 0$ we have

$$E[\widehat{u}(\alpha f)] < +\infty \qquad \Longleftrightarrow \qquad E[u(-\alpha|f|)] > -\infty, \tag{14}$$

a characterization that will be repeatedly used in what follows. For instance, using (14) it is easy to see that

- *a positive random variable W is strongly compatible (resp. compatible) with the utility function u if and only if $W \in M^{\widehat{u}}$ (resp. $W \in L^{\widehat{u}}$).*

When $W \in L^{\widehat{u}}$, the negative part of each element in K^W belongs to $L^{\widehat{u}}$, but in general we do not have the inclusion $K^W \subseteq L^{\widehat{u}}$.

¿From the definition of Φ we know that $\Phi(0) = u(+\infty)$, Φ is bounded from below and it satisfies $\lim_{y\to+\infty} \frac{\Phi(y)}{y} = +\infty$. This limit is a consequence of u being finite valued on $\mathbb{R}$. Indeed, from the inequality $\Phi(y) \geq u(x) - xy$ for all $x, y \in \mathbb{R}$, we get $\liminf_{y\to+\infty} \frac{\Phi(y)}{y} \geq \liminf_{y\to+\infty} \frac{u(x)}{y} - x = -x$ for all $x \in \mathbb{R}$.

The convex conjugate of $\widehat{u}$, called the *complementary* Young function in the theory of Orlicz spaces, is denoted here by $\widehat{\Phi}$, since it admits the representation

$$\widehat{\Phi}(y) = \begin{cases} 0 & \text{if } |y| \leq \beta \\ \Phi(|y|) - \Phi(\beta) & \text{if } |y| > \beta \end{cases} \tag{15}$$

where $\beta \geq 0$ is the right derivative of $\widehat{u}$ at 0, namely $\beta = D^+\widehat{u}(0) = D^-u(0)$, and $\Phi(\beta) = u(0)$. If u is differentiable, note that $\beta = u'(0)$ and it is the unique solution of the equation $\Phi'(y) = 0$.

¿From (15) it then follows that $\widehat{\Phi}$ is also a Young function, which induces the Orlicz space $L^{\widehat{\Phi}}$ endowed with the Orlicz (dual) norm

$$\|g\|_{\widehat{\Phi}} = \sup\{E[|fg|] \mid E[\widehat{u}(g)] \leq 1\}.$$

As before, $L^\infty \subseteq L^{\widehat{\Phi}} \subseteq L^1$. Moreover, L^Φ is a dual space, as

$$(M^{\widehat{u}})^* = L^{\widehat{\Phi}}, \tag{16}$$

in the sense that if $Q \in (M^{\widehat{u}})^*$ is a continuous linear functional on $M^{\widehat{u}}$, then there exists a unique $g \in L^{\widehat{\Phi}}$ such that

$$Q(f) = \int_\Omega fg\,dP, \qquad f \in M^{\widehat{u}},$$

with

$$\|Q\|_{(M^{\widehat{u}})^*} := \sup_{N_{\widehat{u}}(f)\leq 1} |Q(f)| = \|g\|_{\widehat{\Phi}}.$$

The characterization of the topological dual for the larger space $L^{\widehat{u}}$ is more demanding than (16). For the complementary pair of Young functions $(\widehat{u}, \widehat{\Phi})$, it follows from [RR91, Cor. IV.2.9] that each element $Q \in (L^{\widehat{u}})^*$ can be uniquely expressed as

$$Q = Q^r + Q^s,$$

where the *regular* part Q^r is given by

$$Q^r(f) = \int_\Omega fg\,dP, \qquad f \in L^{\widehat{u}},$$

for a unique $g \in L^{\widehat{\Phi}}$, and the *singular* part Q^s satisfies

$$Q^s(f) = 0, \qquad \forall f \in M^{\widehat{u}}. \tag{17}$$

In other words,

$$(L^{\widehat{u}})^* = (M^{\widehat{u}})^* \oplus (M^{\widehat{u}})^{\perp}$$

where $(M^{\widehat{u}})^{\perp} = \{z \in (L^{\widehat{u}})^* \mid z(f) = 0, \forall f \in M^{\widehat{u}}\}$ denotes the annihilator of $M^{\widehat{u}}$.

Consider now the concave integral functional $I_u : L^{\widehat{u}} \to [-\infty, \infty)$ defined as

$$I_u(f) := E[u(f)].$$

As usual, its effective domain is denoted by

$$\mathrm{Dom}(I_u) := \left\{ f \in L^{\widehat{u}} \mid E[u(f)] > -\infty \right\}.$$

It was shown in [BF06, Lemma 17] that thanks to the selection of the appropriate Young function $\widehat{u}$ associated with the utility function u, the norm of a *nonnegative* singular element $z \in (M^{\widehat{u}})^{\perp}$ satisfies

$$\|z\|_{(L^{\widehat{u}})^*} := \sup_{N_{\widehat{u}}(f) \le 1} z(f) = \sup_{f \in \mathrm{Dom}(I_u)} z(-f). \tag{18}$$

2.3 Loss and dual variables

¿From now on, the loss controls W are assumed suitable and compatible, i.e. $W \in \mathbb{S} \cap L^{\widehat{u}}$, and will simply be referred to as *loss variables.* Given such W, the cone

$$C^W = (K^W - L^0_+) \cap L^{\widehat{u}},$$

corresponds to random variables that can be super–replicated by trading strategies in $\mathcal{H}^W$ and that satisfy the same type of integrability condition of W. The polar cone of C^W, which will play a role in the dual problem, is

$$(C^W)^0 := \left\{ Q \in (L^{\widehat{u}})^* \mid Q(f) \le 0, \quad \forall f \in C^W \right\}, \tag{19}$$

and it satisfies $(C^W)^0 \subseteq (L^{\widehat{u}})^*_+$, since $(-L^{\widehat{u}}_+) \subseteq C^W$. Therefore, all the functionals of interest are positive and the decomposition $Q = Q^r + Q^s$ enables the identification of Q^r with a measure with density $\frac{dQ^r}{dP} \in L^{\widehat{\Phi}}_+ \subseteq L^1_+$. The subset of normalized functionals in $(C^W)^0$ is defined by

$$\mathcal{M}^W := \{Q \in (C^W)^0 \mid Q(\mathbf{1}_\Omega) = 1\}. \tag{20}$$

Using the notation above, we see that this normalization condition reduces to $Q^r(\mathbf{1}_\Omega) = 1$, since $Q^s \in (M^{\widehat{u}})^{\perp}$ and thus vanishes on any bounded random variable. In other words, *the regular part of any element in $\mathcal{M}^W$ is a true probability measure with density in $L^{\widehat{\Phi}}_+$.* Moreover, it was shown in [BF06, Proposition 19], that

$$\mathcal{M}^W \cap L^1 = \mathbb{M}_\sigma \cap L^{\widehat{\Phi}}, \tag{21}$$

where

$$\mathbb{M}_\sigma = \left\{ Q(\mathbf{1}_\Omega) = 1, \frac{dQ}{dP} \in L^1_+ \mid S \text{ is a } \sigma - \text{martingale w.r.t. } Q \right\}$$

consists of all the $P-$absolutely continuous σ-martingale measures for S, i.e. of those $Q \ll P$ for which there exists a process $\eta \in L(S)$ such that $\eta > 0$ and the stochastic integral $\int \eta dS$ is a Q–martingale. Such probabilities Q were introduced in the context of Mathematical Finance by

Delbaen and Schachermayer in the seminal [DS98], which the reader is referred to for a thorough analysis of their financial significance as pricing measures.

From (21) it follows that the regular elements of the normalized set $\mathcal{M}^W$ coincide with the σ-martingale measures for S that belong to $L^{\widehat{\Phi}}$. In particular *this shows that the (possibly empty) set $\mathcal{M}^W \cap L^1$ does not depend on the particular loss variable W.*

2.4 Utility optimization with no random endowment

The following theorem is a reformulation of [BF06, Theorem 21]. When S is locally bounded, a duality formula similar to (22) - but with no singular components - holds true for all utility functions in the class considered in this paper. This latter fact is well known and was first shown in [BeF02].

Theorem 2.7. *Suppose that there exists a loss variable W satisfying*

$$\sup_{H\in\mathcal{H}^W} E\left[u\left(x+\int_0^T H_t dS_t\right)\right] < u(+\infty).$$

Then $\mathcal{M}^W$ is not empty and

$$\sup_{H\in\mathcal{H}^W} E\left[u\left(x+\int_0^T H_t dS_t\right)\right] = \min_{\lambda>0,\ Q\in\mathcal{M}^W} \left\{\lambda x + E\left[\Phi\left(\lambda\frac{dQ^r}{dP}\right)\right] + \lambda\|Q^s\|\right\}. \tag{22}$$

When $W \in M^{\widehat{u}}$, then the set $\mathcal{M}^W$ can be replaced by $\mathbb{M}_\sigma \cap L^{\widehat{\Phi}}$ and no singular term appears in the duality formula above.

The last statement in the theorem follows from the observation that when $W \in M^{\widehat{u}}$ then the regular component Q^r of $Q \in \mathcal{M}^W$ is already in $\mathcal{M}^W$ (see [BF06, Lemma 41]). Since $\|Q^s\| \geq 0$ this immediately implies that the minimum in (22) is reached on the set $\left\{Q^r \mid Q \in \mathcal{M}^W\right\} = \mathbb{M}_\sigma \cap L^{\widehat{\Phi}}$.

3 Utility optimization with random endowment

3.1 Conditions on the claim

We now turn to the right–hand side of (1) and consider the optimization problem

$$\sup_{H\in\mathcal{H}^W} E\left[u\left(x+\int_0^T H_t dS_t - B\right)\right], \tag{23}$$

where $B \in \mathcal{F}_T$ is a liability faced at terminal T.

Without loss of generality, let $x = 0$. The case with non null initial endowment can clearly be recovered by replacing B with $(B-x)$. In view of the substitution of terminal wealths $\int_0^T H_t dS_t \in K^W$ by random variables $f \in C^W \subset L^{\widehat{u}}$, we require that B satisfies

$$E[u(f-B)] < +\infty, \qquad \forall f \in L^{\widehat{u}}, \tag{24}$$

so that the concave functional $I_u^B : L^{\widehat{u}} \to [-\infty,\infty)$ given by

$$I_u^B(f) := E[u(f-B)]$$

is well defined for such claims.

Remark 3.1. The set of claims satisfying this condition is quite large. In fact, by monotonicity and concavity of u,

$$E[u(f-B)] = E[u(f-(B^+-B^-))] \leq E[u(f+B^-)] \leq u(E[f]+E[B^-]),$$

where the last step follows from Jensen's inequality. Therefore, since $f \in L^{\hat{u}} \subset L^1$, one obtains that a simple sufficient condition for (24) is that $B^- \in L^1$.

Obviously, when the utility function is bounded above (as for example in the exponential case) the condition (24) is satisfied by *any* claim.

A second natural condition on B is that it does not lead to prohibitive punishments when the agent chooses the trading strategy $H \equiv 0 \in \mathcal{H}^W$. In other words, we would like to impose that $E[u(-B)] > -\infty$. Since the utility function is finite and increasing, this is equivalent to

$$E[u(-B^+)] > -\infty, \tag{25}$$

which in turn implies that $-B^+ \in \mathrm{Dom}(I_u)$ and consequently $B^+ \in L^{\hat{u}}$, in view of (14). Be aware that $B^+ \in L^{\hat{u}}$ does not necessarily imply (25).

However, for the main duality result we also need that the claim B satisfies:

$$E[u(-(1+\epsilon)B^+)] > -\infty, \qquad \text{for some } \epsilon > 0. \tag{26}$$

This condition, stronger than (25), is equivalent to requiring that the random variable $(-B^+)$ belongs to $\mathrm{int}(\mathrm{Dom}(I_u))$, the interior in $L^{\widehat{u}}$ of the effective domain of I_u. This is a consequence of Lemma 30 in [BF06], which in turn is based on the definition of the Luxemburg norm on $L^{\widehat{u}}$ and on a simple convexity argument. In addition to its technical relevance (shown in Lemma 3.4), another reason for adopting (26) is explained in Remark 3.5.

Definition 3.2. The set of *admissible* claims $\mathcal{A}_u$ consists of $\mathcal{F}_T$ measurable random variables B satisfying (24) and (26).

The conditions (24), (26) do not really capture the risks corresponding to B^-, which are *gains* for the seller of the claim. For example, it is quite possible to have $B \in \mathcal{A}_u$ and $E[u(-\varepsilon B^-)] = -\infty$ for all $\varepsilon > 0$ (simply take $B^- \in L^1 \backslash L^{\hat{u}}$). This would mean that a *buyer* using the same utility function u, investment opportunities S and control random variable W, would incur losses leading to an infinitely negative expected utility simply by holding any fraction of the claim and doing no other investment. Such undesirable outcome can be avoided by the more stringent condition

$$E[u(-\varepsilon B^-)] > -\infty, \qquad \text{for some } \varepsilon > 0, \tag{27}$$

which is equivalent to $B^- \in L^{\hat{u}}$. Since the focus is on the problem faced by the seller of the claim B, **we refrain from assuming** (27)**, until Section 4 where** B **will belong to the set**

$$\mathcal{B} := \mathcal{A}_u \cap L^{\widehat{u}} = \{B \in L^{\widehat{u}} \mid E[u(-(1+\epsilon)B^+)] > -\infty \text{ for some } \epsilon > 0\}. \tag{28}$$

In any event, the potential buyers for B will likely not have the same investment opportunities, utility function and loss tolerance as the seller, leading to entirely different versions of (1).

For example, suppose that the seller has an exponential utility $u_s(x) = -e^{-x}$ and the buyer has quadratic utility $u_b(x) = -x^2$ for $x \leq -1$, then prolonged so that it is bounded above and satisfies all the other requirements. Then take B so that B^+ has an exponential distribution with parameter $\lambda = 2$ and B^- has a density $\frac{c}{1+x^4}, x \geq 0$, where c is the normalizing constant. It is easy to check that B satisfies (24) and (26) for u_s and that selling B is very attractive, since the tail of the distribution of B^- (the gain for the seller) is much bigger than that of B^+ (the loss for the seller). B^- has no finite exponential moment, therefore it violates (27) and would clearly be unacceptable if the buyer had exponential preferences. However the quadratic tolerance of the losses of u_b accounts for a well posed maximization problem with B even for the buyer.

3.2 The maximization

The first step in our program consists in showing that optimizing over the cone C^W leads to the same expected utility as optimizing over the set of terminal wealths K^W.

Lemma 3.3. *If B satisfies (24) and (25) then*

$$\sup_{k \in K^W} E[u(k-B)] = \sup_{f \in C^W} E[u(f-B)]. \tag{29}$$

Proof. Since $C^W \subset (K^W - L^0_+)$ and the utility function is monotone increasing,

$$\sup_{f \in C^W} E[u(f-B)] \leq \sup_{g \in (K^W - L^0_+)} E[u(g-B)] \leq \sup_{k \in K^W} E[u(k-B)]. \tag{30}$$

Since $k \equiv 0 \in K^W$,

$$\sup_{k \in K^W} E[u(k-B)] \geq E[u(-B)] > -\infty.$$

by (25). Pick any $k \in K^W$ satisfying $E[u(k-B)] > -\infty$. Consider $k_n = k \wedge n$, which is in C^W since $W \in L^{\widehat{u}}$ (this is the only assumption needed on W here). Then

$$u(k_n - B) = u(k^+ \wedge n - B) I_{\{k \geq 0\}} + u(-k^- - B) I_{\{k<0\}} \geq u(-B) + u(-k^- - B) I_{\{k<0\}}$$

and the latter is integrable. An application of the monotone convergence theorem gives $E[u(k_n - B)] \nearrow E[u(k-B)]$, which implies that

$$\sup_{k \in K^W} E[u(k-B)] \leq \sup_{f \in C^W} E[u(f-B)],$$

and completes the proof. □

The next step in the program is to establish that the functional I_u^B has a norm continuity point contained in the cone of interest C^W.

Lemma 3.4. *Suppose that B satisfies (24) and (25). Then the concave functional I_u^B is norm continuous on the interior of its effective domain. Moreover, if $B \in \mathcal{A}_u$ then there exists a norm continuity point of I_u^B that belongs to C^W.*

Proof. Since $I_u^B < +\infty$, the functional I_u^B is proper, monotone and concave. The first sentence in the Lemma thus follows from the Extended Namioka-Klee Theorem (see [RS06] or [BF07]). Denoting the unit ball in $L^{\hat{u}}$ by $\mathcal{S}_1$, it follows rather easily from a convexity argument that the hypothesis (26) on B^+ implies that

$$-B^+ + \frac{\epsilon}{1+\epsilon}\mathcal{S}_1 \subset \mathrm{Dom}(I_u), \tag{31}$$

and therefore $\frac{\epsilon}{1+\epsilon}\mathcal{S}_1 \subset \mathrm{Dom}(I_u^B)$. Therefore, any element of $\frac{\epsilon}{2(1+\epsilon)}\mathcal{S}_1 \cap (-L_+^{\widehat{u}})$ is then in C^W and a continuity point for I_u^B. □

Remark 3.5. At first sight, the condition (26) on the positive part B^+ appears to be an ad-hoc hypothesis imposed for the sake of proving the previous technical lemma. We argue, however, that (26) is in fact a natural condition to impose on a financial liability in this context. Indeed, if the claim B satisfies only $E[u(-B^+)] > -\infty$, it may happen - contrary to (31) - that

$$E[u(-B^+ - c)] = -\infty \text{ for all constants } c > 0$$

as shown in the example below, which would restrict the possibility of any significative trading.

Example 3.6. Consider the smooth function

$$u(x) = \begin{cases} -e^{(x-1)^2} & x \leq 0 \\ -2e^{-x+1} + e & x > 0 \end{cases}$$

as utility function u (the particular expression of u for $x > 0$ is however irrelevant). Consider now a (positive) claim B with distribution $d\mu_B = k\frac{e^{-x^2-2x}}{x^2+1}I_{\{x>0\}}dx$, where k is the normalizing constant. Then,

$$E[u(-B)] = \int_0^{+\infty} -e^{(-x-1)^2}k\frac{e^{-x^2-2x}}{x^2+1}dx > -\infty$$

but for any $c > 0$,

$$E[u(-B-c)] = \int_0^{+\infty} -\frac{e^{2cx+(c+1)^2}k}{x^2+1}dx = -\infty.$$

3.3 Conjugate functionals

As already discussed, the condition $B \in \mathcal{A}_u$ on the claim B does not necessarily imply that $B \in L^{\widehat{u}}$. Therefore, we need appropriate extensions of linear functionals on $L^{\widehat{u}}$.

Though we sketch the proof for the sake of completeness, this extension is morally straightforward. In fact, it is defined in the same way as the expectation $E[g]$ is defined when g is bounded from below, instead of bounded. In this case, $g^- \in L^\infty$ and

$$E[g] := \sup\{E[f] \mid f \in L^\infty, f \leq g\} = \lim_n E[g \wedge n]$$

Accordingly, let us consider the convex cone of random variables with negative part in $L^{\widehat{u}}$:

$$L_{neg}^{\widehat{u}} := \left\{f \in L^0 \mid f^- \in L^{\widehat{u}}\right\} = \left\{f \in L^0 \mid E[u(-\alpha f^-)] > -\infty, \text{ for some } \alpha > 0\right\},$$

and notice that this cone contains K^W, for any loss variable W. For any $Q \in (L^{\widehat{u}})_+^*$ we define $\widehat{Q} : L_{neg}^{\widehat{u}} \to \mathbb{R} \cup \{+\infty\}$ by

$$\widehat{Q}(g) \triangleq \sup\left\{Q(f) \mid f \in L^{\widehat{u}} \text{ and } f \leq g\right\}. \tag{32}$$

Lemma 3.7. *If $Q \in (L^{\widehat{u}})^*_+$ then*

1. *$\widehat{Q}$ is a well-defined extension of Q. It is a positively homogenous, additive (with the convention $+\infty + c = +\infty$ for $c \in \mathbb{R} \cup \{+\infty\}$), monotone functional on the cone $L^{\widehat{u}}_{neg}$. In particular, if $g \in L^{\widehat{u}}_{neg}, h \in L^{\widehat{u}}$ then $\widehat{Q}(g+h) = \widehat{Q}(g) + Q(h)$.*

2. *$Q \in (C^W)^0$ if and only if $\widehat{Q}(k) \le 0$ for all $k \in K^W$.*

3. *If $g \in L^{\widehat{u}}_{neg}$ is such that $E[u(g)] > -\infty$, then*

$$\|Q^s\| \ge -\widehat{Q^s}(g) \tag{33}$$

4. *If $\widehat{Q}(g)$ is finite, then $\widehat{Q}(g) = E[\frac{dQ^r}{dP} g] + \widehat{Q^s}(g)$.*

Proof. The first two statements and item 4 follow rather directly from the definitions of $\widehat{Q}$ and C^W. We only prove additivity of $\widehat{Q}$. Fix then $g_1, g_2 \in L^{\widehat{u}}_{neg}$. We want to show that $\widehat{Q}(g_1+g_2) = \widehat{Q}(g_1) + \widehat{Q}(g_2)$. When $f_i \in L^{\widehat{u}}$, $i = 1,2$, satisfy $f_i \le g_i$,

$$\widehat{Q}(g_1) + \widehat{Q}(g_2) = \sup_{f_i \le g_i} \{Q(f_1) + Q(f_2)\} \le \sup_{f \le g_1 + g_2} Q(f) = \widehat{Q}(g_1 + g_2).$$

To show the opposite inequality, assume first that $g_i \ge 0$. Fix $f \in L^{\widehat{u}}_+$, $f \le g_1 + g_2$. Then $f \wedge g_i \in L^{\widehat{u}}$, $i = 1,2$, and, moreover, $f \le f \wedge g_1 + f \wedge g_2$. Therefore

$$Q(f) \le Q(f \wedge g_1) + Q(f \wedge g_2) \le \widehat{Q}(g_1) + \widehat{Q}(g_2) \text{ for all } f \in L^{\widehat{u}}_+,\ f \le g_1 + g_2$$

so that $\widehat{Q}(g_1 + g_2) \le \widehat{Q}(g_1) + \widehat{Q}(g_2)$. To treat the case g_1 and g_2 not necessarily positive, observe that when $g \in L^{\widehat{u}}_{neg}, h \in L^{\widehat{u}}$:

$$\begin{aligned} \widehat{Q}(g+h) &= \sup\Big\{Q(f) \mid f \in L^{\widehat{u}},\ f \le g + h\Big\} \\ &= \sup\Big\{Q(f) \mid f \in L^{\widehat{u}},\ f \le g\Big\} + Q(h) = \widehat{Q}(g) + Q(h), \end{aligned}$$

As a consequence, $\widehat{Q}(g_i) = \widehat{Q}(g_i^+) - Q(g_i^-)$ and $\widehat{Q}(g_1 + g_2) = \widehat{Q}(g_1^+ + g_2^+) - Q(g_1^- + g_2^-)$. Collecting these relations,

$$\widehat{Q}(g_1+g_2) = \widehat{Q}(g_1^+ + g_2^+) - Q(g_1^- + g_2^-) \le \widehat{Q}(g_1^+) + \widehat{Q}(g_2^+) - Q(g_1^- + g_2^-) = \widehat{Q}(g_1) + \widehat{Q}(g_2).$$

Item 3 follows from $-g^- \in \mathrm{Dom}(I_u)$ and equation (18):

$$\|Q^s\| \ge Q^s(g^-) \ge Q^s(g^-) - \widehat{Q^s}(g^+) = -\widehat{Q^s}(g).$$

□

Finally, when B satisfies (25) the *convex conjugate* $J^B_u : (L^{\widehat{u}})^* \to \mathbb{R} \cup \{+\infty\}$ of the concave functional I^B_u is defined as

$$J^B_u(Q) := \sup_{f \in L^{\widehat{u}}} \{E[u(f - B)] - Q(f)\}, \quad Q \in (L^{\widehat{u}})^*. \tag{34}$$

The following Lemma gives a representation of J^B_u.

Lemma 3.8. *1. If $B \in L^{\widehat{u}}$ and $Q \in (L^{\widehat{u}})^*_+$ then*

$$J_u^B(Q) = Q(-B) + E\left[\Phi\left(\frac{dQ^r}{dP}\right)\right] + \|Q^s\|. \tag{35}$$

2. If B satisfies (24) and (25) and $Q \in (L^{\widehat{u}})^*_+$, then

$$J_u^B(Q) = \widehat{Q}(-B) + E\left[\Phi\left(\frac{dQ^r}{dP}\right)\right] + \|Q^s\|.$$

Proof. 1. This is an elementary consequence of the representation result proved in [K79, Theorem 2.6]. In fact, from $B \in L^{\widehat{u}}$,

$$\begin{aligned} J_u^B(Q) &= \sup_{f \in L^{\widehat{u}}} \{E[u(f-B)] - Q(f)\} \\ &= \sup_{g \in L^{\widehat{u}}} \{E[u(g)] - Q(g)\} - Q(B) \end{aligned}$$

and in the cited Theorem, Kozek proved that

$$\sup_{g \in L^{\widehat{u}}} \{E[u(g)] - Q(g)\} = E\left[\Phi\left(\frac{dQ^r}{dP}\right)\right] + \sup_{g \in \mathrm{Dom}(I_u)} Q^s(-g)$$

so that the thesis in (35) is enabled by (18).

2. The thesis follows from the equality

$$\begin{aligned} &\sup_{G \geq B, G \in L^{\widehat{u}}} \left\{ \sup_{f \in L^{\widehat{u}}} \{E[u(f-G)] - Q(f)\} \right\} \\ =\ &\sup_{f \in L^{\widehat{u}}} \left\{ \sup_{G \geq B, G \in L^{\widehat{u}}} \{E[u(f-G)] - Q(f)\} \right\}. \end{aligned}$$

Indeed, thanks to (35), the left hand side gives

$$\begin{aligned} &\sup_{G \geq B, G \in L^{\widehat{u}}} \left\{ \sup_{f \in L^{\widehat{u}}} \{E[u(f-G)] - Q(f)\} \right\} \\ =\ &\sup_{G \geq B, G \in L^{\widehat{u}}} J_u^G(Q) = \widehat{Q}(-B) + E\left[\Phi\left(\frac{dQ^r}{dP}\right)\right] + \|Q^s\|, \end{aligned}$$

while the right hand side gives

$$\sup_{f \in L^{\widehat{u}}} \left\{ \sup_{G \geq B, G \in L^{\widehat{u}}} \{E[u(f-G)] - Q(f)\} \right\} = \sup_{f \in L^{\widehat{u}}} \{E[u(f-B)] - Q(f)\} = J_u^B(Q). \tag{36}$$

The first equality in (36) holds thanks to the following approximation argument. For each $f \in L^{\widehat{u}}$ such that $E[u(f-B)] > -\infty$ let $G_n := B^+ - (f^- + n) \wedge B^- \in L^{\widehat{u}}$ and $A_n := \{B^- \leq f^- + n\}$. Our assumptions imply that $u(f-B)$ and $u(-B^+)$ are integrable and so from

$$u(f-G_n) = u(f-B)1_{A_n} + u(f-B^+ + f^- + n)1_{A_n^c} \geq u(f-B)1_{A_n} + u(-B^+)1_{A_n^c}$$

we deduce $E[u(f-G_n)] > -\infty$. Since $-G_n \uparrow -B$, the monotone convergence theorem guarantees $\sup_n E[u(f-G_n)] = E[u(f-B)]$.

□

3.4 The dual optimization and a new primal domain

Before establishing the main duality result, let us focus on dual optimizations of the form

$$\inf_{\lambda>0, Q\in\mathcal{N}} \left\{ E\left[\Phi\left(\lambda\frac{dQ^r}{dP}\right)\right] + \lambda\widehat{Q}(-B) + \lambda\|Q^s\| \right\}, \tag{37}$$

where $\mathcal{N}$ is a convex subset of $(L^{\widehat{u}})^*_+$. Problems of this type (but for $\mathcal{N} \subseteq L^1_+$ and $B=0$) were originally solved by Ruschendorf [R84]. A general strategy for tackling such problems is to consider the minimizations over λ and over Q separately. Accordingly, in the next Proposition we fix $\lambda > 0$ and explore the consequences of optimality in Q. The result is the analogue of [BF06, Prop 25] but in the presence of the claim B and the corresponding extended functionals $\widehat{Q}$.

The presence of the scaling factor λ in the expectation term in (37) leads us to consider the convex set:

$$\mathcal{L}_\Phi = \{Q \text{ probab}, Q \ll P \mid E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right] < +\infty \text{ for some } \lambda > 0\}.$$

Clearly $\left\{\frac{dQ}{dP} \mid Q \in \mathcal{L}_\Phi\right\} \subseteq L^{\widehat{\Phi}}_+ \subseteq L^1_+$, but the condition $\frac{dQ}{dP} \in L^{\widehat{\Phi}}_+$ does not in general imply $E\left[\Phi(\frac{dQ}{dP})\right] < +\infty$, nor $Q \in \mathcal{L}_\Phi$. Indeed, the utility function may be unbounded from above, so that $\Phi(0) = +\infty$ is possible.

Remark 3.9. The fact that the set $\mathcal{L}_\Phi$ is convex requires a brief explanation, since $\Phi(0) = +\infty$ is possible. Let $Q_y = yQ_1 + (1-y)Q_2$, $y \in (0,1)$, be the convex combination of any couple of elements in $\mathcal{L}_\Phi$, take $\lambda_i > 0$ satisfying $E\left[\Phi\left(\lambda_i\frac{dQ_i}{dP}\right)\right] < \infty$, $i = 1,2$, and define z_y as the convex combination of $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$, i.e.: $z_y := y\frac{1}{\lambda_1} + (1-y)\frac{1}{\lambda_2} \in (0,\infty)$. As a consequence of the convexity of the function $(z,k) \to z\Phi(\frac{1}{z}k)$ on $\mathbb{R}_+ \times \mathbb{R}_+$(which has been pointed out by [SW05], Section 3) we deduce

$$E\left[z_y\Phi\left(\frac{1}{z_y}\frac{dQ_y}{dP}\right)\right] \le y\frac{1}{\lambda_1}E\left[\Phi\left(\lambda_1\frac{dQ_1}{dP}\right)\right] + (1-y)\frac{1}{\lambda_2}E\left[\Phi\left(\lambda_2\frac{dQ_2}{dP}\right)\right] < \infty.$$

Assumption (A) *The utility function $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing, strictly concave, continuously differentiable and it satisfies the conditions*

$$\lim_{x\downarrow-\infty} u'(x) = +\infty, \ \lim_{x\uparrow\infty} u'(x) = 0 \tag{38}$$

$$\mathcal{L}_\Phi = \{Q \text{ probab}, Q \ll P \mid E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right] < +\infty \text{ for all } \lambda > 0\}. \tag{39}$$

The condition expressed in (39) coincides with assumption (3) in [BF06]. A detailed discussion on assumption (A) and the relationship of (39) with the condition of Reasonable Asymptotic Elasticity introduced by Schachermayer [S01] can be found in [BF06], [BF05]. We stress that this assumption is needed only when dealing with the existence of optimal solutions (i.e. only in Propositions 3.10, 3.11, 3.18, and in the second part of Theorem 3.15).

Proposition 3.10. *Suppose that the utility u satisfies assumption A and that the claim B satisfies (24) and (25) . Fix $\lambda > 0$ and suppose that $\mathcal{N} \subseteq (L^{\widehat{u}})^*_+$ is a convex set such that for any $Q \in \mathcal{N}$ we have $Q^r \in \mathcal{L}_\Phi$. If $Q_\lambda \in \mathcal{N}$ is optimal for*

$$\inf_{Q\in\mathcal{N}} \left\{ E\left[\Phi\left(\lambda\frac{dQ^r}{dP}\right)\right] + \lambda\widehat{Q}(-B) + \lambda\|Q^s\| \right\} < +\infty \tag{40}$$

then, $\forall Q \in \mathcal{N}$ *with* $\widehat{Q}(-B) < +\infty$

$$E_{Q^r_\lambda}\left[\Phi'\left(\lambda\frac{dQ^r_\lambda}{dP}\right)\right] + \widehat{Q}_\lambda(-B) + \|Q^s_\lambda\| \leq E_{Q^r}\left[\Phi'\left(\lambda\frac{dQ^r_\lambda}{dP}\right)\right] + \widehat{Q}(-B) + \|Q^s\|. \qquad (41)$$

Proof. If Q_λ is optimal then $\widehat{Q_\lambda}(-B)$ must be finite. We can assume $\lambda = 1$, the case with general λ being analogous, since condition (39) holds true. Denoting the optimal functional by Q_1, fix any Q with $\widehat{Q}(-B)$ finite and consider $Q_x = xQ_1 + (1-x)Q$. Also denote by $V(Q)$ the objective function to be minimized in (40) when $\lambda = 1$. Consider the convex function of x

$$F(x) := E\left[\Phi\left(\frac{dQ^r_x}{dP}\right)\right] + x\widehat{Q_1}(-B) + (1-x)\widehat{Q}(-B) + \|Q^s_x\|$$

then $F(1) = V(Q_1)$ and $F(x) \geq V(Q_x)$ since $\widehat{Q_x}(-B)$ is convex in x. Taking this inequality into account and given that Q_1 is a minimizer of $V(Q)$,

$$F'(1_-) \leq V'(Q_{1_-}) \leq 0.$$

Now, as in [BF06, Prop 25], it can be shown that

$$F'(1_-) = E\left[\left(\frac{dQ^r_1}{dP} - \frac{dQ^r}{dP}\right)\Phi'\left(\frac{dQ^r_1}{dP}\right)\right] + \widehat{Q_1}(-B) - \widehat{Q}(-B) + \|Q^s_1\| - \|Q^s\|$$

and since this quantity must be non positive we conclude the proof. □

Next we fix Q and explore the consequences of optimality in λ. The result is identical to [BF06, Prop 26], which we reproduce here for readability:

Proposition 3.11. *Suppose that the utility* u *satisfies assumption A. If* Q *is a probability measure in* $\mathcal{L}_\Phi$ *then for all* $c \in \mathbb{R}$ *the optimal* $\lambda(c; Q)$ *solution of*

$$\min_{\lambda>0}\left\{E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right] + \lambda c\right\} \qquad (42)$$

is the unique positive solution of the first order condition

$$E\left[\frac{dQ}{dP}\Phi'\left(\lambda\frac{dQ}{dP}\right)\right] + c = 0. \qquad (43)$$

The random variable $f^* := -\Phi'(\lambda(c;Q)\frac{dQ}{dP}) \in \left\{f \in L^1(Q) \mid E_Q[f] = c\right\}$ *satisfies* $u(f^*) \in L^1(P)$ *and*

$$\begin{aligned}&\min_{\lambda>0}\left\{E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right] + \lambda c\right\}\\ =\ &\sup\left\{E[u(f)] \mid f \in L^1(Q) \text{ and } E_Q[f] \leq c\right\} = E[u(f^*)] < u(\infty)\end{aligned} \qquad (44)$$

Therefore, whenever $\widehat{Q}(-B)$ is finite, we can set $c = \widehat{Q}(-B) + \|Q^s\|$ and conclude from (44) that the minimization of the objective function in (37) with respect to $\lambda > 0$ for a fixed Q leads to the same value of a utility maximization over integrable functions satisfying $E_Q[f] \leq \widehat{Q}(-B) + \|Q^s\|$. Motivated by these results, we define the following set of functionals and corresponding domain for utility maximization:

Definition 3.12. For any B satisfying (24) and (25) let

$$\mathcal{N}_B^W := \{Q \in \mathcal{M}^W \mid Q^r \in \mathcal{L}_\Phi,\ \widehat{Q}(-B) \in \mathbb{R}\} \tag{45}$$

and

$$K_B^W := \{f \in L^0 \mid f \in L^1(Q^r),\ E_{Q^r}[f] \leq \widehat{Q^s}(-B) + \|Q^s\|,\ \ \forall Q \in \mathcal{N}_B^W\}, \tag{46}$$

with the corresponding optimization problem

$$U_B^W := \sup_{f \in K_B^W} E[u(f - B)]. \tag{47}$$

Remark 3.13. (i) Note that the sets $\mathcal{N}_B^W$ and K_B^W depend also on Φ (and thus on the utility u), but the dependence is omitted for convenience of notation. In the particular case $B \in L^{\widehat{u}}$ however $\widehat{Q}(-B) = Q(-B)$, so that $\mathcal{N}_B^W$ does not depend on B and it coincides with the set of dual functionals used in [BF06]:

$$\mathcal{N}^W = \{Q \in \mathcal{M}^W \mid Q^r \in \mathcal{L}_\Phi\}. \tag{48}$$

While we are going to treat the utility maximization with random endowment for general B, we will focus on $B \in L^{\widehat{u}}$ in the indifference price section, where the set of dual functionals will be simply $\mathcal{N}^W$. Note also that each element in $\mathcal{N}_B^W$ has non zero regular part.

(ii) If $\Phi(0) < +\infty$, then $Q \in \mathcal{L}_\Phi$ iff Q is a probability s.t. $\frac{dQ}{dP} \in L_+^{\widehat{\Phi}}$. As explained in Section 2.3 the regular part of each element in $\mathcal{M}^W$ is already in $L_+^{\widehat{\Phi}}$, so that from (45) we get:

$$\mathcal{N}_B^W := \{Q \in \mathcal{M}^W \mid Q^r \neq 0,\ \widehat{Q}(-B) \in \mathbb{R}\}.$$

(iii) When Assumption (A) is satisfied then

$$\mathcal{N}_B^W := \{Q \in \mathcal{M}^W \mid\ Q^r \neq 0,\ E\left[\Phi\left(\frac{dQ^r}{dP}\right)\right] < +\infty,\ \widehat{Q}(-B) \in \mathbb{R}\}.$$

The utility optimization over the modified domain K_B^W can be easily related with the original utility optimization over terminal wealths K^W.

Lemma 3.14. *Suppose that B satisfies (24) and (25) and that $\mathcal{N}_B^W \neq \emptyset$. Then $K^W \subset K_B^W$ and the following chain of inequalities holds true*

$$\sup_{k \in K^W} E[u(k - B)] \leq U_B^W \leq \inf_{\lambda > 0, Q \in \mathcal{N}_B^W} \left\{\lambda \widehat{Q}(-B) + E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] + \lambda \|Q^s\|\right\} < \infty$$

Proof. For the first inequality, fix a $k \in K^W$ such that $E[u(k - B)] > -\infty$. By Lemma 3.7 item 2, $\widehat{Q}(k) \leq 0$ for all $Q \in \mathcal{M}^W$. The assumptions on B imply that $B^+ \in L^{\widehat{u}}$ so that $(-B) \in L_{neg}^{\widehat{u}}$. In addition, $K^W \subseteq L_{neg}^{\widehat{u}}$ implies $k - B \in L_{neg}^{\widehat{u}}$. Applying Lemma 3.7, item 1, we have for each $Q \in \mathcal{N}_B^W$

$$\widehat{Q}(k - B) = \widehat{Q}(k) + \widehat{Q}(-B) \leq \widehat{Q}(-B) < +\infty.$$

Then $\widehat{Q}(k - B)$ is finite and, by Lemma 3.7 item 4, the above inequality becomes:

$$E_{Q^r}[k - B] + \widehat{Q^s}(k - B) \leq E_{Q^r}[-B] + \widehat{Q^s}(-B).$$

Given that $-\widehat{Q^s}(k - B) \leq \|Q^s\|$ from (33)

$$E_{Q^r}[k - B] \leq E_{Q^r}[-B] + \widehat{Q^s}(-B) + \|Q^s\|$$

and thus, cancelling $E_{Q^r}[-B]$, we get $k \in K_B^W$.

To prove the second inequality, the (pointwise) Fenchel inequality gives

$$u(k - B) \leq Z(k - B) + \Phi(Z)$$

for every positive random variable Z and $k \in L^0$. Let $Q \in \mathcal{N}_B^W$ and take any $\lambda > 0$. By setting $Z = \lambda \frac{dQ^r}{dP}$, fixing $k \in K_B^W$ and taking expectations we have

$$E[u(k - B)] \leq \lambda E_{Q^r}[k - B] + E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right].$$

¿From the definition of K_B^W

$$E_{Q^r}[k] \leq \widehat{Q^s}(-B) + \|Q^s\|$$

whence

$$E[u(k - B)] \leq \lambda(\widehat{Q}(-B) + \|Q^s\|) + E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right].$$

The expression $E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right]$ may be equal to $+\infty$, but for each $Q \in \mathcal{N}_B^W$ there is a positive λ for which it is finite. The thesis then follows. $\square$

3.5 The main duality result

Theorem 3.15. *Fix a loss variable W and a liability $B \in \mathcal{A}_u$. If*

$$\sup_{H \in \mathcal{H}^W} E\left[u\left(\int_0^T H_t dS_t - B\right)\right] < u(+\infty) \tag{49}$$

then $\mathcal{N}_B^W$ is not empty and

$$\begin{aligned} \sup_{H \in \mathcal{H}^W} \ E\left[u\left(\int_0^T H_t dS_t - B\right)\right] &= U_B^W \\ &= \min_{\lambda > 0,\ Q \in \mathcal{N}_B^W} \left\{\lambda \widehat{Q}(-B) + E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] + \lambda \|Q^s\|\right\}. \end{aligned} \tag{50}$$

The minimizer λ_B is unique, while the minimizer Q_B is unique only in the regular part Q_B^r.

Suppose in addition that the utility u satisfies assumption A. Then,

$$U_B^W = E[u(f_B - B)], \tag{51}$$

where the unique maximizer is

$$f_B = \left(-\Phi'(\lambda_B \frac{dQ_B^r}{dP}) + B\right) \in K_B^W \tag{52}$$

and satisfies

$$E_{Q_B^r}[f_B] = \widehat{Q_B^s}(-B) + \|Q_B^s\|. \tag{53}$$

Proof. First observe that it follows from (29) that

$$\sup_{H \in \mathcal{H}^W} E\left[u\left(\int_0^T H_t dS_t - B\right)\right] = \sup_{k \in K^W} E[u(k - B)] = \sup_{f \in C^W} E[u(f - B)].$$

Moreover, Lemma 3.4 enables the application of Fenchel duality theorem to get

$$\begin{aligned}\sup_{f\in C^W} E[u(f-B)] &= \sup_{f\in C^W} I_u^B(f) = \min_{Q\in (C^W)^0} J_u^B(Q) \\ &= \min_{Q\in (C^W)^0} \left\{ E[\Phi(Q^r)] + \widehat{Q}(-B) + \|Q^s\| \right\}\end{aligned}$$

where the last equality is guaranteed by Lemma 3.8. Now if the optimal Q had $Q^r = 0$, then we would have

$$\sup_{f\in C^W} E[u(f-B)] = \Phi(0) + \widehat{Q}^s(-B) + \|Q^s\| \geq u(+\infty),$$

since $\widehat{Q^s}(-B) + \|Q^s\| \geq 0$, according to (33), and $\Phi(0) = u(\infty)$. Because this contradicts condition (49), $Q^r \neq 0$ and a re-parametrization of the domain of minimization in terms of $\mathcal{N}_B^W$ leads to

$$\sup_{f\in C^W} E[u(f-B)] = \min_{\lambda>0,\ Q\in\mathcal{N}_B^W} \left\{ \lambda\widehat{Q}(-B) + E\left[\Phi\left(\lambda\frac{dQ^r}{dP}\right)\right] + \lambda\|Q^s\| \right\}.$$

Uniqueness of λ_B and Q_B^r follow from strict convexity of the dual objective function in λ and Q^r. However, the dependence of the dual objective function on Q^s is mixed: it is linear in the norm $\|\cdot\|$-part due to (18) (see [BF06, Proposition 10] and generally convex in the term $\widehat{Q^s}(-B)$, although this term may also reduce to a linear one in the special case $B \in L^{\widehat{u}}$. Therefore, the optimal singular functional might not be unique. Thanks to Lemma 3.14, the equalities

$$\sup_{k\in K^W} E[u(f-B)] = U_B^W = \min_{\lambda>0,\ Q\in\mathcal{N}_B^W} \left\{ \lambda\widehat{Q}(-B) + E\left[\Phi\left(\lambda\frac{dQ^r}{dP}\right)\right] + \lambda\|Q^s\| \right\}$$

are immediate. Under assumption A, the expression for f_B can be derived by observing that any minimizer Q^B is obtained as the minimizer of

$$\min_{Q\in\mathcal{N}_B^W} \left\{ \lambda_B\widehat{Q}(-B) + E\left[\Phi\left(\lambda_B\frac{dQ^r}{dP}\right)\right] + \lambda_B\|Q^s\| \right\}$$

and from a standard combination of the results in Propositions 3.10 and 3.11. □

Corollary 3.16. *Whenever $B \in \mathcal{B} = \mathcal{A}_u \cap L^{\widehat{u}}$ we have that $\widehat{Q}(-B) = -Q(B)$ in* (50) *and $\mathcal{N}_B^W = \mathcal{N}^W$. Moreover, if both W and B are in $M^{\widehat{u}}$, then $\mathcal{N}_B^W$ can be replaced by the set $\mathbb{M}_\sigma \cap \mathcal{L}_\Phi$ of σ- martingale probabilities with finite generalized entropy and no singular term appears in* (50).

Proof. The first statement is clear from the definition of $\widehat{Q}$. For the second statement, notice that when $W \in M^{\widehat{u}}$ then the regular component Q^r of $Q \in \mathcal{M}^W$ is already in $\mathcal{M}^W$ (see [BF06, Lemma 41]). If B is in $M^{\widehat{u}}$ as well, then $Q_s(B) = 0$. Since $\|Q^s\| \geq 0$, the minimum must be achieved on the set $\left\{Q^r \mid Q \in \mathcal{N}_B^W\right\} = \mathbb{M}_\sigma \cap \mathcal{L}_\Phi$. □

We can see from (50) that the singular part Q^s in the dual objective function plays a double role. Its norm $\|Q^s\|$ sums up the generic risk of the high exposure in the market generated by S. When the agent sells B, there is obviously an extra idiosyncratic exposure. Given our very general assumptions on B, this extra exposure may also be extremely risky, and this is expressed by the term $\widehat{Q^s}(-B)$. Of course, the presence of "high exposure" terms in the dual does not imply that the actual minimizer Q_B *must* have a non-zero singular part. However, in the next section we construct some examples displaying the more interesting situation where Q^s_B is necessarily non-zero. In view of (53), a sufficient condition for this is $E_{Q^r_B}[f_B] > 0$. The condition is by no means necessary, since it could happen that $Q^s_B \neq 0$ but $\|Q^s_B\| + \widehat{Q^s_B}(-B) = 0$ in (53).

It is interesting to investigate and possibly derive more accurate bounds for $\widehat{Q^s}(-B)$. The next Proposition gives *a priori* good bounds for this singular contribution.

Proposition 3.17. *For any $B \in \mathcal{A}_u$ set*

$$L = \sup\{\beta > 0 \mid E[\widehat{u}(\beta B^+)] < +\infty\}$$

and fix any $Q \in \mathcal{N}^W_B$. Then

$$\widehat{Q^s}(-B) \geq -\frac{1}{L}\|Q^s\|. \tag{54}$$

If B^- is also in $L^{\widehat{u}}$, set

$$l = \sup\{\alpha > 0 \mid E[\widehat{u}(\alpha B^-)] < +\infty\}.$$

Then

$$-\frac{1}{L}\|Q^s\| \leq Q^s(-B) \leq \frac{1}{l}\|Q^s\| \tag{55}$$

and in particular we recover again $Q^s(B) = 0$ when $B \in M^{\widehat{u}}$.

Proof. ¿From (26), $E[u(-(1+\varepsilon)B^+)] < +\infty$, so $L \geq 1 + \epsilon$. For any $b < L$, (33) gives $\|Q^s\| \geq b\,Q^s(B^+)$ and therefore

$$\widehat{Q^s}(-B) \geq -Q^s(B^+) \geq -\frac{1}{b}\|Q^s\|$$

whence the desired $\widehat{Q^s}(-B) \geq -\frac{1}{L}\|Q^s\|$. To prove the right inequality in (55), observe that the additional hypothesis on B^- means $l > 0$ and $-\alpha B^- \in \mathrm{Dom}(I_u)$ for any $\alpha < l$. Hence

$$Q^s(-B) \leq Q^s(B^-) = \frac{1}{\alpha}Q^s(\alpha B^-) \leq \frac{1}{\alpha}\|Q^s\| \quad \text{for all } \alpha < l$$

□

The result in Theorem 3.15 does not guarantee in full generality that the optimal random variable $f_B \in K^W_B$ can be represented as terminal value from an investment strategy in $L(S)$, that is, $f_B = \int_0^T H_t dS_t$. The next proposition presents a partial result in this direction.

Proposition 3.18. *Suppose that the utility u satisfies assumption A. Under the same hypotheses of Theorem 3.15, if $Q^s_B = 0$ and $Q^r_B \sim P$, then f_B can be represented as terminal wealth from a suitable strategy H.*

Proof. It follows from Theorem 3.15 that f_B must satisfy

$$E_{Q^r}[f_B] \leq \widehat{Q^s}(-B) + \|Q^s\| \ \ \forall Q \in \mathcal{N}^W_B$$

and equality must hold at any optimal Q_B, according to (53). When the optimal Q_B has zero singular part, then it is a σ-martingale measure with finite entropy, according to (21). This being the case, it is easy to see that the dual problem could be reformulated as a minimum over $M_\sigma \cap \mathcal{L}_\Phi$. In this simplified setup, one can show exactly as in [BF05, Therem 4, Theorem 1 (d)] that the optimal f_B belongs in fact to

$$\bigcap_{Q \in M_\sigma \cap \mathcal{L}_\Phi} \{f \mid f \in L^1 Q, E_Q[f] \leq 0\}$$

and can be represented as terminal wealth from a suitable strategy H. □

3.6 Exponential utility

For an exponential utility function $u(x) = -e^{-\gamma x}$, $\gamma > 0$, we have

$$\begin{aligned}\Phi(y) &= \frac{y}{\gamma}\log\frac{y}{\gamma} - \frac{y}{\gamma} \\ \widehat{u}(x) &= e^{\gamma|x|} - 1\end{aligned}$$

Using (14), we see that in this case $M^{\widehat{u}}$ consists of those random variables that have *all* the (absolute) exponential moments finite, while the larger space $L^{\widehat{u}}$ corresponds to random variables that have *some* finite exponential moment.

Moreover, since $\widehat{\Phi}(y) = (\Phi(|y|) - \Phi(\gamma))I_{\{|y|>\gamma\}}$ and $\Phi(0) < \infty$, we have that

$$E[\widehat{\Phi}(f)] < \infty \quad \Longleftrightarrow \quad E[\Phi(|f|)] < \infty. \tag{56}$$

Finally, since $\widehat{\Phi}$ in this case satisfies the Δ_2–growth condition (see [RR91, pp 22, 77]), the subspace $M^{\widehat{\Phi}}$ coincides with $L^{\widehat{\Phi}}$, that is, $E[\widehat{\Phi}(\alpha f)] < \infty$ for *some* $\alpha > 0$ if and only if $E[\widehat{\Phi}(\alpha f)] < \infty$ for *all* $\alpha > 0$.

The duality result for an exponential utility, which clearly satisfies Assumption (A), follows directly as a corollary of our main Theorem 3.15. Since $u(\infty) = 0$, the condition (24) automatically holds for all $\mathcal{F}_T$ measurable random variables B and furthermore,

$$\mathcal{L}_\Phi = \{Q \text{ probab, } Q \ll P \mid E\left[\Phi\left(\frac{dQ}{dP}\right)\right] < +\infty\} = \{Q \text{ probab, } Q \ll P \mid E\left[\widehat{\Phi}\left(\frac{dQ}{dP}\right)\right] < +\infty\}.$$

Corollary 3.19. *Suppose that the random endowment $B \in L^0(\Omega, \mathcal{F}_T, P)$ satisfies*

$$E[e^{\gamma(1+\epsilon)B}] < +\infty \text{ for some } \epsilon > 0$$

and suppose that there exists a loss variable W satisfying

$$\sup_{H\in\mathcal{H}^W} E\left[-e^{-\gamma(\int_0^T H_t dS_t - B)}\right] < 0. \tag{57}$$

Then $\mathcal{N}_B^W$ is not empty and

$$\sup_{H\in\mathcal{H}^W} E\left[-e^{-\gamma(\int_0^T H_t dS_t - B)}\right] = \\ -\exp\left\{-\min_{Q\in\mathcal{N}_B^W}\left(H(Q^r|P) + \gamma\widehat{Q}(-B) + \gamma\|Q^s\|\right)\right\}, \tag{58}$$

where $H(Q^r|P) = E\left[\frac{dQ^r}{dP}\log\left(\frac{dQ^r}{dP}\right)\right]$ denotes the relative entropy of Q^r with respect to P. The minimizer $Q_B \in \mathcal{N}_B^W$ is unique only in the regular part Q_B^r. In addition,

$$\sup_{H\in\mathcal{H}^W} E\left[-e^{-\gamma(\int_0^T H_t dS_t - B)}\right] = E[-e^{-\gamma(f_B - B)}],$$

where the optimal claim is

$$f_B = -\frac{1}{\gamma}\ln\left(\frac{\lambda_B}{\gamma}\frac{dQ_B^r}{dP}\right) + B,$$

where $\lambda_B = \gamma\exp(H(Q_B^r|P) + \gamma\widehat{Q_B}(-B) + \gamma\|Q_B^s\|) = -\frac{1}{\gamma}U_B^W$, and it satisfies

1. $f_B \in L^1(Q^r)$, $E_{Q^r}[f_B] \leq \widehat{Q^s}(-B) + \|Q^s\|$ for all $Q \in \mathcal{N}_B^W$ (i.e. it belongs to K_B^W)

2. $E_{Q_B^r}[f_B] = \widehat{Q_B^s}(-B) + \|Q_B^s\|$

Whenever B has some exponential (absolute) moments finite, $\widehat{Q}(-B) = -Q(B)$. Also, if both W and B have all the exponential moments finite, then $\mathcal{N}_B^W$ can be replaced by the "classic" set of probabilities $Q \in \mathbb{M}_\sigma$ that have finite relative entropy, i.e. $E[\frac{dQ}{dP}\ln(\frac{dQ}{dP})] < +\infty$, and no singular term appears in (58).

Proof. The conditions on B are exactly those in Theorem 3.15, adapted to the exponential case. So, directly from Theorem 3.15

$$\sup_{H\in\mathcal{H}^W} E\left[-e^{-\gamma(\int_0^T H_t dS_t - B)}\right] =$$

$$\min_{\lambda>0,\ Q\in\mathcal{N}_B^W}\left\{\lambda\widehat{Q}(-B) + E\left[\frac{\lambda}{\gamma}\frac{dQ^r}{dP}\log\left(\frac{\lambda}{\gamma}\frac{dQ^r}{dP}\right) - \frac{\lambda}{\gamma}\frac{dQ^r}{dP}\right] + \lambda\|Q^s\|)\right\},$$

and an explicit minimization over $\lambda > 0$ leads to the duality formula (58). The remaining assertions follow as in the proof of Theorem 3.15. □

3.6.1 Examples with nonzero singular parts

We now explore the case of an exponential utility to construct two examples where the existence of a nonzero singular part in the dual optimizer can be asserted explicitly.

Example 3.20. Consider a one period model with $S_0 = 0$ and $S_1 = YZ$ where Y is an exponential random variable with density $f(y) = e^{-y}$, $y \geq 0$ and Z is a discrete random variable taking the values $\{1, -\frac{1}{2}, \ldots, \frac{1}{n} - 1, \ldots\}$. Assume that Y and Z are independent and let

$$\begin{aligned} p_1 :=& \quad P(Z=1) > 0 \\ p_n :=& \quad P\left(Z = \frac{1}{n} - 1\right) > 0, \quad n \geq 2 \end{aligned}$$

be the probability distribution of Z. For an investor with exponential utility $u(x) = -e^{-x}$, it is clear that the random variable $W = 1 + Y$ is suitable and compatible. Suppose now that $B = \alpha(Y, Z)$, where α is a bounded Borel function, so that the seller of the claim B faces the problem

$$\sup_{h\in\mathbb{R}} E\left[-e^{-hS_1+B}\right] = \sup_{h\in\mathbb{R}} E\left[-e^{-hZY+\alpha(Y,Z)}\right].$$

Because Y is exponentially distributed with parameter 1, α is bounded, $-1 < Z \leq 1$ and independent from Y, a necessary condition for the expectation above to be finite is that $-1 < h \leq 1$. Now the function

$$g(h) = E\left[-e^{-hS_1+B}\right],$$

has a formal derivative given by

$$g'(h) = E\left[S_1 e^{-hS_1+B}\right] = p_1 E\left[Ye^{-hY+\alpha(Y,Z)}\right] + \sum_{n\geq 2} p_n z_n E\left[Ye^{-hz_nY+\alpha(Y,z_n)}\right].$$

Since $-1 < z_n < 0$ for $n \geq 2$, we have that

$$g'(h) \geq p_1 E\left[Ye^{-Y+B}\right] - \sum_{n\geq 2} p_n E\left[Ye^{-z_nY+\alpha(Y,z_n)}\right].$$

When $p_n \to 0$ sufficiently fast, this expression is not only well defined but strictly positive. Therefore, by adjusting the distribution of Z, we can guarantee that $0 < g'(h) < \infty$ for all $-1 < h \le 1$. Therefore, the function $g(h)$ is strictly increasing and attains its maximum at $h = 1$. But this implies that

$$\sup_{h \in \mathbb{R}} E\left[-e^{-hS_1+B}\right] = E\left[-e^{-S_1+B}\right],$$

so that the optimizer for the primal problem is $f_B = S_1$. From the identity

$$u'(f_B - B) = \lambda_B \frac{dQ_B^r}{dP},$$

we obtain that the optimizer for the dual problem has a regular part given by

$$\frac{dQ_B^r}{dP} = \frac{e^{-S_1+B}}{E[e^{-S_1+B}]}. \tag{59}$$

Using (59) to calculate the expectation of f_B with respect to Q_B^r, we conclude from (53) that

$$Q_B^s(-B) + \|Q_B^s\| = E_{Q_B^r}[f_B] = \frac{E[S_1 e^{-S_1+B}]}{E[e^{-S_1+B}]} = \frac{g'(1)}{E[e^{-S_1+B}]} > 0,$$

which implies that $Q_B^s \neq 0$.

Observe that a proper selection of the probabilities p_n also guarantees that setting $B = 0$ in the expressions above does not alter the domain of the function $g(h)$ and the remaining calculations. In particular, the maximum of $E[-e^{-hS_1}]$ would be still attained at $h = 1$, which implies that the optimizers f_0 and f_B for the primal problem with and without the claim coincide. This means that the investor does not use the underlying market to hedge the claim, despite the fact that $B = \alpha(Y, Z)$ is explicitly correlated with $S_1 = YZ$. Such behavior stems from the fact that the risk associated with the unboundedness of the underlying outweighs the risk associated with the bounded claim. This should be contrasted with the case of locally bounded markets, where even a bounded claim leads to a different optimizer for the primal problem.

Example 3.21. Consider now the same setting as in the previous example, but with a claim of the form $B = \delta Y$, $0 < \delta < 1$, so that the investor faces the problem

$$\sup_{h \in \mathbb{R}} E\left[-e^{-(hS_1 - \delta Y)}\right] = \sup_{h \in \mathbb{R}} E\left[-e^{-(hZ - \delta)Y}\right].$$

A necessary condition for the expectation above to be finite is $-(1-\delta) < h \le (1-\delta)$, since $-1 < Z \le 1$. Define the function

$$g(h) = E\left[-e^{-hS_1 + \delta Y}\right],$$

with derivative

$$g'(h) = E\left[S_1 e^{-hS_1+\delta Y}\right] = p_1 E\left[Y e^{-(h-\delta)Y}\right] + \sum_{n \ge 2} p_n z_n E\left[Y e^{-(hz_n - \delta)Y}\right].$$

As before,

$$g'(h) \ge p_1 E\left[Y e^{-(1-\delta)Y}\right] - \sum_{n \ge 2} p_n E\left[Y e^{-(z_n+\delta)Y}\right],$$

which can be made strictly positive for $p_n \to 0$ sufficiently fast (as a consequence, we can assume $p_1 \gg p_n$). Therefore, $0 < g'(h) < \infty$ for all $-(1-\delta) < h \leq (1-\delta)$ and the function $g(h)$ attains its maximum at $h = 1-\delta$. We then obtain that $f_B = (1-\delta)S_1$, which implies that

$$\frac{dQ_B^r}{dP} = \frac{e^{-(1-\delta)S_1+\delta Y}}{E[e^{-(1-\delta)S_1+\delta Y}]}, \tag{60}$$

in view of the identity

$$u'(f_B - B) = \lambda_B \frac{dQ_B^r}{dP}.$$

As before, inserting this in (53)

$$Q_B^s(-B) + \|Q_B^s\| = E_{Q_B^r}[f_B] = \frac{E[(1-\delta)S_1 e^{-(1-\delta)S_1+\delta Y}]}{E[e^{-(1-\delta)S_1+\delta Y}]} = \frac{g'(1)}{E[e^{-(1-\delta)S_1+\delta Y}]} > 0,$$

which implies that $Q_B^s \neq 0$.

Apart from the appearance of a nonzero singular part in the pricing measure, an interesting feature of this example is the excess hedge $f_B - f_0 = -\delta S_1$ induced by the presence of the claim B. Observe that the selection $p_1 \gg p_n$ guarantees that B is positively correlated with S_1, since

$$\mathrm{Cov}(B, S_1) = \delta E[Z]\mathrm{Var}[Y],$$

and $E[Z]$ is positive when p_1 is sufficiently larger than p_n. This would suggest that the seller of B should hedge it by *buying* more shares of S. What our analyses indicates is that this intuition is in fact *wrong*, since the excess hedge due to the presence of B consists of *selling* δ shares of S. The explanation for this counterintuitive result relies on the fact that B is not *perfectly* correlated with S. In fact, whenever $Z < 0$, the risks of large downward moves in $S_1 = YZ$ and large upward moves in $B = \delta Y$ are both related to the same exponential random variable Y. Therefore, in the presence of B, the preference structure prohibits to buy more than $1-\delta$ shares, which must be then the new optimum.

4 The indifference price π

4.1 Definition and domain of π

Consider an agent with utility u (not necessarily satisfying Assumption (A)), initial endowment x and investment possibilities given by $\mathcal{H}^W$ who seeks to sell a claim B. As pointed out in Section 1, the indifference price $\pi(B)$ for this claim is defined as the implicit solution to (1). In view of the duality result of Theorem 3.15, we now rephrase this definition in terms of the function

$$U_B^W(x) := \sup_{k \in K^W} E[u(x + k - B)]. \tag{61}$$

Comparing this with (50), we see that the optimal value $U_B^W(0)$ is exactly what has been there denoted by U_B^W. Notice that we could alternatively denote (61) by U_{B-x}^W, which would be consistent with (50) for a claim of the form $(B - x)$. We prefer $U_B^W(x)$ instead, since it better illustrates the different financial roles played by the initial endowment x and the claim B.

Definition 4.1. Provided that the related maximization problems are well–posed, the seller's indifference price $\pi(B)$ of the claim B is the implicit solution of the equation

$$U_0^W(x) = U_B^W(x + \pi(B)) \tag{62}$$

that is, $\pi(B)$ is the additional initial money that makes the optimal utility with the liability B equal to the optimal utility without B.

The next lemma shows that the class

$$\mathcal{B} = \mathcal{A}_u \cap L^{\widehat{u}} = \{B \in L^{\widehat{u}} \mid E[\widehat{u}((1+\epsilon)B^+)] < +\infty \text{ for some } \epsilon > 0\} \tag{63}$$

of claims B, for which we compute indifference prices, is considerably large and has desirable properties. Note that the equivalence (14) says that $E[\widehat{u}((1+\epsilon)B^+)] < +\infty$ if and only if B satisfies (26), so that (28) and (63) agree. In other words, $\mathcal{B}$ consists of the set of claims which, in addition to satisfying the hypotheses of Theorem 3.15, are also in $L^{\widehat{u}}$. Upon fixing the loss variable W, the strengthening assumption $B \in L^{\widehat{u}}$ allows us to use Corollary 3.16 and guarantees that the set of dual functionals $\mathcal{N}_B^W$ does not depend on B and reduces to the set $\mathcal{N}^W$ defined in (48).

Lemma 4.2.

$$\mathcal{B} = \{B \in L^{\widehat{u}} \mid (-B) \in \text{ int}(\text{Dom}(I_u))] \tag{64}$$

and therefore has the properties:

1. *$\mathcal{B}$ is convex and open in $L^{\widehat{u}}$;*
2. *If $B_1 \in \mathcal{B}$ and $B_2 \leq B_1$, then $B_2 \in \mathcal{B}$.*
3. *$\mathcal{B}$ contains $M^{\widehat{u}}$ (and thus L^∞);*
4. *for any given $B \in \mathcal{B}$ and $C \in M^{\widehat{u}}$, we have that $B + C \in \mathcal{B}$. In particular, $B + c \in \mathcal{B}$ for all constants $c \in \mathbb{R}$.*

Proof. As remarked after (26), we already know that B satisfies (26) iff $-B^+ \in \text{int}(\text{Dom}(I_u))$. Under the extra condition $B \in L^{\widehat{u}}$, B satisfies (26) iff $-B \in \text{int}(\text{Dom}(I_u))$, which shows (64).

Then, $\mathcal{B}$ is obviously open and convex (property 1) and property 2 is a consequence of the monotonicity of I_u. It is evident that $M^{\widehat{u}}$ is contained in $\mathcal{B}$, since $C \in M^{\widehat{u}}$ iff $E[\widehat{u}(kC)] < +\infty$ for all $k > 0$ (property 3). In order to prove property 4, fix $B \in \mathcal{B}$ and a convenient ϵ. For any C in $M^{\widehat{u}}$, set $r = \frac{\frac{\epsilon}{2}}{(1+\epsilon)(1+\frac{\epsilon}{2})}$. Then

$$E\left[\widehat{u}\left((1+\frac{\epsilon}{2})(B+C)^+\right)\right] \leq \frac{1+\frac{\epsilon}{2}}{1+\epsilon} E\left[\widehat{u}((1+\epsilon)B^+)\right] + \frac{\epsilon/2}{1+\epsilon} E\left[\widehat{u}\left(\frac{C^+}{r}\right)\right] < +\infty.$$

□

4.2 The properties of π

The next Proposition lists the various properties of the *indifference price functional* π, defined on the set $\mathcal{B} \subseteq L^{\widehat{u}}$. Some results are new, in particular the regularity of the map and the description of the conjugate π^* and of the subdifferential $\partial\pi$. They are nice consequences of the choice of the natural Orlicz framework and the proofs are quite short and easy. The other items are extensions of well–established results to the present general setup (see e.g. [Be03] or the recent [OZ07, Prop. 7.5] and the references therein). A recent reference book for the necessary notions from Convex Analysis is [BZ05].

In the next proposition, the assumption that $U_0^W(x) < u(+\infty)$ can be replaced by $\mathcal{N}^W \neq \emptyset$, whenever the utility function satisfies assumption (A). Indeed, in this case Proposition 3.11 and $\mathcal{N}^W \neq \emptyset$ guarantees that $U_0^W(x) < u(+\infty)$ for all $x \in \mathbb{R}$.

Proposition 4.3. *Fix a loss variable W and an initial wealth $x \in \mathbb{R}$ such that $U_0^W(x) < u(+\infty)$. The seller's indifference price*

$$\pi : \mathcal{B} \to \mathbb{R}$$

verifies the following properties:

1. *π **is well–defined**. The solution to the equation (62) above exists and it is unique.*
2. ***Convexity and monotonicity.** π is a convex, monotone non–decreasing functional.*
3. ***Translation invariance.** Given $B \in \mathcal{B}$, $\pi(B + c) = \pi(B) + c$ for any $c \in \mathbb{R}$.*
4. ***Regularity.** π is norm continuous and subdifferentiable.*
5. ***Dual representation.** π admits the representation*

$$\pi(B) = \max_{Q \in \mathcal{N}^W} (Q(B) - \alpha(Q)) \tag{65}$$

where the (minimal) penalty term $\alpha(Q)$ is given by

$$\alpha(Q) = x + \|Q^s\| + \inf_{\lambda > 0} \left\{ \frac{E[\Phi(\lambda \frac{dQ^r}{dP})] - U_0^W(x)}{\lambda} \right\}.$$

As a consequence, the subdifferential $\partial\pi(B)$ of π at B is given by

$$\partial\pi(B) = \mathcal{Q}_B^W(x + \pi(B)) \tag{66}$$

where $\mathcal{Q}_B^W(x + \pi(B))$ is the set of minimizers of the dual problem associated with the right–hand side of (62).

6. ***Bounds.** π satisfies the bounds*

$$\max_{Q \in \mathcal{Q}_0^W(x)} Q(B) \leq \pi(B) \leq \sup_{Q \in \mathcal{N}^W} Q(B)$$

If $W \in M^{\widehat{u}}$ and $B \in M^{\widehat{u}}$, the bounds above simplify to

$$E_{Q^*}[B] \leq \pi(B) \leq \sup_{Q \in M_\sigma \cap \mathcal{L}_\Phi} E_Q[B]$$

where the probability $Q^* \in M_\sigma \cap \mathcal{L}_\Phi$ is the unique dual minimizer in $\mathcal{Q}_0^W(x)$.

7. ***Volume asymptotics.*** *For any $B \in \mathcal{B}$ we have*

$$\lim_{b\downarrow 0}\frac{\pi(bB)}{b}=\max_{Q\in\mathcal{Q}_0^W(x)}Q(B). \tag{67}$$

If B is in $M^{\widehat{u}}$,

$$\lim_{b\to+\infty}\frac{\pi(bB)}{b}=\sup_{Q\in\mathcal{N}^W}Q(B). \tag{68}$$

If $W\in M^{\widehat{u}}$ and $B\in M^{\widehat{u}}$, the two volume asymptotics above become

$$\lim_{b\downarrow 0}\frac{\pi(bB)}{b}=E_{Q^*}[B],\qquad \lim_{b\to+\infty}\frac{\pi(bB)}{b}=\sup_{Q\in M_\sigma\cap\mathcal{L}_\Phi}E_Q[B]$$

where the probability $Q^*\in M_\sigma\cap\mathcal{L}_\Phi$ is the unique dual minimizer in $\mathcal{Q}_0^W(x)$.

8. ***Price of replicable claims.*** *If $B\in\mathcal{B}$ is replicable in the sense that $B=c+\int_0^T H_t dS_t$ with $H\in\mathcal{H}^W$, but also $-H\in\mathcal{H}^W$, then $\pi(B)=c$.*

Proof. Applying Theorem 2.7, eq. (22), we preliminary observe that the assumption $U_0^W(x)<u(+\infty)$ implies $\mathcal{N}^W\neq\varnothing$.

1. Let $F(p):=U_B^W(x+p)$. By standard arguments it can be shown that $F:\mathbb{R}\to(-\infty,u(+\infty)]$ is concave and monotone non–decreasing, though not necessarily strictly increasing. By monotone convergence we also have

$$\lim_{p\to+\infty}F(p)=u(+\infty). \tag{69}$$

We now show that $\lim_{p\to-\infty}F(p)=-\infty$, so that $F(p)$ is not constantly equal to $u(+\infty)$. Fix $Q\in\mathcal{N}^W$ and take $\lambda>0$ for which $E[\Phi(\lambda\frac{dQ^r}{dP})]$ is finite. ¿From the inclusion $K^W\subseteq K_B^W$, proved in Lemma 3.14, and Fenchel inequality it follows, as in the second part of Lemma 3.14, that for all $k\in K^W$

$$\begin{aligned} E[u(x+p+k-B)] &\leq E\left[(x+p+k-B)\lambda\frac{dQ^r}{dP}\right]+E\left[\Phi\left(\lambda\frac{dQ^r}{dP}\right)\right]\\ &\leq \lambda(x+p-Q(B)+\|Q^s\|)+E\left[\Phi\left(\lambda\frac{dQ^r}{dP}\right)\right]<+\infty, \end{aligned}$$

so that the r.h.s. does not depend on k anymore. Taking the *sup* over k

$$F(p)=U_B^W(x+p)\leq\lambda(x+p-Q(B)+\|Q^s\|)+E\left[\Phi\left(\lambda\frac{dQ^r}{dP}\right)\right]$$

and then, passing to the limit for $p\to-\infty$, one obtains $\lim_{p\to-\infty}F(p)=-\infty$. The well-posedness of the definition of π is now straightforward. In fact, let p_L be the infimum of the set $\{p\in\mathbb{R}\mid F(p)=F(+\infty)=u(+\infty)\}$. ¿From concavity, on $(-\infty,p_L)$ F is continuous and strictly monotone and thus a bijection onto the image $(-\infty,u(+\infty))$. Since $U_0^W(x)<u(+\infty)$, there always exists a unique p such that $F(p)=U_0^W(x)$, namely the indifference price $\pi(B)$.

2. Convexity and monotonicity are consequences of the definition (62), of the concavity and monotonicity of u and of the convexity of $\mathcal{H}^W$.

3. Translation invariance follows directly from the definition (62).

4. For this item, observe that π is a real valued, convex, monotone functional on the convex open subset $\mathcal{B}$ of the Banach lattice $L^{\widehat{u}}$. It then follows from item 2 of Lemma 4.2 that the extension $\widetilde{\pi}$ of π on $L^{\widehat{u}}$ with the value $+\infty$ on $L^{\widehat{u}}\backslash\mathcal{B}$ is still monotone, convex and translation invariant. Trivially, the interior of the proper domain of $\widetilde{\pi}$ coincides with $\mathcal{B}$. Therefore, norm continuity and sub-differentiability of $\widetilde{\pi}$ (and thus of π) on $\mathcal{B}$ follow from an extension of the classic Namioka-Klee theorem for convex monotone functionals (see [RS06], but also [BF07] and [CL07] in the context of Risk Measures). As a consequence, π admits a dual representation on $\mathcal{B}$ as

$$\pi(B)=\widetilde{\pi}(B)=\max_{Q\in(L^{\widehat{u}})^*_+,Q(\mathbf{1}_\Omega)=1}\{Q(B)-\pi^*(Q)\} \tag{70}$$

where π^* is the convex conjugate of $\widetilde{\pi}$, that is $\pi^*:(L^{\widehat{u}})^*\to(-\infty,+\infty]$,

$$\pi^*(z)=\sup_{B'\in L^{\widehat{u}}}\{z(B')-\widetilde{\pi}(B')\}=\sup_{B\in\mathcal{B}}\{z(B)-\pi(B)\}.$$

The normalization condition $Q(\mathbf{1}_\Omega)=1$ in (70) derives from the translation invariance property. The subdifferential of π at B is, as always, given by

$$\partial\pi(B)=\mathrm{argmax}\{Q(B)-\pi^*(Q)\}. \tag{71}$$

Note that, since $\pi(0)=0$, π^* is nonnegative and thus it can be interpreted as a penalty function. The next item presents a characterization of π^* and therefore of $\partial\pi(B)$.

5. A dual representation for π has just been obtained in (70). The current item is proved in two steps: first, we establish representation (65) with the penalty α; second, we prove that $\alpha=\pi^*$, that is α is the *minimal penalty* function, which together with (71) gives (66) and completes the proof.

Step 1. From the definition of $\pi(B)$ and from the dual formula (50)

$$\begin{aligned}U_0^W(x) &= U_B^W(x+\pi(B))\\ &= \min_{\lambda>0,Q\in\mathcal{N}^W}\left\{\lambda Q(-B+x+\pi(B))+\lambda\|Q^s\|+E\left[\Phi\left(\lambda\frac{dQ^r}{dP}\right)\right]\right\}.\end{aligned}$$

Necessarily then

$$\pi(B)\geq Q(B)-\left[x+\|Q^s\|+\frac{E[\Phi(\lambda\frac{dQ^r}{dP})]-U_0^W(x)}{\lambda}\right]\text{ for all }\lambda>0,Q\in\mathcal{N}^W$$

and equality holds for the optimal λ^* and any $Q^*\in\mathcal{Q}_B^W(x+\pi(B))$. Fixing $Q\in\mathcal{N}^W$ and taking first the supremum over $\lambda>0$, we get

$$\pi(B)\geq Q(B)-\inf_{\lambda>0}\left[x+\|Q^s\|+\frac{E[\Phi(\lambda\frac{dQ^r}{dP})]-U_0^W(x)}{\lambda}\right].$$

Taking then the supremum over Q we finally obtain

$$\pi(B)=\max_{Q\in\mathcal{N}^W}\left\{Q(B)-\inf_{\lambda>0}\left[x+\|Q^s\|+\frac{E[\Phi(\lambda\frac{dQ^r}{dP})]-U_0^W(x)}{\lambda}\right]\right\}$$

where equality holds for $\lambda^*, Q^* \in \mathcal{Q}_B^W(x+\pi(B))$. Observe that the following extension, still denoted by α,

$$\alpha(Q) = \begin{cases} \inf_{\lambda>0} \left[x + \|Q^s\| + \frac{E[\Phi(\lambda \frac{dQ^r}{dP})] - U_0^W(x)}{\lambda} \right] & \text{when } Q \in \mathcal{N}^W \\ +\infty & \text{otherwise} \end{cases}$$

is $[0,+\infty]$-valued and satisfies $\inf_{Q\in(L^{\widehat{u}})^*} \alpha(Q) = 0$. Therefore, it is a *grounded* penalty function and clearly

$$\pi(B) = \max_{Q\in(L^{\widehat{u}})^*_+} \{Q(B) - \alpha(Q)\}$$

and the set

$$\operatorname{argmax}\{Q(B) - \alpha(Q)\} \text{ coincides with } \mathcal{Q}_B^W(x+\pi(B)). \tag{72}$$

In particular, when $B = 0$

$$\pi(0) = 0 \quad \text{and} \quad \operatorname{argmax}\{-\alpha(Q)\} = \operatorname{argmin}\{\alpha(Q)\} = \mathcal{Q}_0^W(x). \tag{73}$$

Step 2. As α provides another penalty function, a basic result in convex duality ensures that $\pi^* = \alpha^{**}$, i.e. π^* is the convex, $\sigma((L^{\widehat{u}})^*, L^{\widehat{u}})$–lower semicontinuous hull of α. We want to show that $\pi^* = \alpha$. To this end, we prove that α is *already* convex and lower semicontinuous.

(a) α is convex: Let $Q(y) = yQ_1 + (1-y)Q_2$ be the convex combination of any couple of elements in $\mathcal{N}^W$ (if the Q_i are not in $\mathcal{N}^W$ there is nothing to prove). Given any $\lambda_1, \lambda_2 > 0$, define $\lambda(y) = \frac{1}{(1-y)\frac{1}{\lambda_2} + y\frac{1}{\lambda_1}}$, so that $\frac{1}{\lambda(y)} = (1-y)\frac{1}{\lambda_2} + y\frac{1}{\lambda_1}$. Then

$$\alpha(Q(y)) \le \left[x + \|Q^s(y)\| + \frac{E[\Phi(\lambda(y)\frac{dQ^r(y)}{dP})] - U_0^W(x)}{\lambda(y)} \right] \le$$
$$y\left[x + \|Q_1^s\| + \frac{E[\Phi(\lambda_1 \frac{dQ_1^r}{dP})] - U_0^W(x)}{\lambda_1} \right] + (1-y)\left[x + \|Q_2^s\| + \frac{E[\Phi(\lambda_2 \frac{dQ_2^r}{dP})] - U_0^W(x)}{\lambda_2} \right]$$

where the inequalities follow from the convexity of the norm and of the function $(z,k) \to z\Phi(k/z)$ on $\mathbb{R}_+ \times \mathbb{R}_+$, as already pointed out. Taking the infimum over λ_1 and λ_2

$$\alpha(Q(y)) \le y\,\alpha(Q_1) + (1-y)\,\alpha(Q_2).$$

(b) α is lower semicontinuous: Since α is a convex map on a Banach space, weak lower semicontinuity is equivalent to norm lower semicontinuity. Suppose then that Q_k is a sequence converging to Q with respect to the Orlicz norm. We must prove that

$$\alpha(Q) \le \liminf_k \alpha(Q_k) := L$$

We can assume $L = \liminf_k \alpha(Q_k) < +\infty$, otherwise there is nothing to prove. Now, it is not difficult to see that

$$Q_k \stackrel{\|\cdot\|}{\to} Q \text{ iff } Q_k^r \stackrel{\|\cdot\|}{\to} Q^r, Q_k^s \stackrel{\|\cdot\|}{\to} Q^s \tag{74}$$

so that $Q_k^r \to Q^r$ in $L^{\widehat{\Phi}}$ and henceforth in L^1. We can extract a subsequence, still denoted by Q_k to simplify notation, such that

$$\alpha(Q_k) \to L \text{ and } Q_k^r \to Q^r \ a.s.$$

So these Q_k are (definitely) in $\mathcal{N}^W$, which is closed and therefore the limit $Q \in \mathcal{N}^W$.
For all $k \in \mathbb{N}_+$ there exists $\lambda_k > 0$ such that

$$\alpha(Q_k) \leq x + \|Q_k^s\| + \frac{E[\Phi(\lambda_k \frac{dQ_k^r}{dP})] - U_0^W(x)}{\lambda_k} \leq \alpha(Q_k) + \frac{1}{k}.$$

The next arguments rely on a couple of applications of Fatou Lemma to (a subsequence of) the sequence $\left(\frac{\Phi(\lambda_k \frac{dQ_k^r}{dP}) - U_0^W(x)}{\lambda_k}\right)_k$. Fatou Lemma is enabled here by the condition $U_0^W(x) < u(+\infty)$ and by the convergence of the regular parts $(\frac{dQ_k^r}{dP})_k$. In fact, one can always find an $\widetilde{x}$ such that $u(\widetilde{x}) = U_0^W(x)$ and then the Fenchel inequality gives the required control from below

$$\frac{\Phi(\lambda_k \frac{dQ_k^r}{dP}) - U_0^W(x)}{\lambda_k} + \frac{dQ_k^r}{dP}\widetilde{x} \geq 0. \tag{75}$$

The sequence $(\lambda_k)_k$ cannot tend to $+\infty$. In fact, if $\lambda_k \to +\infty$, then a.s. we would have (remember that Φ is bounded below)

$$\begin{aligned}
& \liminf_k \frac{\Phi\left(\lambda_k \frac{dQ_k^r}{dP}\right) - U_0^W(x)}{\lambda_k} = \liminf_k \frac{\Phi\left(\lambda_k \frac{dQ_k^r}{dP}\right)}{\lambda_k} \\
\geq \; & \lim_k \frac{(\min_y \Phi(y))}{\lambda_k} \mathbf{1}_{\{\frac{dQ_k^r}{dP} \wedge \frac{dQ^r}{dP} = 0\}} + \lim_k \frac{\Phi(\lambda_k \frac{dQ_k^r}{dP})}{\lambda_k} \mathbf{1}_{\{\frac{dQ_k^r}{dP} \wedge \frac{dQ^r}{dP} > 0\}} \\
= \; & \lim_k \frac{\Phi(\lambda_k \frac{dQ_k^r}{dP})}{\lambda_k \frac{dQ_k^r}{dP}} \frac{dQ_k^r}{dP} \mathbf{1}_{\{\frac{dQ_k^r}{dP} > 0\}} \mathbf{1}_{\{\frac{dQ^r}{dP} > 0\}}
\end{aligned} \tag{76}$$

Since $\mathbf{1}_{\{\frac{dQ_k^r}{dP} > 0\}} \mathbf{1}_{\{\frac{dQ^r}{dP} > 0\}} \to \mathbf{1}_{\{\frac{dQ^r}{dP} > 0\}}$ a.s. and, as already checked, $\lim_{y \to +\infty} \frac{\Phi(y)}{y} = +\infty$ the limit in (76) is in fact $+\infty$ on the set $\{\frac{dQ^r}{dP} > 0\}$ which has positive probability as $Q \in \mathcal{N}^W$. But then

$$\begin{aligned}
L = \lim_k \{\alpha(Q_k) + \tfrac{1}{k}\} &= \lim_k \left\{ x + \|Q_k^s\| + E\left[\tfrac{\Phi(\lambda_k \frac{dQ_k^r}{dP}) - U_0^W(x)}{\lambda_k}\right]\right\} \\
&\geq x + \|Q^s\| + E\left[\liminf_k \tfrac{\Phi(\lambda_k \frac{dQ_k^r}{dP}) - U_0^W(x)}{\lambda_k}\right] = +\infty
\end{aligned}$$

where in the inequality we apply (74) and Fatou's Lemma.
Therefore there exists some compact subset of $\mathbb{R}_+$ that contains λ_k for infinitely many k's, so that we can extract a subsequence $\lambda_{k_n} \to \lambda^*$. The inequality (75) ensures that λ^* must be strictly positive. Otherwise, if $\lambda^* = 0$, the numerator of the fraction there tends to $\Phi(0) - U_0^W(x) = u(+\infty) - U_0^W(x) > 0$ and globally the limit random variable would be $+\infty$. Finally,

$$\begin{aligned}
\alpha(Q) &\leq x + \|Q^s\| + \frac{E[\Phi(\lambda^* \frac{dQ^r}{dP})] - U_0^W(x)}{\lambda^*} \\
&\leq x + \liminf_n \left\{ \|Q_{k_n}^s\| + \frac{E[\Phi(\lambda_{k_n} \frac{dQ_{k_n}^r}{dP})] - U_0^W(x)}{\lambda_{k_n}} \right\} = L.
\end{aligned}$$

Therefore, $\alpha = \pi^*$ and the identity $\partial\pi(B) = \mathcal{Q}_B^W(x + \pi(B))$ in (66) follows from (71) and (72).

6. The bounds below are easily proved,

$$\sup_{Q\in\mathcal{Q}_0^W(x)} Q(B) \leq \pi(B) \leq \sup_{Q\in\ \mathcal{N}^W} Q(B) \tag{77}$$

since the first inequality follows from the fact that when $Q \in \mathcal{Q}_0^W(x)$, the penalty $\alpha(Q) = 0$ (see (73)) and the second inequality holds because α is a penalty, i.e. $\alpha(Q) \geq 0$. The first supremum is in fact a maximum, which is a consequence of the"Max Formula" as better explained in item 7 below.
The case $W, B \in M^{\widehat{u}}$ is immediate from (77) and from the special form of the dual as stated in Corollary 3.16.

7. Let $\pi'(C, B)$ indicate the directional derivative of π at C along the direction B, i.e. $\pi'(C, B) = \lim_{b\downarrow 0} \frac{\pi(C+bB)-\pi(C)}{b}$. The so–called Max Formula ([BZ05, Theorem 4.2.7]) states that given a convex function π and a continuity point C, then

$$\pi'(C, B) = \max_{Q\in\partial\pi(C)} Q(B)$$

So the first volume asymptotic becomes a trivial application of the Max Formula with $C = 0$, since $bB \in \mathcal{B}$ if $b \leq 1 + \epsilon$ and

$$\lim_{b\downarrow 0} \frac{\pi(bB)}{b} = \pi'(0, B) = \max_{Q\in\mathcal{Q}_0^W(x)} Q(B),$$

because $\pi(0) = 0$ and $\mathcal{Q}_0^W(x) = \partial\pi(0)$.
For the second volume asymptotic, when $B \in M^{\widehat{u}}$ then $bB \in \mathcal{B}$ for all $b \in \mathbb{R}$. So, $\pi(bB)$ is well–defined and for all $b > 0$ we have that $\pi(bB) \leq \sup_{Q\in\mathcal{N}^W} Q(bB)$. Therefore

$$\limsup_{b\to+\infty} \frac{\pi(bB)}{b} \leq \sup_{Q\in\mathcal{N}^W} Q(B).$$

If we fix $Q \in \mathcal{N}^W$, the penalty $\alpha(Q)$ is finite and

$$\frac{\pi(bB)}{b} \geq Q(B) - \frac{\alpha(Q)}{b} \quad \text{for all } b > 0$$

so that

$$\liminf_{b\to+\infty} \frac{\pi(bB)}{b} \geq Q(B) \text{ for all } Q \in \mathcal{N}^W$$

so that

$$\lim_{b\to+\infty} \frac{\pi(bB)}{b} = \sup_{Q\in\mathcal{N}^W} Q(B).$$

Finally, the case $W, B \in M^{\widehat{u}}$ follows from the asymptotics just proved and Corollary 3.16.

8. If B and $-B$ are replicable with admissible strategies, then $Q(B) = c$ for all $Q \in \mathcal{N}^W$, whence in particular for the "zero penalty functionals" $Q \in \mathcal{Q}_0^W(x)$. Therefore $\pi(B) = \max_Q\{Q(B) - \alpha(Q)\} = c$

□

Remark 4.4. As already noted in [Be03, Remark 2.6], if B is not in $\mathcal{B}$ (e.g. a call option in a Black-Scholes model for an investor with exponential preferences) but satisfies

$$B = B^* + \int_0^T H_s^* dS_s$$

where $B^* \in \mathcal{B}$ and the strategy H^* is such that $\{H + H^* \mid H \in \mathcal{H}^W\} = \mathcal{H}^W$, then one can apply the analysis to B^* and *define* $\pi(B) = \pi(B^*)$.

To better compare our results with the current literature, in the next Corollary we specify the formula for π in the exponential utility case.

Corollary 4.5. *Let $u(x) = -e^{-\gamma x}$, fix a loss variable W and assume that $\mathcal{N}^W \neq \emptyset$. If $B \in \mathcal{B}$ then:*

$$\pi_\gamma(B) = \max_{Q \in \mathcal{N}^W} \left[Q(B) - \frac{1}{\gamma} \mathbb{H}(Q, P) \right], \tag{78}$$

where the penalty term is given by:

$$\begin{aligned} \mathbb{H}(Q, P) := & \quad \gamma \|Q^s\| + H(Q^r|P) - U_0^W \\ = & \quad \gamma \|Q^s\| + H(Q^r|P) - \min_{Q \in \mathcal{N}^W} \{H(Q^r|P) + \gamma \|Q^s\|\}. \end{aligned} \tag{79}$$

Observe that, apart from the presence of the singular term $\|Q^s\|$, this result coincides with equation (5.6) of [6Au02]. For a possible interpretation of this term, both in (79) and in the general representation (65) let us define a *catastrophic event* as a random variable χ such that

$$E[u(\chi)] > -\infty \text{ but } E[u(\alpha\chi)] = -\infty \text{ for some } \alpha > 0. \tag{80}$$

In other words, catastrophic events are given by random variables in the set

$$\widehat{\mathcal{D}} := \{f \in L^{\widehat{u}} \backslash M^{\widehat{u}} \text{ and } E[u(f)] > -\infty\}. \tag{81}$$

Since Q^s vanishes on $M^{\widehat{u}}$, we conclude that

$$\|Q^s\| = \sup_{f \in \mathcal{D}} Q^s(-f) = \sup_{f \in \widehat{\mathcal{D}}} Q^s(-f), \tag{82}$$

so that the singular component is only relevant when computing $Q(f)$ for a catastrophic $f \in \widehat{\mathcal{D}}$. Therefore, if $Q \in \mathcal{M}^W$ is a "pricing measure" for which $\|Q^s\| > 0$, then it might happen that $E_{Q^r}[f] > 0$ for a catastrophic random variable in the domain of optimization, despite the fact that $Q(f) \leq 0$ for all $f \in C^W$. Such Q should then be used with caution. When pricing the claim B using the formula (78) or (65), the pricing measures $Q \in \mathcal{M}^W$ that allow this unnatural behavior are penalized with a penalization proportional to the relevance that Q^s attributes to the catastrophic events according to (82).

We conclude this section with some considerations on the risk measure induced by π.

Corollary 4.6. *Under the same hypotheses of Proposition 4.3 , the seller's indifference price π defines a convex risk measure on $\mathcal{B}$, with the following representation:*

$$\rho(B) = \pi(-B) = \max_{Q \in \mathcal{N}^W} \{Q(-B) - \alpha(Q)\}. \tag{83}$$

If both the loss control W and the claim B are in $M^{\widehat{u}}$, then this risk measure has the Fatou property. In terms of π, this means

$$B_n \uparrow B \Rightarrow \pi(B_n) \uparrow \pi(B) \tag{84}$$

Proof. The first part is a consequence of the above Proposition and the second part follows from the fact that we have often stressed that when W, B are in $M^{\widehat{u}}$ there is a version of the dual problem only with regular elements $Q \in \mathcal{N}^W \cap L^1 = \mathbb{M}_\sigma \cap \mathcal{L}_\Phi$. Consequently there is a representation $\rho(B) = \max_{Q \in \mathcal{N}^W \cap L^1}\{Q(-B) - \alpha(Q)\}$ on the order continuous dual. But this implies the Fatou property (see e.g. [BF07, Prop. 26]). □

5 Comparison with existing literature

The results above extend the literature on utility maximization with random endowment when u is finite on the entire real line. In fact, we allow the semimartingale S to be non locally bounded and as far as we know ours is the first paper in this direction.

Also, the conditions we put on the claim B, that is $B \in \mathcal{A}_u$, are extremely weak - for the exponential utility $B \in \mathcal{A}_u$ simply means that B satisfies (26). The following list compares our conditions on B with those in the cited papers, *which are all formulated in the S locally bounded case.*

To better compare these works, we stress that when S is locally bounded, we may select $W = 1$ and therefore (see Corollary 3.16) the dual problem can be formulated totally free of singular parts, as soon as $B \in M^{\widehat{u}}$, and we also get the representation of the optimal f_B as terminal value of an S-stochastic integral (Proposition 3.18).

1. The first paper where a duality result of the type (2)-(3) appeared - obviously with no singular components - is Bellini and Frittelli [BeF02] Corollaries 2.2, 2.3, 2.4. In this paper, u is finite on the entire real line, B is bounded, $W = 1$, so that the admissible set of trading strategies is $\mathcal{H}^1$ and $\mathcal{M}^1$ is the set of local martingale measures.

2. The six Authors paper [6Au02] (see also the related work by Kabanov and Stricker [KS02]) considers only the exponential u. They extended the results [BeF02] in two respect. First they consider four different classes of trading strategies (including $\mathcal{H}^1$) and secondly, they assume condition (26) **plus** B bounded from below. These conditions clearly imply that $B \in \mathcal{B} = L^{\widehat{u}} \cap \mathcal{A}_u$.

3. Becherer's paper [Be03] also consider only the exponential case and extend further the results in 1) and 2) above. His Assumption 2.4

$$E[e^{(\gamma+\varepsilon)B}] < +\infty, \; E[e^{-\varepsilon B}] < +\infty$$

is however equivalent to saying that conditions (26) and (27) hold, i.e. that $B \in \mathcal{B} = L^{\widehat{u}} \cap \mathcal{A}_u$.

4. For general utility u finite on $\mathbb{R}$, the Assumption 1.6 on B in Owen and Zitkovich [OZ07] is on a different level, since it is a joint condition on B and the admissible strategies. This condition is not easy to verify in practice, since it requires the prior knowledge of the dual measures. Also, for economic reasons, we believe that it is better to state the conditions on the claim only in terms of the compatibility with the utility function.

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