THE ELLIPTIC CURVE IN THE S-DUALITY THEORY AND EISENSTEIN SERIES FOR KAC-MOODY GROUPS

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Abstract. We establish a relation between the generating functions appearing in the S-duality conjecture of Vafa and Witten and geometric Eisenstein series for Kac-Moody groups. For a pair consisting of a surface and a curve on it, we consider a refined generating function (involving G-bundles with parabolic structures along the curve) which depends on the elliptic as well as modular variables and prove its functional equation with respect to the affine Weyl group, thus establishing the elliptic behavior. When the curve is $\mathbb{P}^1$, we calculate the Eisenstein-Kac-Moody series explicitly and it turns out to be a certain deformation of the irreducible Kac-Moody character, more precisely, an analog of the Hall- Littlewood polynomial for the affine root system. We also get an explicit formula for the universal blowup function for any simply connected structure group.

Introduction

(0.1) The goal of this paper is to develop a certain mathematical framework underlying the S-duality conjecture of Vafa and Witten [VW]. Let us recall the formulation. Let S be a smooth projective surface over $\mathbb{C}$ and G a semisimple algebraic group. Denote by $M_G(S,n)$ the moduli space of semistable principal G-bundles on S with the (second) Chern number equal to n and form the generating function of their topological Euler characteristics:

$$F_G(q) = \sum_n \chi(M_G(S,n))q^n, \quad q = e^{2\pi i\tau}. \tag{1}$$

Then, the conjecture says that up to simple factors, $F_G(q)$ is a modular form with respect to a certain congruence subgroup in $SL_2(\mathbb{Z})$ and, moreover, relates the image of F_G under the transformation $\tau \mapsto -(1/\tau)$ with F_{G^L}, where G^L is the Langlands dual group. In fact, this formulation is correct only for surfaces which are not of general type and has to be modified in general.

Important work [Go] [LQ1-2] [Y] has been done on the verification of this conjecture in various particular cases (mostly restricted to $G = SL_2$ or PGL_2). A treatment of the general case has been prevented by the lack of some fundamental understanding of the problem. More precisely, underlying the whole theory (and logically preceding any finer details of the transformation properties) is the following immediate question.

Question 1. *Is there a purely mathematical conceptual reason for F_G to have anything to do with modular functions at all? In other words, why should the variable q in (1), which is just a formal variable in the generating function, be thought of as parametrizing the elliptic curves $\mathcal{E}_q = \mathbb{C}^*/q^{\mathbb{Z}}$?*

Let $D = \{0 < |q| < 1\}$ be the punctured unit disk and $\mathcal{E} \to D$ be the family of the curves $\mathcal{E}_q$. Our first step is to introduce a new generating function which depends on variables living on (a cover of) $\mathcal{E}$ and not just on D. Suppose S is equipped with a smooth irreducible curve X. We consider pairs (P, π) where P is a principal G-bundle as before and π is a parabolic structure in $P|_X$, i.e., a reduction of the structure group from G to B, the Borel subgroup. A B-bundle Q on X has a vector of degrees $\deg(Q)$ belonging to L, the lattice of coweights of G. We denote by $M_G(n, a)$ the moduli space of semistable parabolic G-bundles P on S with $c_2(P) = n$ and $\deg(P|_X) = a$. Here we must presume fixed the "polarization parameters" [MY] neccesary to fix the meaning of semistability. We now form the generating function

$$E_G(q, z) = \sum_{n \in \mathbb{Z}, a \in L} \chi(M_G(n, d)) q^n z^a, \quad q \in D, z \in T^\vee. \tag{2}$$

Here $T^\vee = \mathrm{Spec}(\mathbb{C}[L])$ is the "dual torus". The product $D \times T^\vee$ is a natural cover of the relative abelian variety $\mathcal{E}_L = \mathcal{E} \otimes_{\mathbb{Z}} L$, much as $D \times \mathbb{C}^*$ is a cover of $\mathcal{E}$.

It is natural then to conjecture that in the cases when F_G is expected to have a modular behavior, E_G should be, up to simple factors, a Jacobi form [EZ], i.e., exhibit both modular behavior in q and elliptic behavior in z.

(0.2) The first main result of this paper is the proof the elliptic behavior of the function E_G in a wide range of "relative situations" (allowing, for example, G to be an arbitrary simply connected group). More precisely, we fix a G-bundle P° on $S - X$ and consider the moduli space $M_{G,P^\circ}(n, a)$ formed by triples (P, τ, π) where (P, π) are as above and τ is an isomorphism of $P|_{S-X}$ with P°. When the self-intersection index $X_S^2 = -d$ is negative, we prove (Theorem 2.2.1) that this space is a scheme of finite type over k without imposing any stability conditions. One can form then the relative analogs of the functions F_G and E_G, which we denote $F_{G,P^\circ}(q)$ and $E_{G,P^\circ}(q, z)$. When $X = \mathbb{P}^1$ and $d = 1$, we have the blowup situation considered in [LQ1-2].

Instead of the Euler characteristic, we work with any kind of "motivic measure", i.e., any invariant of algebraic varieties additive with respect to cutting and pasting, such as, e.g., the number of points over the finite field. We prove (Theorem 3.4.4) that after multiplying $E_{G,P^\circ}(q, z)$ with a certain product over affine roots of G (an analog of the Weyl-Kac denominator), we get a series which is a regular section of the dth power of a natural theta-bundle Θ on $\mathcal{E}_L$. This bundle is well-known in the theory of Kac-Moody groups: among its sections one finds characters of level d integrable representations of the Kac-Moody group associated to the Langlands dual group G^L (note that $T^\vee$ is the maximal torus for G^L).

The second main result of the paper is a complete determination of the functions $E_{G,P^\circ}(q,z)$ and $F_{G,P^\circ}(q)$ in the case when the curve X is $\mathbb{P}^1$. In this case, we first show (Theorem 7.1.2) that the type of P° on the punctured formal neighborhood of X can be encoded by an antidominant coweight b of the Kac-Moody group of G, or, if one prefers, an antidominant weight of the Kac-Moody group of G^L. Next we identify (Theorem 7.3.1) the function $E_{G,P^\circ}(q,z)$ with the analog of the Hall-Littlewood polynomial [Mac] corresponding to the affine root system of G^L. In other words, this function is a certain deformation of the character of the irreducible representation of the Kac-Moody group with the highest weight $(-b)$. The parameter of deformation, denoted $\mathbb{L}$, is interpreted as the "Tate motive" (the value of our motivic invariant on the affine line). In particular, we get an explicit formula for the universal blowup function for an arbitrary simply connected structure group.

(0.3) The main idea behind our approach is that E_{G,P° is an analog, for the Kac-Moody groups, of the unramified Eisenstein series familiar in the Langlands' program [Lan] and studied in detail for function fields by Harder [H2].

More precisely, if X is a curve over a finite field $\mathbb{F}_l$ and Q is a principal G-bundle over X, then we have the associated flag fibration $\mathcal{F}_Q$ on X with fiber G/B. A section s of $\mathcal{F}_Q$ has a natural degree lying in L, and all sections of given degree a form a scheme of finite type $\Gamma_a(Q)$. The Eisenstein series is just the generating function

$$E_{G,Q}(z) = \sum_{a\in L} \#(\Gamma_a(Q)(\mathbb{F}_l))z^a, \quad z \in T^\vee. \tag{3}$$

This series is known to represent a rational function which satisfies a functional equation for each element of W, the Weyl group. When we replace G by its Kac-Moody group, we get, essentially, the series (2) and in fact Theorem 3.4.4 says that it satisfies a functional equation for each element of $\widehat{W}$, the affine Weyl group. The lattice L is a subgroup of $\widehat{W}$, so we get the elliptic behavior as a particular case.

Similarly, the explicit calculation of the functions E_{G,P° and F_{G,P° in Theorems 7.3.1 and 7.4.6 can be seen of the analogs, in the Kac-Moody situation of Langlands' calculation of the constant term of the Eisenstein series [Lan].

The analogy between E_{G,P° and the "classical" Eisenstein series $E_{G,Q}$ is naturally understood in the context of Atiyah's work on instantons in two and four dimensions. [A].

(0.4) We can now formulate succinctly our proposed answer to Question 1: *the elliptic curve appearing in the theory of G-bundles on surfaces is the same curve as appears in the theory of Kac-Moody groups.* More precisely, $\widehat{G}$, the full Kac-Moody group associated to G (i.e., the group whose Lie algebra is associated to the affine Cartan matrix of G by the procedure of [Ka]) is not the loop group $G((\lambda))$, nor its central extension, but the semidirect product of the central extension with the group $\mathbb{C}_q^*$ of "rotations of the loop". Accordinly the maximal torus is $\widehat{T} = \mathbb{C}_c^* \times T \times \mathbb{C}_q^*$, where $\mathbb{C}_c^*$ is the center. The quotient of the open part $\mathbb{C}_c^* \times T \times D$ by $L \subset \widehat{W}$ is a $\mathbb{C}^*$-bundle over $\mathcal{E}_L$ which corresponds to the theta-bundle, see [Lo]. So, for instance, characters of $\widehat{G}$, being homogeneous in $\mathbb{C}_c^*$ and $\widehat{W}$-invariant, are authomatically sections of a power of the theta-bundle.

In our case, however, we have a similar situation but applied to the dual group G^L. Formally, the Langlands dual group to $\widehat{G}$ should be $\widehat{G^L}$, and the center of $\widehat{G}$ corresponds to the loop rotation subgroup of $\widehat{G^L}$ and vice versa. Now, the concept of the second Chern class (for vector bundles) is known since S. Bloch [Bl] to be intimately related to central extensions of matrix groups, and in our case this relation descends to the central extension of $G((\lambda))$. More precisely, in our case λ has the meaning of a local equation of X in S, and since we consider bundles identified with P° on $S - X$, the Chern class lives naturally in the cohomology with support in X.

So the variable n in (1) corresponds naturally to $\mathbb{C}_c^*$ in $\widehat{G}$. On the other hand, the self-intersection index $(-d)$ of X in S in an invariant of the normal bundle of X in S and so corresponds to $\mathbb{C}_q^*$ in $\widehat{G}$. When we pass to the Eisenstein series, we are transferred to the dual group, so d behaves now as if it was the central charge of a character of $\widehat{G^L}$. In fact, out Theorem 7.3.1 identifies the Eisenstein series with the $\mathbb{L}$-deformation of a character of central charge d.

(0.5) Let us now describe the contents of the paper in more detail. In Section 1 we review the general concept of "motivic measures" (such as the Euler characteristic) and of integration of such measures. Section 2 is devoted to the finiteness theorem 2.2.1 for the moduli space $M_{G,P^\circ}(n)$. This result can be seen as an analog of the reduction theory developed by H. Garland [Ga] for loop groups over number fields. Garland's twist by e^{-tD}, $t > 0$, corresponds to our condition that the self-intersection index of the curve is negative. In Section 3 we formulate our results about the structure of the generating function $E_{G,P^\circ}(q,z)$. In Section 4 we give a self-contained treatment of motivic Eisenstein series for finite-dimensional groups. In Section 5 we give a treatment of the Kac-Moody group as a functor on the category of smooth varieties. Our aim in this section was to make straightforward the connection with the second Chern class which appears in the generating functions (1) and (2). Next, in Section 6 we develop the formalism of Eisenstein series for Kac-Moody groups and prove the results of Section 3. Finally, Section 7 is devoted to explicit calculations of Eisenstein-Kac-Moody series. in the case $X = \mathbb{P}^1$. We first establish a version of Grothendieck's theorem [Gro] on G-bundles on $\mathbb{P}^1$ in the case when G is replaced by the sheaf $\mathcal{G}$ of Kac-Moody groups; as a result, we let labelling of isomorphism classes of $\mathcal{G}$-torsors by antidominant affine coweights of G. Then, we calculate the Eisenstein series in terms of the affine analog of the Hall polynomial and also give a "parahoric" version of this calculation which identifies F_{G,P°.

(0.6) This work was started during my stay in Max-Planck Institut für Mathematik in 1996 and was reported at the Oberwolfach conference on conformal field theory in June 1996. In writing the present text, I was fortunate to be able to use remarks and advice of several people. Thus, the idea of applying the method of torus actions, used in Examples 2.4.2 and 3.1.2, I owe to H. Nakajima, who uses it in a similar (but different) situation in the work in progress with K. Yoshioka. The purely algebraic approach to the proof of functional equations for the Eisenstein series, presented in n. 4.3, I learned from a seminar talk of V. Drinfeld. I am grateful to A. Beilinson and V. Drinfeld for numerous remarks on the previous version and to I. Cherednik and A. Kirillov, Jr. for discussions of Macdonald's theory and its affine generalization. The revised version was prepared during my visit to MSRI in January-February 2000. The financial support by MPI and MSRI is hereby gratefully acknowledged. The research on this paper was partly supported by the NSF.

§1. Motivic measures and zeta functions.

We will use a version of the formalism of "motivic integration" from [DL].

(1.1) Motivic measures. Throughout this paper we fix a field k and denote by Sch_k the category of schemes of finite type over k.

Let A a commutative ring. An A-valued *motivic measure* on Sch_k is a function μ which associates to any scheme $Z \in \mathrm{Sch}_k$ an element $\mu(Z) \in A$ depending only on the isomorphism class of Z such that the two conditions hold:

(1.1.1) If $Y \subset Z$ is a closed subscheme, then $\mu(Z) = \mu(Y) + \mu(Z - Y)$.

(1.1.2) $\mu(Z_1 \times Z_2) = \mu(Z_1) \times \mu(Z_2)$.

Note that (1.1.1) implies that $\mu(Z)$ coincides with $\mu(Z_{red})$.

(1.1.3) Examples: (a) $k = \mathbb{C}$, $A = \mathbb{Z}$ and $\mu(V) = \chi_c(V) = \sum_i (-1)^i \dim H^i_c(V, \mathbb{C})$ is the topological Euler characteristic with compact supports.

(b) $k = \mathbb{F}_l$, a finite field with l elements, $A = \mathbb{Z}$, and $\mu(V) = \# V(\mathbb{F}_l)$ is the number of $\mathbb{F}_l$-points.

(c) $k = \mathbb{C}, A = \mathbb{Z}[l^{1/2}]$ is the polynomial ring, and μ takes V to its Serre (or virtual Hodge) polynomial $S_V(l) = \sum (-1)^i l^{i/2} \chi(\mathrm{gr}^W_i (H^\bullet(X(\mathbb{C}), \mathbb{C})))$. Here W is the weight filtration and χ is the Euler characteristic of the graded vector space. If V is smooth projective, then $S_V(l) = \sum_i (-1)^i l^{i/2} \dim(H^i(V(\mathbb{C}), \mathbb{C}))$ is the Poincaré polynomial of V; in fact S_V is uniquely determined by this condition and the fact that it is a motivic measure. Specializing at $l = 1$, we get $S_V(1) = \chi_c(V(\mathbb{C}))$.

(d) There is the universal motivic measure μ_u whose target A_u is the ring generated by the symbols $[Z]$, where Z is a quasi-projective variety, and which are subject to the relations $[Z] = [Y] + [Z - Y]$, for a closed $Y \subset Z$ and $[Z_1] \cdot [Z_2] = [Z_1 \times Z_2]$.

We denote by $\mathbb{L}$ the value of μ at the affine line $\mathbb{A}^1_k$. Thus, in the Example 1.1.3(b) $\mathbb{L} = l$ is the number of the elements in $\mathbb{F}_l$, and in (1.1.3)(c) $\mathbb{L} = l$ is the square of the generator of the polynomial ring $A = \mathbb{Z}[l^{1/2}]$.

(1.2) Motivic integration. By $\bar{k}$ we denote a fixed separable closure of k. Fix an A-valued motivic measure μ. Let Z be a scheme of finite type over k. If $W \subset Z$ is a closed subscheme (defined over k), we denote by $\mathbf{1}_W : Z(\bar{k}) \to A$ the characteristic function of $Z(\bar{k})$. More generally, by a constructible (A-valued) function on Z we mean a function $f : Z(\bar{k}) \to A$ which can be represented as a finite linear combination of characteristic functions of closed subschemes $f = \sum_i a_i \mathbf{1}_{W_i}$. The set of such functions will be denoted $\mathrm{Const}_A(Z)$. For $f \in \mathrm{Const}_A(Z)$ one defines its integral (with respect to μ) as

$$\int_Z f(z) d\mu := \sum a_i \mu(W_i), \quad \text{if} \quad f = \sum_i a_i \mathbf{1}_{W_i}. \tag{1.2.1}$$

It is standard to see that this definition is independent on the way of writing f as $\sum a_i \mathbf{1}_{W_i}$. Further, let $\varphi : X \to Y$ be a morphism of schemes and $f \in \mathrm{Const}_A(X)$. Then the direct image (with respect to μ) of f is the function

$$\varphi_*(f) = \int_{X/Y} f \cdot d\mu, \quad y \mapsto \int_{\varphi^{-1}(y)} f(x) d\mu. \tag{1.2.2}$$

The following is straightforward.

(1.2.3) Proposition. *Suppose that there is a finite stratification of Y by locally closed subschemes Y_α such that over each Y_α the morphism φ is a (Zariski) locally trivial fibration. Then for a constructible function $f \in \mathrm{Const}_A(X)$ the function $\varphi_*(f)$ is also constructible, and*

$$\int_X f(x) d\mu = \int_Y \varphi_*(f)(y) d\mu.$$

(1.3) Motivic zeta. Let X be a smooth algebraic variety over k. We denote by $X^{(n)}$ the n-fold symmetric product of X. The motivic zeta-function of X (associated to μ) is the formal series

$$\zeta_\mu(X, u) = \sum_{n=0}^{\infty} \mu(X^{(n)}) u^n \in A[[u]]. \tag{1.3.1}$$

(1.3.2) Examples. (a) If $k = \mathbb{F}_l$ and μ is given by the number of $\mathbb{F}_l$-points, then we get the usual Hasse-Weil zeta-function of X.

(b) Consider the situation of Example 1.1.3(c) and assume that X is projective. Then it is easy to see that

$$\zeta_\mu(X, u) = \prod_i (1 - l^{i/2} u)^{(-1)^{i+1} b_i(X)}, \quad b_i(X) = \dim H^i(X, \mathbb{C}).$$

This formula has the same shape as the Hasse-Weil zeta function in the $\mathbb{F}_l$-case, with all the eigenvalues of the Frobenius on H^i being replaced by $l^{i/2}$.

The following fact generalizes the well known properties of zeta functions of curves over $\mathbb{F}_l$.

(1.1.9) Theorem. *Let X be a smooth projective irreducible curve of genus g. Suppose that A is a field and $\mathbb{L} \neq 0$. Suppose further that there exists a line bundle on X of degree 1. Then:*

(a) The series $\zeta_\mu(X, u)$ represents a rational function. In fact, $\Phi_X(u) = (1 - u)(1 - \mathbb{L}u)\zeta_\mu(X, u)$ is a polynomial of degree $2g$.

(b) The function $\zeta_\mu(X, u)$ satisfies the functional equation

$$\zeta_\mu(X, 1/\mathbb{L}u) = \mathbb{L}^{1-g}u^{2-2g}\zeta_\mu(X, u).$$

Proof: This is analogous to Artin's classical proof for the $\mathbb{F}_l$-case. Let $\mathrm{Pic}_n(X)$ be the variety of line bundles of degree n on X and $p_n : X^{(n)} \to \mathrm{Pic}_n(X)$ be the natural projection. Clearly $p_n^{-1}(L) = \mathbb{P}(H^0(X, L))$, so Proposition 2.1.6 is applicable and yields

$$\zeta_\mu(X, u) = \sum_{n\in\mathbb{Z}} u^n \int_{L\in\mathrm{Pic}_n(X)} \frac{\mathbb{L}^{h^0(L)-1}}{\mathbb{L} - 1} d\mu_L.$$

This means that

$$\zeta_\mu(X, 1/\mathbb{L}u) = \sum_{n\in\mathbb{Z}} u^n\mathbb{L}^n \int_{L\in\mathrm{Pic}_{-n}(X)} \frac{\mathbb{L}^{h^0(L)-1}}{\mathbb{L} - 1} d\mu_L,$$

while

$$\mathbb{L}^{1-g}u^{2-2g}\zeta_\mu(X, t) = \sum_n u^n \int_{M\in\mathrm{Pic}_{2g-2+n}(X)} \frac{\mathbb{L}^{h^0(M)-1}}{\mathbb{L} - 1}\mathbb{L}^{1-g} d\mu.$$

Consider the isomorphism

$$\sigma : \mathrm{Pic}_{-n}(X) \to \mathrm{Pic}_{2g-2+n}(X), \quad L \mapsto \omega_X \otimes L^*.$$

By the Riemann-Roch theorem, for $M = \sigma(L)$ we have $h^0(L) - h^0(M) = -n + 1 - g$ and thus

$$\mathbb{L}^n\frac{\mathbb{L}^{h^0(L)-1}}{\mathbb{L} - 1} - \mathbb{L}^{1-g}\frac{\mathbb{L}^{h^0(M)-1}}{\mathbb{L} - 1} = \frac{\mathbb{L}^{1-g} - \mathbb{L}^n}{\mathbb{L} - 1}.$$

So the difference between the two sides of the putative functional equation, considered as a formal series in both positive and negative powers of u, is

$$\frac{1}{\mathbb{L} - 1}\sum_{n\in\mathbb{Z}} \mu(\mathrm{Pic}_n(X))(\mathbb{L}^{1-g} - \mathbb{L}^n)u^n.$$

Since there exists a line bundle of degree 1 on X, the multiplication by the nth tensor power of this bundle identifies $\mathrm{Pic}_n(X)$ with $\mathrm{Pic}_0(X)$ and the above series is equal to

$$\frac{\mu(\mathrm{Pic}_0(X))}{\mathbb{L}-1}\left(\sum_{n\in\mathbb{Z}}\mathbb{L}^{1-g}u^n - \sum_{n\in\mathbb{Z}}\mathbb{L}^n u^n\right) = \mu(\mathrm{Pic}_0(X))\left(\frac{\mathbb{L}^{1-g}}{\mathbb{L}-1}\delta(u) - \frac{1}{\mathbb{L}-1}\delta(\mathbb{L}u)\right),$$

where $\delta(u) = \sum_{n\in\mathbb{Z}} u^n$ is the Fourier series of the delta-function at 1. Now we quote the following elementary algebraic fact, which implies our statement.

(1.3.4) Lemma. *Let A be a field and $g_0(u) \in A((u))$, $g_\infty(u) \in A((u^{-1}))$ be two formal Laurent series in powers of u, resp. u^{-1}. Let $D = \sum n_i a_i$ be a positive divisor on the affine line over A, with $a_i \in A, n_i \geq 0$. Suppose that we have an equality of formal series in positive and negative powers of t:*

$$g_0(u) - g_\infty(u) = \sum_i \sum_{\nu=1}^{n_i} c_{i\nu}\delta^{(\nu)}(u/a_i),$$

where $\delta^{(\nu)}(u)$ is the νth formal derivative of $\delta(u)$. Then there exists a rational function $g \in A(u)$ whose expansion at 0 is g_0, at ∞ is g_∞ and whose divisor of poles is bounded by D.

(1.3.5) Remarks. (a) If we drop the assumption $\mathrm{Pic}_1(X)(k) \neq \emptyset$, then the above arguments still show that $\zeta_\mu(X,u)$ is rational and satisfies the same functional equation, but it may have more poles. The poles will then lie at the points u satisfying $u^d = 1$ and $\mathbb{L}u^d = 1$, where d is such that $\mathrm{Pic}_d(X)(k) \neq \emptyset$.

(b) It is natural to expect that the motivic zeta-functions of higher-dimensional varieties are rational and satisfy similar functional equations.

§2. Relative moduli spaces of G-bundles.

(2.1) The second Chern class. For a smooth variety S over k we denote by $CH^m(S)$ the Chow group of codimension m cycles on S modulo rational equivalence. Thus $CH^m(S) = H^m(S, \underline{K}_{m,S})$, where $\underline{K}_{m,S} = K_m(\mathcal{O}_S)$ is the sheaf of Quillen K-functors [Q].

Let G be a split simple, simply connected algebraic group over k and $\underline{G}_S$ be the sheaf of G-valued regular functions on S. By the work of Steinberg, Moore and Matsumoto [Ma2], as developed in [BD] [EKLV], we have a natural (in S) central extension of sheaves of groups on S:

$$1 \to \underline{K}_{2,S} \to \widetilde{\underline{G}}_S \to \underline{G}_S \to 1. \tag{2.1.1}$$

This extension comes from a canonical element in $H^2(B_\bullet G, \underline{K}_2)$ which, in its turn, is represented by a multiplicative $\underline{K}_{2,G}$-torsor Φ on G, defined uniquely up to isomorphism of multiplicative torsors, see [BD]. In the sequel, we will assume Φ fixed. This fixes (2.1.1) uniquely up to a unique isomorphism.

If P is a principal G-bundle over S, then (2.1.1) gives a class $c_2(P) \in H^2(S, \underline{K}_{2,S}) = CH^2(S)$ called the second Chern class of P. More precisely (see [Bl]), the sheaf $\underline{P}$ of regular sections of P is a sheaf of $\underline{G}_S$-torsors and the (local) liftings of it to a sheaf of $\underline{\widetilde{G}}_S$-torsors form a $\underline{K}_{2,S}$-gerbe on S, and $c_2(P)$ is the class corresponding to this gerbe by Giraud's theory [Gi].

We now describe the properties of c_2 which we need. Let T be the maximal torus in G and $L = \mathrm{Hom}(\mathbb{G}_m, T)$ be the lattice of coweights. Let also W be the Weyl group of G and $\Psi : L \times L \to \mathbb{Z}$ be the minimal W-invariant integral negative definite scalar product on L. It is proportional to the Killing form, see [Ka], Ch. 6. The quadratic form $\Psi(x, x)$ is even, so $\Psi(x, x)/2$ is the minimal W-invariant quadratic form on L.

If V is a representation of G with $\dim(V) = N$, then it induces a homomorphism $T \to \mathbb{G}_m^N$ of tori and at the level of 1-parameter subgroups, a homomorphism of Abelian groups $\lambda_V : L \to \mathbb{Z}^N$. Let φ be the quadratic form on $\mathbb{Z}^N$ given by $\varphi(a_1, ..., a_N) = \sum_{i<j} a_i a_j$ amd $\lambda_V^* \varphi$ be the induced form on L. It is then an integer multiple of the form $\Psi(x, x)/2$ and we denote by $\varkappa_V \in \mathbb{Z}$ the coefficient of proportionality: $\varphi(\lambda_V(x)) = \varkappa_V \Psi(x, x)/2$. We also denote by ${}^P V$ the vector bundle on S associated to P and the representation V.

(2.1.2) Proposition. *(a) The usual Chern class $c_2({}^V P)$ is equal to $\varkappa_V c_2(P)$.*

(b) Let Q be a principal T-bundle on S and $c_1(Q) \in H^1(S, \underline{Q}) = \mathrm{Pic}(S) \otimes_{\mathbb{Z}} L$ be the class of Q. If P is the G-bundle obtained from Q, then

$$c_2(P) = (m \otimes \Psi)(c_1(Q), c_1(Q)),$$

where $m : \mathrm{Pic}(S) \times \mathrm{Pic}(S) \to CH^2(S)$ is the intersection product.

Proof: Both properties follow easily from the construction of the central extension in [BD], because the form Ψ is used first to construct the central extension over the torus. .

(2.2) The relative moduli space. Let now S be a smooth projective surface. We have then the degree homomorphism $\deg : CH^2(S) \to \mathbb{Z}$. For a principal G-bundle P on S we will abbreviate $\deg(c_2(P))$ to simply $c_2(P)$ and call it the (second) Chern number of P. Let $X \subset S$ be a smooth irreducible curve. We denote X_S^2 the self-intersection index of X in S and assume that $X_S^2 < 0$. We denote $S^\circ = S - X$. Let P° be a principal G-bundle on S°.

(2.2.1) Theorem. *There exists a fine moduli space $M_{G,P^\circ}(n)$ of pairs (P, τ) where P is a principal G-bundle on S with $c_2(P) = n$ and $\tau : P|_{S^\circ} \to P^\circ$ is an isomorphism. This space is a scheme of finite type over k which is empty for $n \ll 0$.*

Proof: Let $\mathcal{M}_{G,P^\circ}(n)$ be the moduli functor on the category of k-schemes which corresponds to our problem. Note that any (P,τ) as above has no nontrivial automorphism. This means that $\mathcal{M}_{G,P^\circ}(n)$, coming as it does, from a stack, is in fact a sheaf with respect to the flat topology.

(2.2.2) Proposition. *The functor $\mathcal{M}_{G,P^\circ}(n)$ is represented by an ind-scheme $M_{G,P^\circ}(n)$ which is an inductive limit of quasiprojective schemes over k.*

Proof: First consider the case $G = SL_r$. We have then a rank r bundle V° on S° with $\det(V^\circ) \simeq \mathcal{O}_{S^\circ}$. Let $j : S^\circ \to S$ be the embedding. Then pairs (P,τ) as in the theorem are in bijection with locally free subsheaves $V \subset j_*V^\circ$, $\det(V^\circ) \simeq \mathcal{O}_S$. let us assume that there exists at least one such subsheaf, say V_0 (otherwise $\mathcal{M}_{G,P^\circ}(n)$ is the functor identically equal to the empty set and there is nothing to prove). Let $\mathbf{m} \subset \mathcal{O}_S$ be the sheaf of ideals of X. Then any V as above is contained in $\mathbf{m}^{-N}V_0$ and contains $\mathbf{m}^N V_0$ for some $N \gg 0$. This means that $\mathcal{M}_{G,P^\circ}(n)$ is an inductive limit, over N, of functors represented by locally closed subschemes in $Quot(\mathbf{m}^{-N}V_0/\mathbf{m}^N V_0)$, and the case $G = SL_r$ is proved.

The case of arbitrary G is reduced to the above by taking a sufficient number of representations $V_1, ..., V_M$ of G and realizing $M_{G,P^\circ}(n)$ as a closed sub-ind-scheme in the product $\prod_i M_{SL_{r_i}, P^\circ V_i}(\varkappa_{V_i} n)$.

Let Y be the formal neighborhood of X in S and $\mathrm{Bun}_G(Y)$, $\mathrm{Bun}_G(X)$ be the moduli stacks of G-bundles on Y and X. We have then the restriction maps

$$\mathrm{Bun}_{G,P^\circ}(n) \xrightarrow{\varphi} \mathrm{Bun}_G(Y) \xrightarrow{\psi} \mathrm{Bun}_G(X). \tag{2.2.3}$$

Here we regard $\mathrm{Bun}_{G,P^\circ}(n)$ as a trivial stack (a sheaf of sets).

(2.2.4) Proposition. *For any principal G-bundle Q on X the stack $\psi^{-1}(Q)$ is bounded (i.e., dominated by a k-scheme of finite type).*

Proof: Let $X^{(d)} = \mathrm{Spec}(\mathcal{O}_S/\mathbf{m}^{d+1})$ be the dth infinitesimal neighborhood of X in S. Given an extension $Q^{(d)}$ of Q to $X^{(d)}$, all its further extensions to $X^{(d+1)}$, if exist, form a homogeneous space over $H^1(X, \mathrm{ad}(Q) \otimes (\mathbf{m}^{d+1}/\mathbf{m}^{d+2}))$. But $\mathbf{m}^{d+1}/\mathbf{m}^{d+2} = (N^*_{X/S})^{\otimes(d+1)}$ and since $X_S^2 = \deg(N_{X/S}) < 0$, for large d the cohomology in question vanishes. Thus all the extensions of Q to a bundle on Y are determined by their restrictions to $X^{(d)}$ for some d, and the latter form a bounded family.

(2.2.5) Proposition. *For any principal F-bundle $\widehat{P}$ on Y, the stack $\varphi^{-1}(\widehat{P})$ is bounded.*

Proof: Let $Y^\circ = Y \cap S^\circ$ be the punctured formal neighborhood of X in S. Supposing $\varphi^{-1}(\widehat{P}) \neq \emptyset$ (otherwise there is nothing to prove), there is a G-bundle P on S whose restriction to Y is isomorphic to $\widehat{P}$ and whose restriction to S° is isomorphic to P°. Any other object of $\varphi^{-1}(\widehat{P})$ is then obtained by gluing $\widehat{P}$ and $\widehat{P}^\circ$ over Y° in a different way. This new gluing function is an element of $\mathrm{Aut}(P|_{Y^\circ})$. If $g, g' \in \mathrm{Aut}(P|_{Y^\circ})$ are such that $g' = gh$ with $h \in \mathrm{Aut}(\widehat{P})$, then the gluings by g and g' give the same object (element) of $M_{G,P^\circ}(n)$. Thus it is enough to prove the following.

(2.2.6) Proposition. *Let P be any principal G-bundle on S. Then the quotient* $\mathrm{Aut}(P|_{Y^\circ})/\mathrm{Aut}(P|_Y)$ *is bounded.*

Proof: We first consider the case $G = SL_r$, so P is given by a rank r bundle V on S. As $X_S^2 < 0$, the curve X can be blown down to a (possibly singular) point p on a new surface S'. Let $\sigma : S \to S'$ be the projection. Because σ^{-1} identifies $S' - \{p\}$ with $S^\circ = S - X$, we have the embeddings $j : S^\circ \to S$ and $j' = \sigma j : S^\circ \to S'$. Let Y' be the formal neighborhood of p in S' and $\widehat{\sigma} : Y \to Y'$ be the projection. Let also ${Y'}^\circ = Y' - \{p\}$ be the punctured formal neighborhood. Denote $\mathbf{m} \subset \mathcal{O}_S$ and $\widehat{\mathbf{m}} \subset \mathcal{O}_Y$ the sheaves of ideals corresponding to X. The morphism σ is projective, in fact

$$S = \mathrm{Proj}\Big(\bigoplus_{d\geq 0} \sigma_*(\mathbf{m}^d)\Big)$$

and the relative sheaf $\mathcal{O}(1)$ on the projective spectrum is identified with $\mathbf{m}$. The same is true for $\widehat{\sigma}$. Now, let us aply the relative version of Serre's theorem (about the equivalence of the categories of coherent sheaves and graded modules) to the coherent sheaf $\widehat{V} = V|_Y$ on Y. We conclude that for $d \gg 0$ we have

$$\mathrm{Hom}_Y(\widehat{V}, \widehat{V}) = \mathrm{Hom}_{Y'}\big(\widehat{\sigma}_*(\widehat{V}(d)), \widehat{\sigma}_*(\widehat{V}(d))\big),$$

where $\widehat{V}(d) = \widehat{V} \otimes \mathcal{O}(d) = \mathbf{m}^d \widehat{V}$. Hence $\mathrm{Aut}(\widehat{V}) = \mathrm{Aut}(\widehat{\sigma}_* \widehat{V}(d))$. Since $\mathrm{Aut}(\widehat{V}) = \mathrm{Aut}(\widehat{V}(d))$, we can replace $\widehat{V}$ by $\widehat{V}(d)$ and assume that $\mathrm{Aut}(\widehat{V}) = \mathrm{Aut}(\widehat{\sigma}_*(\widehat{V}))$. On the other hand,

$$\mathrm{Aut}(\widehat{V}|_{Y^\circ}) = \mathrm{Aut}((\widehat{\sigma}_* \widehat{V})|_{Y'^\circ}) \subset \mathrm{Aut}((j'_* {j'}^* \sigma_* V)|_{Y'}).$$

Since j' is an embedding of the complement of a point into a surface, $j'_* {j'}^* \sigma_* V$ is a coherent sheaf on S' which contains the (torsion-free) sheaf $\sigma_* V$ and coincides with it outside p. We have therefore an exact sequence

$$0 \to \widehat{\sigma}_* \widehat{V} \to (j'_* {j'}^* \sigma_* V)|_{Y'} \to \mathcal{F} \to 0, \tag{2.2.7}$$

where $\mathcal{F}$ is a coherent sheaf supported at p. It follows that the quotient of the automorphism group of the middle sheaf by the automorphism group of the left sheaf is bounded. Thus Proposition 2.2.6 is proved for $G = SL_r$. The case of general G is deduced from this by applying the above reasoning to vector bundles associated to sufficiently many representations of G.

(2.3) End of the proof of Theorem 2.2.1. because of Propositions 2.2.4 and 2.2.5, we are reduced to the following fact.

(2.3.1) Proposition. *The image of $M_{G,P^\circ}(n)$ under the map $\psi\varphi$ in (2.2.3) is bounded.*

Proof: As before, we start with the case $G = SL_r$, so we work with rank r bundles V on S with $\det(V) \simeq \mathcal{O}$, $c_2(V) = n$ equipped with an identification $V|_{S^\circ} \to V^\circ$

where V° is some fixed bundle on S°. We will use the notation introduced in the proof of Proposition 2.2.6, in particular, we will use the global analog of the sequence (2.2.7), which we write in the form

$$0 \to \sigma_* V \to j'_* V^\circ \to \mathcal{F} \to 0 \tag{2.3.2}$$

with $\mathcal{F}$ supported at $p \in S'$. We choose an embedding $f : S' \to \mathbb{P}^N$ and apply the Grothendieck-Riemann-Roch theorem to V and to $f\sigma : S \to \mathbb{P}^N$. From (2.3.2) we infer that

$$ch_N(f_*\sigma_* V) = c - l(\mathcal{F}),$$

where $l(\mathcal{F}) = \dim H^0(\mathcal{F})$ is the length of the 0-dimensional sheaf $\mathcal{F}$ and c is a constant depending only on V° but not on V. The only higher direct image to take into account is $R^1(f\sigma)_* V = f_* R^1 \sigma_* V$. The sheaf $R^1\sigma_* V$ is again supported at p and we have

$$ch_N(f_* R^1 \sigma_* V) = l(R^1\sigma_* V).$$

As $c_1(V) = 0, c_2(V) = n$ are fixed, the GRR theorem implies part (a) of the following fact.

(2.3.3) Lemma. *(a) For $V \in M_{SL_r, V^\circ}(n)$ the sum $l(\mathcal{F}) + l(R^1\sigma_* V)$ is equal to some fixed constant $c = c(V^\circ, n)$. In particular, $l(R^1\sigma_* V)$ is bounded.*

(b) Further, for $n \ll 0$ we have $c(V^\circ, n) < 0$ which implies that $M_{SL_r, V^\circ}(n) = \emptyset$.

Part (b) is true because the dependence of $c(V^\circ, n)$ on n comes out to be affine linear with a positive coefficient in the linear part.

Now, since $R^1\sigma_* V$ is supported at $p = \sigma(X)$, we have

$$H^0(R^1\sigma_* V) = H^1(Y, V|_Y), \tag{2.3.4}$$

where Y is, as before, the formal neighborhood of X in S. In fact, one can replace in (2.3.4) Y by the infinitesimal neighborhood $X^{(d)}$ for sufficiently large d. Consider the spectral sequence corresponding to the filtration of $V|_Y$ (or $V|_{X^{(d)}}$) by powers of $\mathbf{m}$. Its quotients are $V|_X \otimes (N^*_{X/S})^{\otimes i}$, $i \geq 0$. By analyzing the spectral sequence, we find that the component $H^1(X, V|_X)$ of the E_1-term consists of permanent cycles, by dimension reasons. So $\dim H^1(X, V|_X) \leq l(R^1\sigma_* V) \leq c(V^\circ, n)$ bounded. As $V|_X$ has $c_1 = 0$, we find, by Kleiman's criterion [Kl] that all possible $V|_X$ for $V \in M_{SL_r, V^\circ}(n)$ form a bounded family.

This finishes the proof for $G = SL_r$. The case of arbitrary G is reduced to this one by considering associated vector bundles and applying the following fact which is an easy consequence of the reduction theory of Harder [H1]

(2.3.5) Lemma. *A family of principal G-bundles on a curve is bounded if and only if for any representation of G the corresponding family of vector bundles is bounded.*

(2.4) The generating function. We now fix a ring A and an A-valued motivic measure μ on Sch_k, as in §1. To the data of S, X, G and P° we can, in virtue of

Theorem 2.2.1, associate the generating function

$$F(q) = F_{G,P^\circ}(q) = \sum_{n\in\mathbb{Z}} \mu(M_{G,P^\circ}(n))q^n \quad \in \quad A((q)). \tag{2.4.1}$$

(2.4.2) Example. Let $k = \mathbb{C}$, $A = \mathbb{Z}$ and μ be given by the Euler characteristic, as in Example 1.1.3(a). Assume that P° is trivial. Then we have a G-action on $M_{G,P^\circ}(n)$ given by $g(P,\tau) = (P, g\circ\tau)$. Let us consider the induced T-action and find the fixed locus $M_{G,P^\circ}(n)^T$. By definition, it consists of (P,τ) which are isomorphic (as bundles with trivialization) to $(P, t\circ\tau)$ for any $t\in T$. Since the moduli problem associated to $M_{G,P^\circ}(n)$ has trivial automorphism groups, we find that $M_{G,P^\circ}(n)^T$ consists of (P,τ) which come from T-bundles on S trivialized on S°. In the identification $\mathrm{Bun}_T(S) = \mathrm{Pic}(S)\otimes_{\mathbb{Z}} L$, bundles trivial on S° form a subgroup $\mathbf{m}\otimes L$ where $\mathbf{m}$ is, as before, the (invertible) sheaf of ideals of X. We will denote such bundles by $\mathbf{m}\otimes a$, $a\in L$. The trivialization of $\mathbf{m}\otimes a$ over S° is unique, up to isomorphism in our sense, and the second Chern number of the associated G-bundle is, by Proposition 2.1.2(b), equal to $(-d)\Psi(a,a)$, where $d = -X_S^2$. Therefore

$$\chi(M_{G,P^\circ}(n)) = \chi(M_{G,P^\circ}(n)^T) = \#\{a\in L : -d\Psi(a,a) = m\},$$

and hence

$$F_{G,P^\circ}(q) = \sum_{a\in L} q^{-d\Psi(a,a)/2} = \theta_L(q^d)$$

is the theta-function (or, rather the theta-zero-value) of the lattice L with the positive definite quadratic form $(-\Psi(x,x)/2)$.

§3. The refined generating function and its elliptic behavior.

(3.1) The relative moduli space of parabolic bundles. We keep the notation of §2. A T-bundle U on X has a degree $\deg(U)\in L$ which is the image of the class of U in $\mathrm{Bun}_T(X) = \mathrm{Pic}(X)\otimes L$ under $\deg'\otimes\mathrm{Id}$, where $\deg' : \mathrm{Pic}(X)\to\mathbb{Z}$ is the usual degree of line bundles. Let B be a fixed Borel subgroup in G containing T. As $B/[B,B]\simeq T$, a B-bundle R gives a T-bundle, whose degree will also be denoted $\deg(R)\in L$. We set $d = -X_S^2$. By our assumptions, $d > 0$. Let $L_+\subset L$ be the semigroup spanned by the positive coroots. For a principal G-bundle Q on X and $a\in L$ let $\Gamma_a(Q)$ be the scheme of all B-structures in Q of degree a. The following result is due to G. Harder [H2].

(3.1.1) Proposition. *$\Gamma_a(Q)$ is a quasiprojective scheme over k. There is $a_0 in L$ such that $\Gamma_q(Q) = \emptyset$ unless $a \in a_0 + L_+$.*

Together with Theorem 2.2.1 this implies the following.

(3.1.2) Corollary. *For any $a \in L, n \in \mathbb{Z}$ there exists a k-scheme $M_{G,P^\circ}(n,a)$ of finite type, which is a fine moduli space of triples (P, τ, π) where:*
- *P is a G-bundle on S with $c_2(P) = n$;*
- *$\tau : P|_{S^\circ} \to P^\circ$ is an isomorphism;*
- *π is a B-structure in $P|_X$ of degree a.*

We now form the generating function

$$E_{G,P^\circ}(q,z,v) = \sum_{n\in\mathbb{Z},\, a\in L} \mu(M_{G,P^\circ}(n,a)) q^n z^a v^d. \tag{3.1.3}$$

Here μ is a fixed A-valued motivic measure on Sch_k and z is a formal variable running in the "dual torus" $T^\vee = \mathrm{Spec}\ \mathbb{Z}[L]$ whose lattice of characters is L. The variable v (in which E_{G,P° is homogeneous of degree d) is added for convenience of future formulations.

(3.1.4) Example. Consider the situation of Example 2.4.2. As before, we have a G-action on $M_{G,P^\circ}(n,a)$. The set of T-fixed points consists of one element, the G-bundle corresponding to $\mathbf{m} \otimes a$, if $\Psi(a,a) = n$, and is empty otherwise. This implies that

$$E_{G,P^\circ}(q,z,v) = \sum_{a\in L} q^{-d\Psi(a,a)/2} z^a v^d = \theta_L(q^d, z) v^d,$$

where $\theta_L(q,z)$ is now the full theta-function, depending on the elliptic variables as well as on the modular one.

(3.2) The affine root system. We denote by $L^\vee = \mathrm{Hom}(L, \mathbb{Z})$ the weight lattice of G and by $\Delta_{\mathrm{sim}} \subset \Delta_+ \subset \Delta \subset L^\vee$ the sets of simple, resp. positive, resp. all roots of G. Let $\rho \in L^\vee$ is the half-sum of the positive roots. Let $L^\vee_{\mathrm{aff}} = \mathbb{Z} \oplus L^\vee \oplus \mathbb{Z}$ be the lattice of affine weights and

$$\widehat{\Delta} = \{(0, \alpha, n), \alpha \in \Delta, n \in \mathbb{Z}\} \subset L^\vee_{\mathrm{aff}},$$

$$\widehat{\Delta}_+ = (\{0\} \times \Delta_+ \times \{0\}) \ \cup\ (\{0\} \times \Delta \times \mathbb{Z}_{\geq 0})$$

be the system of affine roots of G, resp. positive affine roots. The set of simple affine roots will be denoted by

$$\widehat{\Delta}_{\mathrm{sim}} = (\{0\} \times \Delta_{\mathrm{sim}} \times \{0\}) \ \cup\ \{(0, -\theta, 1,$$

where $\theta \in \Delta_+$ is the maximal root. We denote $\widehat{\rho} = (0, \rho, h^\vee) \in L^\vee_{\mathrm{aff}}$, where $h^\vee$ is the dual Coxeter number of G, see [Ka].

As G is assumed simply connected, its coweight lattice L coincides with the coroot lattice. For any $\alpha \in \Delta$ let $\alpha^\vee \in L$ be the corresponding coroot and $\rho^\vee = (1/2)\sum_{\alpha\in\Delta_+} \alpha \in (1/2)L$. We set $L_{\text{aff}} = \mathbb{Z}\oplus L\oplus\mathbb{Z}$. This is the lattice dual to $L^\vee_{\text{aff}}$. Denote by $\widehat{\Delta}^\vee \subset L_{\text{aff}}$ the system of affine coroots. Again, we use the notation $\alpha^\vee$ to denote the affine coroot corresponding to $\alpha \in \widehat{\Delta}$. Thus, if $\alpha = (0, \beta, n)$ with $\beta \in \Delta$ and $n \in \mathbb{Z}$, then $\alpha^\vee = (n, \beta^\vee, 0)$. We also write $\widehat{\rho}^\vee = (h, \rho^\vee, 0)$, where h is the Coxeter number of G.

(3.3) The abelian variety $\mathcal{E}_L$. We now recall the standard appearance of elliptic curves in the Kac-Moody theory, see [Lo].

Let $\widehat{W} = W \ltimes L$ be the affine Weyl group of G. It is generated by the reflections $s_\alpha, \alpha \in \widehat{\Delta}_{\text{sim}}$. This group acts on $L_{\text{aff}} = \mathbb{Z}\oplus L\oplus\mathbb{Z}$ by

$$w \circ (n, a, m) = (n, w(a), m), \quad w \in W,$$

$$b \circ (n, a, m) = \left(n + \Psi(a, b) + \frac{1}{2}\Psi(b, b)m, a + mb, m\right), \quad b \in L, \tag{3.3.1}$$

see [PS]. Note that this action preserves the subgroup $\mathbb{Z}\oplus L \subset L_{\text{aff}}$ given by $m = 0$. Accordingly, we have the $\widehat{W}$-action on $T^\vee_{\text{aff}} = \mathbb{G}_m \times T^\vee \times \mathbb{G}_m = \text{Spec } \mathbb{Z}[L_{\text{aff}}]$. We will denote a typical point of $T^\vee_{\text{aff}}$ by $t = (q, z, v)$, where $q, v \in \mathbb{G}_m$ and $z \in T^\vee$. because the sugroup $\{m = 0\}$ is preserved by $\widehat{W}$, we have a $\widehat{W}$-action on the torus $\mathbb{G}_m \times T^\vee = \text{Spec } \mathbb{Z}[\mathbb{Z}\oplus L]$ with coordinates q, z, so that the projection $T^\vee_{\text{aff}} \to \mathbb{G}_m \times T^\vee$ given by $(q, z, v) \mapsto (q, z)$, is $\widehat{W}$-equivariant. In other words, $T^\vee_{\text{aff}}$ is a $\widehat{W}$-equivariant $\mathbb{G}_m$-bundle on $\mathbb{G}_m \times T^\vee$. We denote by θ the corresponding $\widehat{W}$-equivariant line bundle on $\mathbb{G}_m \times T^\vee$ (so sections of θ are functions $f(q, z, v)$ homogeneous in v of degree 1).

Assume now that the ring A where our motivic measure μ takes values, is a field. Then $A((q))$ is a complete discrete valued field. Let $\mathcal{E}$ be the Tate elliptic curve over $A((q))$. Thus $\mathcal{E}^{\text{an}}$, the rigid analytic $A((q))$-space corresponding to $\mathcal{E}$, is the quotient $\mathcal{E}^{\text{an}} = \mathbb{G}^{\text{an}}_{m,A((q))}/q^{\mathbb{Z}}$. Consider also the abelian variety $\mathcal{E}_L$ over $A((q))$ such that

$$\mathcal{E}_L^{\text{an}} = T^{\vee,\text{an}}_{A((q))}/q^{\Psi(L)}. \tag{3.3.2}$$

Here we regard Ψ as a homomorphism $L \to L^\vee$ and view $L^\vee$ as the lattice of 1-parameter subgroups in $T^\vee$, so q^λ, $\lambda \in L^\vee$, is the value at q of the 1-parameter subgroup λ. Note that $T^\vee_{A((q))}$ can be viewed as a completion of $(\mathbb{G}_m \times T^\vee)_A$, with q being the coordinate in $\mathbb{G}_m$. The action of $L \subset \widehat{W}$ on $T^\vee_{A((q))}$ coming, via this identification, from the action on $\mathbb{G}_m \times T^\vee$ induced by (3.3.1), is the same as the one used in (3.3.2). In particular, W acts on $\mathcal{E}_L$ and

$$(\mathcal{E}_L/W)^{\text{an}} = T^{\vee,\text{an}}_{A((q))}/\widehat{W}. \tag{3.3.3}$$

The $\widehat{W}$-equivariant line bundle θ on $\mathbb{G}_m \times T^\vee$ gives a W-equivariant line bundle on $\mathcal{E}_L$, which we denote by Θ. Similarly, 1-cocycles of $\widehat{W}$ with coefficients in $A[T^\vee_{\rm aff}]^*$, the group of invertible regular functions, give rise to line bundles on $\mathcal{E}_L/W$ and hence to W-equivariant line bundles on $\mathcal{E}_L$. We will need the following two cocycles:

(3.3.4) The cocycle $w \mapsto (-1)^{l(w)}$, where $l(w)$ is the length of w. This cocycle is trivial on $L \subset \widehat{L}$.

(3.3.5) The cocycle

$$w \mapsto t^{w(\widehat{\rho}^\vee)-\widehat{\rho}^\vee} = \prod_{\alpha\in\widehat{\Delta}_+\cap w^{-1}(\widehat{\Delta}_-)} t^{\alpha^\vee}, \quad t = (q,z,v) \in T^\vee_{\rm aff}.$$

As $2\widehat{\rho}^\vee \in L_{\rm aff}$, the square of this cocycle is a coboundary. We denote by $\mathcal{L}$ the line bundle on $\mathcal{E}_L$ corresponding to this cocycle. Thus $\mathcal{L}^{\otimes 2} \simeq \mathcal{O}$.

(3.4) The functional equation. We now proceed to formulate the first main result of this paper. We write $\zeta(u)$ for $\zeta_\mu(X,u)$, the motivic zeta function of X and assume from now on that $\mathrm{Pic}_1(X)(k) \neq \emptyset$. This condition, added for simplicity of formulations, always holds when k is algebraically closed or finite. Thus $\Phi(u) = \Phi_X(u)$, the numerator of $\zeta(u)$, is a polynomial if degree $2g$ with constant term 1. As before, we denote $d = -X_S^2$ and assume $d > 0$. We also assume that A is a field and $\mathbb{L} \neq 0$.

(3.4.1) Theorem. *The series $E_{G,P^\circ}(t) = E_{G,P^\circ}(q,z,v)$ converges to a meromorphic function on $(T^\vee \times \mathbb{G}_m)^{\rm an}_{A((q))}$ (homogeneous of degree d in the variable v) which satisfies, for any $w \in \widehat{W}$, the functional equation*

$$E_{G,P^\circ}(t) = \Lambda_w(t) E_{G,P^\circ}(\mathbb{L}^{\widehat{\rho}-w(\widehat{\rho})}t),$$

where

$$\Lambda_w(t) = \mathbb{L}^{l(w)(1-g)} \prod_{\alpha\in\widehat{\Delta}_+\cap w^{-1}(\widehat{\Delta}_-)} \frac{\zeta(\mathbb{L}t^{\alpha^\vee})}{\zeta(t^{\alpha^\vee})}.$$

Further, the singularities of $E_{G,P^\circ}(t)$ are contained among the singularities of the rational functions $\Lambda_w(t)$.

It is convenient to reformulate this theorem as follows. Let

$$D(t) = \prod_{\alpha\in\widehat{\Delta}_+} (1-\mathbb{L}^2 t^{\alpha^\vee})\Phi(t^{\alpha^\vee}). \tag{3.4.2}$$

As with the Weyl-Kac denominator [Ka], it is clear that $D(t)$ is an analytic function on $T^{\vee,{\rm an}}_{A((q))}$ (it does not depend on v), a kind of theta-function. Then set

$$N_{G,P^\circ}(t) = E_{G,P^\circ}(t)D(t) \tag{3.4.3}$$

and call $N_{G,P^\circ}(t)$ the *numerator* of E_{G,P°. Taking into account the functional equation for $\zeta(u)$ together with the fact that $D(t)$ dominates all the poles of all the $\Lambda_w(t)$, we can reformulate Theorem 3.4.1 as follows.

(3.4.4) Theorem. *The series $N_{G,P^\circ}(t)$ is well-defined and represents an analytic function on $T^{\vee,\mathrm{an}}_{A((q))}$ which, for any $w \in \widehat{W}$, satisfies the functional equation*

$$N_{G,P^\circ}(t) = (-1)^{l(w)} \big((\mathbb{L}^{\widehat{\rho}} t)^{w(\widehat{\rho}^\vee) - \widehat{\rho}^\vee} \big)^{1+2g} N_{G,P^\circ}(\mathbb{L}^{\widehat{\rho} - w(\widehat{\rho})} w(t)).$$

In other words, up to a shift and a multiplication by a monomial, N_{G,P° is a regular section of the line bundle $\mathcal{L} \otimes \Theta^{\otimes d}$ on $\mathcal{E}_L$, which is antisymmetric with respect to the $\widehat{W}$-action on this equivariant bundle.

This identifies N_{G,P° (and therefore E_{G,P°) as an element of a finite-dimensional $A((q))$-vector space depending only on G and d.

The proofs of Theorems 3.4.1 and 3.4.4 will be given in §6.

§4. Motivic Eisenstein series for G.

We first summarize the classical theory of unramified geometric Eisenstein series [H2] putting it into the motivic framework. In this section G will be assumed to be any split reductive group over k such that $[G, G]$ is simply connected. The notations pertaining to the root system remain the same.

(4.1) Rationality and functional equation. Let Q be a principal G-bundle on X. The motivic Eisenstein series associated to Q, is the generating function

(4.1.1) $$E_{G,Q}(z) = \sum_{a \in L} \mu(\Gamma_a(Q)) z^a, \quad z \in T^\vee,$$

where $\Gamma_a(Q)$ is the scheme of B-structures on Q of degree a, see (3.1) for this and other notations.

(4.1.2) Theorem. *(a) The series $E_{G,Q}(z)$ represents a rational function on $T^\vee_A$.*

(b) This rational function satisfies, for any $w \in W$, the functional equation

$$E_{G,Q}(z) = M_w(z) E_{G,Q}(\mathbb{L}^\rho w(\mathbb{L}^{-\rho} z)),$$

where

$$M_w(z) = \mathbb{L}^{l(w)(1-g)} \prod_{\substack{\alpha \in \Delta_+ \\ w(\alpha) \in \Delta_-}} \frac{\zeta(\mathbb{L} z^{\alpha^\vee})}{\zeta(z^{\alpha^\vee})}.$$

Further, the singularities of $E_{G,Q}(z)$ are contained among the singularities of the rational functions $M_w(t)$, $w \in W$.

Let us give an equivalent formulation in terms of the "numerator"

(4.1.3) $$N_{G,Q}(z) = E_{G,Q}(z) \cdot \prod_{\alpha \in \Delta_+} (1 - \mathbb{L}^2 z^{\alpha^\vee}) \Phi(z^{\alpha^\vee}).$$

(4.1.4) Theorem. *The series $N_{G,Q}(z)$ is a Laurent polynomial which satisfies, for any $w \in W$, the functional equation*

$$N_{G,Q}(z) = (-1)^{l(w)} \big((\mathbb{L}^\rho z)^{w(\rho^\vee)-\rho^\vee} \big)^{1+2g} N_{G,Q}(\mathbb{L}^{\rho - w(\rho)} w(z)).$$

The equivalence of (4.1.2) and (4.1.4) is a formal consequence of the functional equation for $\zeta(u)$ and is left to the reader.

(4.2) Proof for $G = GL_2$. Assume $G = GL_2$. Thus Q comes from a rank 2 vector bundle V on X. In this case $L = \mathbb{Z}^2$ and a B-structure on Q of degree $a = (a_1, a_2)$ is the same as a rank 1 subbundle $\mathcal{M} \subset V$ with $\deg(\mathcal{M}) = a_1$ and $\deg(V/\mathcal{M}) = a_2$. Thus $\Gamma_a(Q)$ parametrizes such subbundles. Recall that a subbundle is just a coherent subsheaf which is locally a direct summand. Any proper coherent subsheaf $\mathcal{M} \subset V$ (subbundle or not) is automatically locally free of rank 1, so $\deg(\mathcal{M})$ is defined. Further, $V/\mathcal{M}$ is a direct sum $(V/\mathcal{M})_{\mathrm{lf}} \oplus (V/\mathcal{M})_{\mathrm{tors}}$ of a locally free sheaf of rank 1 and a torsion sheaf. We define

$$\deg(V/\mathcal{M}) = \deg((V/\mathcal{M})_{\mathrm{lf}}) + \dim H^0((V/\mathcal{M})_{\mathrm{tors}}).$$

Let $\overline{\Gamma_{a_1,a_2}}(Q)$ be the scheme parametrizing subsheaves $\mathcal{M} \subset V$ with $\deg(M) = a_1$ and $\deg(V/\mathcal{M}) = a_2$. This is just a component of the *Quot* scheme of V.

As $L = \mathbb{Z}^2$, we have $T^\vee = \mathbb{G}_m^2$ and write $E_{G,Q}(z)$ as $E_{G,Q}(z_1, z_2)$.

(4.2.1) Proposition. *(cf.* [Lau]*.) The generating function*

$$\widetilde{E}(z_1, z_2) = \sum_{a_1, a_2 \in \mathbb{Z}} \mu(\overline{\Gamma_{a_1,a_2}}(Q)) z_1^{a_1} z_2^{a_2}$$

is equal to $\zeta(z_2/z_1) E_{G,Q}(z_1, z_2)$.

Proof: Any rank 1 subsheaf $\mathcal{M} \subset V$ can be uniquely written in the form the form $\mathcal{M} = \overline{\mathcal{M}}(-D)$ where $\overline{\mathcal{M}}$ is a rank 1 subbundle and D is a positive divisor on X. This implies the statement.

In view of the functional equation for $\zeta(u)$, Theorem 4.1.2 for the GL_2-bundle Q is equivalent to the following statement.

(4.2.2) Proposition. *The series $\widetilde{E}(z_1, z_2)$ represents a rational function with only poles being simple poles along the lines $z_1 = z_2$ and $z_1 = \mathbb{L}^2 z_2$. It satisfies the functional equation*

$$\widetilde{E}(z_1, z_2) = (\mathbb{L} z_1/z_2)^{2-2g} \widetilde{E}(\mathbb{L} z_2, \mathbb{L}^{-1} z_1).$$

Proof: We will use Lemma 1.3.4 and prove that the difference of the two sides of the proposed equality, considered as a formal series in z_i, z_i^{-1}, is a sum of two delta functions. Consider the projection

$$p_{a_1,a_2} : \overline{\Gamma}_{a_1,a_2}(Q) \to \mathrm{Pic}_{a_1}(X),$$

which takes $\mathcal{M} \subset V$ into the isomorphism class of $\mathcal{M}$ in the Picard group. If $a_1 + a_2 = \deg(V)$ (otherwise $\overline{\Gamma}_{a_1,a_2}(Q) = \emptyset$), then the fiber $p^{-1}_{a_1,a_2}(\mathcal{M})$ is the projective space $\mathbb{P}(\mathrm{Hom}(\mathcal{M}, V))$. Thus the coefficient at $z_1^{a_1} z_2^{a_2}$, $a_1 + a_2 = \deg(V)$ in $\widetilde{E}(z_1, z_2)$ is

$$\int_{\mathcal{M}\in \mathrm{Pic}_{a_1}(X)} \frac{\mathbb{L}^{\dim \mathrm{Hom}(\mathcal{M},V)} - 1}{\mathbb{L} - 1} d\mu_{\mathcal{M}},$$

while the coefficient at the same monomial in $(\mathbb{L}z_1/z_2)^{2-2g}\widetilde{E}(\mathbb{L}z_2, \mathbb{L}^{-1}z_1)$ is

$$\int_{\mathcal{M}'\in \mathrm{Pic}_{a_2+2-2g}(X)} \mathbb{L}^{a_2-a_1+2-2g} \frac{\mathbb{L}^{\dim \mathrm{Hom}(\mathcal{M}',V)} - 1}{\mathbb{L} - 1} d\mu_{\mathcal{M}'}.$$

Consider the isomorphism

$$\sigma : \mathrm{Pic}_{a_1}(X) \to \mathrm{Pic}_{a_2+2-2g}(X), \quad \mathcal{M} \to \mathcal{M}^* \otimes \Lambda^2 V \otimes \omega_X^*,$$

where ω_X is the sheaf of 1-forms. Since for a rank 2 bundle we have $V^* \simeq V \otimes (\Lambda^2 V)^*$, we find from the Riemann-Roch theorem that

$$\dim \mathrm{Hom}(\mathcal{M}, V) - \dim \mathrm{Hom}(\mathcal{M}', V) = a_2 - a_1 + 2 - 2g,$$

whenever $\mathcal{M}' = \sigma(\mathcal{M})$. This means that the difference of the coefficients at $z_1^{a_1} z_2^{a_2}$ in the two sides of (4.2.2) is

$$\int_{\mathcal{M}\in \mathrm{Pic}_{a_1}(X)} \frac{\mathbb{L}^{a_2-a_1+2-2g} - 1}{\mathbb{L} - 1} d\mu_{\mathcal{M}} = \mu(\mathrm{Pic}_{a_1}(X)) \cdot \frac{\mathbb{L}^{a_2-a_1+2-2g} - 1}{\mathbb{L} - 1}.$$

Our assumption that $\mathrm{Pic}_1(X)(k) \neq \emptyset$ implies that $\mu(\mathrm{Pic}_{a_1}(X)) = \mu(\mathrm{Pic}_0(X))$ and we find that the difference between the two series in (4.2.2) is

$$\frac{\mu(\mathrm{Pic}_0(X))}{\mathbb{L} - 1} \sum_{a_1+a_2=\deg(V)} (\mathbb{L}^{a_2-a_1+2-2g} - 1) z_1^{a_1} z_2^{a_2} =$$

$$= \frac{\mu(\mathrm{Pic}_0(X))}{\mathbb{L} - 1} \left(z_2^{\deg(V)} \mathbb{L}^{\deg(V)+2-2g} \delta\left(\frac{z_1}{\mathbb{L}^2 z_2}\right) - z_2^{\deg(V)} \delta\left(\frac{z_1}{z_2}\right)\right).$$

This proves our claim.

(4.2.3) Corollary. *Theorems 4.1.2 and 4.1.4 hold whenever G has semisimple rank 1.*

(4.3) The general case. We now deduce Theorems 4.1.2 and 4.1.4 for general G from the case of semisimple rank 1, following an approach which the author learned from V. Drinfeld (this approach is also implicit in [BG]).

Note that unlike the functional equation (4.1.2) for $E_{G,Q}(z)$ which is to be understood in the sense of analytic continuation (summing a series to a rational function and expanding in a different region), the equation (4.1.4) for $N_{G,Q}(z)$ is supposed to hold at the level of monomials (provided we know that $N_{G,Q}$ is indeed a Laurent polynomial). Conversely, suppose we know that the *series* $N_{G,Q}(z)$ defined by (4.1.3) satisfies the equations (4.1.4) at the level of monomials. Then we can *deduce* that $N_{G,Q}(z)$ is actually a Laurent polynomial. Indeed, the support of $N_{G,Q}$ lies in some translation of $L_+ \subset L$, the cone of dominant coweights of $[G, G]$. The equations (4.1.4), if they hold at the level of monomials, would then imply that the support of $N_{G,Q}(z)$ lies in the intersection, over all $w \in W$, of some translations of $w(L_+)$. But any such intersection is finite.

Further, the group W being generated by simple reflections, we are reduced to the following fact.

(4.3.1) Lemma. *Let $\alpha \in \Delta_{\text{sim}}$. Then the series $N_{G,Q}(z)$ defined by (4.1.3), satisfies the equation (4.1.4) with $w = s_\alpha$, at the level of monomials.*

Proof: Let P_α be the parabolic subgroup in G defined by one negative root $(-\alpha)$ and $M_\alpha \subset P_\alpha$ be the Levi subgroup. So M_α has semisimple rank 1, its maximal torus is T, its Borel subgroup is $B_\alpha = B \cap M_\alpha$ and the Weyl group is $\{1, s_\alpha\}$. Let $P_\alpha^{\text{ab}} = P_\alpha/[P_\alpha, P_\alpha]$ and $L_\alpha = \text{Hom}(\mathbb{G}_m, P_\alpha^{\text{ab}})$, so we have a homomorphism $\lambda : L \to L_\alpha$. Since M_α is isomorphic to the quotient of P_α by its unipotent radical, we have a projection $\varphi : P_\alpha \to M_\alpha$. A B_α-bundle on X has degree lying in L and a P_α-bundle has degree lying in L_α.

Now, a B-structure of degree a on a G-bundle Q gives, in particular, a P_α-structure of degree $\lambda(\alpha)$. Conversely, suppose that π is a P_α-structure on Q and let (Q, π) be the corresponding P_α-bundle. Then a B-structure on Q refining π is the same as a B_α-structure on the M_α-bundle $\varphi_*(Q, \pi)$. Let $\Gamma_b^{P_\alpha}(Q)$, $b \in L_\alpha$, be the scheme of P_α-structures on Q of degree b. We conclude that

$$E_{G,Q}(z) = \sum_{b \in L_\alpha} \int_{\pi \in \Gamma_b^{P_\alpha}(Q)} E_{M_\alpha, \varphi_*(Q,\pi)}(z) d\mu_\pi,$$

the equality being understood at the level of each coefficient. Therefore

$$N_{G,Q}(z) = \left(\prod_{\beta \in \Delta_+ - \{\alpha\}} (1 - \mathbb{L}^2 z^{\beta^\vee}) \Phi(z^{\beta^\vee}) \right) \sum_{b \in L_\alpha} \int_{\pi \in \Gamma_b^{P_\alpha}(Q)} N_{M_\alpha, \varphi_*(Q,\pi)}(z) d\mu_\pi.$$

Since $N_{M_\alpha, \varphi_*(Q,\pi)}(z)$ is a Laurent polynomial, it satisfies the functional equation (4.1.4) with $w = s_\alpha$ at the level of monomials. As $\Delta_+ - \{\alpha\}$ is permuted by s_α, we finally conclude that $N_{G,Q}(z)$ satisfies (4.1.4) with $w = s_\alpha$ at the level of monomials. Lemma is proved.

§5. The Kac-Moody sheaf and c_2.

Let $G((\lambda))$ be the loop group of G. This is an ind-scheme over k such that for any commutative k-algebra R we have

$$G((\lambda))(R) = G\big(R((\lambda))\big).$$

If R is a field, Garland [Ga] has constructed a central extension of $G((\lambda))(R)$ by R^* which is an algebraic version of the minimal central extension of the loop group of a compact Lie group [PS]. We need the more general case when R is replaced by the structure sheaf of a smooth algebraic variety over k. In this section we describe the corresponding central extension in such a way as to make clear its relation to the second Chern class for G-bundles.

Garland's extension is induced from Matsumoto's extension of $G\big(R((\lambda))\big)$ by $K_2\big(R((\lambda))\big)$ by the tame symbol map $K_2\big(R((\lambda))\big) \to R^*$. Similarly to this, we use the sheaf-theoretic extension of G by K_2 constructed by Brylinski and Deligne [BD].

(5.1) Generalities.

(5.1.1) Sheaves on the category of smooth varieties. Let $\mathcal{S}m$ be the category of smooth algebraic varieties over k. We consider it as a Grothendieck site with respect to the Zariski topology. Every scheme, or ind-scheme Z gives rise to a functor (sheaf) on $\mathcal{S}m$ represented by Z, which we will denote $\underline{Z}$. If X is a smooth variety, then the sheaf on the Zariski topology of X formed by Z-valued functions, will be denoted by $\underline{Z}_X$. The sheaf $\underline{A}^1$, represented by the affine line, will be denoted by $\mathcal{O}$.

Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{S}m$. A vector bundle on $\mathcal{F}$ is, by definition, a locally free sheaf of $\mathcal{O}$-modules on $\mathcal{F}$. Thus, when $\mathcal{F} = \underline{Z}$, a vector bundle V on $\mathcal{F}$ is a system of data associating to any morphism $f : X \to Z$, where X is a smooth variety, a vector bundle V_f on X and to any pair of morphisms $X' \xrightarrow{g} X \xrightarrow{f} Z$ (with X, X' being smooth varieties), an isomorphism $g^*V_f \to V_{fg}$. These isomorphisms are required to satisfy the obvious compatibility conditions for any triple of morphisms $X'' \xrightarrow{g'} X' \xrightarrow{g} X \xrightarrow{f} Z$ with X, X', X'' being smooth varieties.

Thus, if Z is a smooth variety, then a vector bundle on $\underline{Z}$ is the same as a vector bundle on Z in the usual sense. The same holds if Z is an ind-scheme which is an inductive limit of smooth varieties.

(5.1.2) A-groupoids. Let A be an Abelian group and A-Tors be the category of A-torsors. This is a symmetric monoidal category with respect to the operation $\otimes$ of tensor product of torsors. We call an A-groupoid a small category $\mathcal{C}$ enriched in A-Tors, i.e., a set Ob $\mathcal{C}$ together with a collection of torsors $\mathrm{Hom}_{\mathcal{C}}(x, y)$, $x, y \in \mathrm{Ob}\ \mathcal{C}$ and composition morphisms

$$\mathrm{Hom}_{\mathcal{C}}(y, z) \otimes \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$$

satisfying the usual axioms.

Let S be a Grothendieck site, and $\mathcal{A}$ a sheaf of Abelian groups on S. By globalizing the above definition to the topos of sheaves on S, we get the concept of a sheaf of $\mathcal{A}$-groupoids. Such a sheaf $\mathcal{C}$ consists of a sheaf of sets $\mathrm{Ob}(\mathcal{C})$ on S together with, for any two local sections $x, y \in \mathrm{Ob}(\mathcal{C})(U)$, a sheaf $\mathcal{H}om_{\mathcal{C}}(x,y)$ of $\mathcal{A}|_U$-torsors and composition morphisms for any three local sections satisfying the usual axioms and compatible with the restrictions.

(5.1.2.1) Example. Let Z be an ind-scheme. We will call sheaves of $\mathcal{O}^*$-groupoids on $\mathcal{S}m$ with the sheaf of objects $\underline{Z}$ groupoid line bundles on $\underline{Z}$. If Z is a smooth variety (or an inductive limit of such), then a groupoid line bundle on $\underline{Z}$ is just a line bundle on $Z \times Z$ in the usual sense, equipped with a kind of multiplicative structure.

(5.1.2.2) Proposition. *Let $\mathcal{B}' \subset \mathcal{B}$ be sheaves of groups on S. Then the category of central extensions of $\mathcal{B}$ by $\mathcal{A}$ trivial on $\mathcal{B}'$, is equivalent to the category of $\mathcal{B}$-equivariant sheaves of $\mathcal{A}$-groupoids with the sheaf of objects $\mathcal{B}/\mathcal{B}'$.*

Proof: Obvious when $S = \{pt\}$. The general case reduces to this.

(5.1.3) $\mathcal{A}$-gerbes. Let $S, \mathcal{A}$ be as before. An $\mathcal{A}$-gerbe is (cf. [Bre] [Bry] [Gi]) a stack $\mathcal{R}$ of (not necessarily small) categories on S in which each sheaf $\mathcal{H}om_{\mathcal{R}}(x,y)$, $x, y \in \mathrm{Ob}(\mathcal{R}(U))$, is made into a sheaf of $\mathcal{A}|_U$-torsors in a way compatible with restrictions and compositions. A trivial example is the stack $\mathcal{A}$-Tors. Any sheaf $\mathcal{C}$ of $\mathcal{A}$-groupoids can be embedded into an $\mathcal{A}$-gerbe, namely $\mathcal{F}un^{\circ}_{\mathcal{A}}(\mathcal{C}, \mathcal{A}\text{-Tors})$, the stack of contravariant functors $\mathcal{C} \to \mathcal{A}$-Tors which preserve the torsor structure on $\mathcal{H}om$-sheaves We will call such functors simply $\mathcal{A}$-functors. The following is a basic fact of Giraud's theory.

(5.1.3.1) Proposition. *(a) Equivalence classes of $\mathcal{A}$-gerbes form a set, naturally identified with $H^2(S, \mathcal{A})$. We denote by $\gamma(\mathcal{R}) \in H^2(S, \mathcal{A})$ the class of an $\mathcal{A}$-gerbe $\mathcal{R}$. The equality $\gamma(\mathcal{R}) = 0$ is equivalent to $\mathcal{R}(S) \neq \emptyset$.*

(b) If $\mathcal{R}, \mathcal{R}'$ are two $\mathcal{A}$-gerbes, then so is $\mathcal{F}un(\mathcal{R}, \mathcal{R}')$ and $\gamma(\mathcal{F}un(\mathcal{R}, \mathcal{R}')) = \gamma(\mathcal{R}') - \gamma(\mathcal{R})$.

We also need some additional properties of this correspondence.

(5.1.3.2) Proposition. *Let S be a topological space, $X \subset S$ a closed subspace, $S^{\circ} = S - X$. Then:*

(a) Equivalence classes of pairs $(\mathcal{R}, x)$, where $\mathcal{R}$ is an $\mathcal{A}$-gerbe and $x \in \mathrm{Ob}(\mathcal{R}(S^{\circ}))$, form a set naturally identified with $H^2_X(S, \mathcal{A})$. We denote by $\gamma(\mathcal{R}, x) \in H^2_X(S, \mathcal{A})$ the class of $(\mathcal{R}, x)$. Its image in $H^2(S, \mathcal{A})$ is $\gamma(\mathcal{R})$.

(b) Assume that the local cohomology sheaves $\mathcal{H}^i_X(S, \mathcal{A})$ vanish for $i \neq 1$ (so $H^2_X(S, \mathcal{A}) = H^1(X, \mathcal{H}^1_X(S, \mathcal{A}))$). Then the 2-category formed by the $(\mathcal{R}, x)$ as in (a), has trivial 2-morphisms, i.e., reduces to a usual category and this category is equivalent to the category of $\mathcal{H}^1_X(S, \mathcal{A})$-torsors. We will denote $\underline{\gamma}(\mathcal{R}, x)$ the torsor corresponding to $(\mathcal{R}, x)$.

Proof: Part (a) being a generality, we restrict ourselves to giving the construction of $\underline{\gamma}(\mathcal{R}, x)$ in (b). As $\mathcal{H}^2_X(S, \mathcal{A}) = 0$, the object x is, locally on on X, extendable to an object defined on a neighborhood of X. Let $j : S^\circ \hookrightarrow S$ be the embedding, so

$$\mathcal{H}^1_X(S, \mathcal{A}) = (j_* j^* \mathcal{A})/\mathcal{A}$$

is the sheaf of "principal parts" of sections of $\mathcal{A}$ on S°. If $U \subset X$ is a small open set and $\widetilde{U} \subset S$ is a small open neighborhood of U, then $x|_{\widetilde{U}-U}$ admits a lifting onto $\widetilde{U}$, i.e., there is $y \in \mathcal{R}(\widetilde{U})$ such that $\mathrm{Hom}(y|_{\widetilde{U}-U}, x|_{\widetilde{U}-U}) \neq \emptyset$. Then, this Hom is an $\mathcal{A}(\widetilde{U} - U)$-torsor. The torsor

$$\mathrm{Hom}\big(y|_{\widetilde{U}-U}, x|_{\widetilde{U}-U}\big)/\mathcal{A}(\widetilde{U})$$

over $\mathcal{A}(\widetilde{U} - U)/\mathcal{A}(U)$ is independent, up to a canonical isomorhism, on the choice of y and this is, by definition, $\underline{\gamma}(\mathcal{R}, x)(U)$. The rest is left to the reader.

(5.2) The affine Grassmannian. Let $G[[\lambda]]$ be the group of Taylor loops in G. This is a group scheme (of infinite type) over k. The affine Grassmannian of G is the ind-scheme $\widehat{\mathrm{Gr}} = G((\lambda))/G[[\lambda]]$. In fact, for every commutative k-algebra R we have

$$\widehat{\mathrm{Gr}}(R) = G\big(R((\lambda))\big)/G\big(R[[\lambda]]\big).$$

Further, it is known that $\widehat{\mathrm{Gr}}$ is an inductive limit of projective algebraic varieties (the closures of the affine Schubert cells). Our aim in this section is to construct a central extension of sheaves of groups on $\mathcal{S}m$

$$1 \to \mathcal{O}^* \to \underline{G}' \to \underline{G((\lambda))} \to 1 \qquad (5.2.1)$$

trivial on $\underline{G[[\lambda]]}$. By (5.2.1-2), this is the same as to construct a $\underline{G((\lambda))}$-equivariant groupoid line bundle C on $\underline{\widehat{\mathrm{Gr}}}$, i.e., to construct, for any smooth variety X and any morphisms $f, f' : X \to \widehat{\mathrm{Gr}}$, a line bundle $C(f, f')$ and for any three morphisms $f, f', f'' : X \to \widehat{\mathrm{Gr}}$, a canonical identification $C(f, f') \otimes C(f', f'') \to C(f, f'')$ satisfying the associativity and compatible with inverse images. We start by recalling a geometric description of morphisms into $\widehat{\mathrm{Gr}}$.

For a scheme X let $X[[\lambda]] = S \times \mathrm{Spf}(k[[\lambda]])$. We will consider it as a ringed space $(X, \mathcal{O}_X[[\lambda]])$. Let $X((\lambda))$ be the ringed space $(X, \mathcal{O}_X((\lambda)))$. Then, the following is an easy consequence of the definitions.

(5.2.2) Proposition. *(a) Morphisms $X \to \widehat{\mathrm{Gr}}$ are in bijection with isomorphism classes of pairs (P, τ), where P is a principal G-bundle on $X[[\lambda]]$ and τ is a trivialization of P over $X((\lambda))$.*

(5.2.3) Definition. *Let X be a k-scheme. We call a ribbon on X a formal scheme Y whose underlying ordinary scheme is X and which is locally isomorphic to $X[[\lambda]]$. An isomorphism of ribbons is an isomorphism of formal schemes identical on X.*

The set of isomorphism classes of ribbons on X is identified with $H^1(X, \mathcal{A}ut(X[[\lambda]]))$, where $\mathcal{A}ut(X[[\lambda]])$ is the sheaf of groups on X formed by (local) automorphisms of the formal scheme $X[[\lambda]]$ identical on X.

For a ribbon Y we denote $Y^\circ = Y - X$. This is a ringed space with the underlying space X and the structure sheaf locally isomorphic to $\mathcal{O}_X((\lambda))$.

Let now S be a smooth algebraic variety and $X \subset S$ a smooth hypersurface. We have then a particular ribbon Y on X, namely the formal neighborhood of X in S. The following is a consequence of the descent lemma of Beauville-Laszlo [BL].

(5.2.4) Proposition. *Let P° be a principal G-bundle on $S^\circ = S - X$. Then the following sets are in natural bijection:*

(i) Isomorphism classes of pairs (P, τ), where P os a G-bundle on S and $\tau : P|_{S^\circ} \to P^\circ$ is an isomorphism.

(ii) Isomorphism classes of pairs $(\widehat{P}, \widehat{\tau})$, where $\widehat{P}$ is a G-bundle on Y and $\widehat{\tau} : \widehat{P}|_{Y^\circ} \to P^\circ|_{Y^\circ}$ is an isomorphism.

(5.3) The relative c_2-bundle. Let S be a smooth algebraic variety and $X \subset S$ be a smooth hypersurface. Recall (2.1) that we have fixed a multiplicative $\underline{K}_2$-torsor Φ on G and this gives rise to a particular central extension (2.1.1) of $\underline{G}_S$ by $\underline{K}_{2,S}$. For a principal G-bundle P on S let $\mathrm{Lift}(P)$ be the $\underline{K}_{2,S}$-gerbe of liftings of the $\underline{G}_S$-torsor $\underline{P}$ to a $\widetilde{G}_S$-torsor, so $\gamma(\mathrm{Lift}(P)) = c_2(P)$.

Let now P, P' be two G-bundles on S and $\phi : P|_{S^\circ} \to P'|_{S^\circ}$ be an isomorphism. Then ϕ gives an object of the $\underline{K}_{2,S}$-gerbe $\mathcal{F}un(\mathrm{Lift}(P), \mathrm{Lift}(P'))$, defined over S°. Further, the Brown-Gersten-Quillen resolution of $\underline{K}_{2,S}$ (see [Q]) implies that

$$\mathcal{H}^1_X(S, \underline{K}_2) = \mathcal{O}^*_X, \quad \mathcal{H}^i_X(S, \underline{K}_2) = 0, i \neq 1. \tag{5.3.1}$$

Therefore, Proposition 5.1.3.2(b) implies the following.

(5.3.2) Proposition. *To every P, P', ϕ as above, there is naturally associated a line bundle $C(P, P', \phi)$ on X. The image of the class of $C(P, P', \phi)$ in $\mathrm{Pic}(X)$ under the direct image homomorphism $\mathrm{Pic}(X) \to \mathrm{CH}^2(S)$ is equal to $c_2(P') - c_2(P)$. Given three G-bundles P, P', P'' on S and isomorphisms $P|_{S^\circ} \xrightarrow{\phi} P'|_{S^\circ} \xrightarrow{\phi'} P''|_{S^\circ}$, there is a natural isomorphism*

$$C(P', P'', \phi') \otimes C(P, P', \phi) \to C(P, P'', \phi'\phi),$$

and these isomorphisms are associative for any four G-bundles on S with compatible identifications over S°.

(5.3.3) Example. To a coherent sheaf $\mathcal{F}$ on S supported on (some infinitesimal neighborhood of) X, there is naturally associated the determinantal line bundle $\det_X(\mathcal{F})$ on X which is uniquely characterized by the two properties:

(a) Multiplicativity in short exact sequences of coherent sheaves on S.

(b) If $\mathcal{F}$ is a vector bundle on X of rank r, then $\det_X(\mathcal{F}) = \Lambda^r(\mathcal{F})$.

Given a vector bundle V on S and a locally free subsheaf $V' \subset V$ coinciding with V over S°, we set $\det_X(V : V') = \det_X(V/V')$. By multiplicativity one extends the construction of $\det_X(V : V')$ to the case when V, V' are two arbitrary vector bundles on S identified over S°. If now V, V' have trivial determinant, so give rise to SL_r-bundles P, P' on S and to an identification $\phi : P|_{S^\circ} \to P'|_{S^\circ}$, then $C(P, P', \phi) = \det_X(V : V')$. This statement (which can be viewed as a kind of Riemann-Roch theorem) follows at once from Quillen's description of the boundary map on K_2 in terms of "lattices" [Gra]. Indeed, it is this boundary map which gives the identification $\mathcal{H}^1_X(S, \underline{K}_2) \simeq \mathcal{O}^*_X$.

Further, let Y be any ribbon on a smooth algebraic variety X. The Gersten conjecture for equicharacteristic regular local rings recently proved by Panin [Pa], implies that we have, similarly to (5.3.1):

$$\mathcal{H}^1_X(Y, \underline{K}_2) = \mathcal{O}^*_X, \quad \mathcal{H}^i_X(Y, K_2) = 0, i \neq 1, \tag{5.3.4}$$

where Y is considered as a scheme $\mathrm{Spec}(\mathcal{O}_Y)$. Because of this, we can generalize the above construction of the relative c_2-bundle $C(P, P', \phi)$ to the case when P, P' are G-bundles over Y and ϕ is their identification over Y°. Moreover, this construction is compatible with the earlier one in the case when Y is the formal neighborhood of X in S and P, P' come from G-bundles on S.

(5.3.5) Definition. *Let P° be a G-bundle on Y°. The c_2-groupoid $\mathfrak{C}_2(P^\circ)$ is the sheaf of $\mathcal{O}^*_X$-groupoids on X defined as follows. An object of $\mathfrak{C}_2(P^\circ)$ over an open $U \subset X$ is a pair (P, τ) where P is a G-bundle on Y_U and $\tau : P|_{Y^\circ_U} \to P^\circ|_{Y^\circ_U}$ is an isomorphism, while $\mathcal{H}om_{\mathfrak{C}_2(P^\circ)}((P, \tau), (P', \tau'))$ is the $\mathcal{O}^*_U$-torsor corresponding to the line bundle $C(P, P', \tau'\tau^{-1})$.*

Take now $Y = X[[\lambda]]$, so $Y^\circ = X((\lambda))$ and let P° be the trivial G-bundle on Y°. In this case an object of $\mathfrak{C}_2(P^\circ)$ is the same a morphism $X \to \widehat{\mathrm{Gr}}$. We obtain therefore the construction of a line bundle $C(f, f')$ on X for any two morphisms $f, f' : X \to \widehat{\mathrm{Gr}}$. The next statement is now obvious.

(5.3.6) Proposition. *The line bundles $C(f, f')$ give rise to a groupoid line bundle C on the sheaf of sets $\underline{\widehat{\mathrm{Gr}}} \times \underline{\widehat{\mathrm{Gr}}}$ on $\mathcal{S}m$. This groupoid line bundle is equivariant with respect to the sheaf of groups $\underline{G((\lambda))}$. In particular, we get a central extension $\underline{G}'$ of $\underline{G((\lambda))}$ by $\underline{\mathbb{G}_m}$ in the category of sheaves on $\mathcal{S}m$.*

(5.3.7) Remark. It is natural to expect that the central extension G' can be constructed as an group ind-scheme, not just as a sheaf on the category of smooth varieties. For this, we need to construct C as a groupoid line bundle on the ind-sheme $\widehat{\mathrm{Gr}} \times \widehat{\mathrm{Gr}}$ (i.e., to construct $C(f, f')$ for any two morphisms of any scheme X into $\widehat{\mathrm{Gr}}$. This does not automatically follow from the above construction because the Schubert varieties forming a direct system representing $\widehat{\mathrm{Gr}}$, are singular. Our construction uses the Gersten resolution and therefore is not directly applicable to the singular case. It is probably possible to push the construction by using the approach of Kumar and Mathieu [Ku] [Ma1] (in particular, the disingularizations of the Schubert varieties). However, it is beside the main purpose of the present paper, where it is enough to consider the sheaves of G'-valued functions on smooth varieties X (in fact, we really need only the case of smooth curves).

(5.3.8) Definition. *Let Y be a ribbon on a smooth algebraic variety X and P° be a principal G-bundle on Y°. A c_2-theory on P° is an object if the category $\mathcal{F}un_{\mathcal{O}_X^*}(\mathfrak{C}_2(P^\circ), \mathcal{O}_X^*\text{-}Tors)$, i.e., a rule C which to any local extension P of P° to a G-bundle on Y_U, $U \subset X$, associates a line bundle $C(P)$ on U and to any two such extensions P, P' an isomorphism $C(P) \otimes C(P, P') \to C(P')$ in a way satisfying the associativity and compatible with the restrictions.*

(5.4) The full Kac-Moody group. Consider the algebraic group $\mathbb{G}_m$ acting on $\mathrm{Spf}(k[[\lambda]])$ by the "rotation of the loop" $\lambda \mapsto z\lambda$, $z \in \mathbb{G}_m$. Because of the naturality of the central extension (2.1.1) and of the c_2-bundles, we have a natural action of $\underline{\mathbb{G}_m}$ on $\underline{G}'$. We define $\widehat{\underline{G}} = \underline{G}' \rtimes \underline{\mathbb{G}_m}$. We regard $\widehat{\underline{G}}$ as the functor on $\mathcal{S}m$ represented by the full Kac-Moody group. In particular, the maximal torus in $\widehat{\underline{G}}$ is $T_{\mathrm{aff}} = \mathbb{G}_m \times T \times \mathbb{G}_m$ where the first $\mathbb{G}_m$ is the center and the second one is the group of rotations of the loop. Accordingly, the affine root system as introduced in (3.2) is just the root system of $\widehat{\underline{G}}$, i.e., the system of weights of T_{aff} in the adjoint representation of $\widehat{\underline{G}}$. Similarly, the action of $\widehat{W}$ on L_{aff} as given in (3.3.1) comes from the action on T_{aff} of the normalized of T_{aff} in $\widehat{\underline{G}}$. Let $I = \{g(\lambda) \in G[[\lambda]] : g(0) \in B\}$ be the Iwahori subgroup in $G((\lambda))$ and let $\widehat{\underline{B}} = \underline{\mathbb{G}_m} \times \underline{I} \rtimes \underline{\mathbb{G}_m} \subset \widehat{\underline{G}}$. Then positive affine roots are the weights on the Lie algebra of $\widehat{\underline{B}}$.

More generally, let a smooth algebraic variety X be fixed. We have then an action of the sheaf of groups $\mathcal{A}ut(X[[\lambda]])$ on $\underline{G}'_X$ by group automorphisms and we define $\mathcal{G}_X = \underline{G}'_X \rtimes \mathcal{A}ut(X[[\lambda]])$. We also define $\mathcal{B}_X$ to be the semidirect product $\underline{\mathbb{G}_m} \times \underline{I}_X \rtimes \mathcal{A}ut(X[[\lambda]])$.

(5.4.1) Proposition. *Let X be a smooth algebraic variety. Then:*

(a) The category of sheaves of $\mathcal{G}_X$-torsors is equivalent to the category of triples (Y, P°, C) where Y is a ribbon on X, P° is a G-bundle on Y° and C is a C_2-theory on P°.

(b) The category of sheaves of $\widehat{\underline{G}}_X$-torsors is equivalent to the category of triples (N, P°, C) where N is a line bundle on X, P° a G-bundle on the punctured formal neighborhood of X in N and C is a c_2-theory on P°.

Proof: This is a more or less straightforward consequence of the definitions. Thus, in (a), a $\mathcal{G}_X$-torsor gives an $\mathcal{A}ut(X[[\lambda]])$-torsor via the homomorphism $\mathcal{G}_X \to \mathcal{A}ut(X[[\lambda]])$, and an $\mathcal{A}ut(X[[\lambda]])$-torsor is the same as a ribbon, say, Y. Lifting an $\mathcal{A}ut(X[[\lambda]])$-torsor corresponding to Y to a torsor over $\underline{G((\lambda))}_X \rtimes \mathcal{A}ut(X[[\lambda]])$ amounts to giving a G-bundle P° on Y°. Further lifting of a torsor over $\underline{G((\lambda))}_X \rtimes \mathcal{A}ut(X[[\lambda]])$ to a torsor over $\mathcal{G}_X$ amounts to fixing a c_2-theory in P° because the central extension is defined, in the first place, in terms of the relative c_2-bundles. We leave the details to the reader. The next proposition is equally straightforward.

(5.4.2) Proposition. *Let $\mathcal{Q}$ be a $\mathcal{G}_X$-torsor corresponding to the data (Y, P°, C) as in (5.4.1)(a). Then, $\mathcal{B}_X$-structures in $\mathcal{Q}$ are naturally identified with isomorphism classes of triples (P, τ, π) where P is a G-bundle on Y, while $\tau : P|_{Y^\circ} \to P^\circ$ is an isomorphism and π is a B-structure in $P|_X$.*

Note that we have a homomorphism of sheaves of groups

$$\mathcal{A}ut(X[[t]]) \to \mathcal{O}_X^* \tag{5.4.3}$$

which takes an automorphism g of $X[[\lambda]]$ identical on X, into the invertible function given by the action of the differential of g on the normal bundle of X in $X[[\lambda]]$. Therefore we get a homorphism $\mathcal{B}_X \to \underline{T}_{\mathrm{aff},X}$; in particular, a $\mathcal{B}_X$-torsor $\mathcal{T}$ has a degree $\deg(\mathcal{T})$ lying in $L_{\mathrm{aff}} = \mathbb{Z} \oplus L \oplus \mathbb{Z}$ which is the group of 1-parameter subgroups in T_{aff}. The above proposition is now easily complemented as follows.

(5.4.4) Proposition. *Let $\mathcal{T}$ be a $\mathcal{B}_X$-torsor and (P, τ, π) be the data corresponding to $\mathcal{T}$ by Proposition 5.4.2. Then*

$$\deg(\mathcal{T}) = \Big(c_2(P), \deg(P|_X, \pi), X_Y^2 \Big).$$

§6. Motivic Eisenstein series for $\widehat{G}$.

(6.1) The Eisenstein series. We now assume that X is a smooth projective curve, as in §2-4. Let Y be a ribbon on X and P° be a principal G-bundle on Y°. Fix a c_2-theory C on P°. If P is an extension of P° to the whole of Y, we will write $c_2(P) = \deg C(P) \in \mathbb{Z}$. We denote by $\mathcal{Q}$ the $\mathcal{G}_X$-torsor corresponding to (Y, P°, C) by Proposition 5.4.1.

(6.1.1) Theorem. *Suppose that X_Y^2, the self-intersection index of X in Y, is negative. Then for each $n \in \mathbb{Z}$ there exists a scheme $\Gamma_n(\mathcal{Q})$ which is a fine moduli space for extensions P of P° to a G-bundle on Y with $c_2(P) = n$. The scheme $\Gamma_n(\mathcal{Q})$ is empty for $n \ll 0$.*

Proof: This is obtained by the same arguments as in Theorem 2.2.1, except that we should use Chern classes with values in local cohomology and the Grothendieck-Riemann-Roch theorem for such classes. We leave the details to the reader.

(6.1.2) Corollary. *For each $n \in \mathbb{Z}$ and $a \in L$ there exists a scheme $\Gamma_{n,a}(\mathcal{Q})$ of finite type, which is a fine moduli space for triples (P, τ, π) where (P, τ) are as in Theorem 6.1.1 and π is a B-structure in $P|_X$ of degree a.*

The motivic Eisenstein series corresponding to $Y, \widehat{G}$ and $\mathcal{P}$ is then defined to be

$$E_{\mathcal{G},\mathcal{Q}}(q,z,v) = \sum_{n\in\mathbb{Z}, a\in L} \mu(\Gamma_{n,a}(\mathcal{Q}))q^n z^a v^d, \quad d = -X_Y^2. \tag{6.1.3}$$

As before, we will write $t \in T_{\mathrm{aff}}^\vee$ for (q, z, v).

(6.1.4) Proposition. *Let S be a smooth projective surface, $X \subset S$ be a smooth curve, P° be a G-bundle on $S - X$ and Y be the formal neighborhood of X in S. Fix some c_2-theory C in $P^\circ|_{Y^\circ}$ and let $\mathcal{Q}$ be the corresponding $\mathcal{G}_X$-torsor. Then the generating function $E_{G,P^\circ}(q,z,v)$ from (3.1.1) differs from $E_{\mathcal{G},\mathcal{Q}}(q,z,v)$ by a factor of q^m for some m.*

Proof: By Proposition 5.2.4, there is a bijection between extensions of P° to a bundle on S and extensions of $P^\circ|_{Y^\circ}$ to a bundle on Y. So unless both E_{G,P° and $E_{\widehat{G},\mathcal{P}}$ are both identically zero, there exists an extension P of P° to a bundle on S. Then, P gives rise to a canonical c_2-theory normalized so that its value on $P|_Y$ is $\mathcal{O}_X$ and thus to a $\mathcal{G}_X$-torsor $\mathcal{Q}'$.

It follows that $E_{\mathcal{G},\mathcal{Q}'}(q,z,v) = q^{-c_2(P)}E_{G,P^\circ}(q,z,v)$, where $c_2(P)$ is the usual second Chern class of P on S. Further, $E_{\mathcal{G},\mathcal{Q}}(q,z,v)$ and $E_{\mathcal{G},\mathcal{Q}'}(q,z,v)$ differ by a factor of a power of q, since the concepts of $c_2 \in \mathbb{Z}$ for G-bundles on Y defined by $\mathcal{Q}$ and $\mathcal{Q}'$, differ by a constant, namely, by deg $C(P)$. Proposition is proved.

In view of the above proposition, Theorems 3.4.1 and 3.4.4 would follow from the next fact.

(6.1.5) Theorem. *For any Y with $X_Y^2 = -d < 0$, any G and any $\mathcal{G}_X$-torsor $\mathcal{Q}$ as above, the motivic Eisenstein series $E_{\mathcal{G},\mathcal{Q}}(q,z,v)$ satisfies the properties claimed in Theorems 3.4.1 and 3.4.4.*

(6.2) Proof of Theorem 6.1.5. Let $D(t)$ be as in (3.4.2) and $N_{\mathcal{G},\mathcal{Q}}(t) = E_{\mathcal{G},\mathcal{Q}}(t)D(t)$, $t = (q, z, v)$. Note that

$$E_{\mathcal{G},\mathcal{Q}}(q,z,v) = v^d \sum_n q^n \int_{P\in\Gamma_n(\mathcal{Q})} E_{G,P|_X}(z)d\mu_P, \tag{6.2.1}$$

we find that for any n the coefficient at $q^n v^d$ in $E_{\mathcal{G},\mathcal{Q}}$ is a series in z whose support (a subset in L) lies is some translation of the cone of dominant coweights. This implies that $N_{\mathcal{G},\mathcal{Q}}(t)$ is well-defined as formal series. Further, the desired functional equation for $N_{\mathcal{G},\mathcal{Q}}(t)$ in the case $w \in W \subset \widehat{W}$ follows from (6.2.1) and Theorem 4.1.4. So it is enough to consider the case $w = s_{\alpha_0}$, where α_0 is the new simple affine root. This is parallel to the proof of Lemma 4.3.1 except that we have to deal with parahoric subgroups in $G((\lambda))$ instead of parabolic subgroups in G.

Let $\Pi_0 \subset G((\lambda))$ be the parahoric subgroup corresponding to the negative affine root $(-\alpha_0)$ and $\underline{\Pi}'_X \subset \underline{G}'_X$ be the preimage of $\underline{\Pi}_{0,X} \subset \underline{G((\lambda))}_X$. Set further $\mathcal{P} = \underline{\Pi}'_X \cdot \mathcal{A}ut(X[[\lambda]]) \subset \mathcal{G}_X$.

Let $M_0 \subset \Pi_0$ be the standard Levi subgroup. We have then the projection $\phi_0 : \Pi_0 \to M_0$. Let $\underline{M}'_X \subset \underline{G}'_X$ be the preimage of $\underline{M}_{0,X}$ and $\underline{M_X} = \underline{M}'_X \cdot \mathcal{O}^*_X$, where $\mathcal{O}^*_X$ is regarded as the subgroup in $\mathcal{A}ut(X[[\lambda]])$ formed by the automorphisms $\lambda \mapsto f(x) \cdot \lambda$, $f(x) \in \mathcal{O}^*_X$. The projection ϕ_0 together with the homomorphism (5.4.3) induce a homomorphism $\phi : \mathcal{P} \to \underline{M}_X$.

Note that the sheaf of groups $\underline{M}_X$ is in fact formed by regular functions with values in a reductive algebraic group M over k, of semisimple rank 1, with maximal torus T_{aff}, root system $\{\pm\alpha_0\}$ and Weyl group $\{1, s_{\alpha_0}\}$. Let $B' \subset M$ be the standard Borel subgroup, so $\underline{B}'_X = \underline{M}_X \cap \mathcal{B}_X$.

(6.2.2) Lemma. *let $\mathcal{Q}$ be a $\mathcal{G}_X$-torsor. Then, defining a $\mathcal{B}_X$-structure in $\mathcal{Q}$ is equivalent to, first, defining a $\mathcal{P}$-structure, say, ϖ and then, defining a $\underline{B}'_X$-structure in the $\underline{M}_X$-torsor $\phi_*(\mathcal{Q}, \varpi)$.*

Proof: Follows from the equalities

$$\mathcal{P} = \underline{M}_X \cdot \mathcal{B}_X, \quad \underline{B}'_X = \underline{M}_X \cap \mathcal{B}_X.$$

Let $L'_{\text{aff}} = \text{Hom}(\mathbb{G}_m, M^{\text{ab}})$, so that we have a homomorphism $L_{\text{aff}} \to L'_{\text{aff}}$ with kernel isomorphic to $\mathbb{Z}$. To finish the proof of the functional equation of $N_{\mathcal{G},\mathcal{Q}}(t)$ for $w = s_{\alpha_0}$ in the manner exactly identical to the proof of Lemma 4.3.1, it is enough to establish the next proposition.

(6.2.3) Proposition. *Let $b' \in \overline{L}'$. Then there exists a scheme $\Gamma^{\mathcal{P}}_{b'}(\mathcal{Q})$ of finite type over k, parametrizing $\mathcal{P}$-structures in $\mathcal{Q}$ of degree b'.*

Proof: Let $w \in \widehat{W}$ be an element taking α_0 to a simple non-affine root $\alpha \in \Delta_{\text{sim}}$ and $f \in G((\lambda))$ be a representative of w. Then the conjugation with f takes Π_0 into the parahoric subgroup Π_α corresponding to $(-\alpha)$. More precisely,

$$\Pi_\alpha = \big\{ g(\lambda) \in G[[\lambda]] \;\big| g(0) \in P_\alpha) \big\},$$

where $P_\alpha \subset G$ is the parabolic subgroup corresponding to $(-\alpha)$, see the proof of Lemma 4.3.1 from which we borrow other notations as well. Let $\mathcal{P}_\alpha$ be the sheaf of groups on X constructed from Π_α int he same was as $\mathcal{P}$ was constructed from Π_0: we first lift $\underline{\Pi}_{\alpha,X}$ to $\underline{G}'_X$ and then multiply with $\mathcal{A}ut(X[[\lambda]])$. Degrees of $\mathcal{P}_\alpha$-torsors lie therefore in $L^\alpha_{\text{aff}} := \mathbb{Z} \oplus L_\alpha \oplus \mathbb{Z}$. Since $\mathcal{P}$- and $\mathcal{P}_\alpha$-structures in $\mathcal{Q}$ are in bijection (via conjugation with f), it is enough to show that for any $b'' = (n, b, -d) \in L^\alpha_{\text{aff}}$, there exists a scheme of finite type $\Gamma^{\mathcal{P}_\alpha}_{b''}(\mathcal{Q})$ parametrizing $\mathcal{P}_\alpha$-structures in $\mathcal{Q}$ of degree b''. But this is trivial: $\Gamma^{\mathcal{P}_\alpha}_{b''}(\mathcal{Q})$ is fibered over the scheme $M_{G,P^\circ}(n)$ with the fiber over (P, τ) being the scheme $\Gamma^{P_\alpha}_{G,P|_X}$ of P_α-structures in $P|_X$ of degree b.

This finishes the proof of Theorem 6.1.5 and thus of Theorems 4.1.2 and 4.1.4.

§7. Explicit calculations for $X = \mathbb{P}^1$.

In this section we assume that $X = \mathbb{P}^1$. Our aim is to classify all $\mathcal{G}_X$-torsors whose associated ribbons have negative self-intersection index and, for each such torsor, to find the corresponding Eisenstein series completely. Our method can be seen as a version of Langlands' calculation of the constant term of the Eisenstein series, but applied to the case of Kac-Moody groups and to the framework of motivic measures.

(7.1) Grothendieck's theorem for $\mathcal{G}_X$-torsors. Recall that we have the following homomorphisms of sheaves of groups on $X = \mathbb{P}^1$:

$$\underline{T}_{\mathrm{aff},X} \hookrightarrow \mathcal{G}_X \to \mathcal{A}ut(X[[\lambda]]) \overset{(5.4.3)}{\longrightarrow} \mathcal{O}_X^*. \tag{7.1.1}$$

If $\mathcal{Q}$ is a $\mathcal{G}_X$-torsor and (Y, P°, C) are the data corresponding to $\mathcal{Q}$ by Proposition 5.4.2, then the image of the class of $\mathcal{Q}$ under the map $H^1(X, \mathcal{G}_X) \to H^1(X, \mathcal{O}_X^*)$ is the class of the normal bundle $N_{X/Y}$. We will say that $\mathcal{Q}$ is of negative index, if $N_{X/Y}$ has negative degree, so $N_{X/Y} = \mathcal{O}(-d)$, $d > 0$. Since $X = \mathbb{P}^1$, isomorphism classes of T_{aff}-bundles on X are in bijection with $L_{\mathrm{aff}} = \mathrm{Hom}(\mathbb{G}_m, T_{\mathrm{aff}})$. For $a \in L_{\mathrm{aff}}$ be denote by $\mathcal{O}^*(a)$ the corresponding T_{aff}-bundle on X. By $\mathcal{O}^*(a)_{\mathcal{G}}$ we will mean the $\mathcal{G}_X$-torsor induced from $\mathcal{O}^*(a)$ by the first homomorphism in (7.1.1).

The following fact can be seen as an analog of the theorem of Grothendieck [Gro] for Kac-Moody groups.

(7.1.2) Theorem. *Suppose k is algebraically closed. A $\mathcal{G}_X$-torsor $\mathcal{Q}$ whose index is negative, has the form $\mathcal{O}^*(b)_{\mathcal{G}}$, where $b \in L_{\mathrm{aff}}$ is an antidominant coweight, which is uniquely defined by $\mathcal{Q}$.*

Proof: Let (Y, P°, C) be the the data corresponding to $\mathcal{Q}$ by Proposition (5.4.2). We start by identifying Y.

(7.1.3) Lemma. *Any ribbon Y on $X = \mathbb{P}^1$ such that X_Y^2 is negative has a linear structure, i.e., is isomorphic to the formal neighborhood of X in the total space of some line bundle N on X (which is identified with $N_{X/Y}$).*

Proof: An automorphism of $X[[\lambda]]$ is the same as an algebra automorphism of $\mathcal{O}_X[[\lambda]]$ which is continuous in the λ-adic topology and is identical on $\mathcal{O}_X$. Such an automorphism is uniquely determined by its values on $\mathcal{O}_X$ and λ which have the form

$$a \mapsto a + \sum_{i=1}^{\infty} D_i(a)\lambda^i, \quad \lambda \to \lambda + \sum_{i=1}^{\infty} b_i \lambda_i, \tag{7.1.4}$$

where $D_i : \mathcal{O}_X \to \mathcal{O}_X$ are morphisms of sheaves of vector spaces and b_i are functions such that b_1 is invertible.

The only constraint on these data is that the first correspondence in (7.1.4) defines an algebra homomorphism $\mathcal{O}_X \to \mathcal{O}_X[[\lambda]]$. This gives a set of Leibniz-type conditions on the D_i which are sometimes expressed by saying that (D_i) forms a higher derivation of $\mathcal{O}_X$, see [Ma3], §27. In particular, D_1 is a derivation (vector field) in the usual sense. Further, if $D_1 = ... = D_{i-1} = 0$, then D_i is a section of a line bundle on X, namely the ith tensor power of the tangent bundle $T_{\mathbb{P}^1} = \mathcal{O}(2)$. For $i \geq 1, j \geq 2$ let F^{ij} be the sheaf of subgroup in $\mathrm{Ker}(\phi)$ defined by $D_1, ..., D_{i-1} = 0$, $b_2, ..., b_{j-1} = 0$. This is a decreasing filtration by normal subgroups with the quotients being Abelian and, more precisely, $\mathrm{gr}_F^{ij} = \mathcal{O}(2i)$. Now, $\mathcal{A}ut(X[[\lambda]])$ is a semidirect product of $\mathcal{O}_X^*$ and $\mathrm{Ker}(\phi)$. The action of $\mathcal{O}_X^*$ on the line bundle gr_F^{ij} induced by the conjugation is via the homomorphism $\mathcal{O}_X^* \to \mathcal{O}_X^*$ given by raising to the $(i+j)$th power. Our lemma says that the preimage in $H^1(X, \mathcal{A}ut(X[[\lambda]]))$ of the class of $\mathcal{O}(-d)$ in $H^1(X, \mathcal{O}^*)$ consists of one element, if $d > 0$. To see this, we use the filtration of $\mathcal{A}ut(X[[\lambda]])$ formed by $G^{ij} = F^{ij} \rtimes \mathcal{O}_X^*$ and find that possible liftings from $\mathcal{A}ut(X[[\lambda]]/G^{ij}$ to $\mathcal{A}ut(X[[\lambda]])/G^{i+1,j}G^{i,j+1}$ form a homogeneous space over $H^1(X, 2i + d(i+j)) = 0$. QED.

We denote by Y_d the formal neighborhood of X in the total space of the line bundle $\mathcal{O}(-d)$. Thus we have a projection $p_d : Y_d^\circ \to X$. Note that we have an isomorphism of line bundles $p_d^* \mathcal{O}(d) \to \mathcal{O}_{Y_d^\circ}$ on Y_d°.

(7.1.5) Lemma. *The Picard group of Y_d° is identified with $\mathbb{Z}/d$ and consists of line bundles $p_d^*\mathcal{O}(i)$, where i is taken modulo d.*

Proof: Consider first the case $d = 1$. Then we have the blow-down morphism $\sigma : Y_1 \to D$ where $D = \mathrm{Spec}\ k[[x, y]]$ is the 2-dimensional formal disk. Let $D^\circ = D - \{0\}$ be the punctured formal disk and $j : D^\circ \hookrightarrow D$ the embedding. Since σ is an isomorphism outside X, we find that a line bundle on Y_1° is the same as a line bundle on D°. But we have the following fact which expresses the well known property that a reflexive sheaf on a smooth surface is a vector bundle.

(7.1.6) Lemma. *If V° is any vector bundle on D°, then $j_* V^*$ is a vector bundle on D. In particular (since any vector bundle on D is trivial), V° is trivial.*

Thus the case $d = 1$ is clear. If d is arbitrary, consider the morphism of the total space of $\mathcal{O}(-1)$ to the total space of $\mathcal{O}(-d) = \mathcal{O}(-1)^{\otimes d}$ given at each fiber by $v \mapsto v \otimes ... \otimes v$ (d times). Let ϕ be the induced morphism $Y_1^\circ \to Y_d^\circ$. This is a Galois covering with the Galois group being the group scheme μ_d of dth roots of unity. If $\mathcal{L}$ is a line bundle on Y_d°, then $\phi^*\mathcal{L}$ is trivial, so $\mathcal{L}$ can be defined, by descent, by defining a μ_d-action in the trivial bundle on Y_1°. By doing so, we get exactly the bundles $p_d^*\mathcal{O}(i)$. QED

To finish the proof of Theorem 7.1.2 it is enough to establish the following fact.

(7.1.7) Proposition. *Let $d > 0$. Then the following sets are naturally identified:*

(i) Isomorphism classes of G-bundles on Y_d°.

(ii) Conjugacy classes of homomorphisms $\mu_d \to G$.

(iii) Dominant affine coweights of G of the form $(0, a, d) \in L_{\mathrm{aff}} = \mathbb{Z} \oplus L \oplus \mathbb{Z}$.

(7.1.8) Remark. Note that the definition of the set (iii) can be phrased in a more suggestive form: as the set of isomorphism classes of level d irreducible projective representations of the loop group $G^L((\lambda))$ where G^L is the Langlands dual group of G. Later in this section we will interpret the Eisenstein-Kac-Moody series corresponding to Y_d and a G-bundle P° on Y_d° as a deformation of the character of the representation corresponding, by the above, to P°. A natural problem is then to construct the representation itself in some algebro-geometric terms. Note the similarity of our situation with the framework of Nakajima [N]: the weight of an irreducible representations is in both cases encoded in the topological type of a bundle on some open part of a variety (neighborhood of the infinity on an ALE space in Nakajima's case, the complement of X in Y_d in our situation).

Proof of (7.1.7): If $d = 1$, all three sets consist of one element, so the statement is true. If $d > 1$, we consider the morphism $\phi_d : Y_1^\circ \to Y_d^\circ$ introduced in the proof of Lemma 7.1.5. Let P° be a G-bundle on Y_d° Then $\phi_d^* P^\circ$ is trivial, as can be seen by applying Lemma 7.1.6 to any vector bundle on Y_d° associated to P° and a representation of G. Thus P° is obtained, by descent, by defining a μ_d-action in the trivial G-bundle on Y_1°. Isomorphism classes of such descent data form exactly the set (ii). Further, any homomorphism $\mu_d \to G$ is factored through T. (See [W] for representation theory of μ_d in finite characteristic which is the same as representation theory of $\mathbb{Z}/d$ in characteristic 0.) This shows that P° has a T-structure. The identification of conjugacy classes (with respect to G or W) of such T-structures, or, what is the same, of the set (ii), with (iii), is straightforward (compare with Kac's classification of automorphisms of finite order of a semisimple Lie algebra [Ka]).

(7.2) The affine Hall polynomials. Let A be a field and $l \in A$ be a nonzero element. We introduce the twisted $\widehat{W}$-action on $T^\vee_{\mathrm{aff},A}$ given by

$$w * t = l^{-\widehat{\rho}} w(l^{\widehat{\rho}} \cdot t), \tag{7.2.1}$$

so that, for example, for $\alpha \in \widehat{\Delta}_{\mathrm{sim}}$ we have $s_\alpha * t = l^{-\alpha} s_\alpha(t)$. Set

$$K(t; l) = \prod_{\alpha \in \widehat{\Delta}_+} \frac{1 - l t^{\alpha^\vee}}{1 - t^{\alpha^\vee}}. \tag{7.2.2}$$

We consider this as a power series in $t = (q, z, v) \in T^\vee_{\mathrm{aff}} = \mathbb{G}_m \times T^\vee \times \mathbb{G}_m$. As in (3.3), we can consider K as a function on $(T^\vee \times \mathbb{G}_m)_{A((q))}$ and as such, it is a meromorphic function.

For an antidominant $b \in L_{\mathrm{aff}}$ we define the affine Hall polynomial to be

$$P_b(t; l) = \sum_{w \in \widehat{W}} w * (t^b K(t)), \quad t \in T_{\mathrm{aff}}. \tag{7.2.3}$$

This definition is entirely similar to the definition of Hall polynomials for reductive groups as given by Macdonald [Mac]. Unlike the finite-dimensional theory, $P_b(t, l)$ is not a polynomial in t any more; however, the following fact holds.

(7.2.4) Proposition. *(a) $P_b(t;l)$ is an analytic function on the rigid analytic space $(T^\vee \times \mathbb{G}_m)^{\mathrm{an}}_{A((q))}$.*

(b) $P_b(t;1)$ is the monomial symmetric function (the result of averaging of t^b over $\widehat{W}$).

(c) If $k = \mathbb{C}$, then $P_b(t;0) = \chi_{-b}(t^{-1})$ where χ_b is the character of the irreducible representation of the Kac-Moody Lie algebra (associated to G^L) with highest weight $(-b)$.

Proof: (a) follows because the denominator of $K(t;l)$ is a $\widehat{W}$-antisymmetric function. Part (b) is obvious, which (c) is just the Weyl-Kac character formula.

(7.2.5) Remark. As shown by Macdonald, see [Mac], any finite root system gives rise to a 2-parameter family of polynomials on the maximal torus invariant with respect to the Weyl group. These include, as special cases, Hall polynomials and Jack polynomials. The extension of the full 2-variable Macdonald theory to affine root systems has not yet been developed. The affine analogs of Jack polynomials have been studied by Etingof and Kirillov [EK].

For future reference we will need the following fact.

(7.2.6) Proposition. *We have the equality*

$$P_b(t;l) = K(t;l) \sum_{w \in \widehat{W}} l^{l(w)} \big(l^{-\hat{\rho}} w(l^{\hat{\rho}} t)\big)^b \prod_{\alpha \in \widehat{\Delta}_+ \cap w^{-1}(\widehat{\Delta}_-)} \frac{1 - t^{\alpha^\vee}}{1 - l^2 t^{\alpha^\vee}}.$$

Proof: Let us, for simplicity, drop l from the notation for P_b and K. What we need to show is that for any $w \in \widehat{W}$ the wth summand in the right hand side of the proposed equality is equal to $K(t)^{-1} w * (t^b K(t))$. We will work out here the case $w = s_\alpha$, $\alpha \in \widehat{\Delta}_{\mathrm{sim}}$, the general case being similar. In this case we claim that

$$l \frac{1 - t^{\alpha^\vee}}{1 - l^2 t^{\alpha^\vee}} = \frac{s_\alpha * \kappa(t)}{\kappa(t)}, \quad \kappa(t) = \frac{1 - l t^{\alpha^\vee}}{1 - t^{\alpha^\vee}}.$$

But this is elementary: by using the equality $\langle \alpha, \alpha^\vee \rangle = 2$, we see that

$$s_\alpha * \kappa(t) = \frac{1 - l(l^{-\alpha} s_\alpha t)^{\alpha^\vee}}{1 - (l^{-\alpha} s_\alpha t)^{\alpha^\vee}} = \frac{1 - l^{-1} t^{-\alpha^\vee}}{1 - l^{-2} t^{-\alpha^\vee}} =$$

$$= \frac{t^{\alpha^\vee} - l^{-1}}{t^{\alpha^\vee} - l^{-2}} = \frac{1 - l t^{\alpha^\vee}}{l^{-1}(1 - l^2 t^{\alpha^\vee})}.$$

(7.3) The identification of the Eisenstein series. Let $\mathcal{Q}$ be a $\mathcal{G}_X$-torsor of negative index of the form $\mathcal{O}^*(b)_{\mathcal{G}}$, where $b \in L_{\mathrm{aff}}$ is an antidominant affine coweight. When k is algebraically closed, any torsor is of this form, by Theorem 7.1.2. We assume fixed a field A and an A-valued motivic measure μ on Sch_k; in particular, $\mathbb{L} = \mu(A^1)$. The second main result of this paper is the following.

(7.3.1) Theorem. *The Eisenstein series $E_{\mathcal{G},\mathcal{Q}}(t)$ associated to $\mathcal{Q}$, is equal to $K(t;\mathbb{L})^{-1}P_b(t;\mathbb{L})$.*

Proof: By our assumptions, $b = (m, a, -d) \in L_{\mathrm{aff}} = \mathbb{Z} \oplus L \oplus \mathbb{Z}$, with $d > 0$. In other words, in the data (Y, P°, C) corresponding to $\mathcal{Q}$ by Proposition 5.4.1(a), the ribbon Y is Y_d, the formal neighbohrood of $X = \mathbb{P}^1$ in the total space of $\mathcal{O}(-d)$. Thus, by (5.4.1)(b), the torsor $\mathcal{Q}$ comes from a $\widehat{G}_X$-torsor which we denote Q. Let $\widehat{F} = G((\lambda))/I$ be the affine flag variety of G. It follows that we have a fibration $\widehat{\mathcal{F}}$ over X with fibers isomorphic to $\widehat{F}$ and $\mathcal{B}$-structures in $\mathcal{Q}$ are just sections of this fibration.

Recall (affine Bruhat decomposition) that for any field $\mathfrak{k}$, any two $\mathfrak{k}$-points (f, f') on $\widehat{F}$ have a uniquely defined relative position $\delta(f, f') \in \widehat{W}$. Given f, we define the affine Schubert cell $U_w(f)$ to consist of f' with $\delta(f, f') = w$. This is an algebraic variety over $\mathfrak{k}$, isomorphic to the affine space of dimension $l(w)$.

Take $\mathfrak{k} = k(X)$, the field of rational functions on X. The fibration $\widehat{\mathcal{F}}$ is trivial over $\mathfrak{k}$, because the bundle P° on Y_d° is trivial over the punctured formal neighborhood of A^1 in Y_d. As $\mathcal{Q}$ comes from a T_{aff}-bundle on X, it comes equipped with a canonical $\mathcal{B}$-structure π_0 which we regard as a distinguished section of $\widehat{\mathcal{F}}$. Recall that the Eisenstein series associated to $\mathcal{Q}$ is

$$E_{\mathcal{G},\mathcal{Q}}(t) = \sum_{a \in L_{\mathrm{aff}}} \mu(\Gamma_a(\mathcal{Q}))t^a,$$

where $\Gamma_a(\mathcal{Q})$ is the scheme of $\mathcal{B}$-structures in $\mathcal{Q}$ of degree a. Consider the stratification

$$\Gamma_a(\mathcal{Q}) = \coprod_{w \in \widehat{W}} \Gamma_a^w(\mathcal{Q}),$$

where $\Gamma_a(\mathcal{Q})$ consists of those $\mathcal{B}$-structures in $\mathcal{Q}$ which, being regarded as sections of $\widehat{\mathcal{F}}$ are, over the generic point of X (i.e., over $\mathfrak{k}$), in relative position w with π_0. Then we have

$$E_{\mathcal{G},\mathcal{Q}}(t) = \sum_{w \in \widehat{W}} E^w(t), \quad E^w(t) := \sum_a \mu(\Gamma_a^w(\mathcal{Q}))t^a.$$

We proceed to find each $E^w(t)$ separately.

Consider a point $x \in X$ and let $\widehat{\mathcal{F}}_x$ be the fiber of $\widehat{\mathcal{F}}$ over x. It contains a distinguished point $\pi_{0,x}$, the value of π_0 at x. Therefore we have the Schubert cell $U_w(\pi_{0,x}) \subset \widehat{\mathcal{F}}_x$.

(7.3.2) Lemma. *The Bruhat decomposition gives a canonical identification of algebraic varieties*

$$U_w(\pi_{0,x}) \longrightarrow \bigoplus_{\alpha \in \widehat{\Delta}_+ \cap w^{-1}(\widehat{\Delta}_-)} \mathcal{O}_{P^1}(-\langle \alpha, b\rangle)_x,$$

where $\mathcal{O}_{P^1}(-\langle \alpha, b\rangle)_x$ is the fiber at x of the line bundle $\mathcal{O}_{P^1}(-\langle \alpha, b\rangle)$.

Proof: An exercise in Bruhat decomposition. Left to the reader.

(7.3.3) Corollary. *We have an identification of sets*

$$\coprod_{a\in L_{\mathrm{aff}}} \Gamma^w_a(\mathcal{Q}) \longrightarrow \prod_{\alpha\in\widehat{\Delta}_+\cap w^{-1}(\widehat{\Delta}_-)} \Gamma_{\mathrm{rat}}(X, \mathcal{O}(-\langle\alpha, b\rangle)),$$

where Γ_{rat} *denotes the space of rational sections of a line bundle.*

The next statement can be seen as a geometric analog of the Gindikin-Karpelevic formula which is the the crucial ingredient in evaluation of the constant term of the Eisenstein series [Lan].

(7.3.4) Proposition. *Let* f_α*,* $\alpha \in \widehat{\Delta}_+\cap w^{-1}(\widehat{\Delta}_-)$*, be rational sections of* $\mathcal{O}_{P^1}(-\langle\alpha, b\rangle)$ *and let* D_α *be the divisor of poles of* f_α*. Then the degree of the* $\mathcal{B}$*-structure corresponding to the system* (f_α) *by Corollary 7.3.3, is equal to*

$$w(b) + \sum_{\alpha\in\widehat{\Delta}_+\cap w^{-1}(\widehat{\Delta}_-)} \deg(D_\alpha)\cdot\alpha^\vee.$$

Proof: This can be viewed as a statement about two arbitrary sections π, π' of $\widehat{\mathcal{F}}$ in a generic relative position w: it expresses how their degrees are related if we use the identification of the Schubert cell $U_w(\pi)$ given by the Bruhat decomposition. Take a reduced decomposition $w = s_{\alpha_1}...s_{\alpha_l}$, $l = l(w)$, $\alpha_i \in \widehat{\Delta}_{\mathrm{sim}}$ and choose intermediate sections $\pi = \pi_0, \pi_1, ..., \pi_l = \pi'$ of $\widehat{\mathcal{F}}$ so that generically $\delta(\pi_i, \pi_{i+1}) = s_{\alpha_i}$ (this is possible because the fibration is rationally trivial). We find that it is enough to prove the required statement only for (π_i, π_{i+1}) for each i. In other words, Proposition 7.3.4 in full generality follows from the particular case of simple reflection $w = s_\alpha$ which we now assume. This case, however, reduces to the case of two sections of a flag fibration corresponding to the group GL_2.

In other words, we need to consider a rank 2 vector bundle $V = \mathcal{O}(b_1)\oplus\mathcal{O}(b_2)$ on $X = \mathbb{P}^1$, and two sections rank 1 subbundles in V, namely $V_0 = \mathcal{O}(d_1)$ and V_1 generically transversal to $\mathcal{O}(d_1)$. The identification of Corollary 7.3.3 in this case means simply that V_1 is the graph of a rational morphism $\phi : \mathcal{O}(d_2) \to \mathcal{O}(d_1)$ and the required particular case of Proposition 7.3.4 is:

(7.3.5) Lemma. *If* D *is the divisor of poles of* ϕ*, then the subbundle* V_1 *corresponding to the graph of* ϕ *has degree* $d_2 - \deg(D)$*.*

Proof: We consider the projection $V_1 \to \mathcal{O}(d_2)$. This is a morphism of line bundles with the divisor of zeroes being exactly D, whence the statement.

(7.3.6) Definition. *If* $\mathcal{L}$ *is a line bundle on* X *and* $D' \le D$ *are positive divisors on* X*, then we denote by* $\Gamma^{D'}(X, \mathcal{L}(D))$ *the open subvariety in the affine space* $\Gamma(X, \mathcal{L}(D))$ *formed by those sections whose divisor of poles (in the sense of meromorphic sections of* $\mathcal{L}$*) is equal to* D'*. We introduce the following generating function*

$$\psi_{\mathcal{L}}(u) = \sum_{n\ge 0}\int_{D\in X^{(n)}} \mu(\Gamma^D(X, \mathcal{L}((D)))u^d \quad \in \quad A[[u]].$$

The geometric Gindikin-Karpelevic formula implies at once:

(7.3.7) Corollary. *We have the equality*

$$E^w(t) = z^{w(b)} \prod_{\alpha\in\widehat{\Delta}_+\cap w^{-1}(\widehat{\Delta}_-)} \psi_{\mathcal{O}(-\langle b,\alpha\rangle)}(t^{\alpha^\vee}).$$

In order to finish the proof of Theorem 7.3.1 it is enough, by the above corollary and Proposition 7.2.6, to establish the following fact.

(7.3.8) Proposition. *If* $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(m)$, $m \geq 0$, *then*

$$\psi_{\mathcal{L}}(u) = \mathbb{L}^{m+1}\frac{\zeta(\mathbb{L}u)}{\zeta(u)}, \quad \textit{where} \quad \zeta(u) = \frac{1}{(1-u)(1-\mathbb{L}u)}$$

is the motivic zeta-function of $\mathbb{P}^1$.

Indeed, we apply (7.3.8) to $m = -\langle b, \alpha\rangle$ (which is nonnegative since b is antidominant) and notice that

$$\frac{\zeta(\mathbb{L}u)}{\zeta(u)} = \frac{1-u}{1-\mathbb{L}^2u},$$

and that Proposition 7.2.6 identifies $P_b(k;\mathbb{L})$ with the sum of products of factors of exactly this kind.

Proof of (7.3.8): Note that $\Gamma(X,\mathcal{L}(D)) = \bigcup_{D'\leq D}\Gamma^{D'}(X,\mathcal{L}(D))$. Now let us write

$$\zeta(u)\psi_{\mathcal{L}}(u) = \left(\sum_{n'\geq 0} u^{n'}\int_{D'\in X^{(n')}} d\mu\right)\left(\sum_{n''\geq 0}\int_{D''\in X^{(n'')}} \mu(\Gamma^{D''}(X,\mathcal{L}(D'')))d\mu\right)$$

$$= \sum_{n\geq 0} u^n \int_{D\in X^{(n)}} \mu(\Gamma(X,\mathcal{L}(D)))d\mu.$$

Because $X = \mathbb{P}^1$ and $\mathcal{L} = \mathcal{O}(m)$ with $m \geq 0$, we find that $\dim(\Gamma(X,\mathcal{L}(D))) = m+n+1$, so the above sum is equal to

$$\sum_{n\geq 0} u^n \mu(X^{(n)})\mathbb{L}^{m+n+1} = \mathbb{L}^{m+1}\zeta(\mathbb{L}u).$$

This finishes the proof of Theorem 7.3.1.

(7.4) The parahoric Eisenstein series and the universal blowup functions. We start with general definitions which make sense for any smooth projective curve X. Let $\mathcal{Q}$ be a $\mathcal{G}_X$-torsor and (Y, P°, C) be the data corresponding to $\mathcal{Q}$ by Proposition 5.4.1(a). The parahoric Eisenstein series associated to $\mathcal{Q}$ is the generating function

$$F_{\mathcal{G},\mathcal{Q}}(q) = \sum_{n\in\mathbb{Z}} \mu(\Gamma_a(\mathcal{Q}))q^n, \tag{7.4.1}$$

where $\Gamma_a(\mathcal{Q})$ is the scheme from Theorem 6.1.1. This is a ribbon analog of the generating function $F_{G,P^\circ}(q)$ from (2.4.1). In fact, we have the following analog of Proposition 6.1.4, which is proved in the same way.

(7.4.2) Proposition. *Let S be a smooth projective surface, $X \subset S$ be a smooth curve, P° be a G-bundle on $S - X$ and Y be the formal neighborhood of X in S. Fix some c_2-theory C in $P^\circ|_{Y^\circ}$ and let $\mathcal{Q}$ be the corresponding $\mathcal{G}_X$-torsor. Then the generating function $F_{G,P^\circ}(q)$ differs from $F_{\mathcal{G},\mathcal{Q}}(q)$ by a factor of q^m for some m.*

The function $F_{\mathcal{G},\mathcal{Q}}(q)$ is related to the Eisenstein series $E_{\mathcal{G},\mathcal{Q}}((q,z,v)$ in the same way as the parahoric subgroup $G[[\lambda]] \subset G((\lambda))$ is related to the Iwahori subgroup I. More precisely, considering the sheaf of subgroups $\mathcal{K}_X = \mathcal{O}_X^* \times \underline{G[[\lambda]]}_X \rtimes \mathcal{A}ut(X[[\lambda]])$ in $\mathcal{G}_X$, we find that the schemes $\Gamma_n(\mathcal{Q})$ parametrize $\mathcal{K}_X$-structures in $\mathcal{Q}$. The interest to this particular version of the Eisenstein series is explained by the following fact.

(7.4.3) Proposition. *Let $X = \mathbb{P}^1$ and $\mathcal{Q}$ be the $\mathcal{G}_X$-torsor corresponding, by Theorem 7.1.2, to the antidominant coweight $(0,0,-1)$. Then Then $F_{\mathcal{G},\mathcal{Q}}(q)$ is the universal blowup function for the S-duality theory with the gauge group G.*

Let us first explain the meaning of the proposition, cf. [LQ1-2]. Consider a smooth projective surface S', a point $p \in S'$ and the blow-up $\sigma : S \to S'$ with the exceptional divisor $X = \sigma^{-1}(p)$. Let H be an ample divisor on S' and $M_G^H(S', n)$ be the moduli space of H-semistable G-bundles on S' with $c_2 = n$. For $i \in \mathbb{Z}$ consider the divisor $H_i = i\sigma^* H - X$ on S. Then it is known that H_i is ample for $i \gg 0$ and further, for any given n the spaces $M_G^{H_i}(S,n)$ are identified. This space is denoted by $M_G^{H_\infty}(S,n)$. The statement of Proposition 7.4.3 is, more precisely, that

$$\sum_n \mu(M_G^{H_\infty}(S,n)) q^n = F_{\mathcal{G},\mathcal{Q}}(q) \cdot \sum_n \mu(M_G^H(S',n)) q^n. \tag{7.4.4}$$

Proof of (7.4.3-4): The data (Y, P°, C) corresponding to our choice of $\mathcal{Q}$ by Proposition 5.4.1(a), are as follows: $Y = Y_1$ is the formal neighborhood of $X = \mathbb{P}^1$ in the total space of $\mathcal{O}(-1)$. The G-bundle P° on Y° is trivial, and C is the c_2-theory on P° normalized so that its value on the trivial G-bundle on Y is $\mathcal{O}_X$.

On the other hand, identifying X with the exceptional fiber $\sigma^{-1}(p) \subset S$, we identify the formal neighborhood of X in S with $Y = Y_1$. If P is any G-bundle on S, then its restriction to Y° is trivial by Proposition 7.1.7. Thus P is glued from a G-bundle P' on S' and a bundle P'' on Y via an identification (trivialization) $\tau : P''|_{Y^\circ} \to \sigma^{-1}(P')|_{Y^\circ}$. Recalling the definition of $\Gamma_n(\mathcal{Q})$, we find that $M_G^{H_\infty}(S,n)$ is a union of strata $\Sigma_{i,j}$, $i+j = n$, such that Σ_{ij} is a (Zariski locally trivial) fibration over $M_G^H(S,j)$ with fiber $\Gamma_i(\mathcal{Q})$. On the level of generating functions this gives the equality (7.4.4).

We now proceed to give an explicit formula for $F_{\mathcal{G},\mathcal{Q}}(q)$ when $X = \mathbb{P}^1$ and $\mathcal{Q}$ corresponds to an arbitraty antidominant weight $b \in L_{\mathrm{aff}}$. For $a \in L$ we denote

$$\widehat{\Delta}(a) = \left\{ (n, \alpha \in \mathbb{Z} \times \Delta \,\middle|\, 1 \le n \le \langle \alpha, a \rangle \right\} \tag{7.4.5}$$

and set $\lambda(a) = |\widehat{\Delta}(a)|$.

(7.4.6) Theorem. *If $X = \mathbb{P}^1$ and $\mathcal{Q}$ corresponds to $b = (m, f, -d) \in \mathbb{Z} \oplus L \oplus \mathbb{Z}$, then*

$$F_{\mathcal{G},\mathcal{Q}}(q) = \sum_{a \in L} q^{\Psi(a,f) - d\Psi(a,a)/2} \prod_{(n,\alpha) \in \widehat{\Delta}(a)} \mathbb{L}^{-\langle f, \alpha \rangle + mn + 1} \frac{1 - q^n}{1 - \mathbb{L}^2 q^n}.$$

In particular, the universal blowup function ($m = f = 0, d = 1$) is equal to

$$\sum_{a \in L} q^{-\Psi(a,a)/2} \mathbb{L}^{\lambda(a)} \prod_{(n,\alpha) \in \widehat{\Delta}(a)} \frac{1 - q^n}{1 - \mathbb{L}^2 q^n}.$$

In the case when μ is given by the Euler characteristic we have $\mathbb{L} = 1$ and the last formula specializes to the theta-zero-value of Example 2.4.2.

Proof: This is proved by the same method as Theorem 7.3.1 except that we use the decomposition of the affine Grassmannian and not the affine flag variety, into Schubert cells. So we just indicate the main steps. Along with the affine flag fibration $\widehat{\mathcal{F}} \to X$ we have a fibration $\widehat{\mathcal{G}r} \to X$ with fibers isomorphic to $\widehat{\mathrm{Gr}}$. The section π_0 of $\widehat{\mathcal{F}}$ defines the decomposition of the fiber of $\widehat{\mathcal{G}r}$ over any $x \in X$, into Schubert cells. These cells are labelled by $\widehat{W}/W = L$; the cell corresponding to $a \in L$ is identified, similarly to Lemma 7.3.2, with the fiber at x of $\bigoplus_{(n,\alpha) \in \widehat{\Delta}(a)} \mathcal{O}(-\langle f, \alpha \rangle + mn)_x$. More precisely, we need to consider the roots entering the root decomposition of the nilpotent radical of the parahoric subalgebra $\mathfrak{g}[[\lambda]]$, and these are (n, α) such that $n \geq 1$ and $\alpha \in \Delta$. Then we need to consider those of such roots which are taken into negative affine roots by a considered as an element of $\widehat{W}$. These form precisely the set $\widehat{\Delta}(a)$. The parahoric analog of the geometric Gindikin-Karpelevic formula (7.3.4) says that the degree (i.e., the second Chern class) of the $\mathcal{K}_X$-structure corresponding to a family $(f_{(n,\alpha)})$, $(\alpha, n) \in \widehat{\Delta}(a)$, of rational sections of $\mathcal{O}(-\langle f, \alpha \rangle + mn)$ is equal to

$$(a \circ b)_1 + \sum_{(n,\alpha)} \deg(D_{(n,\alpha)}) \cdot n,$$

where $(a \circ b)_1$ is the first component (lying in $\mathbb{Z}$) of the result of action of $a \in \widehat{W}$ on $b = (m, f, -d) \in L_{\mathrm{aff}}$ and $D_{(n,\alpha)}$ is the divisor of poles of $f_{(n,\alpha)}$. Recalling the rule (3.3.1) describing the a-action on L_{aff}, we establish the theorem.

(7.4.7) Remark. In this paper we always consider the generating functions for the motivic invariants of the uncompactified moduli spaces. Accordingly, in the case $G = SL_2$ our blowup function differs slightly from the function obtained in [LQ1-2] where the Uhlenbeck and Gieseker compactifications are used.

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