A Remark on the Chisini Conjecture

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In this note, we establish the following consequence of Kulikov's results on the Chisini conjecture [4].

Theorem 1 *A generic ramified covering $f : S \to \mathbb{P}^2$ of degree at least* 12 *is uniquely determined by its branch curve in $\mathbb{P}^2$. In other words, the Chisini conjecture holds for generic morphisms of degree ≥ 12.*

We discuss the assumptions of this theorem and the Chisini conjecture itself in section 1. Then, in section 2, we present a proof of Theorem 1 and observe a few related applications of our approach.

1. Generic morphisms of surfaces and the Chisini conjecture. A ramified covering $f : S \to \mathbb{P}^2$ is a finite morphism of a non-singular irreducible projective surface onto the projective plane. The *branch curve* of f is defined as the set of points over which f is not étale.

A finite morphism $f : S \to \mathbb{P}^2$ of degree $\deg f \geq 3$ is said to be *generic* if the following holds:

i) the branch curve $B \subset \mathbb{P}^2$ is irreducible and has ordinary cusps and nodes only;

ii) $f^*B = 2R + C$, where the *ramification divisor* R is irreducible and non-singular, and C is reduced;

iii) $f|_R : R \to B$ is the normalization of B.

Two generic morphisms $f_1 : S_1 \to \mathbb{P}^2$ and $f_2 : S_2 \to \mathbb{P}^2$ are called equivalent if there exists an isomorphism $\varphi : S_1 \to S_2$ such that $f_1 = f_2 \circ \varphi$.

The precise assertion of Theorem 1 is therefore that two generic morphisms (of *a priori* different surfaces!) having the same branch curve are equivalent provided that at least one of them has degree ≥ 12.

The above definition of genericity is parallel to the case of Riemann surfaces. Recall that, according to Riemann and Hurwitz, a ramified covering $f : \Sigma \to \mathbb{P}^1$ is called generic if over each point in $\mathbb{P}^1$ there is at most one quadratic ramification point of f. On the other hand, Theorem 1 shows that the complex surface case is essentially rigid.

Note that every algebraic surface S admits a generic morphism $f : S \to \mathbb{P}^2$ or, in other words, can be represented as a generic ramified covering over $\mathbb{P}^2$. For instance, if $S \subset \mathbb{P}^r$, then almost every projection $\mathbb{P}^r \to \mathbb{P}^2$ yields a generic morphism $p : S \to \mathbb{P}^2$. Therefore Theorem 1 might be used to understand the moduli spaces of complex surfaces in terms of the geometry of plane curves.

The conjecture that a generic morphism of degree at least 5 is completely determined by its branch curve was proposed by O. Chisini [2], who also gave an alleged proof of this statement.

The assumption $\deg f \geq 5$ is necessary because of the following example due to Chisini and Catanese [1]. Let $B = C^*$ be the dual curve of a non-singular plane cubic C. (B is a sextic with 9 cusps). Then there exist four non-equivalent generic morphisms with branch curve B. Three of them, of degree 4, are maps from $\mathbb{P}^2$ given by projections of the Veronese surface. The fourth map, of degree 3, is the projection on $\mathbb{P}^{2*}$ of the elliptic ruled surface obtained as the preimage of C in the incidence variety $\mathbb{P}^2 \times \mathbb{P}^{2*}$. So far this example and its fiber products with ramified coverings are the only known examples of non-uniqueness of generic morphisms with a given branch curve.

In the important paper [5], Moishezon proved the Chisini conjecture for branch curves of generic projections of smooth hypersurfaces in $\mathbb{P}^3$ by introducing and analyzing the braid presentations of the fundamental group of $\mathbb{P}^2 \setminus B$.

Recently this approach was substantially developed by Vik. Kulikov [4]. He proved that, for a given branch curve B, the generic morphism is unique provided that its degree is greater than a certain function depending on the curve B (the explicit expression is given in formula (1) below). This allowed him to prove the Chisini conjecture for a wide class of generic morphism (for instance, for pluri-canonical morphisms of surfaces of general type).

Our (rather modest) contribution is that one can estimate Kulikov's function from above by using the Bogomolov–Miyaoka–Yau (BMY) inequalities.

In what follows we say that the Chisini conjecture holds for a class of generic morphisms if every morphism in this class is determined by its branch curve up to equivalence.

2. Cusps of branch curves and BMY inequalities. Consider a generic morphism $f : S \to \mathbb{P}^2$ of degree $\deg f = N$ with branch curve $B \subset \mathbb{P}^2$. Denote by $2d$ the degree of B (it is always even), by g the genus of the desingularization of B, and by c the number of cusps of B.

It was proved in [4] that the morphism $f : S \to \mathbb{P}^2$ is uniquely determined by B if

$$N > \frac{4(3d+g-1)}{2(3d+g-1)-c}. \tag{1}$$

We wish to apply the Bogomolov-Miyaoka-Yau inequality on the algebraic surface S to estimate the number of cusps of the branch curve B via g and d, which leads to an *a priori* upper bound for the right hand side of (1). To this end we shall need the following formulas (cf. Lemmas 6 and 7 in [4]) for the self-intersection of the canonical class and the topological Euler characteristic of S:

$$\begin{array}{rcl} K_S^2 & = & 9N - 9d + g - 1, \\ e(S) & = & 3N + 2(g-1) - c. \end{array} \tag{2}$$

Proof of Theorem 1. Let us assume first that S satisfies the BMY inequality

$$K_S^2 \leq 3e(S). \tag{3}$$

(This means essentially that S is not a blow-up of an irrational ruled surface of irregularity > 1.) Plugging (2) into the BMY inequality we obtain

$$9N - 9d + g - 1 \leq 9N + 6(g-1) - 3c,$$

and therefore

$$c \leq 3d + \frac{5}{3}(g-1).$$

It follows that

$$\frac{4(3d+g-1)}{2(3d+g-1)-c} \leq \frac{12d + 4(g-1)}{3d + \frac{1}{3}(g-1)} = 4 + \frac{8(g-1)}{9d + (g-1)} < 12, \tag{4}$$

which proves the Theorem in this case.

Now, if S is (a blow-up of) an irrational ruled surface, it satisfies the inequality $K_S^2 \leq 2e(S)$. (The BMY inequality used above does not follow from this because both sides may be negative.) Arguing in the same way as before, we obtain

$$c \leq -\frac{3}{2}N + \frac{9}{2}d + \frac{3}{2}(g-1) < \frac{3}{2}(3d + g - 1).$$

This gives us the (sharper) estimate

$$\frac{4(3d+g-1)}{2(3d+g-1)-c} < 8,$$

which completes the proof. $\square$

In fact, all surfaces of non-general type except $\mathbb{P}^2$ satisfy $K_S^2 \leq 2e(S)$. Moreover, by Theorem 3 in [4] the Chisini conjecture holds for generic endomorphisms of $\mathbb{P}^2$ of degree ≥ 5. Hence, the second part of our proof yields the following result.

Theorem 2 *The Chisini conjecture holds for generic morphisms of degree at least* 8 *of surfaces of non-general type.*

Returning to the case of surfaces satisfying the "general" BMY inequality (3), we see that, in certain cases, the last inequality in (4) can be improved. From the formula for K_S^2 we have

$$9d = -K_S^2 + 9N + (g-1).$$

Plugging this into (4) we deduce the following result (including by the way the case of $\mathbb{P}^2$).

Corollary. *The Chisini conjecture holds for generic morphisms* $f : S \to \mathbb{P}^2$ *of degree* ≥ 8 *such that* $K_S^2 < 9 \deg f$.

Another complementary result can be obtained under additional assumptions on the branch curve.

Corollary. *The Chisini conjecture holds for generic morphisms of degree* ≥ 8 *such that the genus of the branch curve is less than* 64.

Proof. It follows from the Riemann–Hurwitz formula applied to the preimage of a projective line that $\deg f \leq d+1$ (cf. Lemma 1 in [4]). However, if $d \geq 7$ and $g \leq 63$ (so that $9d > g-1$), then estimate (4) improves to < 8. $\square$

Remarks

1° Viktor Kulikov has also obtained the following result: the Chisini conjecture holds for generic morphisms of degree ≥ 5 whose branch curves are cuspidal, i. e. have no nodes.

2° The estimates for the number of cusps of branch curves of generic morphisms obtained above are slightly stronger than the bounds known for arbitrary curves with simple singularities. It should be noted that the best estimates so far were obtained by applying the logarithmic BMY inequality to double coverings (see [3]).

Acknowledgements. The author is deeply grateful to Viktor Kulikov for sharing his results and expertise. This paper was written during the author's stay at Max-Planck Institute für Mathematik in Bonn. It is a pleasure to acknowledge the support of this hospitable institution. The author was also supported by RFBR grant 99-01-00969.

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